06 Reflection Split of Dirichlet Series

In the previous chapter, we considered the basic split of Dirichlet series and the composite split combining them. Among the composite split, there are composite splits whose sums are represented by special values of the elementary function regardless the degree. Moreover, these special values are obtained by algebraic solution, except for the circle ratio \( \pi \).

6.1 Reflection Formula & Reflection Split

As first, we repeat the definition and the formula of the previous chapter here. Then, we describe the reflection formula for a polygamma function, and, finally we define reflection split of Dirichlet series.

**Definition 6.1.0 (Basic Split) (Repeat)**

When we can create positive or negative term series \( A_k \) \( (k=1, 2, \cdots m) \) by choosing the terms from the \( k \) th term \( (k=1, 2, \cdots m) \) with \( (m-1) \) skipping in a series, We call this a basic \( m \)-split (or simply \( m \)-split).

**Formula 6.1.1 (Repeat)**

Let \( \zeta(n, z) \) be Hurwitz zeta function, \( \psi_n(z) \) be polygamma function, and Dirichlet lambda series \( \lambda(n) \) and the \( m \)-split series \( A_k \) are as follows respectively.

\[
\lambda(n) = \sum_{r=0}^{\infty} \frac{1}{(2r+1)^n}
\]

\[
A_k = \sum_{r=0}^{\infty} \frac{1}{(2mr+2k-1)^n} \quad k=1, 2, \cdots, m
\]

Then, the following expressions hold for \( k=1, 2, \cdots, m \).

\[
\sum_{r=0}^{\infty} \frac{1}{(2mr+2k-1)^n} = \frac{1}{(2m)^n} \zeta\left(n, \frac{2k-1}{2m}\right)
\]

\[
= \frac{(-1)^n}{(2m)^n (n-1)!} \psi_{n-1}\left(\frac{2k-1}{2m}\right)
\]

**Formula 6.1.2 (Polygamma Reflection Formula)**

When \( \psi_n(z) \) \( (n=1, 2, 3, \cdots) \) is the Polygamma function, the following expressions hold.

\[
\psi_{2n-1}(z) + \psi_{2n-1}(1-z) = -\pi \frac{d^{2n-1}}{dz^{2n-1}} \cot(\pi z)
\]

\[
\psi_{2n-2}(z) - \psi_{2n-2}(1-z) = -\pi \frac{d^{2n-2}}{dz^{2n-2}} \cot(\pi z)
\]

**Proof**

According to MathWorld (http://mathworld.wolfram.com/PolygammaFunction.html), the reflection formula for polygamma functions is as follows.

\[
\psi_n(1-z) + (-1)^{n+1} \psi_n(z) = (-1)^n \pi \frac{d^n}{dz^n} \cot(\pi z)
\]

Multiplying \( (-1)^{n+1} \) by both sides

\[
\psi_n(z) + (-1)^{n+1} \psi_n(1-z) = -\pi \frac{d^n}{dz^n} \cot(\pi z)
\]
The first few for (1.21) and (1.22) are as follows respectively.

(1.21)

\[
\psi_1(z) + \psi_1(1-z) = \pi^2 \csc^2(\pi z)
\]

\[
\psi_3(z) + \psi_3(1-z) = \pi^4 \left\{ 4 \cot^2(\pi z) \csc^2(\pi z) + 2 \csc^4(\pi z) \right\}
\]

\[
\psi_5(z) + \psi_5(1-z) = \pi^6 \left\{ 16 \cot^4(\pi z) \csc^2(\pi z) + 88 \cot^2(\pi z) \csc^4(\pi z) + 16 \csc^6(\pi z) \right\}
\]

\[
\psi_7(z) + \psi_7(1-z) = \pi^8 \left\{ 64 \cot^6(\pi z) \csc^2(\pi z) + 1824 \cot^4(\pi z) \csc^4(\pi z) + 2880 \cot^2(\pi z) \csc^6(\pi z) + 272 \csc^8(\pi z) \right\}
\]

(1.22)

\[
\psi_0(z) - \psi_0(1-z) = -\pi^1 \cot(\pi z)
\]

\[
\psi_2(z) - \psi_2(1-z) = -2 \pi^3 \cot(\pi z) \csc^2(\pi z)
\]

\[
\psi_4(z) - \psi_4(1-z) = -\pi^5 \left\{ 8 \cot^3(\pi z) \csc^2(\pi z) + 16 \cot(\pi z) \csc^4(\pi z) \right\}
\]

\[
\psi_6(z) - \psi_6(1-z) = -\pi^7 \left\{ 32 \cot^5(\pi z) \csc^2(\pi z) + 416 \cot^3(\pi z) \csc^4(\pi z) + 272 \cot(\pi z) \csc^6(\pi z) \right\}
\]

In addition, these coefficients are listed in OEIS (https://oeis.org) A008303.

**Note 1**

As seen in the previous chapter, the sum of n-degree Dirichlet series was given by the following.

\[
\frac{(-1)^n}{m^n (n-1)!} \left\{ \psi_{n-1} \left( \frac{p}{m} \right) \pm \psi_{n-1} \left( \frac{q}{m} \right) \right\}
\]

Among these, Formula 6.1.2 is applicable to those with \( q = p - 1 \).

And, considering the sign and degree of Formula 6.1.2, it becomes as follows.

(1) (1.21) is applicable to even-order positive Dirichlet series.

(2) (1.22) is applicable to odd-order alternating Dirichlet series.

Conversely, this formula can not be applied to odd-degree positive Dirichlet series or even-order alternating Dirichlet series.

**Note 2**

The right-hand side of this formula is a polynomial of elementary functions.

**Definition of Reflection Split of Dirichlet Series**

**Definition 6.1.3 (Comprehensive)**

Create positive or negative term series \( A_k \ (k=1, 2, \cdots, 2m) \) by choosing the terms from the \( k \) th term \( (k=1, 2, \cdots, 2m) \) with \( (m-1) \) skipping in a series according to Definition 6.1.0. Then, among the composite
\( m \)-splits composed of these, there are combinations satisfying the following reflection relation:

\[(A_1 + A_{2m}) , (A_2 + A_{2m-1}) , \ldots , (A_k + A_{2m-k+1}) , \ldots , (A_m + A_{2m-m+1})\]

We call these composite split especially reflection \( m \)-split.

**Note**

Here, the split which satisfies the reflection relation means the relationship \( k + (2m - k + 1) = 2m + 1 \) in \((A_k + A_{2m-k+1})\). An example with this definition is shown in Section 2.

Reflection Split can be defined also somewhat restrictively as follows.

**Definition 6.1.3' (Restrictive)**

(1) Let Dirichlet lambda series \( \tilde{\lambda}(2n) \) be

\[
\tilde{\lambda}(2n) = 1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \cdots
\]

When this series is split as follows, we call this reflection \( m \)-split of Dirichlet lambda series.

\[
ak = \sum_{r=0}^{\infty} \left\{ \frac{1}{(4mr+2k-1)^{2n}} + \frac{1}{(4mr+4m-2k+1)^{2n}} \right\} \quad k=1, 2, \ldots, m
\]

\[a_1 + a_2 + \cdots + a_m = \tilde{\lambda}(2n)
\]

(2) Let Dirichlet beta series \( \beta(2n-1) \) be

\[
\beta(2n-1) = 1 - \frac{1}{3^{2n-1}} + \frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{11^{2n-1}} + \frac{1}{13^{2n-1}} - \cdots
\]

When this series is split as follows, we call this reflection \( m \)-split of Dirichlet beta series.

\[
ak = (-1)^{k-1} \sum_{r=0}^{\infty} \left\{ \frac{1}{(4mr+2k-1)^{2n-1}} - \frac{1}{(4mr+4m-2k+1)^{2n-1}} \right\} \quad k=1, 2, \ldots, m
\]

\[a_1 + a_2 + \cdots + a_m = \beta(2n-1)
\]

The following formula holds for Definition 6.1.3':

**Formula 6.1.4**

The sum of reflection split series \( a_k \) in Definition 6.1.3' is given by the following expression.

(1) For Dirichlet lambda series \( \tilde{\lambda}(2n) \),

\[
a_k = -\frac{\pi}{(4m)^{2n-1}} \frac{d^{2n-1}}{dz} \cot(\pi z) \frac{\pi}{(2n-1)!} \left. \frac{d^{2k-1}}{dz} \cot(\pi z) \right|_{\frac{2k-1}{4m}} \quad k=1, 2, \ldots, m \tag{1.4}_1
\]

(2) For Dirichlet beta series \( \beta(2n-1) \),

\[
a_k = \frac{(-1)^{k-1} \pi}{(4m)^{2n-1}} \frac{d^{2n-2}}{dz} \cot(\pi z) \frac{\pi}{(2n-2)!} \left. \frac{d^{2k-1}}{dz} \cot(\pi z) \right|_{\frac{2k-1}{4m}} \quad k=1, 2, \ldots, m \tag{1.4}_2
\]

**Proof**

Applying Formula 6.1.1 (1.1') to the reflection split series \( a_k \) in Definition 6.1.3',
(1) For Dirichlet lambda series $\lambda(2n)$,

$$a_k = \frac{1}{(4m)^{2n}(2n-1)!} \left\{ \psi_{2n-1} \left( \frac{2k-1}{4m} \right) + \psi_{2n-1} \left( \frac{4m-2k+1}{4m} \right) \right\} \quad k=1, 2, \ldots, m$$

From Formula 6.1.2,

$$\psi_{2n-1}(z) + \psi_{2n-1}(1-z) = -\pi \frac{d^{2n-1}}{dz^{2n-1}} \cot(\pi z) \quad (1.2a)$$

Substituting $z = (2k-1)/4m$ for this,

$$\psi_{2n-1} \left( \frac{2k-1}{4m} \right) + \psi_{2n-1} \left( \frac{4m-2k+1}{4m} \right) = -\pi \frac{d^{2n-1}}{dz^{2n-1}} \cot(\pi z) \bigg|_{\frac{2k-1}{4m}}$$

Substituting this for the right side of the above, we obtain (1.41).

(2) For Dirichlet beta series $\beta(2n-1)$,

In a way similar to (1), we obtain (1.42).

The following theorem is derived immediately from this formula.

**Theorem 6.1.5**

The sum of reflection split series $a_k$ in Definition 6.1.3 is a special value of an elementary function.

**Proof**

The higher order derivative of an elementary function is an elementary function. Then, the right side of (1.41) and (1.42) are special values of the elementary functions.
6.2 Reflection 2-split of Dirichlet Series

6.2.0 Basic 4-split of $\lambda(2n)$

Dirichlet lambda series $\lambda(2n)$ is

$$\lambda(2n) = 1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \cdots$$

The basic 4-split is

$$A_1 = 1 + \frac{1}{9^{2n}} + \frac{1}{17^{2n}} + \frac{1}{25^{2n}} + \frac{1}{33^{2n}} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+1)^{2n}}$$

$$A_2 = \frac{1}{2} + \frac{1}{11^{2n}} + \frac{1}{19^{2n}} + \frac{1}{27^{2n}} + \frac{1}{35^{2n}} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+3)^{2n}}$$

$$A_3 = \frac{1}{5} + \frac{1}{13^{2n}} + \frac{1}{21^{2n}} + \frac{1}{29^{2n}} + \frac{1}{37^{2n}} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+5)^{2n}}$$

$$A_4 = \frac{1}{7} + \frac{1}{15^{2n}} + \frac{1}{23^{2n}} + \frac{1}{31^{2n}} + \frac{1}{39^{2n}} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+7)^{2n}}$$

Expression by polygamma function

According to Formula 6.1.1 (1.1') the sums of the series $A_1, A_2, A_3, A_4$ are given by

$$A_1 = \sum_{r=0}^{\infty} \frac{1}{(8r+1)^{2n}} = \frac{1}{8^{2n}(2n-1)!!} \psi_{2n-1} \left( \frac{1}{8} \right)$$

$$A_2 = \sum_{r=0}^{\infty} \frac{1}{(8r+3)^{2n}} = \frac{1}{8^{2n}(2n-1)!!} \psi_{2n-1} \left( \frac{3}{8} \right)$$

$$A_3 = \sum_{r=0}^{\infty} \frac{1}{(8r+5)^{2n}} = \frac{1}{8^{2n}(2n-1)!!} \psi_{2n-1} \left( \frac{5}{8} \right)$$

$$A_4 = \sum_{r=0}^{\infty} \frac{1}{(8r+7)^{2n}} = \frac{1}{8^{2n}(2n-1)!!} \psi_{2n-1} \left( \frac{7}{8} \right)$$

6.2.1 Reflection 2-split of $\lambda(2n)$

Although there are 7 combinations in composing the series $A_1, A_2, A_3, A_4$ into 2 series, here we choose $(A_1 + A_4), (A_2 + A_3)$ such that the total of subscripts is 5. Then,

$$a_1 = 1 + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} + \frac{1}{15^{2n}} + \frac{1}{17^{2n}} + \cdots = \sum_{r=0}^{\infty} \left\{ \frac{1}{(8r+1)^{2n}} + \frac{1}{(8r+7)^{2n}} \right\}$$

$$a_1 = \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \frac{1}{19^{2n}} + \cdots = \sum_{r=0}^{\infty} \left\{ \frac{1}{(8r+3)^{2n}} + \frac{1}{(8r+5)^{2n}} \right\}$$

$$\lambda(2n) = a_1 + a_2$$

The sums of the series $a_1, a_2$ are given by

$$1 + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} + \frac{1}{15^{2n}} + \frac{1}{17^{2n}} + \cdots = \frac{1}{8^{2n}(2n-1)!!} \left\{ \psi_{2n-1} \left( \frac{1}{8} \right) + \psi_{2n-1} \left( \frac{7}{8} \right) \right\}$$

$$\frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \frac{1}{19^{2n}} + \cdots = \frac{1}{8^{2n}(2n-1)!!} \left\{ \psi_{2n-1} \left( \frac{3}{8} \right) + \psi_{2n-1} \left( \frac{5}{8} \right) \right\}$$
We can see that these right sides satisfy the requirements of Formula 6.1.2 (1.2) respectively. That is
\[ \psi_{2n-1}(z) + \psi_{2n-1}(1-z) = -\pi \frac{d^{2n-1}}{dz^{2n-1}} \cot (\pi z) \] (1.2)

Substituting \( z = 1/8, 3/8 \) for this respectively,
\[ \psi_{2n-1}\left(\frac{1}{8}\right) + \psi_{2n-1}\left(\frac{7}{8}\right) = -\pi \frac{d^{2n-1}}{dz^{2n-1}} \cot (\pi z) \mid_{1/8} \]
\[ \psi_{2n-1}\left(\frac{3}{8}\right) + \psi_{2n-1}\left(\frac{5}{8}\right) = -\pi \frac{d^{2n-1}}{dz^{2n-1}} \cot (\pi z) \mid_{3/8} \]

Substituting this for the above, we obtain the general expressions of reflection 2-split of \( \lambda(2n) \) series.

Example \( n=2 \)
\[ 1 + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} + \frac{1}{15^{2n}} + \frac{1}{17^{2n}} + \cdots = -\frac{\pi}{8^{2n}(2n-1)!} \frac{d^{2n-1}}{dz^{2n-1}} \cot (\pi z) \mid_{1/8} \]
\[ 1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \frac{1}{19^{2n}} + \cdots = -\frac{\pi}{8^{2n}(2n-1)!} \frac{d^{2n-1}}{dz^{2n-1}} \cot (\pi z) \mid_{3/8} \]

That is,
\[ a_1 = 1 + \frac{1}{7^4} + \frac{1}{9^4} + \frac{1}{15^4} + \frac{1}{17^4} + \cdots = \frac{\pi^4(16+11\sqrt{2})}{3072} = 1.000610446\cdots \]
\[ a_2 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{11^4} + \frac{1}{13^4} + \frac{1}{19^4} + \cdots = \frac{\pi^4(16-11\sqrt{2})}{3072} = 0.01406758545\cdots \]
\[ a_1 + a_2 = \frac{\pi^4(16+11\sqrt{2})}{3072} + \frac{\pi^4(16-11\sqrt{2})}{3072} = \frac{\pi^4}{96} = \lambda(4) = 1.014678031\cdots \]
6.2.2 Reflection 2-split of $\beta(2n-1)$

Dirichlet beta series $\beta(2n-1)$ is

$$\beta(2n-1) = 1 - \frac{1}{3^{2n-1}} + \frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{11^{2n-1}} + \frac{1}{13^{2n-1}} - \cdots$$

The basic 4-split is

$$A_1 = \frac{1}{9^{2n-1}} + \frac{1}{17^{2n-1}} + \frac{1}{25^{2n-1}} + \frac{1}{33^{2n-1}} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+1)^{2n-1}}$$

$$A_2 = -\left(\frac{1}{3^{2n-1}} + \frac{1}{11^{2n-1}} + \frac{1}{9^{2n-1}} + \frac{1}{7^{2n-1}} + \frac{1}{5^{2n-1}} + \frac{1}{3^{2n-1}} + \cdots \right) = -\sum_{r=0}^{\infty} \frac{1}{(8r+3)^{2n-1}}$$

$$A_3 = \frac{1}{5^{2n-1}} + \frac{1}{13^{2n-1}} + \frac{1}{21^{2n-1}} + \frac{1}{29^{2n-1}} + \frac{1}{37^{2n-1}} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+5)^{2n-1}}$$

$$A_4 = -\left(\frac{1}{7^{2n-1}} + \frac{1}{15^{2n-1}} + \frac{1}{31^{2n-1}} + \frac{1}{39^{2n-1}} + \frac{1}{37^{2n-1}} + \cdots \right) = -\sum_{r=0}^{\infty} \frac{1}{(8r+7)^{2n-1}}$$

Expression by polygamma function

According to Formula 6.1.1 (1.1) the sums of the series $A_1, A_2, A_3, A_4$ are given by

$$A_1 = \sum_{r=0}^{\infty} \frac{1}{(8r+1)^{2n-1}} = -\frac{1}{8^{2n-1}(2n-2)!} \psi_{2n-2} \left( \frac{1}{8} \right)$$

$$A_2 = -\sum_{r=0}^{\infty} \frac{1}{(8r+3)^{2n-1}} = \frac{1}{8^{2n-1}(2n-2)!} \psi_{2n-2} \left( \frac{3}{8} \right)$$

$$A_3 = \sum_{r=0}^{\infty} \frac{1}{(8r+5)^{2n-1}} = -\frac{1}{8^{2n-1}(2n-2)!} \psi_{2n-2} \left( \frac{5}{8} \right)$$

$$A_4 = -\sum_{r=0}^{\infty} \frac{1}{(8r+7)^{2n-1}} = \frac{1}{8^{2n-1}(2n-2)!} \psi_{2n-2} \left( \frac{7}{8} \right)$$

Reflection 2-split of $\beta(2n-1)$

Although there are 7 combinations in composing the series $A_1, A_2, A_3, A_4$ into 2 series, here we choose $(A_1 + A_4), (A_2 + A_3)$ such that the total of subscripts is 5. Then,

$$a_1 = 1 - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{15^{2n-1}} + \frac{1}{17^{2n-1}} - \frac{1}{23^{2n-1}} + \frac{1}{25^{2n-1}} - \cdots$$

$$a_2 = -\left(\frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} + \frac{1}{11^{2n-1}} - \frac{1}{13^{2n-1}} + \frac{1}{19^{2n-1}} - \frac{1}{21^{2n-1}} + \cdots \right)$$

$$\beta(2n-1) = a_1 + a_2$$

The sums of the series $a_1, a_2$ are given by

$$1 - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{15^{2n-1}} + \cdots = -\frac{1}{8^{2n-1}(2n-2)!} \left\{ \psi_{2n-2} \left( \frac{1}{8} \right) - \psi_{2n-2} \left( \frac{7}{8} \right) \right\}$$

$$\left(\frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} + \frac{1}{11^{2n-1}} - \frac{1}{13^{2n-1}} + \cdots \right) = \frac{1}{8^{2n-1}(2n-2)!} \left\{ \psi_{2n-2} \left( \frac{3}{8} \right) - \psi_{2n-2} \left( \frac{5}{8} \right) \right\}$$
We can see that these right sides satisfy the requirements of Formula 6.1.2 (1.2) respectively. That is

\[
\psi_{2n-2}(z) - \psi_{2n-2}(1-z) = -\pi \frac{d^{2n-2}}{dz^{2n-2}} \cot (\pi z) \quad (1.2)
\]

Substituting \( z = 1/8, 3/8 \) for this respectively,

\[
\psi_{2n-2}\left(\frac{1}{8}\right) - \psi_{2n-2}\left(\frac{7}{8}\right) = -\pi \frac{d^{2n-2}}{dz^{2n-2}} \cot (\pi z) \bigg|_{1/8}
\]

\[
\psi_{2n-2}\left(\frac{3}{8}\right) - \psi_{2n-2}\left(\frac{5}{8}\right) = -\pi \frac{d^{2n-2}}{dz^{2n-2}} \cot (\pi z) \bigg|_{3/8}
\]

Substituting this for the above, we obtain the general expressions of reflection 2-split of \( \beta (2n-1) \) series.

\[
1 - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{15^{2n-1}} + \cdots = \frac{\pi}{8^{2n-1} (2n-1)!} \frac{d^{2n-2}}{dz^{2n-2}} \cot (\pi z) \bigg|_{1/8}
\]

\[
- \left( \frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} + \frac{1}{11^{2n-1}} - \frac{1}{13^{2n-1}} + \cdots \right) = -\frac{\pi}{8^{2n-1} (2n-2)!} \frac{d^{2n-2}}{dz^{2n-2}} \cot (\pi z) \bigg|_{3/8}
\]

**Example \( n = 3 \)**

\[
1 - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{15^5} + \frac{1}{17^5} - \frac{1}{23^5} + \cdots = \frac{\pi}{8^5 4!} \frac{d^4}{dz^4} \cot (\pi z) \bigg|_{1/8}
\]

\[
= \frac{\pi^5}{8^5 4!} \left\{ 8 \cot^3 (\pi z) \csc^2 (\pi z) + 16 \cot (\pi z) \csc^4 (\pi z) \right\}
\]

\[
= \frac{\pi^5}{8^5 4!} 8 \left( \frac{\sqrt{2} + 1}{3} \right)^3 \left( \sqrt{4 + 2\sqrt{2}} \right)^2 + 16 (\sqrt{2}+1) \left( \sqrt{4+2\sqrt{2}} \right)^4
\]

\[
- \left( \frac{1}{3^5} - \frac{1}{5^5} + \frac{1}{11^5} - \frac{1}{13^5} + \frac{1}{19^5} - \frac{1}{21^5} + \cdots \right) = -\frac{\pi}{8^5 4!} \frac{d^4}{dz^4} \cot (\pi z) \bigg|_{3/8}
\]

\[
= -\frac{\pi^5}{8^5 4!} \left\{ 8 \cot^3 (\pi z) \csc^2 (\pi z) + 16 \cot (\pi z) \csc^4 (\pi z) \right\}
\]

\[
= -\frac{\pi^5}{8^5 4!} 8 \left( \frac{\sqrt{2} - 1}{3} \right)^3 \left( \sqrt{4 - 2\sqrt{2}} \right)^2 + 16 (\sqrt{2}-1) \left( \sqrt{4-2\sqrt{2}} \right)^4
\]

That is,

\[
a_1 = 1 - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{15^5} + \frac{1}{17^5} - \frac{1}{23^5} + \cdots = \frac{\pi^5 (80+57\sqrt{2})}{49152}
\]

\[
a_2 = - \left( \frac{1}{3^5} - \frac{1}{5^5} + \frac{1}{11^5} - \frac{1}{13^5} + \frac{1}{19^5} - \frac{1}{21^5} + \cdots \right) = \frac{\pi^5 (80 - 57\sqrt{2})}{49152}
\]
\[ a_1 + a_2 = \frac{\pi^5 (80 + 57\sqrt{2})}{49152} + \frac{\pi^5 (80 - 57\sqrt{2})}{49152} = \frac{5\pi^5}{1536} = \beta(5) \]

\[ = 0.9999567572\ldots - 0.003798929159\ldots = 0.9961578280\ldots \]
6.3 Reflection 3-split of Dirichlet Series

Reflection odd-split of Dirichlet series is also possible. In this section, we take up reflection 3-split of Dirichlet beta series as an example.

6.3.0 Basic 6-split of $\beta(2n-1)$

According to Definition 6.1.3 (2),

$$a_k = (-1)^{k-1} \sum_{r=0}^{\infty} \left( \frac{1}{(4mr+2k-1)^{2n-1}} - \frac{1}{(4mr+4m-2k+1)^{2n-1}} \right) \quad k=1, 2, \ldots, m$$

Putting $m=3$,

$$a_1 = \sum_{r=0}^{\infty} \left[ \frac{1}{(12r+1)^{2n-1}} - \frac{1}{(12r+11)^{2n-1}} \right]$$

$$a_2 = -\sum_{r=0}^{\infty} \left[ \frac{1}{(12r+3)^{2n-1}} - \frac{1}{(12r+9)^{2n-1}} \right]$$

$$a_3 = \sum_{r=0}^{\infty} \left[ \frac{1}{(12r+5)^{2n-1}} - \frac{1}{(12r+7)^{2n-1}} \right]$$

Expanding these,

$$a_1 = 1 - \frac{1}{12^{2n-1}} + \frac{1}{13^{2n-1}} - \frac{1}{23^{2n-1}} + \frac{1}{25^{2n-1}} - \frac{1}{35^{2n-1}} + \ldots$$

$$a_2 = -\left( \frac{1}{3^{2n-1}} - \frac{1}{9^{2n-1}} + \frac{1}{15^{2n-1}} - \frac{1}{21^{2n-1}} + \frac{1}{27^{2n-1}} - \frac{1}{33^{2n-1}} + \ldots \right)$$

$$a_3 = \frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \frac{1}{17^{2n-1}} - \frac{1}{19^{2n-1}} + \frac{1}{29^{2n-1}} - \frac{1}{31^{2n-1}} + \ldots$$

$$\beta(2n-1) = a_1 + a_2 + a_3$$

According to Formula 6.1.1 (1.1'), the sums of the series $a_1, a_2, a_3$ are given by

$$a_1 = -\left( \frac{1}{12^{2n-1}} \psi_{2n-2} \left( \frac{1}{12} \right) - \psi_{2n-2} \left( \frac{11}{12} \right) \right)$$

$$a_2 = \left( \frac{1}{12^{2n-1}} \psi_{2n-2} \left( \frac{3}{12} \right) - \psi_{2n-2} \left( \frac{9}{12} \right) \right)$$

$$a_3 = -\left( \frac{1}{12^{2n-1}} \psi_{2n-2} \left( \frac{5}{12} \right) - \psi_{2n-2} \left( \frac{7}{12} \right) \right)$$

We can see that these right sides satisfy the requirements of Formula 6.1.2 (1.2) respectively. That is,

$$\psi_{2n-2}(z) - \psi_{2n-2}(1-z) = -\pi \left( \cot(\pi z) \right)^{2n-2}$$

Substituting $z=\frac{1}{12}, \frac{3}{12}, \frac{5}{12}$ for this respectively,

$$\psi_{2n-2} \left( \frac{1}{12} \right) - \psi_{2n-2} \left( \frac{11}{12} \right) = -\pi \left( \frac{d}{dz} \frac{1}{2n-2} \cot(\pi z) \right)_{1/12}$$

$$\psi_{2n-2} \left( \frac{3}{12} \right) - \psi_{2n-2} \left( \frac{9}{12} \right) = -\pi \left( \frac{d}{dz} \frac{1}{2n-2} \cot(\pi z) \right)_{3/12}$$

$$\psi_{2n-2} \left( \frac{5}{12} \right) - \psi_{2n-2} \left( \frac{7}{12} \right) = -\pi \left( \frac{d}{dz} \frac{1}{2n-2} \cot(\pi z) \right)_{5/12}$$
Substituting this for the above, we obtain the general expressions of reflection 3-split of $\beta(2n-1)$ series.

$$1 - \frac{1}{11^{2n-1}} + \frac{1}{13^{2n-1}} - \frac{1}{23^{2n-1}} + \frac{1}{25^{2n-1}} + \cdots = -\frac{\pi}{12^{2n-1}(2n-2)!} \frac{d^{2n-2}}{dz^{2n-2}} \cot(z)$$

$$-\left(\frac{1}{3^{2n-1}} - \frac{1}{9^{2n-1}} + \frac{1}{15^{2n-1}} - \frac{1}{21^{2n-1}} + \cdots\right) = \frac{\pi}{12^{2n-1}(2n-2)!} \frac{d^{2n-2}}{dz^{2n-2}} \cot(z)$$

$$\frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \frac{1}{17^{2n-1}} - \frac{1}{19^{2n-1}} + \cdots = -\frac{\pi}{12^{2n-1}(2n-2)!} \frac{d^{2n-2}}{dz^{2n-2}} \cot(z)$$

Example $n=3$

$$1 - \frac{1}{11^5} + \frac{1}{13^5} - \frac{1}{23^5} + \frac{1}{25^5} - \frac{1}{35^5} + \cdots = -\frac{\pi}{12^74!} \frac{d^4}{dz^4} \cot(z)$$

$$-\left(\frac{1}{3^5} - \frac{1}{9^5} + \frac{1}{15^5} - \frac{1}{21^5} + \frac{1}{27^5} - \frac{1}{33^5} + \cdots\right) = \frac{\pi}{12^74!} \frac{d^4}{dz^4} \cot(z)$$

$$\frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{17^5} - \frac{1}{19^5} + \frac{1}{29^5} - \frac{1}{31^5} + \cdots = -\frac{\pi}{12^54!} \frac{d^4}{dz^4} \cot(z)$$

Calculating in the same way as in the previous section, we obtain

$$1 - \frac{1}{11^5} + \frac{1}{13^5} - \frac{1}{23^5} + \frac{1}{25^5} - \frac{1}{35^5} + \cdots = \frac{\pi^5(305+176\sqrt{3})}{186624}$$

$$-\left(\frac{1}{3^5} - \frac{1}{9^5} + \frac{1}{15^5} - \frac{1}{21^5} + \frac{1}{27^5} - \frac{1}{33^5} + \cdots\right) = -\frac{5\pi^5}{373248}$$

$$\frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{17^5} - \frac{1}{19^5} + \frac{1}{29^5} - \frac{1}{31^5} + \cdots = \frac{\pi^5(305-176\sqrt{3})}{186624}$$

$$\frac{\pi^5(305+176\sqrt{3})}{186624} - \frac{5\pi^5}{373248} + \frac{\pi^5(305-176\sqrt{3})}{186624} = \frac{5\pi^5}{1536} = \beta(5)$$

**Composite reflection 2-split**

Thus, $a_1, a_2, a_3$ are all special values of elementary functions. So, when 2-split series of $\beta(5)$ are composed from these, $(a_1, a_2 + a_3), (a_2, a_1 + a_3), (a_3, a_1 + a_2)$ can be made. Summations of these series are of course special values of elementary functions. And, these are different from the reflection 2-split series in the previous section. That is, 4 sets of the reflection 2-split series have been obtained.

If such operations are performed on more than basic 8-split series, an infinite set of reflection split series can be obtained.

**Infinity of composite reflection split**

In a similar way, an infinite set of reflection m-split series ($m=3, 4, 5, \cdots$) can be obtained.
6.4 Algebraic Solvability of Reflection Split

Each sum of split series satisfying the reflection formula for polygamma function is obtained by algebraic solution, except circle ratio $\pi$. In this section, we prove and exemplify it.

6.4.1 Reflection Formula for Trigonometric Functions

Formula 6.4.1 (Trigonometric Reflection Formula)

For natural number $n$ and $k=1, 2, 3, \ldots, n-1$, the following expressions hold.

\[
\cos \frac{k\pi}{n} = \frac{1}{2}
\left\{ (-1)^{k/n} - (-1)^{1-k/n} \right\}
\]

\[
\sin \frac{k\pi}{n} = \frac{1}{2i}
\left\{ (-1)^{k/n} + (-1)^{1-k/n} \right\}
\]

Proof

\[
\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}
\]

Putting $z = \frac{k\pi}{n}$

\[
\cos \frac{k\pi}{n} = \frac{1}{2}
\left( e^{\frac{k\pi i}{n}} + e^{-\frac{k\pi i}{n}} \right), \quad \sin \frac{k\pi}{n} = \frac{1}{2i}
\left( e^{\frac{k\pi i}{n}} - e^{-\frac{k\pi i}{n}} \right)
\]

Since $e^{\pi i} = -1$,

\[
\cos \frac{k\pi}{n} = \frac{1}{2}
\left\{ (-1)^{k/n} + (-1)^{1-k/n} \right\}, \quad \sin \frac{k\pi}{n} = \frac{1}{2i}
\left\{ (-1)^{k/n} - (-1)^{1-k/n} \right\}
\]

Further,

\[
(-1)^{1-k/n} = -(-1)^{1-k/n} = -(-1)^{1-k/n}
\]

Substituting this for the above, we obtain

\[
\cos \frac{k\pi}{n} = \frac{1}{2}
\left\{ (-1)^{k/n} - (-1)^{1-k/n} \right\}, \quad \sin \frac{k\pi}{n} = \frac{1}{2i}
\left\{ (-1)^{k/n} + (-1)^{1-k/n} \right\}
\]

Example $\cos 2\pi / 7, \sin 2\pi / 7$

\[
\begin{align*}
\mathbb{N}\left\{ \cos \frac{2\pi}{7}, \frac{1}{2} \left( (-1)^{\frac{2}{7}} - (-1)^{\frac{5}{7}} \right) \right\}, 8 & \quad \mathbb{N}\left[ \left\{ \sin \frac{2\pi}{7}, \frac{1}{2i} \left( (-1)^{\frac{2}{7}} - (-1)^{\frac{5}{7}} \right) \right\}, 8 
\right. \\
\{0.62348980, 0.62348980 \times 10^{-10} i\} & \quad \{0.78183148, 0.78183148 \times 10^{-10} i\}
\end{align*}
\]

Corollary 6.4.1

For natural number $n$ and $k=1, 2, 3, \ldots, n-1$, the following expressions hold.

\[
\cot \frac{k\pi}{n} = i\left\{ \frac{k}{n} - (-1)^{1-k/n} \right\}
\]
\[
\csc \frac{k\pi}{n} = \frac{2i}{(\frac{k}{n})^n + (\frac{1}{n})^{1 - \frac{k}{n}}}
\]

### 6.4.2 Algebraic Solvability of Reflection Split

Using Corollary 6.4.1, we can prove that each sum of split series is expressed with addition, multiplication and rational powers, except circle ratio \( \pi \).

**Theorem 6.4.2 (Algebraic Solvability)**

The sum of reflection split series \( a_k \) in Definition 6.1.3' is expressed with addition, multiplication and rational powers, except circle ratio \( \pi \).

**Proof**

According to Formula 6.1.4, the \( a_k \) in Definition 6.1.3' is given by the following expression.

\[
(1) \quad a_k = -\frac{\pi}{(4m)^{2n}(2n-1)!} \frac{d^{2n-1}}{dz^{2n-1}} \cot (\pi z) \left|_{\frac{2k-1}{4m}} \right. \\
\]

As seen in the calculation example of Formula 6.1.2, this higher order difference quotient is a polynomial of \( \cot \frac{(2k-1)\pi}{4m} \) and \( \csc \frac{(2k-1)\pi}{4m} \). Then, by Corollary 6.4.1, this is expressed with addition, multiplication and rational powers of \( (-1)^k \).

(2) For Dirichlet beta series \( \beta (2n-1) \),

For the same reason as (1), \( a_k \) is expressed with addition, multiplication and rational powers of \( (-1)^k \), except circle ratio \( \pi \).

**Remark**

Certainly, this is an algebraic expression. In the first place, \( (-1)^{1/n}, (-1)^{2/n}, \ldots, (-1)^{(n-1)/n} \) are \( n \)-equally divided points of the unit semicircle drawn on the Gaussian plane. That is, these are solutions of equation \( z^n + 1 = 0 \). Although algebraic equations of 5th degree or higher cannot be solved algebraically in general, it is proved by Gauss that such a cyclotomic equation can be solved algebraically.

**Example Reflect 7-split of \( \beta (3) \)**

According to Definition 6.1.3' (2), Dirichlet beta series \( \beta (3) \) and the reflection 7-split series are as follows.

\[
\beta (3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \ldots
\]

\[
a_1 = 1 - \frac{1}{27^3} + \frac{1}{29^3} - \frac{1}{55^3} + \frac{1}{57^3} - \frac{1}{83^3} + \ldots
\]

\[
a_2 = -\left( \frac{1}{3^3} - \frac{1}{25^3} + \frac{1}{31^3} - \frac{1}{53^3} + \frac{1}{59^3} - \frac{1}{81^3} + \ldots \right)
\]

\[
a_3 = \frac{1}{5^3} - \frac{1}{23^3} + \frac{1}{33^3} - \frac{1}{51^3} + \frac{1}{61^3} - \frac{1}{79^3} + \ldots
\]
Calculating the right sides, 

\[ a_4 = -\left( \frac{1}{7^3} - \frac{1}{21^3} + \frac{1}{35^3} - \frac{1}{49^3} + \frac{1}{63^3} - \frac{1}{77^3} + \cdots \right) \]

\[ a_5 = -\left( \frac{1}{9^3} - \frac{1}{19^3} + \frac{1}{37^3} - \frac{1}{47^3} + \frac{1}{65^3} - \frac{1}{75^3} + \cdots \right) \]

\[ a_6 = -\left( \frac{1}{11^3} - \frac{1}{17^3} + \frac{1}{39^3} - \frac{1}{45^3} + \frac{1}{67^3} - \frac{1}{73^3} + \cdots \right) \]

\[ a_7 = \left( \frac{1}{13^3} - \frac{1}{15^3} + \frac{1}{41^3} - \frac{1}{43^3} + \frac{1}{69^3} - \frac{1}{71^3} + \cdots \right) \]

\[ \beta(3) = a_1 + a_2 + a_3 + \cdots + a_7 \]

According to Formula 6.1.1 (1.1), the sums of the series \( a_1, a_2, a_3, \ldots, a_7 \) are given by

\[ a_1 = \sum_{r=0}^{\infty} \frac{1}{(28r+1)^3} - \frac{1}{(28r+27)^3} = -\frac{1}{28^3 2!} \left\{ \psi_2\left( \frac{1}{28} \right) - \psi_2\left( \frac{27}{28} \right) \right\} \]

\[ a_2 = -\sum_{r=0}^{\infty} \frac{1}{(28r+3)^3} - \frac{1}{(28r+25)^3} = \frac{1}{28^3 2!} \left\{ \psi_2\left( \frac{3}{28} \right) - \psi_2\left( \frac{25}{28} \right) \right\} \]

\[ \vdots \]

\[ a_6 = -\sum_{r=0}^{\infty} \frac{1}{(28r+11)^3} - \frac{1}{(28r+17)^3} = \frac{1}{28^3 2!} \left\{ \psi_2\left( \frac{11}{28} \right) - \psi_2\left( \frac{17}{28} \right) \right\} \]

\[ a_7 = \sum_{r=0}^{\infty} \frac{1}{(28r+13)^3} - \frac{1}{(28r+15)^3} = -\frac{1}{28^3 2!} \left\{ \psi_2\left( \frac{13}{28} \right) - \psi_2\left( \frac{15}{28} \right) \right\} \]

We can see that these right sides satisfy the requirements of Formula 6.1.2 (1.2) respectively. That is,

\[ \psi_{2n-2}(z) - \psi_{2n-2}(1-z) = -\pi \frac{d^{2n-2}}{dz^{2n-2}} \cot(\pi z) \]

Substituting \( z=1/28, 3/28, \ldots, 13/28 \) for this respectively,

\[ 1 - \frac{1}{27^3} + \frac{1}{29^3} - \frac{1}{55^3} + \frac{1}{57^3} - \frac{1}{83^3} + \cdots = \frac{\pi}{28^3 2!} \left\{ \frac{d^2}{dz^2} \cot(\pi z) \right\} \Bigg|_{1/28} \]

\[ -\left( \frac{1}{3^3} - \frac{1}{25^3} + \frac{1}{31^3} - \frac{1}{53^3} + \frac{1}{59^3} - \frac{1}{81^3} + \cdots \right) = -\frac{\pi}{28^3 2!} \left\{ \frac{d^2}{dz^2} \cot(\pi z) \right\} \Bigg|_{3/28} \]

\[ \vdots \]

\[ -\left( \frac{1}{11^3} - \frac{1}{17^3} + \frac{1}{39^3} - \frac{1}{45^3} + \frac{1}{67^3} - \frac{1}{73^3} + \cdots \right) = -\frac{\pi}{28^3 2!} \left\{ \frac{d^2}{dz^2} \cot(\pi z) \right\} \Bigg|_{11/28} \]

\[ \frac{1}{13^3} - \frac{1}{15^3} + \frac{1}{41^3} - \frac{1}{43^3} + \frac{1}{69^3} - \frac{1}{71^3} + \cdots = \frac{\pi}{28^3 2!} \left\{ \frac{d^2}{dz^2} \cot(\pi z) \right\} \Bigg|_{13/28} \]

Calculating the right sides,

\[ 1 - \frac{1}{27^3} + \frac{1}{29^3} - \frac{1}{55^3} + \frac{1}{57^3} - \frac{1}{83^3} + \cdots = \frac{\pi^3}{28^3 2!} \left\{ 2\cot(\frac{\pi}{28}) \csc^2 \frac{3\pi}{28} \right\} \]

\[ -\left( \frac{1}{3^3} - \frac{1}{25^3} + \frac{1}{31^3} - \frac{1}{53^3} + \frac{1}{59^3} - \frac{1}{81^3} + \cdots \right) = -\frac{\pi^3}{28^3 2!} \left\{ 2\cot(\frac{3\pi}{28}) \csc^2 \frac{3\pi}{28} \right\} \]

\[ \vdots \]
Applying Corollary 6.4.1 to these, we obtain

\[- \left( \frac{1}{11^3} - \frac{1}{17^3} + \frac{1}{39^3} - \frac{1}{45^3} + \frac{1}{67^3} - \frac{1}{73^3} + \cdots \right) = -\frac{\pi^3}{28^3} \left\{ \frac{\text{cot} \frac{11\pi}{28} \csc^2 \frac{11\pi}{28}}{28^2!} \right\} \]

\[- \left( \frac{1}{13^3} - \frac{1}{15^3} + \frac{1}{41^3} - \frac{1}{43^3} + \frac{1}{69^3} - \frac{1}{71^3} + \cdots \right) = \frac{\pi^3}{28^3} \left\{ \frac{\text{cot} \frac{13\pi}{28} \csc^2 \frac{13\pi}{28}}{28^2!} \right\} \]

The calculation results of the 1st and the 2nd series by Mathematica are as follows.

\[\text{a1}[\_m_] := \sum_{r=0}^{N} \left\{ \frac{1}{(28 r + 1)^3} - \frac{1}{(28 r + 27)^3} \right\} \]

\[\text{N[a1[10000], 10]} \quad 0.9999893925 \]

\[\text{a2}[\_m_] := -\sum_{r=0}^{N} \left\{ \frac{1}{(28 r + 3)^3} - \frac{1}{(28 r + 25)^3} \right\} \]

\[\text{N[a2[10000], 10]} \quad -0.03700417387 \]
6.5 Reflection $m 2^n$-split of Dirichlet Series

Among the reflection split, in particular, the sum of the $m 2^n$-split series is expressed by the nested radical of $m$ or $2$ and their addition and multiplication. In this section, we prove and exemplify it.

**Formula 6.5.1**

When $T_n$ is Chebyshev Polynomial of the 1st kind, the following expressions hold for natural numbers $m, k$.

\[
\cos \left(\frac{(2k-1)\pi}{m}\right) = T_{2k-1} \left(\cos \frac{\pi}{m}\right) \quad (5.1c)
\]

\[
\sin \left(\frac{(2k-1)\pi}{m}\right) = (-1)^{k-1} T_{2k-1} \left(\sin \frac{\pi}{m}\right) \quad (5.1s)
\]

**Proof**

According [http://mathworld.wolfram.com/Multiple-AngleFormulas.html](http://mathworld.wolfram.com/Multiple-AngleFormulas.html), the multiple-angle formulas for trigonometric functions are represented using Chebyshev Polynomial of the 1st kind as follows.

\[
cos nx = T_n(\cos x)
\]

\[
\sin nx = (-1)^{(n-1)/2} T_n(\sin x) \quad \text{for } n \text{ odd}
\]

Substituting $n = 2k - 1, x = \pi / m$ for these, we obtain the desired expressions.

**Example**

\[
\sin \left(\frac{5\pi}{32}\right) = (-1)^{3-1} T_{2^{3-1}} \left(\sin \frac{\pi}{32}\right) = 5 \sin \left(\frac{\pi}{32}\right) - 20 \sin^3 \left(\frac{\pi}{32}\right) + 16 \sin^5 \left(\frac{\pi}{32}\right)
\]

**Formula 6.5.2**

The following expressions hold for natural number $n$.

\[
\begin{align*}
\cos \frac{\pi}{2^n} &= \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} \\
(\text{n-1-nests}) \\
\sin \frac{\pi}{2^n} &= \frac{1}{2} \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}
\end{align*}
\]

\[
\begin{align*}
\cos \frac{\pi}{3 \cdot 2^n} &= \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{3}}}} \\
(\text{n-nests}) \\
\sin \frac{\pi}{3 \cdot 2^n} &= \frac{1}{2} \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{3}}}}
\end{align*}
\]

\[
\begin{align*}
\cos \frac{\pi}{5 \cdot 2^n} &= \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{5}}}} \\
(\text{n-nests}) \\
\sin \frac{\pi}{5 \cdot 2^n} &= \frac{1}{2} \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{5}}}}
\end{align*}
\]
Proof

\[ \cos \frac{\theta}{2^n} = \sqrt{\frac{1}{2} + \frac{1}{2} \cos \theta} \]

Putting \( \theta = \pi / 2 \),

\[ \cos \frac{\pi}{2^n} = \sqrt{\frac{1}{2} + \frac{1}{2} \cos \frac{\pi}{2}} = \sqrt{\frac{1}{2} + 0} = \frac{1}{2} \sqrt{2} \]

\[ \cos \frac{\pi}{2^3} = \sqrt{\frac{1}{2} + \frac{1}{2} \cos \frac{\pi}{2}} = \frac{1}{2} \sqrt{2 + \sqrt{2}} \]

\[ \cos \frac{\pi}{2^4} = \sqrt{\frac{1}{2} + \frac{1}{2} \cos \frac{\pi}{2}} = \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}} \]

\[ \vdots \]

Hereafter, by induction, we obtain \( \cos \left( \frac{\pi}{2^n} \right) \). \( \sin \left( \frac{\pi}{2^n} \right) \) is obtained using \( \sqrt{1 - \cos^2 z} \).

(5.23) and (5.25) are also derived in a similar way.

**Theorem 6.5.3**

The sum of reflection split series \( a_k \quad k = 1, 2, \ldots, 2^n \) in Definition 6.1.3' is expressed with addition, multiplication and the nested radical of 2, except circle ratio \( \pi \).

**Proof**

According to Formula 6.1.4, the sum \( a_k \) of the \( 2^n \)-split series is given by the following expression.

1. For Dirichlet lambda series \( \lambda(2h) \),

\[ a_k = -\frac{\pi}{(4 \cdot 2^n)^{2h} (2h-1)!} \frac{d^{2h-1} \cot (\pi z)}{dz^{2h-1} \cot (\pi z)} \quad k = 1, 2, \ldots, 2^n \]

As seen in the calculation example of Formula 6.1.2, this higher order difference quotient is a polynomial of \( \cot \left( \frac{2k-1}{2^n+2} \pi \right) \) and \( \csc \left( \frac{2k-1}{2^n+2} \pi \right) \). Since \( \cot z = \cos z / \sin z \), \( \csc z = 1 / \sin z \), this is expressed with a polynomial of \( \cos \left( \frac{\pi}{2^n+2} \right) \) and \( \sin \left( \frac{\pi}{2^n+2} \right) \) from Formula 6.5.1. And, from Formula 6.5.2 (5.23), this higher order difference quotient results in a polynomial with the following elements.

\[ \frac{1}{2} \sqrt{2 \pm \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}} \quad (n+1)-\text{nests} \]

**Corollary 6.5.3**

1. The sum of reflection split series \( a_k \quad k = 1, 2, \ldots, 3 \cdot 2^n \) is expressed with addition, multiplication and the nested radical of 2 and 3, except circle ratio \( \pi \).

2. The sum of reflection split series \( a_k \quad k = 1, 2, \ldots, 5 \cdot 2^n \) is expressed with addition, multiplication and the nested radical of 2 and 5, except circle ratio \( \pi \).
Example 1: Reflection 8-split of \( \lambda (2) \)

Dirichlet lambda series \( \lambda (2) \) is

\[
\lambda (2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots
\]

According to Definition 6.1.3 (1)

\[
a_k = \sum_{r=0}^{\infty} \left\{ \frac{1}{(4mr+2k-1)^{2n}} + \frac{1}{(4m+4m-2k+1)^{2n}} \right\} \quad k=1, 2, \ldots, m
\]

\[
a_1 + a_2 + \cdots + a_m = \lambda (2n)
\]

Putting \( n=1, m=8 \),

\[
a_k = \sum_{r=0}^{\infty} \left\{ \frac{1}{(32r+2k-1)^{2n}} + \frac{1}{(32r+32-2k+1)^{2n}} \right\} \quad k=1, 2, \ldots, 8
\]

Expanding these,

\[
a_1 = 1 + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{63^2} + \frac{1}{65^2} + \frac{1}{95^2} + \cdots
\]

\[
a_2 = \frac{1}{3^2} + \frac{1}{29^2} + \frac{1}{31^2} + \frac{1}{61^2} + \frac{1}{67^2} + \frac{1}{93^2} + \cdots
\]

\[\vdots\]

\[
a_8 = \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{79^2} + \frac{1}{81^2} + \cdots
\]

\[
\lambda (2) = a_1 + a_2 + a_3 + \cdots + a_8
\]

From Formula 6.1.4 (1),

\[
a_k = -\frac{\pi}{32^2 1!} \frac{d}{dz} \cot (\pi z) \bigg| _{2k-1 \frac{32}{32}} k=1, 2, \ldots, 8
\]

Here,

\[
\frac{d}{dz} \cot (\pi z) = -\pi \csc^2 (\pi z)
\]

So,

\[
a_k = \frac{\pi^2}{32^2 1!} \csc^2 \left( \frac{(2k-1) \pi}{32} \right) = \frac{\pi^2}{32^2 1!} \left\{ \sin \left( \frac{(2k-1) \pi}{32} \right) \right\}^{-2} \quad k=1, 2, \ldots, 8
\]

From Formula 6.5.1, the following expression holds for \( k=1, 2, \ldots, 8 \).

\[
\sin \left( \frac{(2k-1) \pi}{32} \right) = (-1)^{k-1} T_{2k-1} (\beta) \quad , \quad \beta = \sin \left( \frac{\pi}{32} \right)
\]

Using Formula 6.5.2 (5.22), let

\[
s_k = (-1)^{k-1} T_{2k-1} (\beta) \quad , \quad \beta = \frac{1}{2} \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}
\]

Then,

\[
a_k = \frac{\pi^2}{32^2 1!} \frac{1}{s_k} =: b_k \quad k=1, 2, \ldots, 8
\]

After calculating and simplifying \( b_k \) using Mathematica, if the results are written together with \( a_k \), it is as follows. Of course, both sides are numerically consistent.
\[a_1 = 1 + \frac{1}{3^2} + \frac{1}{33^2} + \frac{1}{63^2} + \frac{1}{65^2} + \frac{1}{95^2} + \cdots = \frac{\pi^2}{256\left(2\sqrt{2} + \sqrt{2+\sqrt{2}}\right)}\]

\[a_2 = \frac{1}{3^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{61^2} + \frac{1}{67^2} + \frac{1}{93^2} + \cdots = \frac{\pi^2}{256\left(2\sqrt{2} - \sqrt{2+\sqrt{2}}\right)}\]

\[a_3 = \frac{1}{5^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{59^2} + \frac{1}{69^2} + \frac{1}{91^2} + \cdots = \frac{\pi^2}{256\left(2\sqrt{2} - \sqrt{2-\sqrt{2}}\right)}\]

\[a_4 = \frac{1}{7^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{57^2} + \frac{1}{71^2} + \frac{1}{89^2} + \cdots = \frac{\pi^2}{256\left(2\sqrt{2} + \sqrt{2-\sqrt{2}}\right)}\]

\[a_5 = \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{41^2} + \frac{1}{55^2} + \frac{1}{73^2} + \frac{1}{87^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2} - \sqrt{2+\sqrt{2}}\right)}\]

\[a_6 = \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{43^2} + \frac{1}{53^2} + \frac{1}{75^2} + \frac{1}{85^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2} + \sqrt{2-\sqrt{2}}\right)}\]

\[a_7 = \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{45^2} + \frac{1}{51^2} + \frac{1}{77^2} + \frac{1}{83^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2+\sqrt{2}}\right)}\]

\[a_8 = \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{79^2} + \frac{1}{81^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}\]

\[\lambda(2) = a_1 + a_2 + a_3 + \cdots + a_8\]

**Note**

When \(\lambda(2)\) is split in \(2^n\) series, the denominator on the right side is the \(n\)-nested square root of \(2\).

\[a_k = \sum_{r=0}^{\infty} \left\{ \frac{1}{(4^2 r + 2k-1)^2} + \frac{1}{(4^2 r + 4^2 r - 2k+1)^2} \right\} = \frac{\pi^2}{2^{2n+2}\left(2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2}}}\right)}\]

where, the sign differs according to \(k\)

**Example 2: Reflection 6-split of \(\beta(3)\)**

Dirichlet beta series \(\beta(3)\) is

\[\beta(3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + \cdots\]

According to **Definition 6.1.3 (2)**,

\[a_k = (-1)^{k-1} \sum_{r=0}^{\infty} \left\{ \frac{1}{(4mr+2k-1)^{2n-1}} - \frac{1}{(4mr+4m-2k+1)^{2n-1}} \right\} \quad k=1, 2, \ldots, m\]

\[a_1 + a_2 + \cdots + a_m = \beta(2n-1)\]

Putting \(n=2, m=6\),

\[a_k = (-1)^{k-1} \sum_{r=0}^{\infty} \left\{ \frac{1}{(24r+2k-1)^{3}} - \frac{1}{(24r+24-2k+1)^{3}} \right\} \quad k=1, 2, \ldots, 6\]

Expanding these,
\[ a_1 = \frac{1}{23^3} - \frac{1}{25^3} + \frac{1}{47^3} + \frac{1}{49^3} - \frac{1}{71^3} + \cdots \]

\[ a_2 = -\left( \frac{1}{3^3} - \frac{1}{21^3} + \frac{1}{27^3} - \frac{1}{45^3} + \frac{1}{51^3} - \frac{1}{69^3} + \cdots \right) \]

\[ a_5 = \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{33^3} - \frac{1}{39^3} + \frac{1}{57^3} - \frac{1}{63^3} + \cdots \]

\[ a_6 = -\left( \frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{35^3} - \frac{1}{37^3} + \frac{1}{59^3} - \frac{1}{61^3} + \cdots \right) \]

\[ a_1 + a_2 + \cdots + a_6 = \beta(3) \]

From Formula 6.1.4 (2),

\[ a_k = \left( -\frac{1}{24^3} \right)^{k-1} \pi \frac{d^2}{dz^2} \cot(\pi z) \bigg|_{2^{k-1}}^{2^k} \quad k=1, 2, \cdots, 6 \]

Here,

\[ \frac{d^2}{dz^2} \cot(\pi z) = 2\pi^2 \cot(\pi z) \csc^2(\pi z) \]

So,

\[ a_k = \left( -\frac{1}{24^3} \right)^{k-1} \pi \cot(2k-1) \pi \frac{\pi}{24} \csc^2(2k-1) \pi \frac{\pi}{24} \]

From Formula 6.5.1 the following expressions hold for \( k=1, 2, \cdots, 6 \).

\[ \cos \left( \frac{(2k-1)}{24} \pi \right) = T_{2k-1}(\alpha), \quad \alpha = \cos \frac{\pi}{24} \]

\[ \sin \left( \frac{(2k-1)}{24} \pi \right) = (-1)^{k-1} T_{2k-1}(\beta), \quad \beta = \sin \frac{\pi}{24} \]

From Formula 6.5.2 (6.23),

\[ \cos \frac{\pi}{24} = \cos \frac{\pi}{3 \cdot 2^3} = \frac{1}{2} \sqrt{2+\sqrt{2+\sqrt{3}}} \]

\[ \sin \frac{\pi}{24} = \sin \frac{\pi}{3 \cdot 2^3} = \frac{1}{2} \sqrt{2-\sqrt{2+\sqrt{3}}} \]

So, let

\[ c_k = T_{2k-1}(\alpha), \quad \alpha = \frac{1}{2} \sqrt{2+\sqrt{2+\sqrt{3}}} \]

\[ s_k = (-1)^{k-1} T_{2k-1}(\beta), \quad \beta = \frac{1}{2} \sqrt{2-\sqrt{2+\sqrt{3}}} \]

Substituting these for the above,

\[ a_k = \left( -\frac{1}{24^3} \right)^{k-1} \frac{\pi^3}{2^3} \frac{c_k}{s_k} = b_k \quad k=1, 2, \cdots, 6 \]

After calculating and simplifying \( b_k \) using Mathematica, if the results are written together with \( a_k \), it is as follows. Where, \( b_c \) is drawn using \( \cos(\pi - \theta) = -\cos \theta, \sin(\pi - \theta) = \sin \theta \) because rationalization of
the denominator is difficult. Of course, both sides are numerically consistent.

\[ a_1 = 1 - \frac{1}{23^3} + \frac{1}{25^3} - \frac{1}{47^3} + \frac{1}{49^3} - \frac{1}{71^3} + \cdots \]

\[ a_2 = -\left( \frac{1}{3^3} - \frac{1}{21^3} + \frac{1}{27^3} - \frac{1}{45^3} + \frac{1}{51^3} - \frac{1}{69^3} + \cdots \right) = \frac{(56+39\sqrt{2}+32\sqrt{3}+23\sqrt{6})\pi^3}{6912} \]

\[ a_3 = \frac{1}{5^3} - \frac{1}{19^3} + \frac{1}{29^3} - \frac{1}{43^3} + \frac{1}{53^3} - \frac{1}{67^3} + \cdots \]

\[ a_4 = -\left( \frac{1}{7^3} - \frac{1}{17^3} + \frac{1}{31^3} - \frac{1}{41^3} + \frac{1}{53^3} - \frac{1}{65^3} + \cdots \right) = \frac{(56-39\sqrt{2}-32\sqrt{3}+23\sqrt{6})\pi^3}{6912} \]

\[ a_5 = \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{33^3} - \frac{1}{39^3} + \frac{1}{57^3} - \frac{1}{63^3} + \cdots \]

\[ a_6 = -\left( \frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{35^3} - \frac{1}{37^3} + \frac{1}{59^3} - \frac{1}{61^3} + \cdots \right) \]

\[ a_1 + a_2 + \cdots + a_6 = \beta(3) \]
6.6 Reflection $m 2^n$-split of $\lambda(2)$

In this section, we deal with reflection $m 2^n$-split of Dirichlet lambda series $\lambda(2)$ chiefly, which is as follows.

$$
\lambda(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots
$$

If 2-split, 4-split, 6-split, 8-split and 10-split of this are calculated by the method mentioned in the previous section, it is as follows.

### 2-split

- $a_1 = 1 + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{23^2} + \cdots = \frac{\pi^2}{16(2 - \sqrt{2})}$
- $a_2 = \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{19^2} + \frac{1}{21^2} + \cdots = \frac{\pi^2}{16(2 + \sqrt{2})}$

### 4-split

- $a_1 = 1 + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{47^2} + \cdots = \frac{\pi^2}{64(2 - \sqrt{2} + \sqrt{2})}$
- $a_2 = \frac{1}{3^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{45^2} + \cdots = \frac{\pi^2}{64(2 + \sqrt{2} - \sqrt{2})}$
- $a_3 = \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{43^2} + \cdots = \frac{\pi^2}{64(2 + \sqrt{2} - \sqrt{2})}$
- $a_4 = \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{41^2} + \cdots = \frac{\pi^2}{64(2 + \sqrt{2} - \sqrt{2})}$

### 6-split

- $a_1 = 1 + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{71^2} + \cdots = \frac{\pi^2}{144(2 - \sqrt{2} + \sqrt{3})}$
- $a_2 = \frac{1}{3^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{45^2} + \frac{1}{51^2} + \frac{1}{69^2} + \cdots = \frac{\pi^2}{3^2 \times 16(2 - \sqrt{2})}$
- $a_3 = \frac{1}{5^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{43^2} + \frac{1}{53^2} + \frac{1}{67^2} + \cdots = \frac{\pi^2}{144(2 - \sqrt{2} - \sqrt{3})}$
- $a_4 = \frac{1}{7^2} + \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{41^2} + \frac{1}{55^2} + \frac{1}{65^2} + \cdots = \frac{\pi^2}{144(2 + \sqrt{2} - \sqrt{3})}$
- $a_5 = \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{33^2} + \frac{1}{39^2} + \frac{1}{57^2} + \frac{1}{63^2} + \cdots = \frac{\pi^2}{3^2 \times 16(2 + \sqrt{2})}$
- $a_6 = \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{35^2} + \frac{1}{37^2} + \frac{1}{59^2} + \frac{1}{61^2} + \cdots = \frac{\pi^2}{144(2 + \sqrt{2} + \sqrt{3})}$

Where, $a_2$, $a_5$ are $\frac{1}{3^2}$ times the reflection 2-split series of $\lambda(2)$.
$$a_1 = 1 + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{63^2} + \frac{1}{65^2} + \frac{1}{95^2} + \cdots = \frac{\pi^2}{256 \left( 2 - \sqrt{2} + \sqrt{2 + \sqrt{2}} \right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{61^2} + \frac{1}{67^2} + \frac{1}{93^2} + \cdots = \frac{\pi^2}{256 \left( 2 - \sqrt{2} + 2 + \sqrt{2} \right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{59^2} + \frac{1}{69^2} + \frac{1}{91^2} + \cdots = \frac{\pi^2}{256 \left( 2 - \sqrt{2} - 2 - \sqrt{2} \right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{57^2} + \frac{1}{71^2} + \frac{1}{89^2} + \cdots = \frac{\pi^2}{256 \left( 2 - \sqrt{2} - 2 + \sqrt{2} \right)}$$

$$a_5 = \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{41^2} + \frac{1}{55^2} + \frac{1}{73^2} + \frac{1}{87^2} + \cdots = \frac{\pi^2}{256 \left( 2 + \sqrt{2} - 2 + \sqrt{2} \right)}$$

$$a_6 = \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{43^2} + \frac{1}{53^2} + \frac{1}{75^2} + \frac{1}{85^2} + \cdots = \frac{\pi^2}{256 \left( 2 + \sqrt{2} - 2 - \sqrt{2} \right)}$$

$$a_7 = \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{45^2} + \frac{1}{51^2} + \frac{1}{77^2} + \frac{1}{83^2} + \cdots = \frac{\pi^2}{256 \left( 2 + \sqrt{2} + 2 - \sqrt{2} \right)}$$

$$a_8 = \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{79^2} + \frac{1}{81^2} + \cdots = \frac{\pi^2}{256 \left( 2 + \sqrt{2} + 2 + \sqrt{2} \right)}$$

$$a_1 = 1 + \frac{1}{39^2} + \frac{1}{41^2} + \frac{1}{79^2} + \frac{1}{81^2} + \frac{1}{119^2} + \cdots = \frac{\pi^2}{400 \left( 2 - \sqrt{2} + \sqrt{2 + \frac{1 + \sqrt{2}}{2}} \right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{37^2} + \frac{1}{43^2} + \frac{1}{77^2} + \frac{1}{83^2} + \frac{1}{117^2} + \cdots = \frac{\pi^2}{400 \left( 2 - \sqrt{2} + \sqrt{2 + \frac{1 - \sqrt{2}}{2}} \right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{35^2} + \frac{1}{45^2} + \frac{1}{75^2} + \frac{1}{85^2} + \frac{1}{115^2} + \cdots = \frac{\pi^2}{5^2 \cdot 16 \left( 2 - \sqrt{2} \right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{33^2} + \frac{1}{47^2} + \frac{1}{73^2} + \frac{1}{87^2} + \frac{1}{113^2} + \cdots = \frac{\pi^2}{400 \left( 2 - \sqrt{2} - \sqrt{2 + \frac{1 - \sqrt{2}}{2}} \right)}$$

$$a_5 = \frac{1}{9^2} + \frac{1}{31^2} + \frac{1}{49^2} + \frac{1}{71^2} + \frac{1}{89^2} + \frac{1}{111^2} + \cdots = \frac{\pi^2}{400 \left( 2 - \sqrt{2} - \sqrt{2 + \frac{1 + \sqrt{2}}{2}} \right)}$$

$$a_6 = \frac{1}{11^2} + \frac{1}{29^2} + \frac{1}{51^2} + \frac{1}{69^2} + \frac{1}{91^2} + \frac{1}{109^2} + \cdots = \frac{\pi^2}{400 \left( 2 + \sqrt{2} - \sqrt{2 + \frac{1 + \sqrt{2}}{2}} \right)}$$
$a_7 = \frac{1}{13^2} + \frac{1}{27^2} + \frac{1}{53^2} + \frac{1}{67^2} + \frac{1}{93^2} + \frac{1}{107^2} + \ldots = \frac{\pi^2}{400 (2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \frac{1 + \sqrt{2}}{2}}}})}$

$a_8 = \frac{1}{15^2} + \frac{1}{25^2} + \frac{1}{55^2} + \frac{1}{65^2} + \frac{1}{95^2} + \frac{1}{105^2} + \ldots = \frac{\pi^2}{5^2 16 (2 + \sqrt{2})}$

$a_9 = \frac{1}{17^2} + \frac{1}{23^2} + \frac{1}{57^2} + \frac{1}{63^2} + \frac{1}{97^2} + \frac{1}{103^2} + \ldots = \frac{\pi^2}{400 (2 + \sqrt{2} + \sqrt{2 + \sqrt{2 + \frac{1 + \sqrt{2}}{2}}}})$

$a_{10} = \frac{1}{19^2} + \frac{1}{21^2} + \frac{1}{59^2} + \frac{1}{61^2} + \frac{1}{99^2} + \frac{1}{101^2} + \ldots = \frac{\pi^2}{400 (2 + \sqrt{2} + \sqrt{2 + \frac{1 + \sqrt{2}}{2}}}}$

Where, $a_2, a_5$ are $1/5^2$ times the reflection 2-split series of $\lambda(2)$.

**Remark**

The denominators are not rationalized purposely. If this expression is used, the sum of each split series is determined by the combination of $+$ and $-$ in the nested square roots. However, in 6-split and 10-split, it is impossible to represent all the sums with only these combinations. Interestingly, the shortages are filled up by reflection 2-split series of $\lambda(2)$.

So, the following theorem holds.

**Theorem 6.6.1**

Let $p$ be a prime number greater than 2 and reflection $2p$-split series $a_k$ of $\lambda(2)$ are as follows.

$$a_k = \sum_{r=0}^{\infty} \left\{ \frac{1}{(8pr + 2k - 1)^2} + \frac{1}{(8pr + 8p - 2k + 1)^2} \right\} \quad k=1, 2, \ldots, 2p$$

$$a_1 + a_2 + \ldots + a_{2p} = \lambda(2)$$

Then, following expressions hold.

$$a_{\frac{p+1}{2}} = \frac{\pi^2}{p^2 16 (2 - \sqrt{2})}, \quad a_{\frac{3p+1}{2}} = \frac{\pi^2}{p^2 16 (2 + \sqrt{2})}$$

**Proof**

According to [Formula 6.1.4 (1)], the sum $b_k$ of the reflection 2-split series and the sum $a_k$ of the reflection $2p$-split series are given as follows respectively.

$$b_1 = \frac{\pi^2}{8^2 \csc^2 \frac{1\pi}{8}} = \frac{\pi^2}{16 (2 - \sqrt{2})}, \quad b_2 = \frac{\pi^2}{8^2 \csc^2 \frac{3\pi}{8}} = \frac{\pi^2}{16 (2 + \sqrt{2})}$$

$$a_k = \frac{\pi^2}{(8p)^2 \csc^2 \frac{(2k - 1)\pi}{8p}} \quad k=1, 2, \ldots, 2p$$

When $2k - 1 = p$, i.e., $k = (p + 1)/2$,

$$a_{\frac{p+1}{2}} = \frac{\pi^2}{(8p)^2 \csc^2 \frac{p\pi}{8p}} = \frac{\pi^2}{p^2 8^2 \csc^2 \frac{\pi}{8}} = \frac{\pi^2}{p^2 16 (2 - \sqrt{2})}$$

When $2k - 1 = 3p$, i.e., $k = (3p + 1)/2$,

$$a_{\frac{3p+1}{2}} = \frac{\pi^2}{(8p)^2 \csc^2 \frac{3p\pi}{8p}} = \frac{\pi^2}{p^2 8^2 \csc^2 \frac{3\pi}{8}} = \frac{\pi^2}{p^2 16 (2 + \sqrt{2})}$$

**- 24 -**
Example: 14-split

For the purpose, expression by the nested radicals of \( \cos \left( \frac{\pi}{7} \right) \) is required first. According to Wikipedia (https://en.wikipedia.org/wiki/Trigonometric_constants_expressed_in_real_radicals), it is as follows.

\[
\cos \left( \frac{\pi}{7} \right) = \frac{1}{24} \sqrt{3 \left( 80 + 3 \sqrt{14336 + \sqrt{-5549064193}} + \sqrt{14336 - \sqrt{-5549064193}} \right)}
\]

We have to apply this half-angle formula 3 times to find \( \sin \left( \frac{\pi}{56} \right) \). Then, we must calculate the sum \( a_k \ (k=1, 2, 3, \ldots, 2p) \) of each split series. However, they become very long formulas and it is difficult to describe here.

Nevertheless, if placed with \( p=7 \) in the theorem, only \( a_4 \), \( a_{11} \) are expressed simply exceptionally. That is

\[
a_4 = \frac{1}{7^2} + \frac{1}{49^2} + \frac{1}{63^2} + \frac{1}{105^2} + \frac{1}{119^2} + \frac{1}{161^2} + \cdots = \frac{\pi^2}{7^2 \cdot 16 \left( 2 - \sqrt{2} \right)}
\]

\[
a_{11} = \frac{1}{21^2} + \frac{1}{35^2} + \frac{1}{77^2} + \frac{1}{91^2} + \frac{1}{133^2} + \frac{1}{147^2} + \cdots = \frac{\pi^2}{7^2 \cdot 16 \left( 2 + \sqrt{2} \right)}
\]

In fact, the calculation results of both sides by Mathematica are as follows.

\[
a_k := \sum_{r=0}^{200000} \left( \frac{1}{(56r + 2k - 1)^2} + \frac{1}{(56r + 56 - 2k + 1)^2} \right)
\]

\[
\{ a_4, \frac{\pi^2}{7^2 \cdot 16 \left( 2 - \sqrt{2} \right)} \} \quad \{ a_{11}, \frac{\pi^2}{7^2 \cdot 16 \left( 2 + \sqrt{2} \right)} \}
\]

\{0.0214904, 0.0214904\} \quad \{0.00368717, 0.00368717\}

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**Alien's Mathematics**

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