09 Power Series of Riemann Zeta etc by Real & Imaginary Parts

Abstract

In this chapter, the real parts and the imaginary parts of the Dirichlet eta function, Dirichlet beta function, Riemann zeta function and hlomorphized zeta function are expanded into power series, respectively.

Preliminaries

For the purpose, Theorem 14.1.2 in " **14 Taylor Expansion by Real Part & Imaginary Part** " (A la carte) is used. If reprinted, it is as follows.

Theorem 14.1.2 (Reprint)

Suppose that a complex function $f(z)$ $(z=x+iy)$ is expanded around a real number a into a Taylor series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a)\frac{(z-a)^s}{s!}$$

Then, when the real and imaginary parts are $f_r(x,y)$, $f_i(x,y)$ respectively, the following expressions hold.

$$f_r(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$f_i(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$.

Treatment of 0^0 in *Mathematica*

In this paper, formula manipulation soft ***Mathematica*** is used for drawing and calculation. The following options are specified prior to calculation .

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Unprotect [Power] ;  Power [0,0] = 1 ;
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9.1 Maclaurin series of $\eta(z)$

Dirichlet eta function $\eta(z)$ is defined on the half plane $Re(z)>0$ as follows.

$$\eta(z) = \sum_{t=1}^{\infty}(-1)^{t-1}e^{-z\log t} = \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} +- \cdots \tag{1.0}$$

This function can be expanded into Maclaurin series as follows.

Formula 9.1.1

When the Dirichlet eta function is $\eta(z)$ $(z=x+iy)$, the following expressions hold on the half plane $Re(z)>0$

$$\eta(z) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^s(t+1)\frac{(-1)^s z^s}{s!} \tag{1.z}$$

$$Re\{\eta(z)\} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^{2r+s}(t+1)\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{1.r}$$

$$Im\{\eta(z)\} = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^{2r+s+1}(t+1)\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.i}$$

Where, $0^0 = 1$

Proof

According to Formula 26.3.2h in " **26 Higher and Super Calculus of Zeta Function etc** " (Higher Calculus & Super Calculus), when $Re(z)>0$, the higher order derivative of $\eta(z)$ was as follows.

$$\eta^{(s)}(z) = \frac{z^{-s}}{\Gamma(1-s)} + (-1)^{-s}\sum_{t=2}^{\infty}(-1)^{t-1}\frac{\log^s t}{t^z} \qquad s=0,1,2,\cdots$$

Changing the initial value of t from 2 to 1,

$$\eta^{(s)}(z) = \frac{z^{-s}}{\Gamma(1-s)} + (-1)^{-s}\sum_{t=1}^{\infty}(-1)^{t}\frac{\log^s (t+1)}{(t+1)^z} \qquad s=0,1,2,\cdots$$

If the possibility of s being a non-integer is not considered,

$$\frac{z^{-s}}{\Gamma(1-s)} = (-1)^0\frac{\log^s(0+1)}{(0+1)^z} = \begin{cases} 1 & for\ s=0 \quad (\because 0^0=1) \\ 0 & for\ s=1,2,3,\cdots \end{cases}$$

So, we can change the initial value of t from 1 to 0,

$$\eta^{(s)}(z) = (-1)^s\sum_{t=0}^{\infty}\frac{(-1)^t\log^s(t+1)}{(t+1)^z} \qquad s=0,1,2,\cdots \quad \text{where, } 0^0=1$$

Substituting $z=0$ for this,

$$\eta^{(s)}(0) = (-1)^s\sum_{t=0}^{\infty}(-1)^t\log^s(t+1) \qquad s=0,1,2,\cdots \quad \text{where, } 0^0=1 \tag{1.s}$$

Therefore, the Maclaurin series of $\eta(z)$ is

$$\eta(z) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^s(t+1)\frac{(-1)^s z^s}{s!} \tag{1.z}$$

Next, from (1.s) ,

$$\eta^{(2r+s)}(0) = (-1)^s \sum_{t=0}^{\infty}(-1)^t log^{2r+s}(t+1) \qquad r,s=0,1,2,\cdots$$

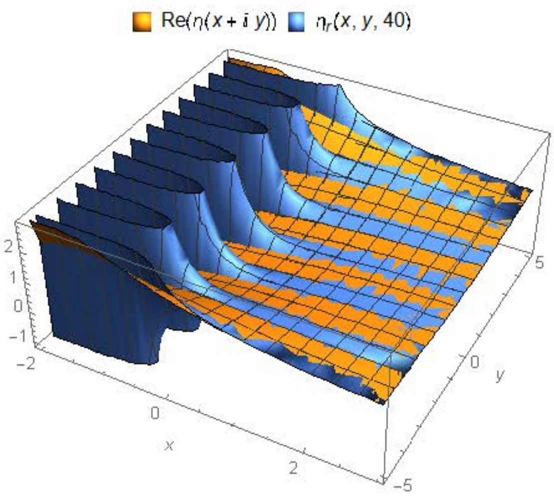
$$\eta^{(2r+s+1)}(0) = (-1)^{s+1} \sum_{t=0}^{\infty}(-1)^t log^{2r+s+1}(t+1) \qquad r,s=0,1,2,\cdots$$

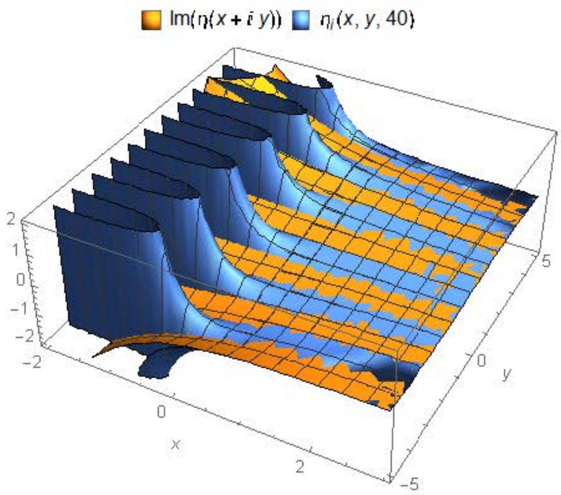
Applying Theorem 14.1.2 to these,

$$Re\{\eta(z)\} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t log^{2r+s}(t+1)\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{1.r}$$

$$Im\{\eta(z)\} = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t log^{2r+s+1}(t+1)\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.i}$$

Both sides of (1.r) and (1.i) are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. The line of convergence is seen at $x=0$.





Since these series converge very slowly, the overlapping state of both sides is not so good. In fact, the function values of (1.r) and (1.i) at point $(3,5)$ are calculated as follows.

```
N[{Re[η[3+ⅈ 5]], ηr[3, 5, 40]}]        N[{Im[η[3+ⅈ 5]], ηi[3, 5, 59]}]
     {1.13283, 1.13283}                    {-0.00976753, -0.00976591}
```

Acceleration form of Formula 9.1.1 (coefficients only)

So we try to accelerate the convergence of (1.r) and (1.i) . Acceleration of the entire triple series is the usual way, but here, only the coefficient part with the slowest convergence is accelerated by Euler Transform. It is as follows. In addition, see " **10 Convergence Acceleration & Summation Method** "(A la carte) for Euler transform.

$$\eta(z) = \sum_{s=0}^{\infty}\left\{\sum_{k=0}^{\infty}\sum_{t=0}^{k}\binom{k}{t}\frac{(-1)^t}{2^{k+1}}log^s(t+1)\right\}\frac{(-1)^s z^s}{s!}$$

$$Re\{\eta(z)\} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left\{\sum_{k=0}^{\infty}\sum_{t=0}^{k}\binom{k}{t}\frac{(-1)^t}{2^{k+1}}log^{2r+s}(t+1)\right\}\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$Im\{\eta(z)\} = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left\{\sum_{k=0}^{\infty}\sum_{t=0}^{k}\binom{k}{t}\frac{(-1)^t}{2^{k+1}}log^{2r+s+1}(t+1)\right\}\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Both sides of *Re* and *Im* are drawn as follows.

```
Unprotect[Power]; Power[0, 0] = 1;     Cm[n_, r_] := Binomial[n, r]
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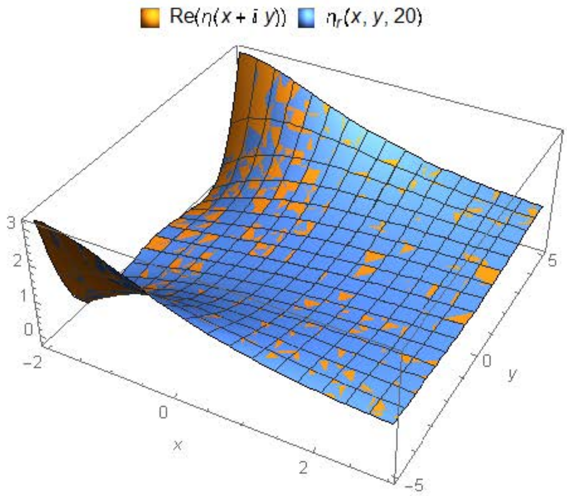
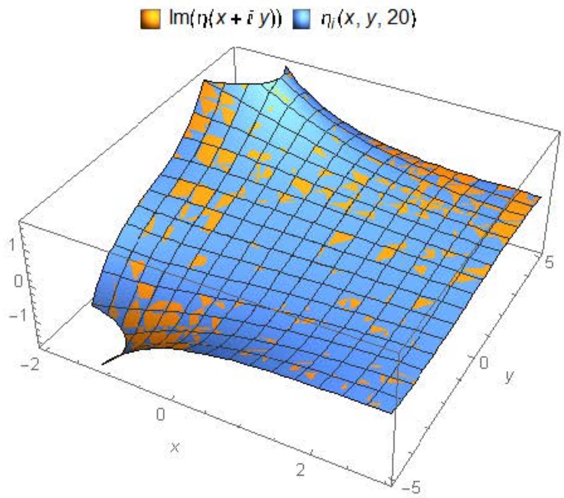
$$\eta_r[x\_, y\_, m\_] := \sum_{r=0}^{m}\sum_{s=0}^{m}\left(\sum_{k=0}^{m}\sum_{t=0}^{k}\mathrm{Cm}[k,t]\,\frac{(-1)^t}{2^{k+1}}\,(\mathrm{Log}[t+1])^{2r+s}\right)\frac{(-1)^s x^s}{s!}\,\frac{(-1)^r y^{2r}}{(2r)!}$$

$$\eta_i[x\_, y\_, m\_] := -\sum_{r=0}^{m}\sum_{s=0}^{m}\left(\sum_{k=0}^{m}\sum_{t=0}^{k}\mathrm{Cm}[k,t]\,\frac{(-1)^t}{2^{k+1}}\,(\mathrm{Log}[t+1])^{2r+s+1}\right)\frac{(-1)^s x^s}{s!}\,\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$





The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. Both sides overlap exactly and look like spots. Further, we can see that the convergence area is analytically continued from the half plane to the entire plane.

9.2 Maclaurin series of $\beta(z)$

Dirichlet beta function $\beta(z)$ is a kind of Dirichlet L-function and is defined on the half plane $Re(z) > 0$ as follows.

$$\beta(z) = \sum_{t=0}^{\infty} \frac{(-1)^t}{(2t+1)^z} = 1 - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + - \cdots \tag{2.0}$$

This function can be expanded into Maclaurin series as follows.

Formula 9.2.1

When the Dirichlet beta function is $\beta(z)$ ($z = x+iy$), the following expressions hold on the half plane $Re(z) > 0$

$$\beta(z) = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^t \log^s (2t+1) \frac{(-1)^s z^s}{s!} \tag{2.z}$$

$$Re\{\beta(z)\} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^t \log^{2r+s}(2t+1) \frac{(-1)^s x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!} \tag{2.r}$$

$$Im\{\beta(z)\} = -\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^t \log^{2r+s+1}(2t+1) \frac{(-1)^s x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{2.i}$$

Where, $0^0 = 1$

Proof

According to Formula 26.4.2h in " **26 Higher and Super Calculus of Zeta Function etc** " (Higher Calculus & Super Calculus), when $Re(z) > 0$, the higher order derivative of $\beta(z)$ was as follows.

$$\beta^{(s)}(z) = \frac{z^{-s}}{\Gamma(1-s)} + (-1)^{-s} \sum_{t=2}^{\infty} (-1)^{t-1} \frac{\log^s(2t-1)}{(2t-1)^z} \qquad s=0,1,2,\cdots$$

Changing the initial value of t from 2 to 1,

$$\beta^{(s)}(z) = \frac{z^{-s}}{\Gamma(1-s)} + (-1)^{-s} \sum_{t=1}^{\infty} (-1)^{t} \frac{\log^s(2t+1)}{(2t+1)^z} \qquad s=0,1,2,\cdots$$

If the possibility of s being a non-integer is not considered,

$$\frac{z^{-s}}{\Gamma(1-s)} = (-1)^0 \frac{\log^s(0+1)}{(0+1)^z} = \begin{cases} 1 & for\ s=0 \quad (\because\ 0^0=1) \\ 0 & for\ s=1,2,3,\cdots \end{cases}$$

So, we can change the initial value of t from 1 to 0,

$$\beta^{(s)}(z) = (-1)^{-s} \sum_{t=0}^{\infty} (-1)^{t} \frac{\log^s(2t+1)}{(2t+1)^z} \qquad s=0,1,2,\cdots \qquad \text{where, } 0^0 = 1$$

Substituting $z=0$ for this,

$$\beta^{(s)}(0) = (-1)^{-s} \sum_{t=0}^{\infty} (-1)^{t} \log^s(2t+1) \qquad s=0,1,2,\cdots \qquad \text{where, } 0^0 = 1 \tag{2.s}$$

Therefore, the Maclaurin series of $\beta(z)$ is

$$\beta(z) = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^t \log^s (2t+1) \frac{(-1)^s z^s}{s!} \tag{2.z}$$

Next, from (2.s),

$$\beta^{(2r+s)}(0) = (-1)^{-s}\sum_{t=0}^{\infty}(-1)^t \log^{2r+s}(2t+1) \qquad r,s=0,1,2,\cdots$$

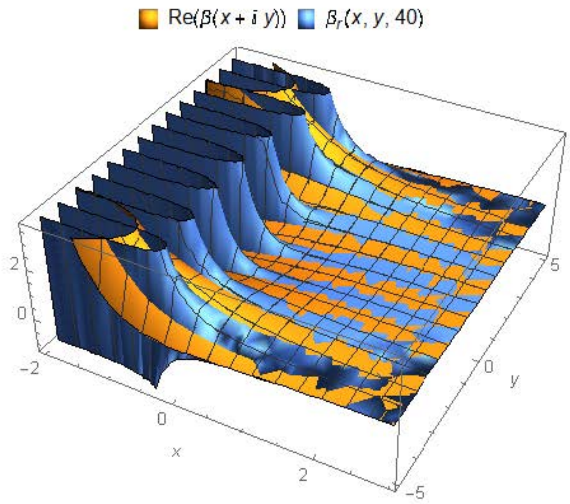
$$\beta^{(2r+s+1)}(0) = (-1)^{-s-1}\sum_{t=0}^{\infty}(-1)^t \log^{2r+s+1}(2t+1) \qquad r,s=0,1,2,\cdots$$

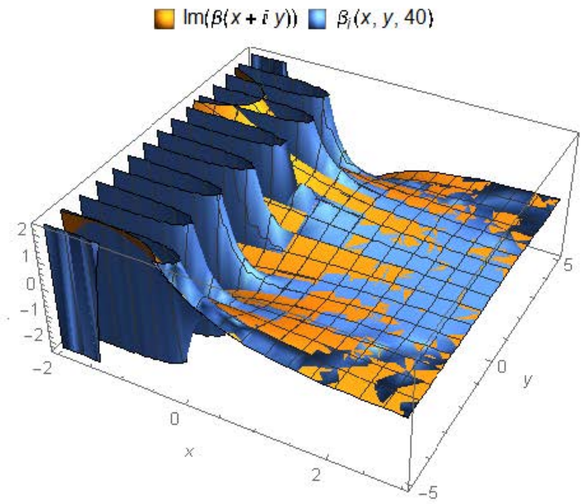
Applying Theorem 14.1.2 to these,

$$Re\{\beta(z)\} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^{2r+s}(2t+1)\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{2.r}$$

$$Im\{\beta(z)\} = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(-1)^t \log^{2r+s+1}(2t+1)\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{2.i}$$

Both sides of (2.r) and (2.i) are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. The line of convergence is seen at $x=0$.





Since these series converge very slowly, the overlapping state of both sides is not so good. In fact, the function values of (2.r) and (2.i) at point $(3,5)$ are calculated as follows.

```
N[{Re[β[3 + i 5]], βr[3, 5, 51]}]        N[{Im[β[3 + i 5]], βi[3, 5, 44]}]
   {0.9748363, 0.974837}                    {-0.03407, -0.034072}
```

Acceleration form of Formula 9.2.1 (coefficients only)

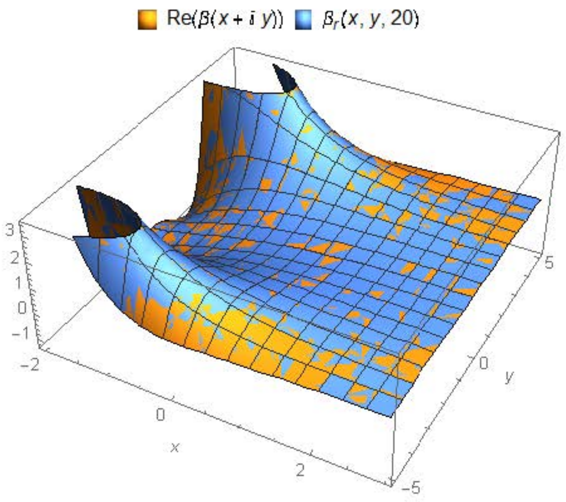
So we try to accelerate the convergence of (2.r) and (2.i). Acceleration of the entire triple series is the usual way, but here, only the coefficient part with the slowest convergence is accelerated by Euler Transform. It is as follows.

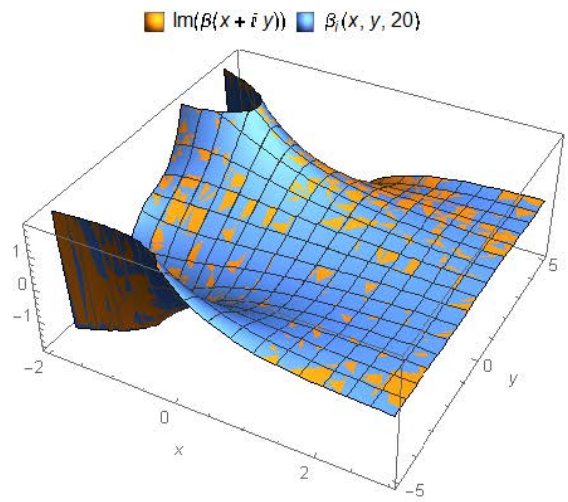
$$\beta(z) = \sum_{s=0}^{\infty}\left\{\sum_{k=0}^{\infty}\sum_{t=0}^{k}\binom{k}{t}\frac{(-1)^t}{2^{k+1}}\log^s(2t+1)\right\}\frac{(-1)^s z^s}{s!}$$

$$Re\{\beta(z)\} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left\{\sum_{k=0}^{\infty}\sum_{t=0}^{k}\binom{k}{t}\frac{(-1)^t}{2^{k+1}}\log^{2r+s}(2t+1)\right\}\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$Im\{\beta(z)\} = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left\{\sum_{k=0}^{\infty}\sum_{t=0}^{k}\binom{k}{t}\frac{(-1)^t}{2^{k+1}}\log^{2r+s+1}(2t+1)\right\}\frac{(-1)^s x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Both sides of Re and Im are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. Both sides overlap exactly and look like spots. Further, we can see that the convergence area is analytically continued from the half plane to the entire plane.





9.3 Laurent series of $\zeta(z)$

Riemann zeta function $\zeta(z)$ is defined on the half plane $Re(z) > 0$ by the following called p - series.

$$\zeta(z) = \sum_{t=1}^{\infty} e^{-z \log t} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots \tag{3.0}$$

This function can be expanded into Laurent series around 1 as follows.

Formula 9.3.1

When the Riemann zeta function is $\zeta(z)$ ($z = x + iy$) and Stieltjes constansts are γ_s $s = 0, 1, 2, \cdots$, the following expressions hold on the whole complex plane except $z = 1$.

$$\zeta(z) = \frac{1}{z-1} + \sum_{s=0}^{\infty} \gamma_s \frac{(-1)^s (z-1)^s}{s!} \tag{3.z}$$

$$Re\{\zeta(z)\} = \frac{x-1}{(x-1)^2+y^2} + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \gamma_{2r+s} \frac{(-1)^s (x-1)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!} \tag{3.r}$$

$$Im\{\zeta(z)\} = -\frac{y}{(x-1)^2+y^2} - \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \gamma_{2r+s+1} \frac{(-1)^s (x-1)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{3.i}$$

Where, $0^0 = 1$

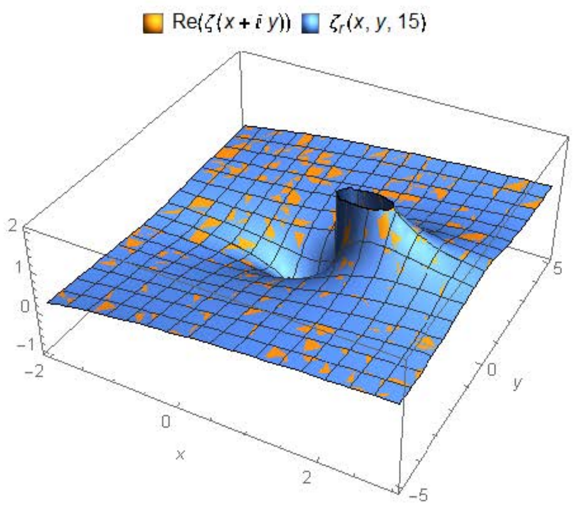
Proof

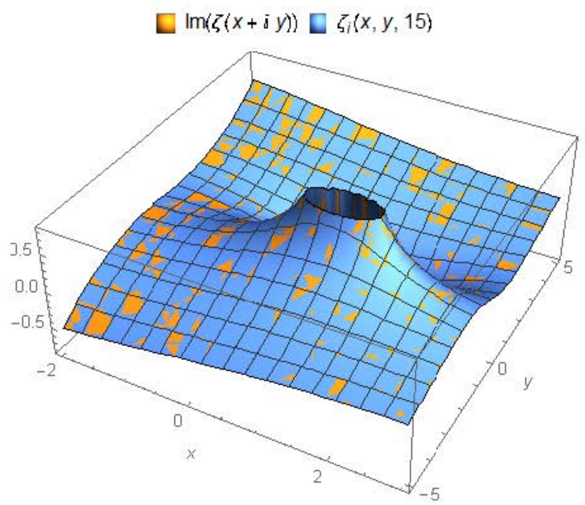
It is known that the following expression holds on the complex plane except $z = 1$.

$$\zeta(z) = \frac{1}{z-1} + \sum_{s=0}^{\infty} \gamma_s \frac{(-1)^s (z-1)^s}{s!} \tag{3.z}$$

Separating the first term into the real and imaginary parts and applying Theorem 14.1.2 to the second term, we obtain the desired expressions.

Both sides of (3.r) and (3.i) are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. Both sides overlap exactly and look like spots. Further, we can see that the convergence area is analytically continued from $Re(z) > 1$ to the whole plane.





9.4 Taylor series of $(z-1)\zeta(z)$

As seen in the previous section, Riemann zeta function $\zeta(z)$ has a singular point at $z=1$. Therefore, let us make a function $(z-1)\zeta(z)$ by multiplying this by $(z-1)$. Then, the function is holomorphic on the whole complex plane. This holomophized function can be expanded into Taylo series around 1 as follows.

Formula 9.4.1

When the Riemann zeta function is $\zeta(z)$ ($z=x+iy$) and Stieltjes constansts are γ_s $s=0,1,2,\cdots$, the following expressions hold on the whole complex plane.

$$(z-1)\zeta(z) = 1 - \sum_{s=1}^{\infty} s\gamma_{s-1}\frac{(-1)^s(z-1)^s}{s!} \tag{4.z}$$

$$Re\{(z-1)\zeta(z)\} = 1 - \sum_{s=1}^{\infty} s\gamma_{s-1}\frac{(-1)^s(x-1)^s}{s!}$$

$$- \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(2r+s)\gamma_{2r+s-1}\frac{(-1)^s(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{4.r}$$

$$Im\{(z-1)\zeta(z)\} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1)\gamma_{2r+s}\frac{(-1)^s(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{4.i}$$

Where, $0^0=1$

Proof

It is known that the following expression holds on the complex plane except $z=1$.

$$\zeta(z) = \frac{1}{z-1} + \sum_{s=0}^{\infty}\gamma_s\frac{(-1)^s(z-1)^s}{s!} \tag{3.z}$$

From this,

$$(z-1)\zeta(z) = 1 - \sum_{s=1}^{\infty} s\gamma_{s-1}\frac{(-1)^s(z-1)^s}{s!}$$

Here, let $f(z)$ be

$$f(z) = (z-1)\zeta(z) = 1 - \sum_{s=1}^{\infty} s\gamma_{s-1}\frac{(-1)^s(z-1)^s}{s!}$$

Then,

$$f^{(s)}(1) = \begin{cases} 1 & s=0 \\ -(-1)^s s\gamma_{s-1} & s=1,2,3,\cdots \end{cases}$$

From this,

$$f^{(2r+s)}(1) = -(-1)^s(2r+s)\gamma_{2r+s-1} \qquad \begin{cases} r=1,2,3,\cdots \\ s=0,1,2,\cdots \end{cases}$$

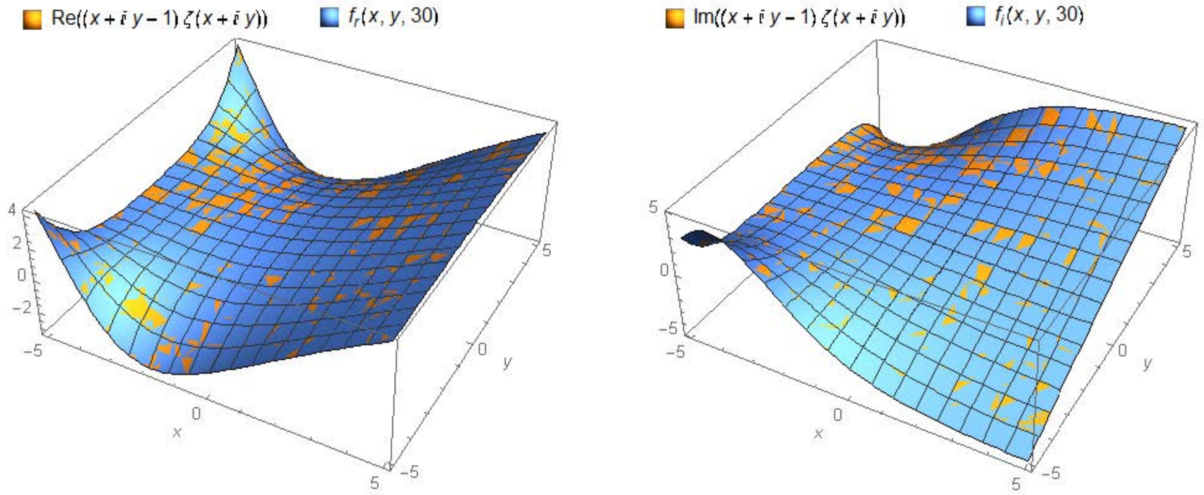
$$f^{(2r+s+1)}(1) = -(-1)^{s+1}(2r+s+1)\gamma_{2r+s} \qquad \begin{cases} r=0,1,2,\cdots \\ s=0,1,2,\cdots \end{cases}$$

Applying Theorem 14.1.2 to these,

$$f_r(x,y) = 1 - \sum_{s=1}^{\infty} s\gamma_{s-1}\frac{(-1)^s(x-1)^s}{s!}$$
$$- \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2r+s)\gamma_{2r+s-1}\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$
$$f_i(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s+1}(2r+s+1)\gamma_{2r+s}\frac{(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Rewriting $f_r(x,y)$, $f_i(x,y)$ in $Re\{(z-1)\zeta(z)\}$, $Im\{(z-1)\zeta(z)\}$ respectively, we obtain the desired expressions.

Both sides of (4.r) and (4.i) are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. Both sides overlap exactly and look like spots. Further, it is also clear that these hold on the whole complex plane.



2020.01.10

Kano Kono

Alien's Mathematics