18 Power Series with the signs of the terms inverted equidistantly

In **" 16 Split of Power Series "**, the function representing the split series extracted equidistantly from a power series was shown as a formula. This formula was a little less interesting on its own.

Recently, however, I have noticed that the signs of the terms of original series can be inverted equidistantly using this formula as a tool. So, in this chapter, we present it as a new formula and show some examples.

18.1 Formula and Verification Method

First, we reprint the formula used as a tool from **" 16 Split of Power Series "**.

Formula 16.2.1 (n-split) (reprint)

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

And let the n-split series $f(k,n,z)$ $k=0,1,2,\cdots,n-1$ are as follows.

$$f(0,n,z) = \sum_{r=0}^{\infty} a_{nr+0} z^{nr+0} = a_0 z^0 + a_n z^n + a_{2n} z^{2n} + a_{3n} z^{3n} + \cdots$$

$$f(1,n,z) = \sum_{r=0}^{\infty} a_{nr+1} z^{nr+1} = a_1 z^1 + a_{n+1} z^{n+1} + a_{2n+1} z^{2n+1} + a_{3n+1} z^{3n+1} + \cdots$$

$$f(2,n,z) = \sum_{r=0}^{\infty} a_{nr+2} z^{nr+2} = a_2 z^2 + a_{n+2} z^{n+2} + a_{2n+2} z^{2n+2} + a_{3n+2} z^{3n+2} + \cdots$$

$$\vdots$$

$$f(n-1,n,z) = \sum_{r=0}^{\infty} a_{nr+n-1} z^{nr+n-1} = a_{n-1} z^{n-1} + a_{2n-1} z^{2n-1} + a_{3n-1} z^{3n-1} + a_{4n-1} z^{4n-1} + \cdots$$

Then, the following expressions hold for $n = 2,3,4,\cdots$, $k = 0,1,2,\cdots,n-1$.

$$f(k,n,z) = \frac{f(z) - \lambda_n (-1)^k f(-z)}{n} + \frac{1}{n} \sum_{s=1}^{\lfloor n/2 \rfloor} \left[(-1)^{-\frac{2sk}{n}} f\left\{ (-1)^{\frac{2s}{n}} z \right\} + (-1)^{\frac{2sk}{n}} f\left\{ (-1)^{-\frac{2s}{n}} z \right\} \right]$$

Where, $\lambda_n = \{1+(-1)^n\}/2$, $\lfloor x \rfloor$ is the floor function.

Sign inversion of power series at equal intervals

Using the above formula, we derive the following formula that equidistantly inverts the signs of the terms of power series

Formula 18.1.1

For integers $n=2,3,4,\cdots$, $k=0,1,2,\cdots$, suppose that the series $f(z)$ and the split series $f(k,n,z)$ are as follows respectively.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

$$f(k,n,z) = \sum_{r=0}^{\infty} a_{nr+k} z^{nr+k} = a_k z^k + a_{n+k} z^{n+k} + a_{2n+k} z^{2n+k} + a_{3n+k} z^{3n+k} + \cdots$$

Then, the series $g(k,n,z)$ in which the signs of the terms $a_{nr+k} z^{nr+k}$ $r=0,1,2,\cdots$ of $f(z)$ are inverted is given by the following expression.

$$g(k,n,z) = \frac{n-2}{n} f(z) + \frac{2}{n} \left\{ \lambda_n (-1)^k f(-z) \right\} - \frac{2}{n} \sum_{s=1}^{\lfloor n/2 \rfloor} \left[(-1)^{-\frac{2sk}{n}} f\left\{ (-1)^{\frac{2s}{n}} z \right\} + (-1)^{\frac{2sk}{n}} f\left\{ (-1)^{-\frac{2s}{n}} z \right\} \right]$$

Where, $\lambda_n = \{1+(-1)^n\}/2$, $\lfloor x \rfloor$ is the floor function.

Proof

The desired series $g(k,n,z)$ is given by

$$g(k,n,z) = f(z) - 2f(k,n,z)$$

So, substituting $f(k,n,z)$ in Formula 16.2.1 for the right-hand side, we obtain the desired expression.

Example

When

$$f(z) = 1 + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \frac{z^8}{8!} + \cdots$$

$$f(2,3,z) = \frac{z^2}{2!} + \frac{z^5}{5!} + \frac{z^8}{8!} + \frac{z^{11}}{11!} + \frac{z^{14}}{14!} + \cdots$$

from these,

$$g(2,3,z) = 1 + \frac{z^1}{1!} - \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} - \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} - \frac{z^8}{8!} ++-\cdots$$

$$= 1 + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \frac{z^8}{8!} + \cdots$$

$$- 2\left(\frac{z^2}{2!} + \frac{z^5}{5!} + \frac{z^8}{8!} + \frac{z^{11}}{11!} + \frac{z^{14}}{14!} + \cdots\right)$$

$$= f(z) - 2f(2,3,z)$$

Verification Method

In the case of the above example, we perform the following calculation using formula manipulation software ***Mathematica***.

```
Sgn_r_ [k_, n_] := If[Mod[r, n] == k, -1, 1]
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$$gs[k\_, n\_, z\_] := \sum_{r=0}^{200} Sgn_r[k, n] \frac{z^r}{r!}$$

$$g[2, 3, z\_] := \frac{e^z}{3} - \frac{2}{3}\left((-1)^{-4/3} e^{(-1)^{2/3} z} + (-1)^{4/3} e^{(-1)^{-2/3} z}\right)$$

```
N[{gs[2, 3, 1], g[2, 3, 1]}, 10]
```

$$\{1.701565508, 1.701565508 + 0. \times 10^{-10}\, i\}$$

$Sgn_r(k,n)$ is a function that starts from the k th order and inverts the sign at n th order intervals. Multiplying this by each term of the series, the sign inversion series $gs(k,n,z)$ is created. $g(k,n,z)$ is the corresponding function. Computing the series value and the function value respectively at the appropriate value $z = 1$, we confirm that they are equal.

18.2 Sign Inversion at 3rd order Intervals

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

From Formula 18.1.1 , the function $g(k,3,z)$ with the signs of the terms $a_{3r+k}\, z^{3r+k}$ $(k=0,1,2)$ inverted is

$$g(k,3,z) = \frac{3-2}{3} f(z) + \frac{2}{3}\left\{\lambda_3 (-1)^k f(-z)\right\} \quad \lambda_3 = \left\{1+(-1)^3\right\}/2$$

$$-\frac{2}{3}\sum_{s=1}^{\lfloor 3/2 \rfloor}\left[(-1)^{-\frac{2sk}{3}} f\left\{(-1)^{\frac{2s}{3}} z\right\} + (-1)^{\frac{2sk}{3}} f\left\{(-1)^{-\frac{2s}{3}} z\right\}\right]$$

If this is written for each $k=0,1,2$,

$$g(0,3,z) = \frac{1}{3} f(z) - \frac{2}{3}\left[f\left\{(-1)^{2/3} z\right\} + f\left\{(-1)^{-2/3} z\right\}\right]$$

$$g(1,3,z) = \frac{1}{3} f(z) - \frac{2}{3}\left[(-1)^{-2/3} f\left\{(-1)^{2/3} z\right\} + (-1)^{2/3} f\left\{(-1)^{-2/3} z\right\}\right]$$

$$g(2,3,z) = \frac{1}{3} f(z) - \frac{2}{3}\left[(-1)^{-4/3} f\left\{(-1)^{2/3} z\right\} + (-1)^{4/3} f\left\{(-1)^{-2/3} z\right\}\right]$$

Example 1 Sign inversion of exponential series at 3rd-order intervals

Original series

$$1 + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \cdots = e^z$$

Series with inverted signs

$$-1 + \frac{z^1}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} - \frac{z^6}{6!} + + - \cdots = \frac{e^z}{3} - \frac{2}{3}\left\{e^{(-1)^{2/3} z} + e^{(-1)^{-2/3} z}\right\}$$

$$= \frac{e^z}{3} - \frac{4}{3\sqrt{e^z}} \cos\frac{\sqrt{3}\, z}{2}$$

$$1 - \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} - \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} - + + \cdots = \frac{e^z}{3} - \frac{2}{3}\left\{(-1)^{-2/3} e^{(-1)^{2/3} z} + (-1)^{2/3} e^{(-1)^{-2/3} z}\right\}$$

$$= \frac{e^z}{3} + \frac{2}{3\sqrt{e^z}}\left(\cos\frac{\sqrt{3}\, z}{2} - \sqrt{3} \sin\frac{\sqrt{3}\, z}{2}\right)$$

$$1 + \frac{z^1}{1!} - \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} - \frac{z^5}{5!} + \frac{z^6}{6!} + - + \cdots = \frac{e^z}{3} - \frac{2}{3}\left\{(-1)^{-4/3} e^{(-1)^{2/3} z} + (-1)^{4/3} e^{(-1)^{-2/3} z}\right\}$$

$$= \frac{e^z}{3} + \frac{2}{3\sqrt{e^z}}\left(\cos\frac{\sqrt{3}\, z}{2} + \sqrt{3} \sin\frac{\sqrt{3}\, z}{2}\right)$$

Especially when $z=1$,

$$-1 + \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + + - \cdots = 0.38216520\cdots$$

$$1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} - + + \cdots = 0.63455111\cdots$$

$$1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} + - + \cdots = 1.70156550\cdots$$

Example 2 Sign inversion of Logarithmic series at 3rd-order intervals ($|z| < 1,\ z \ne 1$)

Original series

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}+\cdots = -\log(1-z)$$

Series with inverted signs

$$\frac{z^1}{1}+\frac{z^2}{2}-\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}-\frac{z^6}{6}+\frac{z^7}{7}+-+\cdots = -\frac{1}{3}\log(1-z)$$
$$+\frac{2}{3}\left[\log\{1-(-1)^{2/3}z\}+\log\{1-(-1)^{-2/3}z\}\right]$$

$$-\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}-\frac{z^7}{7}++-\cdots = -\frac{1}{3}\log(1-z)$$
$$+\frac{2}{3}\left[(-1)^{-2/3}\log\{1-(-1)^{2/3}z\}+(-1)^{2/3}\log\{1-(-1)^{-2/3}z\}\right]$$

$$\frac{z^1}{1}-\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}-\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}-++\cdots = -\frac{1}{3}\log(1-z)$$
$$+\frac{2}{3}\left[(-1)^{-4/3}\log\{1-(-1)^{2/3}z\}+(-1)^{4/3}\log\{1-(-1)^{-2/3}z\}\right]$$

When $z=1/2$,

$$\frac{1}{1\cdot2}+\frac{1}{2\cdot2^2}-\frac{1}{3\cdot2^3}+\frac{1}{4\cdot2^4}+\frac{1}{5\cdot2^5}-\frac{1}{6\cdot2^6}+\frac{1}{7\cdot2^7}+-+\cdots = \frac{1}{3}\log\frac{49}{8} = 0.60412625\cdots$$

$$-\frac{1}{1\cdot2}+\frac{1}{2\cdot2^2}+\frac{1}{3\cdot2^3}-\frac{1}{4\cdot2^4}+\frac{1}{5\cdot2^5}+\frac{1}{6\cdot2^6}-\frac{1}{7\cdot2^7}++-\cdots = \frac{1}{3}\left(\log\frac{8}{7}-2\sqrt{3}\arctan\frac{\sqrt{3}}{5}\right)$$
$$= -0.34055118\cdots$$

$$\frac{1}{1\cdot2}-\frac{1}{2\cdot2^2}+\frac{1}{3\cdot2^3}+\frac{1}{4\cdot2^4}-\frac{1}{5\cdot2^5}+\frac{1}{6\cdot2^6}+\frac{1}{7\cdot2^7}-++\cdots = \frac{1}{3}\left(\log\frac{8}{7}+2\sqrt{3}\arctan\frac{\sqrt{3}}{5}\right)$$
$$= 0.42957211\cdots$$

18.3 Sign Inversion at 4th order Intervals

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

From Formula 18.1.1 , the function $g(k,4,z)$ with the signs of the terms $a_{4r+k} z^{4r+k}$ $(k=0,1,2,3)$ inverted is

$$g(k,4,z) = \frac{4-2}{4} f(z) + \frac{2}{4}\left\{\lambda_4 (-1)^k f(-z)\right\} \quad \lambda_4 = \left\{1+(-1)^4\right\}/2$$

$$-\frac{2}{4}\sum_{s=1}^{\lfloor 4/2 \rfloor}\left[(-1)^{-\frac{2sk}{4}} f\left\{(-1)^{\frac{2s}{4}} z\right\} + (-1)^{\frac{2sk}{4}} f\left\{(-1)^{-\frac{2s}{4}} z\right\}\right]$$

If this is written for each $k=0,1,2,3$,

$$g(0,4,z) = \frac{f(z)+f(-z)}{2} - \frac{f\left\{(-1)^{2/4}z\right\} + f\left\{(-1)^{-2/4}z\right\}}{2} - \frac{f\left\{(-1)^{4/4}z\right\} + f\left\{(-1)^{-4/4}z\right\}}{2}$$

$$g(1,4,z) = \frac{f(z)-f(-z)}{2} - \frac{(-1)^{-2/4} f\left\{(-1)^{2/4}z\right\} + (-1)^{2/4} f\left\{(-1)^{-2/4}z\right\}}{2} - \frac{(-1)^{-4/4} f\left\{(-1)^{4/4}z\right\} + (-1)^{4/4} f\left\{(-1)^{-4/4}z\right\}}{2}$$

$$g(2,4,z) = \frac{f(z)+f(-z)}{2} - \frac{(-1)^{-4/4} f\left\{(-1)^{2/4}z\right\} + (-1)^{4/4} f\left\{(-1)^{-2/4}z\right\}}{2} - \frac{(-1)^{-8/4} f\left\{(-1)^{4/4}z\right\} + (-1)^{8/4} f\left\{(-1)^{-4/4}z\right\}}{2}$$

$$g(3,4,z) = \frac{f(z)-f(-z)}{2} - \frac{(-1)^{-6/4} f\left\{(-1)^{2/4}z\right\} + (-1)^{6/4} f\left\{(-1)^{-2/4}z\right\}}{2} - \frac{(-1)^{-12/4} f\left\{(-1)^{4/4}z\right\} + (-1)^{12/4} f\left\{(-1)^{-4/4}z\right\}}{2}$$

Example 1 Sign inversion of Binomial series at 4th-order intervals ($|z| < 1,\ z \neq 1$)

Original series

$$1+\frac{1!!}{2!!}z^1+\frac{3!!}{4!!}z^2+\frac{5!!}{6!!}z^3+\frac{7!!}{8!!}z^4+\frac{9!!}{10!!}z^5+\frac{11!!}{12!!}z^6+\frac{13!!}{14!!}z^7+\cdots = \frac{1}{\sqrt{1-z}}$$

Series with inverted signs

$$-1+\frac{1!!}{2!!}z^1+\frac{3!!}{4!!}z^2+\frac{5!!}{6!!}z^3-\frac{7!!}{8!!}z^4+\frac{9!!}{10!!}z^5+\frac{11!!}{12!!}z^6+\frac{13!!}{14!!}z^7-+++\cdots$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{1-z}}+\frac{1}{\sqrt{1+z}}\right) - \frac{1}{2}\left(\frac{1}{\sqrt{1-(-1)^{2/4}z}}+\frac{1}{\sqrt{1-(-1)^{-2/4}z}}\right) - \frac{1}{2}\left(\frac{1}{\sqrt{1-(-1)^{4/4}z}}+\frac{1}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

$$1-\frac{1!!}{2!!}z^1+\frac{3!!}{4!!}z^2+\frac{5!!}{6!!}z^3+\frac{7!!}{8!!}z^4-\frac{9!!}{10!!}z^5+\frac{11!!}{12!!}z^6+\frac{13!!}{14!!}z^7+-++\cdots$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{1-z}}-\frac{1}{\sqrt{1+z}}\right) - \frac{1}{2}\left(\frac{(-1)^{-2/4}}{\sqrt{1-(-1)^{2/4}z}}+\frac{(-1)^{2/4}}{\sqrt{1-(-1)^{-2/4}z}}\right)$$

$$-\frac{1}{2}\left(\frac{(-1)^{-4/4}}{\sqrt{1-(-1)^{4/4}z}}+\frac{(-1)^{4/4}}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

$$1+\frac{1!!}{2!!}z^1-\frac{3!!}{4!!}z^2+\frac{5!!}{6!!}z^3+\frac{7!!}{8!!}z^4+\frac{9!!}{10!!}z^5-\frac{11!!}{12!!}z^6+\frac{13!!}{14!!}z^7++-+\cdots$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{1-z}}+\frac{1}{\sqrt{1+z}}\right)-\frac{1}{2}\left(\frac{(-1)^{-4/4}}{\sqrt{1-(-1)^{2/4}z}}+\frac{(-1)^{4/4}}{\sqrt{1-(-1)^{-2/4}z}}\right)$$

$$-\frac{1}{2}\left(\frac{(-1)^{-8/4}}{\sqrt{1-(-1)^{4/4}z}}+\frac{(-1)^{8/4}}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

$$1+\frac{1!!}{2!!}z^1+\frac{3!!}{4!!}z^2-\frac{5!!}{6!!}z^3+\frac{7!!}{8!!}z^4+\frac{9!!}{10!!}z^5+\frac{11!!}{12!!}z^6-\frac{13!!}{14!!}z^7+++-\cdots$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{1-z}}-\frac{1}{\sqrt{1+z}}\right)-\frac{1}{2}\left(\frac{(-1)^{-6/4}}{\sqrt{1-(-1)^{2/4}z}}+\frac{(-1)^{6/4}}{\sqrt{1-(-1)^{-2/4}z}}\right)$$

$$-\frac{1}{2}\left(\frac{(-1)^{-12/4}}{\sqrt{1-(-1)^{4/4}z}}+\frac{(-1)^{12/4}}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

When $z=1/2$,

$$-1+\frac{1!!}{2^1 2!!}+\frac{3!!}{2^2 4!!}+\frac{5!!}{2^3 6!!}-\frac{7!!}{2^4 8!!}+\frac{9!!}{2^5 10!!}+\frac{11!!}{2^6 12!!}+\frac{13!!}{2^7 14!!}-+++\cdots=-0.62158357$$

$$=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}}-\sqrt{\frac{1}{\sqrt{5}}+\frac{2}{5}}$$

$$1-\frac{1!!}{2^1 2!!}+\frac{3!!}{2^2 4!!}+\frac{5!!}{2^3 6!!}+\frac{7!!}{2^4 8!!}-\frac{9!!}{2^5 10!!}+\frac{11!!}{2^6 12!!}+\frac{13!!}{2^7 14!!}+-++\cdots=0.89806817$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}-\sqrt{\frac{1}{\sqrt{5}}-\frac{2}{5}}$$

$$1+\frac{1!!}{2^1 2!!}-\frac{3!!}{2^2 4!!}+\frac{5!!}{2^3 6!!}+\frac{7!!}{2^4 8!!}+\frac{9!!}{2^5 10!!}-\frac{11!!}{2^6 12!!}+\frac{13!!}{2^7 14!!}++-+\cdots=1.21930055$$

$$=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}}+\sqrt{\frac{1}{\sqrt{5}}+\frac{2}{5}}$$

$$1+\frac{1!!}{2^1 2!!}+\frac{3!!}{2^2 4!!}-\frac{5!!}{2^3 6!!}+\frac{7!!}{2^4 8!!}+\frac{9!!}{2^5 10!!}+\frac{11!!}{2^6 12!!}-\frac{13!!}{2^7 14!!}+++-\cdots=1.33264196$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}+\sqrt{\frac{1}{\sqrt{5}}-\frac{2}{5}}$$

Example 2 Sign inversion of Logarithmic series at 4th-order intervals ($|z|<1,\ z\neq 1$)

Original series

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}+\frac{z^8}{8}+\cdots=-\log(1-z)$$

Series with inverted signs

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}-\frac{z^8}{8}+++-\cdots=-\frac{\log(1-z)+\log(1+z)}{2}$$

$$+\frac{1}{2}\left[\log\{1-(-1)^{2/4}z\}+\log\{1-(-1)^{-2/4}z\}\right]$$

$$+\frac{1}{2}\left[\log\{1-(-1)^{4/4}z\}+\log\{1-(-1)^{-4/4}z\}\right]$$

$$= \arctan z + \frac{\log(1-iz) + \log(1+iz)}{2}$$

$$-\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}-\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}+\frac{z^8}{8}-+++\cdots = -\frac{\log(1-z)-\log(1+z)}{2}$$

$$+\frac{1}{2}\left[(-1)^{-2/4}\log\{1-(-1)^{2/4}z\}+(-1)^{2/4}\log\{1-(-1)^{-2/4}z\}\right]$$

$$+\frac{1}{2}\left[(-1)^{-4/4}\log\{1-(-1)^{4/4}z\}+(-1)^{4/4}\log\{1-(-1)^{-4/4}z\}\right]$$

$$= -\arctan z - \frac{\log(1-z)+\log(1+z)}{2}$$

$$\frac{z^1}{1}-\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}-\frac{z^6}{6}+\frac{z^7}{7}+\frac{z^8}{8}+-++\cdots = -\frac{\log(1-z)+\log(1+z)}{2}$$

$$+\frac{1}{2}\left[(-1)^{-4/4}\log\{1-(-1)^{2/4}z\}+(-1)^{4/4}\log\{1-(-1)^{-2/4}z\}\right]$$

$$+\frac{1}{2}\left[(-1)^{-8/4}\log\{1-(-1)^{4/4}z\}+(-1)^{8/4}\log\{1-(-1)^{-4/4}z\}\right]$$

$$= \arctan z - \frac{\log(1-iz) + \log(1+iz)}{2}$$

$$\frac{z^1}{1}+\frac{z^2}{2}-\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}-\frac{z^7}{7}+\frac{z^8}{8}++-+\cdots = -\frac{\log(1-z)-\log(1+z)}{2}$$

$$+\frac{1}{2}\left[(-1)^{-6/4}\log\{1-(-1)^{2/4}z\}+(-1)^{6/4}\log\{1-(-1)^{-2/4}z\}\right]$$

$$+\frac{1}{2}\left[(-1)^{-12/4}\log\{1-(-1)^{4/4}z\}+(-1)^{12/4}\log\{1-(-1)^{-4/4}z\}\right]$$

$$= \arctan z - \frac{\log(1-z)+\log(1+z)}{2}$$

When $z=1/2$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}-\frac{1}{8\cdot 2^8}+++-\cdots = \frac{1}{2}\log\frac{15}{4} = 0.66087792\cdots$$

$$-\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}-\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}-+++\cdots = -\operatorname{arccot}2 + \log 2 - \frac{\log 3}{2}$$

$$= -0.31980657\cdots$$

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}-\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}+-++\cdots = \frac{1}{2}\log\frac{12}{5} = 0.43773436\cdots$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}-\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}-\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}++-+\cdots = \operatorname{arccot}2 + \log 2 - \frac{\log 3}{2}$$

$$= 0.60748864\cdots$$

Example 3 Sign inversion of exponential series at 4th-order intervals

Original series

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}+\cdots = e^z$$

Series with inverted signs

Here, we do not use the above formula, but use the 2-split series of $\cosh z$, $\sinh z$, which are obtained in the previous chapter 16.3.2 . They were as follows.

$$1+\frac{z^4}{4!}+\frac{z^8}{8!}+\frac{z^{12}}{12!}+\frac{z^{16}}{16!}+\frac{z^{20}}{20!}+\cdots = \frac{\cosh z + \cos z}{2}$$

$$\frac{z^2}{2!}+\frac{z^6}{6!}+\frac{z^{10}}{10!}+\frac{z^{14}}{14!}+\frac{z^{18}}{18!}+\cdots = \frac{\cosh z - \cos z}{2}$$

$$\frac{z^1}{1!}+\frac{z^5}{5!}+\frac{z^9}{9!}+\frac{z^{13}}{13!}+\frac{z^{17}}{17!}+\cdots = \frac{\sinh z + \sin z}{2}$$

$$\frac{z^3}{3!}+\frac{z^7}{7!}+\frac{z^{11}}{11!}+\frac{z^{15}}{15!}+\frac{z^{19}}{19!}+\cdots = \frac{\sinh z - \sin z}{2}$$

These are 4-split series of exponential series.

So, substituting these for $g(k,4,z) = f(z) - 2f(k,4,z) \quad k=0,1,2,3$,

$$-1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}-\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}-\frac{z^8}{8!}+++-\cdots = e^z - \cosh z - \cos z$$

$$1-\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}-\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}-+++\cdots = e^z - \sinh z - \sin z$$

$$1+\frac{z^1}{1!}-\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}-\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}+-++\cdots = e^z - \cosh z + \cos z$$

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}-\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}-\frac{z^7}{7!}+\frac{z^8}{8!}++-+\cdots = e^z - \sinh z + \sin z$$

Especially when $z=1$,

$$-1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}-\frac{1}{8!}+++-\cdots = 0.63489888\cdots$$

$$1-\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}-+++\cdots = 0.70160965\cdots$$

$$1+\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}-\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+-++\cdots = 1.71550350\cdots$$

$$1+\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+\frac{1}{8!}++-+\cdots = 2.38455162\cdots$$

18.4 Sign Inversion at 5th order Intervals

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

From Formula 18.1.1 , the function $g(k,5,z)$ with the signs of the terms $a_{5r+k}\, z^{5r+k}$ $(k=0,1,\cdots,4)$ inverted is

$$g(k,5,z) = \frac{5-2}{5} f(z) + \frac{2}{5}\left\{\lambda_5 (-1)^k f(-z)\right\} \quad \lambda_5 = \left\{1+(-1)^5\right\}/2$$

$$-\frac{2}{5}\sum_{s=1}^{\lfloor 5/2 \rfloor}\left[(-1)^{-\frac{2sk}{5}} f\left\{(-1)^{\frac{2s}{5}} z\right\} + (-1)^{\frac{2sk}{5}} f\left\{(-1)^{-\frac{2s}{5}} z\right\}\right]$$

If this is written for each $k=0,1,\cdots,4$,

$$g(0,5,z) = \frac{3}{5} f(z) - \frac{2}{5}\left[f\left\{(-1)^{2/5} z\right\} + f\left\{(-1)^{-2/5} z\right\}\right]$$
$$-\frac{2}{5}\left[f\left\{(-1)^{4/5} z\right\} + f\left\{(-1)^{-4/5} z\right\}\right]$$

$$g(1,5,z) = \frac{3}{5} f(z) - \frac{2}{5}\left[(-1)^{-2/5} f\left\{(-1)^{2/5} z\right\} + (-1)^{2/5} f\left\{(-1)^{-2/5} z\right\}\right]$$
$$-\frac{2}{5}\left[(-1)^{-4/5} f\left\{(-1)^{4/5} z\right\} + (-1)^{4/5} f\left\{(-1)^{-4/5} z\right\}\right]$$

$$g(2,5,z) = \frac{3}{5} f(z) - \frac{2}{5}\left[(-1)^{-4/5} f\left\{(-1)^{2/5} z\right\} + (-1)^{4/5} f\left\{(-1)^{-2/5} z\right\}\right]$$
$$-\frac{2}{5}\left[(-1)^{-8/5} f\left\{(-1)^{4/5} z\right\} + (-1)^{8/5} f\left\{(-1)^{-4/5} z\right\}\right]$$

$$g(3,5,z) = \frac{3}{5} f(z) - \frac{2}{5}\left[(-1)^{-6/5} f\left\{(-1)^{2/5} z\right\} + (-1)^{6/5} f\left\{(-1)^{-2/5} z\right\}\right]$$
$$-\frac{2}{5}\left[(-1)^{-12/5} f\left\{(-1)^{4/5} z\right\} + (-1)^{12/5} f\left\{(-1)^{-4/5} z\right\}\right]$$

$$g(4,5,z) = \frac{3}{5} f(z) - \frac{2}{5}\left[(-1)^{-8/5} f\left\{(-1)^{2/5} z\right\} + (-1)^{8/5} f\left\{(-1)^{-2/5} z\right\}\right]$$
$$-\frac{2}{5}\left[(-1)^{-16/5} f\left\{(-1)^{4/5} z\right\} + (-1)^{16/5} f\left\{(-1)^{-4/5} z\right\}\right]$$

Example 1 Sign inversion of exponential series at 5th-order intervals

Original series

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}+\frac{z^9}{9!}+\frac{z^{10}}{10!}+\cdots = e^z$$

Series with inverted signs

$$-1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}-\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}+\frac{z^9}{9!}-\frac{z^{10}}{10!}++++-\cdots = \frac{3}{5} e^z$$
$$-\frac{2}{5}\left\{ e^{(-1)^{2/5} z} + e^{(-1)^{-2/5} z}\right\} - \frac{2}{5}\left\{ e^{(-1)^{4/5} z} + e^{(-1)^{-4/5} z}\right\}$$

$$1-\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}-\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}+\frac{z^9}{9!}+\frac{z^{10}}{10!}-++++\cdots = \frac{3}{5} e^z$$
$$-\frac{2}{5}\left\{ (-1)^{-2/5} e^{(-1)^{2/5} z} + (-1)^{2/5} e^{(-1)^{-2/5} z}\right\}$$
$$-\frac{2}{5}\left\{ (-1)^{-4/5} e^{(-1)^{4/5} z} + (-1)^{4/5} e^{(-1)^{-4/5} z}\right\}$$

$$1+\frac{z^1}{1!}-\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}-\frac{z^7}{7!}+\frac{z^8}{8!}+\frac{z^9}{9!}+\frac{z^{10}}{10!}+-+++\cdots = \frac{3}{5}e^z - \frac{2}{5}\left\{(-1)^{-4/5}e^{(-1)^{2/5}z}+(-1)^{4/5}e^{(-1)^{-2/5}z}\right\} - \frac{2}{5}\left\{(-1)^{-8/5}e^{(-1)^{4/5}z}+(-1)^{8/5}e^{(-1)^{-4/5}z}\right\}$$

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}-\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}-\frac{z^8}{8!}+\frac{z^9}{9!}+\frac{z^{10}}{10!}++-++\cdots = \frac{3}{5}e^z - \frac{2}{5}\left\{(-1)^{-6/5}e^{(-1)^{2/5}z}+(-1)^{6/5}e^{(-1)^{-2/5}z}\right\} - \frac{2}{5}\left\{(-1)^{-12/5}e^{(-1)^{4/5}z}+(-1)^{12/5}e^{(-1)^{-4/5}z}\right\}$$

$$1+\frac{z^1}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}-\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}+\frac{z^8}{8!}-\frac{z^9}{9!}+\frac{z^{10}}{10!}+++-+\cdots = \frac{3}{5}e^z - \frac{2}{5}\left\{(-1)^{-8/5}e^{(-1)^{2/5}z}+(-1)^{8/5}e^{(-1)^{-2/5}z}\right\} - \frac{2}{5}\left\{(-1)^{-16/5}e^{(-1)^{4/5}z}+(-1)^{16/5}e^{(-1)^{-4/5}z}\right\}$$

Especially when $z=1$,

$$-1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+\frac{1}{9!}-\frac{1}{10!}++++-\cdots = 0.70161461\cdots$$

$$1-\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}-\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+\frac{1}{9!}+\frac{1}{10!}-++++\cdots = 0.71550400\cdots$$

$$1+\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+\frac{1}{8!}+\frac{1}{9!}+\frac{1}{10!}+-+++\cdots = 1.71788499\cdots$$

$$1+\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}-\frac{1}{8!}+\frac{1}{9!}+\frac{1}{10!}++-++\cdots = 2.38489889\cdots$$

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}-\frac{1}{9!}+\frac{1}{10!}+++-+\cdots = 2.63494298\cdots$$

Example 2 Sign inversion of Logarithmic series at 5th-order intervals ($|z|<1,\ z\neq 1$)

Original series

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}+\frac{z^8}{8}+\frac{z^9}{9}+\frac{z^{10}}{10}+\cdots = -\log(1-z)$$

Series with inverted signs

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}-\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}+\frac{z^8}{8}+\frac{z^9}{9}-\frac{z^{10}}{10}++++-\cdots = -\frac{3}{5}\log(1-z) + \frac{2}{5}\left[\log\{1-(-1)^{2/5}z\}+\log\{1-(-1)^{-2/5}z\}\right] + \frac{2}{5}\left[\log\{1-(-1)^{4/5}z\}+\log\{1-(-1)^{-4/5}z\}\right]$$

$$-\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}-\frac{z^6}{6}+\frac{z^7}{7}+\frac{z^8}{8}+\frac{z^9}{9}+\frac{z^{10}}{10}-++++\cdots = -\frac{3}{5}\log(1-z) + \frac{2}{5}\left[(-1)^{-2/5}\log\{1-(-1)^{2/5}z\}+(-1)^{2/5}\log\{1-(-1)^{-2/5}z\}\right]$$

$$+\frac{2}{5}\left[(-1)^{-4/5}\log\left\{1-(-1)^{4/5}z\right\}+(-1)^{4/5}\log\left\{1-(-1)^{-4/5}z\right\}\right]$$

$$\frac{z^1}{1}-\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}-\frac{z^7}{7}+\frac{z^8}{8}+\frac{z^9}{9}+\frac{z^{10}}{10}+-+++\cdots=-\frac{3}{5}\log(1-z)$$

$$+\frac{2}{5}\left[(-1)^{-4/5}\log\left\{1-(-1)^{2/5}z\right\}+(-1)^{4/5}\log\left\{1-(-1)^{-2/5}z\right\}\right]$$

$$+\frac{2}{5}\left[(-1)^{-8/5}\log\left\{1-(-1)^{4/5}z\right\}+(-1)^{8/5}\log\left\{1-(-1)^{-4/5}z\right\}\right]$$

$$\frac{z^1}{1}+\frac{z^2}{2}-\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}-\frac{z^8}{8}+\frac{z^9}{9}+\frac{z^{10}}{10}++-++\cdots=-\frac{3}{5}\log(1-z)$$

$$+\frac{2}{5}\left[(-1)^{-6/5}\log\left\{1-(-1)^{2/5}z\right\}+(-1)^{6/5}\log\left\{1-(-1)^{-2/5}z\right\}\right]$$

$$+\frac{2}{5}\left[(-1)^{-12/5}\log\left\{1-(-1)^{4/5}z\right\}+(-1)^{12/5}\log\left\{1-(-1)^{-4/5}z\right\}\right]$$

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\frac{z^7}{7}+\frac{z^8}{8}-\frac{z^9}{9}+\frac{z^{10}}{10}+++-+\cdots=-\frac{3}{5}\log(1-z)$$

$$+\frac{2}{5}\left[(-1)^{-8/5}\log\left\{1-(-1)^{2/5}z\right\}+(-1)^{8/5}\log\left\{1-(-1)^{-2/5}z\right\}\right]$$

$$+\frac{2}{5}\left[(-1)^{-16/5}\log\left\{1-(-1)^{4/5}z\right\}+(-1)^{16/5}\log\left\{1-(-1)^{-4/5}z\right\}\right]$$

When $z=1/2$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}-\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}+\frac{1}{9\cdot 2^9}-\frac{1}{10\cdot 2^{10}}++++-\cdots=\frac{2}{5}\log 31-\log 2$$

$$=0.68044770\cdots$$

$$-\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}-\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}+\frac{1}{9\cdot 2^9}+\frac{1}{10\cdot 2^{10}}-++++\cdots=-0.31215188\cdots$$

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}-\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}+\frac{1}{9\cdot 2^9}+\frac{1}{10\cdot 2^{10}}+-+++\cdots=0.44087342\cdots$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}-\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}-\frac{1}{8\cdot 2^8}+\frac{1}{9\cdot 2^9}+\frac{1}{10\cdot 2^{10}}++-++\cdots=0.60881807\cdots$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\frac{1}{8\cdot 2^8}-\frac{1}{9\cdot 2^9}+\frac{1}{10\cdot 2^{10}}+++-+\cdots=0.66145422\cdots$$

Note

It is possible to represent $(-1)^{m/n}$ $(m,n=1,2,3,\cdots)$ in the formula with elementary transcendental functions and radicals. However, in the case of the 5th order or higher, it becomes very complicated.

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Alien's Mathematics