06 Reflection Split of Dirichlet Series

In the previous chapter, we considered the basic split of Dirichlet series and the composite split combining them. Amoung the composite split, there are composit splits whose sums are represented by special values of the elementary function regardless the degree. Moreover, these special values are obtained by algebraic solution, except for the circle ratio π .

6.1 Reflection Formula & Reflection Split

As first, we repeat the definition and the formula of the previous chapter here. Then, we describe the reflection formula for a polygamma function, and, finally we define reflection split of Dirichlet series.

Definition 6.1.0 (Basic Split) (Repeat)

When we can create positive or negative term series A_k $(k=1,2,\cdots m)$ by choosing the terms from the k th term $(k=1,2,\cdots m)$ with $(m-1)$ skipping in a series, We call this a **basic m -split** (or simply **m -split**).

Formula 6.1.1 (Repeat)

Let $\zeta(n,z)$ be Hurwitz zeta function, $\psi_n(z)$ be polygamma function, and Dirichlet lambda series $\lambda(n)$ and the m -split series A_k are as follows respectively.

$$\lambda(n) = \sum_{r=0}^{\infty}\frac{1}{(2r+1)^n}$$

$$A_k = \sum_{r=0}^{\infty}\frac{1}{(2mr+2k-1)^n} \qquad k=1,2,\cdots,m$$

Then, the following expressions hold for $k=1,2,\cdots,m$.

$$\sum_{r=0}^{\infty}\frac{1}{(2mr+2k-1)^n} = \frac{1}{(2m)^n}\zeta\left(n,\frac{2k-1}{2m}\right) \tag{1.1}$$

$$= \frac{(-1)^n}{(2m)^n(n-1)!}\psi_{n-1}\left(\frac{2k-1}{2m}\right) \tag{1.1'}$$

Formula 6.1.2 (Polygamma Reflection Formula)

When $\psi_n(z)$ $(n=1,2,3,\cdots)$ is the Polygamma function, the following expressions hold.

$$\psi_{2n-1}(z)+\psi_{2n-1}(1-z) = -\pi\frac{d^{2n-1}}{dz^{2n-1}}cot(\pi z) \tag{1.2$_1$}$$

$$\psi_{2n-2}(z)-\psi_{2n-2}(1-z) = -\pi\frac{d^{2n-2}}{dz^{2n-2}}cot(\pi z) \tag{1.2$_2$}$$

Proof

According to ***MathWorld*** (http://mathworld.wolfram.com/PolygammaFunction.html) , the reflection formula for polygamma functions is as follows.

$$\psi_n(1-z)+(-1)^{n+1}\psi_n(z) = (-1)^n\pi\frac{d^n}{dz^n}cot(\pi z)$$

Multiplying $(-1)^{n+1}$ by both sides

$$\psi_n(z)+(-1)^{n+1}\psi_n(1-z) = -\pi\frac{d^n}{dz^n}cot(\pi z)$$

(1.2_1) is obtained by replacing n with $2n-1$, and (1.2_2) is obtained by replacing n with $2n-2$.

The first few for (1.2_1) and (1.2_2) are as follows respectively.

(1.2_1)

$$\psi_1(z)+\psi_1(1-z) = \pi^2 csc^2(\pi z)$$
$$\psi_3(z)+\psi_3(1-z) = \pi^4\{4cot^2(\pi z)csc^2(\pi z) + 2csc^4(\pi z)\}$$
$$\psi_5(z)+\psi_5(1-z) = \pi^6\{16cot^4(\pi z)csc^2(\pi z) + 88cot^2(\pi z)csc^4(\pi z) + 16csc^6(\pi z)\}$$
$$\psi_7(z)+\psi_7(1-z) = \pi^8\{64cot^6(\pi z)csc^2(\pi z) + 1824cot^4(\pi z)csc^4(\pi z) + 2880cot^2(\pi z)csc^6(\pi z) + 272csc^8(\pi z)\}$$
$$\vdots$$

(1.2_2)

$$\psi_0(z)-\psi_0(1-z) = -\pi^1 cot(\pi z)$$
$$\psi_2(z)-\psi_2(1-z) = -2\pi^3 cot(\pi z)csc^2(\pi z)$$
$$\psi_4(z)-\psi_4(1-z) = -\pi^5\{8cot^3(\pi z)csc^2(\pi z) + 16cot(\pi z)csc^4(\pi z)\}$$
$$\psi_6(z)-\psi_6(1-z) = -\pi^7\{32cot^5(\pi z)csc^2(\pi z) + 416cot^3(\pi z)csc^4(\pi z) + 272cot(\pi z)csc^6(\pi z)\}$$
$$\vdots$$

In addition, These coefficients are listed in ***OEIS*** (https://oeis.org) A008303 .

Note1

As seen in the previous chapter , the sum of n-degree Dirichlet series was given by the following.

$$\frac{(-1)^n}{m^n(n-1)!}\left\{\psi_{n-1}\left(\frac{p}{m}\right) \pm \psi_{n-1}\left(\frac{q}{m}\right)\right\}$$

Among these, Formula 6.1.2 is applicable to those with $q=p-1$.

And, considering the sign and degree of Formula 6.1.2 , it becoms as follows.

(1) (1.2_1) is applicable to even-order positive Dirichlet series.

(2) (1.2_2) is applicable to odd-order alternating Dirichlet series.

Conversely, this formula can not be applied to odd-degree positive Dirichlet series or even-order alternating Dirichlet series.

Note2

The right-hand side of this formula is a polynomial of elementary functions.

Definition of Reflection Split of Dirichlet Series

Definition 6.1.3 (Comprehensive)

Create positive or negative term series A_k $(k=1,2,\cdots,2m)$ by choosing the terms from the k th term $(k=1,2,\cdots,2m)$ with $(m-1)$ skipping in a series according to Definition 6.1.0 . Then, among the composite

m -splits composed of these, there are combinations **satisfying the following reflection relation** .

$$(A_1+A_{2m}),\ (A_2+A_{2m-1}),\ \cdots,\ (A_k+A_{2m-k+1}),\ \cdots,\ (A_m+A_{2m-m+1})$$

We call these composite split especially **reflection m -split** .

Note

Here, the split which satisfies the reflection relation means the relationship $k+(2m-k+1)=2m+1$ in (A_k+A_{2m-k+1}) . An example with this definition is shown in Section 2.

Reflection Split can be defined also somewhat restrictively as follows.

Definition 6.1.3' (Restrictive)

(1) Let Dirichlet lambda series $\lambda(2n)$ be

$$\lambda(2n)=1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\cdots$$

When this series is split as follows, we call this **reflection m -split of Dilichlet lambda series.**

$$a_k=\sum_{r=0}^{\infty}\left\{\frac{1}{(4mr+2k-1)^{2n}}+\frac{1}{(4mr+4m-2k+1)^{2n}}\right\}\qquad k=1,2,\cdots,m$$

$$a_1+a_2+\cdots+a_m=\lambda(2n)$$

(2) Let Dirichlet beta series $\beta(2n-1)$ be

$$\beta(2n-1)=1-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\frac{1}{7^{2n-1}}+\frac{1}{9^{2n-1}}-\frac{1}{11^{2n-1}}+\frac{1}{13^{2n-1}}-+\cdots$$

When this series is split as follows, we call this **reflection m -split of Dilichlet beta series.**

$$a_k=(-1)^{k-1}\sum_{r=0}^{\infty}\left\{\frac{1}{(4mr+2k-1)^{2n-1}}-\frac{1}{(4mr+4m-2k+1)^{2n-1}}\right\}\qquad k=1,2,\cdots,m$$

$$a_1+a_2+\cdots+a_m=\beta(2n-1)$$

The following formula holds for Definition 6.1.3' :

Formula 6.1.4

The sum of reflection split series a_k in Definition 6.1.3' is given by the following expression.

(1) For Dirichlet lambda series $\lambda(2n)$,

$$a_k=-\frac{\pi}{(4m)^{2n}(2n-1)!}\left.\frac{d^{2n-1}}{dz^{2n-1}}\cot(\pi z)\right|_{\frac{2k-1}{4m}}\qquad k=1,2,\cdots,m \qquad (1.4_1)$$

(2) For Dirichlet beta series $\beta(2n-1)$,

$$a_k=\frac{(-1)^{k-1}\pi}{(4m)^{2n-1}(2n-2)!}\left.\frac{d^{2n-2}}{dz^{2n-2}}\cot(\pi z)\right|_{\frac{2k-1}{4m}}\qquad k=1,2,\cdots,m \qquad (1.4_2)$$

Proof

Applying Formula 6.1.1 (1.1') to the reflection split series a_k in Definition 6.1.3' ,

(1) For Dirichlet lambda series $\lambda(2n)$,

$$a_k = \frac{1}{(4m)^{2n}(2n-1)!}\left\{\psi_{2n-1}\left(\frac{2k-1}{4m}\right)+\psi_{2n-1}\left(\frac{4m-2k+1}{4m}\right)\right\} \qquad k=1,2,\cdots,m$$

From Formula 6.1.2 ,

$$\psi_{2n-1}(z)+\psi_{2n-1}(1-z) = -\pi\frac{d^{2n-1}}{dz^{2n-1}}cot(\pi z) \tag{1.2$_1$}$$

Substituting $z = (2k-1)/4m$ for this,

$$\psi_{2n-1}\left(\frac{2k-1}{4m}\right)+\psi_{2n-1}\left(\frac{4m-2k+1}{4m}\right) = -\pi\frac{d^{2n-1}}{dz^{2n-1}}cot(\pi z)\Bigg|_{\frac{2k-1}{4m}}$$

Substituting this for the right side of the above , we obtain (1.4_1) .

(2) For Dirichlet beta series $\beta(2n-1)$,

In a way similar to **(1)**, we obtain (1.4_2) .

The following theorem is derived immediately from this formula.

Theorem 6.1.5

The sum of reflection split series a_k in Definition 6.1.3' is the spcial value of elementary function

Proof

The higher order derivative of elementary function is an elementary function. Then, the right side of (1.4_1) and (1.4_2) are special values of the elementary functions.

6.2 Reflection 2-split of Dirichlet Series

6.2.0 Basic 4-split of $\lambda(2n)$

Dirichlet lambda series $\lambda(2n)$ is

$$\lambda(2n) = 1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \cdots$$

The basic 4-split is

$$A_1 = 1 + \frac{1}{9^{2n}} + \frac{1}{17^{2n}} + \frac{1}{25^{2n}} + \frac{1}{33^{2n}} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+1)^{2n}}$$

$$A_2 = \frac{1}{3^{2n}} + \frac{1}{11^{2n}} + \frac{1}{19^{2n}} + \frac{1}{27^{2n}} + \frac{1}{35^{2n}} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+3)^{2n}}$$

$$A_3 = \frac{1}{5^{2n}} + \frac{1}{13^{2n}} + \frac{1}{21^{2n}} + \frac{1}{29^{2n}} + \frac{1}{37^{2n}} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+5)^{2n}}$$

$$A_4 = \frac{1}{7^{2n}} + \frac{1}{15^{2n}} + \frac{1}{23^{2n}} + \frac{1}{31^{2n}} + \frac{1}{39^{2n}} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+7)^{2n}}$$

Expression by polygamma function

According to Formula 6.1.1 (1.1') , the sums of the series A_1 , A_2 , A_3 , A_4 are given by

$$A_1 = \sum_{r=0}^{\infty} \frac{1}{(8r+1)^{2n}} = \frac{1}{8^{2n}(2n-1)!}\psi_{2n-1}\left(\frac{1}{8}\right)$$

$$A_2 = \sum_{r=0}^{\infty} \frac{1}{(8r+3)^{2n}} = \frac{1}{8^{2n}(2n-1)!}\psi_{2n-1}\left(\frac{3}{8}\right)$$

$$A_3 = \sum_{r=0}^{\infty} \frac{1}{(8r+5)^{2n}} = \frac{1}{8^{2n}(2n-1)!}\psi_{2n-1}\left(\frac{5}{8}\right)$$

$$A_4 = \sum_{r=0}^{\infty} \frac{1}{(8r+7)^{2n}} = \frac{1}{8^{2n}(2n-1)!}\psi_{2n-1}\left(\frac{7}{8}\right)$$

6.2.1 Reflection 2-split of $\lambda(2n)$

Although there are 7 combinations in composing the series A_1 , A_2 , A_3 , A_4 into 2 series, here we choose $(A_1 + A_4)$, $(A_2 + A_3)$ such that the total of subscripts is 5 . Then,

$$a_1 = 1 + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} + \frac{1}{15^{2n}} + \frac{1}{17^{2n}} + \cdots = \sum_{r=0}^{\infty}\left\{\frac{1}{(8r+1)^{2n}} + \frac{1}{(8r+7)^{2n}}\right\}$$

$$a_1 = \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \frac{1}{19^{2n}} + \cdots = \sum_{r=0}^{\infty}\left\{\frac{1}{(8r+3)^{2n}} + \frac{1}{(8r+5)^{2n}}\right\}$$

$$\lambda(2n) = a_1 + a_2$$

The sums of the series a_1 , a_2 are given by

$$1 + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} + \frac{1}{15^{2n}} + \frac{1}{17^{2n}} + \cdots = \frac{1}{8^{2n}(2n-1)!}\left\{\psi_{2n-1}\left(\frac{1}{8}\right) + \psi_{2n-1}\left(\frac{7}{8}\right)\right\}$$

$$\frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \frac{1}{19^{2n}} + \cdots = \frac{1}{8^{2n}(2n-1)!}\left\{\psi_{2n-1}\left(\frac{3}{8}\right) + \psi_{2n-1}\left(\frac{5}{8}\right)\right\}$$

We can see that these right sides satisfy the requirements of Formula 6.1.2 (1.2_1) respectively. That is

$$\psi_{2n-1}(z)+\psi_{2n-1}(1-z) = -\pi\frac{d^{2n-1}}{dz^{2n-1}}\cot(\pi z) \qquad (1.2_1)$$

Substituting $z=1/8,\ 3/8$ for this respectively,

$$\psi_{2n-1}\left(\frac{1}{8}\right)+\psi_{2n-1}\left(\frac{7}{8}\right) = -\pi\frac{d^{2n-1}}{dz^{2n-1}}\cot(\pi z)\Big|_{1/8}$$

$$\psi_{2n-1}\left(\frac{3}{8}\right)+\psi_{2n-1}\left(\frac{5}{8}\right) = -\pi\frac{d^{2n-1}}{dz^{2n-1}}\cot(\pi z)\Big|_{3/8}$$

Substituting this for the above, we obtain the general expressions of reflection 2-split of $\lambda(2n)$ series.

$$1+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}+\frac{1}{15^{2n}}+\frac{1}{17^{2n}}+\cdots = -\frac{\pi}{8^{2n}(2n-1)!}\frac{d^{2n-1}}{dz^{2n-1}}\cot(\pi z)\Big|_{1/8}$$

$$\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\frac{1}{19^{2n}}+\cdots = -\frac{\pi}{8^{2n}(2n-1)!}\frac{d^{2n-1}}{dz^{2n-1}}\cot(\pi z)\Big|_{3/8}$$

Example $n=2$

$$\begin{aligned}1+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{15^4}+\frac{1}{17^4}+\cdots &= -\frac{\pi}{8^4 3!}\frac{d^3}{dz^3}\cot(\pi z)\Big|_{1/8}\\ &= \frac{\pi^4\{4\cot^2(\pi z)\csc^2(\pi z)+2\csc^4(\pi z)\}}{8^4 3!}\Big|_{1/8}\\ &= \frac{\pi^4}{8^4 3!}\left\{4\cot^2\frac{\pi}{8}\csc^2\frac{\pi}{8}+2\csc^4\frac{\pi}{8}\right\}\\ &= \frac{\pi^4}{8^4 3!}\left\{4(\sqrt{2}+1)^2\left(\sqrt{4+2\sqrt{2}}\right)^2+2\left(\sqrt{4+2\sqrt{2}}\right)^4\right\}\end{aligned}$$

$$\begin{aligned}\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{11^4}+\frac{1}{13^4}+\frac{1}{19^4}+\cdots &= -\frac{\pi}{8^4 3!}\frac{d^3}{dz^3}\cot(\pi z)\Big|_{3/8}\\ &= \frac{\pi^4\{4\cot^2(\pi z)\csc^2(\pi z)+2\csc^4(\pi z)\}}{8^4 3!}\Big|_{3/8}\\ &= \frac{\pi^4}{8^4 3!}\left\{4\cot^2\frac{3\pi}{8}\csc^2\frac{3\pi}{8}+2\csc^4\frac{3\pi}{8}\right\}\\ &= \frac{\pi^4}{8^4 3!}\left\{4(\sqrt{2}-1)^2\left(\sqrt{4-2\sqrt{2}}\right)^2+2\left(\sqrt{4-2\sqrt{2}}\right)^4\right\}\end{aligned}$$

That is,

$$a_1 = 1+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{15^4}+\frac{1}{17^4}+\cdots = \frac{\pi^4(16+11\sqrt{2})}{3072} = 1.000610446\cdots$$

$$a_2 = \frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{11^4}+\frac{1}{13^4}+\frac{1}{19^4}+\cdots = \frac{\pi^4(16-11\sqrt{2})}{3072} = 0.01406758545\cdots$$

$$a_1+a_2 = \frac{\pi^4(16+11\sqrt{2})}{3072}+\frac{\pi^4(16-11\sqrt{2})}{3072} = \frac{\pi^4}{96} = \lambda(4) = 1.014678031\cdots$$

6.2.2 Reflection 2-split of $\beta(2n-1)$

Dirichlet beta series $\beta(2n-1)$ is

$$\beta(2n-1) = 1 - \frac{1}{3^{2n-1}} + \frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{11^{2n-1}} + \frac{1}{13^{2n-1}} -+\cdots$$

The basic 4-split is

$$A_1 = 1+\frac{1}{9^{2n-1}}+\frac{1}{17^{2n-1}}+\frac{1}{25^{2n-1}}+\frac{1}{33^{2n-1}}+\cdots = \sum_{r=0}^{\infty}\frac{1}{(8r+1)^{2n-1}}$$

$$A_2 = -\left(\frac{1}{3^{2n-1}}+\frac{1}{11^{2n-1}}+\frac{1}{19^{2n-1}}+\frac{1}{27^{2n-1}}+\frac{1}{35^{2n-1}}+\cdots\right) = -\sum_{r=0}^{\infty}\frac{1}{(8r+3)^{2n-1}}$$

$$A_3 = \frac{1}{5^{2n-1}}+\frac{1}{13^{2n-1}}+\frac{1}{21^{2n-1}}+\frac{1}{29^{2n-1}}+\frac{1}{37^{2n-1}}+\cdots = \sum_{r=0}^{\infty}\frac{1}{(8r+5)^{2n-1}}$$

$$A_4 = -\left(\frac{1}{7^{2n-1}}+\frac{1}{15^{2n-1}}+\frac{1}{23^{2n-1}}+\frac{1}{31^{2n-1}}+\frac{1}{39^{2n-1}}+\cdots\right) = -\sum_{r=0}^{\infty}\frac{1}{(8r+7)^{2n-1}}$$

Expression by polygamma function

According to Formula 6.1.1 (1.1') , the sums of the series A_1, A_2, A_3, A_4 are given by

$$A_1 = \sum_{r=0}^{\infty}\frac{1}{(8r+1)^{2n-1}} = -\frac{1}{8^{2n-1}(2n-2)!}\psi_{2n-2}\left(\frac{1}{8}\right)$$

$$A_2 = -\sum_{r=0}^{\infty}\frac{1}{(8r+3)^{2n-1}} = \frac{1}{8^{2n-1}(2n-2)!}\psi_{2n-2}\left(\frac{3}{8}\right)$$

$$A_3 = \sum_{r=0}^{\infty}\frac{1}{(8r+5)^{2n-1}} = -\frac{1}{8^{2n-1}(2n-2)!}\psi_{2n-2}\left(\frac{5}{8}\right)$$

$$A_4 = -\sum_{r=0}^{\infty}\frac{1}{(8r+7)^{2n-1}} = \frac{1}{8^{2n-1}(2n-2)!}\psi_{2n-2}\left(\frac{7}{8}\right)$$

Reflection 2-split of $\beta(2n-1)$

Although there are 7 combinations in composing the series A_1, A_2, A_3, A_4 into 2 series, here we choose (A_1+A_4) , (A_2+A_3) such that the total of subscripts is 5. Then,

$$a_1 = 1 - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{15^{2n-1}} + \frac{1}{17^{2n-1}} - \frac{1}{23^{2n-1}} + \frac{1}{25^{2n-1}} -+\cdots$$

$$a_2 = -\left(\frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} + \frac{1}{11^{2n-1}} - \frac{1}{13^{2n-1}} + \frac{1}{19^{2n-1}} - \frac{1}{21^{2n-1}} +-\cdots\right)$$

$$\beta(2n-1) = a_1 + a_2$$

The sums of the series a_1, a_2 are given by

$$1 - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{15^{2n-1}} +-\cdots = -\frac{1}{8^{2n-1}(2n-2)!}\left\{\psi_{2n-2}\left(\frac{1}{8}\right) - \psi_{2n-2}\left(\frac{7}{8}\right)\right\}$$

$$-\left(\frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} + \frac{1}{11^{2n-1}} - \frac{1}{13^{2n-1}} +-\cdots\right) = \frac{1}{8^{2n-1}(2n-2)!}\left\{\psi_{2n-2}\left(\frac{3}{8}\right) - \psi_{2n-2}\left(\frac{5}{8}\right)\right\}$$

We can see that these right sides satisfy the requirements of Formula 6.1.2 (1.2_2) respectively. That is

$$\psi_{2n-2}(z) - \psi_{2n-2}(1-z) = -\pi \frac{d^{2n-2}}{dz^{2n-2}} \cot(\pi z) \qquad (1.2_2)$$

Substituting $z = 1/8$, $3/8$ for this respectively,

$$\psi_{2n-2}\left(\frac{1}{8}\right) - \psi_{2n-2}\left(\frac{7}{8}\right) = -\pi \frac{d^{2n-2}}{dz^{2n-2}} \cot(\pi z) \Big|_{1/8}$$

$$\psi_{2n-2}\left(\frac{3}{8}\right) - \psi_{2n-2}\left(\frac{5}{8}\right) = -\pi \frac{d^{2n-2}}{dz^{2n-2}} \cot(\pi z) \Big|_{3/8}$$

Substituting this for the above, we obtain the general expressions of reflection 2-split of $\beta(2n-1)$ series.

$$1 - \frac{1}{7^{2n-1}} + \frac{1}{9^{2n-1}} - \frac{1}{15^{2n-1}} + - \cdots = \frac{\pi}{8^{2n-1}(2n-2)!} \frac{d^{2n-2}}{dz^{2n-2}} \cot(\pi z) \Big|_{1/8}$$

$$-\left(\frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} + \frac{1}{11^{2n-1}} - \frac{1}{13^{2n-1}} + - \cdots\right) = -\frac{\pi}{8^{2n-1}(2n-2)!} \frac{d^{2n-2}}{dz^{2n-2}} \cot(\pi z) \Big|_{3/8}$$

Example $n=3$

$$1 - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{15^5} + \frac{1}{17^5} - \frac{1}{23^5} + - \cdots = \frac{\pi}{8^5 4!} \frac{d^4}{dz^4} \cot(\pi z) \Big|_{1/8}$$

$$= \frac{\pi^5 \{8\cot^3(\pi z)\csc^2(\pi z) + 16\cot(\pi z)\csc^4(\pi z)\}}{8^5 4!} \Big|_{1/8}$$

$$= \frac{\pi^5}{8^5 4!} \left\{8\cot^3\frac{\pi}{8}\csc^2\frac{\pi}{8} + 16\cot\frac{\pi}{8}\csc^4\frac{\pi}{8}\right\}$$

$$= \frac{\pi^5}{8^5 4!} \left\{8(\sqrt{2}+1)^3\left(\sqrt{4+2\sqrt{2}}\right)^2 + 16(\sqrt{2}+1)\left(\sqrt{4+2\sqrt{2}}\right)^4\right\}$$

$$-\left(\frac{1}{3^5} - \frac{1}{5^5} + \frac{1}{11^5} - \frac{1}{13^5} + \frac{1}{19^5} - \frac{1}{21^5} + - \cdots\right) = -\frac{\pi}{8^5 4!} \frac{d^4}{dz^4} \cot(\pi z) \Big|_{3/8}$$

$$= -\frac{\pi^5 \{8\cot^3(\pi z)\csc^2(\pi z) + 16\cot(\pi z)\csc^4(\pi z)\}}{8^5 4!} \Big|_{3/8}$$

$$= -\frac{\pi^5}{8^5 4!} \left\{8\cot^3\frac{3\pi}{8}\csc^2\frac{3\pi}{8} + 16\cot\frac{3\pi}{8}\csc^4\frac{3\pi}{8}\right\}$$

$$= -\frac{\pi^5}{8^5 4!} \left\{8(\sqrt{2}-1)^3\left(\sqrt{4-2\sqrt{2}}\right)^2 + 16(\sqrt{2}-1)\left(\sqrt{4-2\sqrt{2}}\right)^4\right\}$$

That is,

$$a_1 = 1 - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{15^5} + \frac{1}{17^5} - \frac{1}{23^5} + - \cdots = \frac{\pi^5(80+57\sqrt{2})}{49152}$$

$$a_2 = -\left(\frac{1}{3^5} - \frac{1}{5^5} + \frac{1}{11^5} - \frac{1}{13^5} + \frac{1}{19^5} - \frac{1}{21^5} + - \cdots\right) = \frac{\pi^5(80-57\sqrt{2})}{49152}$$

$$a_1 + a_2 = \frac{\pi^5\left(80+57\sqrt{2}\right)}{49152} + \frac{\pi^5\left(80-57\sqrt{2}\right)}{49152} = \frac{5\pi^5}{1536} = \beta(5)$$

$$= 0.9999567572\cdots - 0.003798929159\cdots = 0.9961578280\cdots$$

6.3 Reflection 3-split of Dirichlet Series

Reflection odd-split of Dirichlet series is also possible. In this section, we take up reflection 3-split of Dirichlet beta series as an example.

6.3.0 Basic 6-split of $\beta(2n-1)$

According to Definition 6.1.3' (2) ,

$$a_k = (-1)^{k-1}\sum_{r=0}^{\infty}\left\{\frac{1}{(4mr+2k-1)^{2n-1}} - \frac{1}{(4mr+4m-2k+1)^{2n-1}}\right\} \qquad k=1,2,\cdots,m$$

Putting $m=3$,

$$a_1 = \sum_{r=0}^{\infty}\left\{\frac{1}{(12r+1)^{2n-1}} - \frac{1}{(12r+11)^{2n-1}}\right\}$$

$$a_2 = -\sum_{r=0}^{\infty}\left\{\frac{1}{(12r+3)^{2n-1}} - \frac{1}{(12r+9)^{2n-1}}\right\}$$

$$a_3 = \sum_{r=0}^{\infty}\left\{\frac{1}{(12r+5)^{2n-1}} - \frac{1}{(12r+7)^{2n-1}}\right\}$$

Expanding these,

$$a_1 = 1 - \frac{1}{11^{2n-1}} + \frac{1}{13^{2n-1}} - \frac{1}{23^{2n-1}} + \frac{1}{25^{2n-1}} - \frac{1}{35^{2n-1}} + - \cdots$$

$$a_2 = -\left(\frac{1}{3^{2n-1}} - \frac{1}{9^{2n-1}} + \frac{1}{15^{2n-1}} - \frac{1}{21^{2n-1}} + \frac{1}{27^{2n-1}} - \frac{1}{33^{2n-1}} + - \cdots\right)$$

$$a_3 = \frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \frac{1}{17^{2n-1}} - \frac{1}{19^{2n-1}} + \frac{1}{29^{2n-1}} - \frac{1}{31^{2n-1}} + - \cdots$$

$$\beta(2n-1) = a_1 + a_2 + a_3$$

According to Formula 6.1.1 (1.1') , the sums of the series a_1 , a_2 , a_3 are given by

$$a_1 = -\frac{1}{12^{2n-1}(2n-2)!}\left\{\psi_{2n-2}\left(\frac{1}{12}\right) - \psi_{2n-2}\left(\frac{11}{12}\right)\right\}$$

$$a_2 = \frac{1}{12^{2n-1}(2n-2)!}\left\{\psi_{2n-2}\left(\frac{3}{12}\right) - \psi_{2n-2}\left(\frac{9}{12}\right)\right\}$$

$$a_3 = -\frac{1}{12^{2n-1}(2n-2)!}\left\{\psi_{2n-2}\left(\frac{5}{12}\right) - \psi_{2n-2}\left(\frac{7}{12}\right)\right\}$$

We can see that these right sides satisfy the requirements of Formula 6.1.2 (1.2_2) respectively. That is,

$$\psi_{2n-2}(z) - \psi_{2n-2}(1-z) = -\pi\frac{d^{2n-2}}{dz^{2n-2}}\cot(\pi z) \qquad (1.2_2)$$

Substituting $z=1/12$, $3/12$, $5/12$ for this respectively,

$$\psi_{2n-2}\left(\frac{1}{12}\right) - \psi_{2n-2}\left(\frac{11}{12}\right) = -\pi\frac{d^{2n-2}}{dz^{2n-2}}\cot(\pi z)\Bigg|_{1/12}$$

$$\psi_{2n-2}\left(\frac{3}{12}\right) - \psi_{2n-2}\left(\frac{9}{12}\right) = -\pi\frac{d^{2n-2}}{dz^{2n-2}}\cot(\pi z)\Bigg|_{3/12}$$

$$\psi_{2n-2}\left(\frac{5}{12}\right) - \psi_{2n-2}\left(\frac{7}{12}\right) = -\pi\frac{d^{2n-2}}{dz^{2n-2}}\cot(\pi z)\Bigg|_{5/12}$$

Substituting this for the above, we obtain the general expressions of reflection 3-split of $\beta(2n-1)$ series.

$$1-\frac{1}{11^{2n-1}}+\frac{1}{13^{2n-1}}-\frac{1}{23^{2n-1}}+\frac{1}{25^{2n-1}}+-\cdots=-\frac{\pi}{12^{2n-1}(2n-2)!}\frac{d^{2n-2}}{dz^{2n-2}}cot(\pi z)\Bigg|_{1/12}$$

$$-\left(\frac{1}{3^{2n-1}}-\frac{1}{9^{2n-1}}+\frac{1}{15^{2n-1}}-\frac{1}{21^{2n-1}}+-\cdots\right)=\frac{\pi}{12^{2n-1}(2n-2)!}\frac{d^{2n-2}}{dz^{2n-2}}cot(\pi z)\Bigg|_{3/12}$$

$$\frac{1}{5^{2n-1}}-\frac{1}{7^{2n-1}}+\frac{1}{17^{2n-1}}-\frac{1}{19^{2n-1}}+-\cdots=-\frac{\pi}{12^{2n-1}(2n-2)!}\frac{d^{2n-2}}{dz^{2n-2}}cot(\pi z)\Bigg|_{5/12}$$

Example $n=3$

$$1-\frac{1}{11^5}+\frac{1}{13^5}-\frac{1}{23^5}+\frac{1}{25^5}-\frac{1}{35^5}+-\cdots=-\frac{\pi}{12^7 4!}\frac{d^4}{dz^4}cot(\pi z)\Bigg|_{1/12}$$

$$-\left(\frac{1}{3^5}-\frac{1}{9^5}+\frac{1}{15^5}-\frac{1}{21^5}+\frac{1}{27^5}-\frac{1}{33^5}+-\cdots\right)=\frac{\pi}{12^7 4!}\frac{d^4}{dz^4}cot(\pi z)\Bigg|_{3/12}$$

$$\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{17^5}-\frac{1}{19^5}+\frac{1}{29^5}-\frac{1}{31^5}+-\cdots=-\frac{\pi}{12^7 4!}\frac{d^4}{dz^4}cot(\pi z)\Bigg|_{5/12}$$

Calculating in the same way as in the previous section , we obtain

$$1-\frac{1}{11^5}+\frac{1}{13^5}-\frac{1}{23^5}+\frac{1}{25^5}-\frac{1}{35^5}+-\cdots=\frac{\pi^5(305+176\sqrt{3})}{186624}$$

$$-\left(\frac{1}{3^5}-\frac{1}{9^5}+\frac{1}{15^5}-\frac{1}{21^5}+\frac{1}{27^5}-\frac{1}{33^5}+-\cdots\right)=-\frac{5\pi^5}{373248}$$

$$\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{17^5}-\frac{1}{19^5}+\frac{1}{29^5}-\frac{1}{31^5}+-\cdots=\frac{\pi^5(305-176\sqrt{3})}{186624}$$

$$\frac{\pi^5(305+176\sqrt{3})}{186624}-\frac{5\pi^5}{373248}+\frac{\pi^5(305-176\sqrt{3})}{186624}=\frac{5\pi^5}{1536}=\beta(5)$$

Composite reflection 2-split

Thus, a_1, a_2, a_3 are all special values of elementary functions. So, when 2- split series of $\beta(5)$ are composed from these, (a_1, a_2+a_3) , (a_2, a_1+a_3) , (a_3, a_1+a_2) can be made. Summations of these series are of course special values of elementary functions. And, these are different from the reflection 2-split series in the previous section . That is, 4 sets of the reflection 2-split series have been obtained.

If such operations are performed on more than basic 8-split series, an infinite set of reflection split series can be obtained.

Infinity of composite reflection split

In a similar way, an infinite set of reflection m-split series ($m=3,4,5,\cdots$) can be obtained.

6.4 Algebraic Solvability of Reflection Split

Each sum of split series satisfying the reflection formula for polygamma function is obtained by algebraic solution, except circle ratio π . In this section, we prove and exemplify it.

6.4.1 Reflection Formula for Trigonometric Functions

Formula 6.4.1 (Trigonometric Reflection Formula)

For natural number n and $k=1,2,3,\cdots,n-1$, the following expressions hold.

$$\cos\frac{k\pi}{n} = \frac{1}{2}\left\{(-1)^{\frac{k}{n}} - (-1)^{1-\frac{k}{n}}\right\} \tag{4.1c}$$

$$\sin\frac{k\pi}{n} = \frac{1}{2i}\left\{(-1)^{\frac{k}{n}} + (-1)^{1-\frac{k}{n}}\right\} \tag{4.1s}$$

Proof

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad , \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

Putting $z = k\pi/n$

$$\cos\frac{k\pi}{n} = \frac{1}{2}\left(e^{\frac{k\pi i}{n}} + e^{-\frac{k\pi i}{n}}\right) \quad , \quad \sin\frac{k\pi}{n} = \frac{1}{2i}\left(e^{\frac{k\pi i}{n}} - e^{-\frac{k\pi i}{n}}\right)$$

Since $e^{\pi i} = -1$,

$$\cos\frac{k\pi}{n} = \frac{1}{2}\left\{(-1)^{\frac{k}{n}} + (-1)^{-\frac{k}{n}}\right\} \quad , \quad \sin\frac{k\pi}{n} = \frac{1}{2i}\left\{(-1)^{\frac{k}{n}} - (-1)^{-\frac{k}{n}}\right\}$$

Further,

$$(-1)^{-\frac{k}{n}} = -(-1)^{1}(-1)^{-\frac{k}{n}} = -(-1)^{1-\frac{k}{n}}$$

Substituting this for the above, we obtain

$$\cos\frac{k\pi}{n} = \frac{1}{2}\left\{(-1)^{\frac{k}{n}} - (-1)^{1-\frac{k}{n}}\right\} \quad , \quad \sin\frac{k\pi}{n} = \frac{1}{2i}\left\{(-1)^{\frac{k}{n}} + (-1)^{1-\frac{k}{n}}\right\}$$

Example $\cos 2\pi/7 , \sin 2\pi/7$

$$\text{N}\left[\left\{\text{Cos}\left[\frac{2\pi}{7}\right], \frac{1}{2}\left((-1)^{\frac{2}{7}} - (-1)^{\frac{5}{7}}\right)\right\}, 8\right] \qquad \text{N}\left[\left\{\text{Sin}\left[\frac{2\pi}{7}\right], \frac{1}{2i}\left((-1)^{\frac{2}{7}} + (-1)^{\frac{5}{7}}\right)\right\}, 8\right]$$

$$\{0.62348980 , 0.62348980 + 0\times10^{-10}\, i\} \qquad \{0.78183148 , 0.78183148 + 0\times10^{-10}\, i\}$$

Corollary 6.4.1

For natural number n and $k=1,2,3,\cdots,n-1$, the following expressions hold.

$$\cot\frac{k\pi}{n} = \frac{i\left\{(-1)^{\frac{k}{n}} - (-1)^{1-\frac{k}{n}}\right\}}{(-1)^{\frac{k}{n}} + (-1)^{1-\frac{k}{n}}}$$

$$csc\frac{k\pi}{n} = \frac{2i}{(-1)^{\frac{k}{n}} + (-1)^{1-\frac{k}{n}}}$$

6.4.2 Algebraic Solvability of Reflection Split

Using Corollary 6.4.1 , we can prove that each sum of split series is expressed with addition, multiplication and rational powers, except circle ratio π .

Theorem 6.4.2 (Algebraic Solvability)

The sum of reflection split series a_k in Definition 6.1.3' is expressed with addition, multiplication and rational powers, except circle ratio π .

Proof

According to Formula 6.1.4 , the a_k in Definition 6.1.3' is given by the following expression.

(1) For Dirichlet lambda series $\lambda(2n)$,

$$a_k = -\frac{\pi}{(4m)^{2n}(2n-1)!}\frac{d^{2n-1}}{dz^{2n-1}}cot(\pi z)\Bigg|_{\frac{2k-1}{4m}} \qquad k=1,2,\cdots,m \qquad (1.4_1)$$

As seen in the calculation example of Formula 6.1.2, this higher order difference quotient is a polynomial of $cot\frac{(2k-1)\pi}{4m}$ and $csc\frac{(2k-1)\pi}{4m}$. Then, by Corollary 6.4.1, this is expressed with addition, multiplication and rational powers of (-1) .

(2) For Dirichlet beta series $\beta(2n-1)$,

For the same reason as **(1)**, a_k is expressed with addition, multiplication and rational powers of (-1) , except circle ratio π .

Remark

Certainly, this is an algebraic expression. In the first place, $(-1)^{1/n}, (-1)^{2/n}, \cdots , (-1)^{(n-1)/n}$ are n -equally divided points of the unit semicircle drawn on the Gaussian plane. That is, these are solutions of equation $z^n+1 = 0$. Although algebraic equations of 5th degree or higher can not be solved algebraically in general, it is proved by ***Gauss*** that such a cyclotomic equation can be solved algebraically.

Example Reflect 7-split of $\beta(3)$

According to Definition 6.1.3' (2) , Dirichlet beta series $\beta(3)$ and the reflection 7-split series are as follows.

$$\beta(3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} -+\cdots$$

$$a_1 = 1 - \frac{1}{27^3} + \frac{1}{29^3} - \frac{1}{55^3} + \frac{1}{57^3} - \frac{1}{83^3} +-\cdots$$

$$a_2 = -\left(\frac{1}{3^3} - \frac{1}{25^3} + \frac{1}{31^3} - \frac{1}{53^3} + \frac{1}{59^3} - \frac{1}{81^3} +-\cdots\right)$$

$$a_3 = \frac{1}{5^3} - \frac{1}{23^3} + \frac{1}{33^3} - \frac{1}{51^3} + \frac{1}{61^3} - \frac{1}{79^3} +-\cdots$$

$$a_4 = -\left(\frac{1}{7^3} - \frac{1}{21^3} + \frac{1}{35^3} - \frac{1}{49^3} + \frac{1}{63^3} - \frac{1}{77^3} + - \cdots \right)$$

$$a_5 = \frac{1}{9^3} - \frac{1}{19^3} + \frac{1}{37^3} - \frac{1}{47^3} + \frac{1}{65^3} - \frac{1}{75^3} + - \cdots$$

$$a_6 = -\left(\frac{1}{11^3} - \frac{1}{17^3} + \frac{1}{39^3} - \frac{1}{45^3} + \frac{1}{67^3} - \frac{1}{73^3} + - \cdots \right)$$

$$a_7 = \frac{1}{13^3} - \frac{1}{15^3} + \frac{1}{41^3} - \frac{1}{43^3} + \frac{1}{69^3} - \frac{1}{71^3} + - \cdots$$

$$\beta(3) = a_1 + a_2 + a_3 + \cdots + a_7$$

According to Formula 6.1.1 (1.1') , the sums of the series $a_1, a_2, a_3, \cdots, a_7$ are given by

$$a_1 = \sum_{r=0}^{\infty} \left\{ \frac{1}{(28r+1)^3} - \frac{1}{(28r+27)^3} \right\} = -\frac{1}{28^3 2!} \left\{ \psi_2\left(\frac{1}{28} \right) - \psi_2\left(\frac{27}{28} \right) \right\}$$

$$a_2 = -\sum_{r=0}^{\infty} \left\{ \frac{1}{(28r+3)^3} - \frac{1}{(28r+25)^3} \right\} = \frac{1}{28^3 2!} \left\{ \psi_2\left(\frac{3}{28} \right) - \psi_2\left(\frac{25}{28} \right) \right\}$$

$$\vdots$$

$$a_6 = -\sum_{r=0}^{\infty} \left\{ \frac{1}{(28r+11)^3} - \frac{1}{(28r+17)^3} \right\} = \frac{1}{28^3 2!} \left\{ \psi_2\left(\frac{11}{28} \right) - \psi_2\left(\frac{17}{28} \right) \right\}$$

$$a_7 = \sum_{r=0}^{\infty} \left\{ \frac{1}{(28r+13)^3} - \frac{1}{(28r+15)^3} \right\} = -\frac{1}{28^3 2!} \left\{ \psi_2\left(\frac{13}{28} \right) - \psi_2\left(\frac{15}{28} \right) \right\}$$

We can see that these right sides satisfy the requirements of Formula 6.1.2 (1.2_2) respectively. That is,

$$\psi_{2n-2}(z) - \psi_{2n-2}(1-z) = -\pi \frac{d^{2n-2}}{dz^{2n-2}} cot(\pi z) \qquad (1.2_2)$$

Substituting $z = 1/28, 3/28, \cdots, 13/28$ for this respectively,

$$1 - \frac{1}{27^3} + \frac{1}{29^3} - \frac{1}{55^3} + \frac{1}{57^3} - \frac{1}{83^3} + - \cdots = \frac{\pi}{28^3 2!} \frac{d^2}{dz^2} cot(\pi z) \Big|_{1/28}$$

$$-\left(\frac{1}{3^3} - \frac{1}{25^3} + \frac{1}{31^3} - \frac{1}{53^3} + \frac{1}{59^3} - \frac{1}{81^3} + - \cdots \right) = -\frac{\pi}{28^3 2!} \frac{d^2}{dz^2} cot(\pi z) \Big|_{3/28}$$

$$\vdots$$

$$-\left(\frac{1}{11^3} - \frac{1}{17^3} + \frac{1}{39^3} - \frac{1}{45^3} + \frac{1}{67^3} - \frac{1}{73^3} + - \cdots \right) = -\frac{\pi}{28^3 2!} \frac{d^2}{dz^2} cot(\pi z) \Big|_{11/28}$$

$$\frac{1}{13^3} - \frac{1}{15^3} + \frac{1}{41^3} - \frac{1}{43^3} + \frac{1}{69^3} - \frac{1}{71^3} + - \cdots = \frac{\pi}{28^3 2!} \frac{d^2}{dz^2} cot(\pi z) \Big|_{13/28}$$

Calculating the right sides,

$$1 - \frac{1}{27^3} + \frac{1}{29^3} - \frac{1}{55^3} + \frac{1}{57^3} - \frac{1}{83^3} + - \cdots = \frac{\pi^3}{28^3 2!} \left\{ 2cot\frac{\pi}{28} csc^2\frac{\pi}{28} \right\}$$

$$-\left(\frac{1}{3^3} - \frac{1}{25^3} + \frac{1}{31^3} - \frac{1}{53^3} + \frac{1}{59^3} - \frac{1}{81^3} + - \cdots \right) = -\frac{\pi^3}{28^3 2!} \left\{ 2cot\frac{3\pi}{28} csc^2\frac{3\pi}{28} \right\}$$

$$\vdots$$

$$-\left(\frac{1}{11^3}-\frac{1}{17^3}+\frac{1}{39^3}-\frac{1}{45^3}+\frac{1}{67^3}-\frac{1}{73^3}+-\cdots\right)=-\frac{\pi^3}{28^3 2!}\left\{2\cot\frac{11\pi}{28}\csc^2\frac{11\pi}{28}\right\}$$

$$\frac{1}{13^3}-\frac{1}{15^3}+\frac{1}{41^3}-\frac{1}{43^3}+\frac{1}{69^3}-\frac{1}{71^3}+-\cdots=\frac{\pi^3}{28^3 2!}\left\{2\cot\frac{13\pi}{28}\csc^2\frac{13\pi}{28}\right\}$$

Applying Corollary 6.4.1 to these, we obtain

$$1-\frac{1}{27^3}+\frac{1}{29^3}-\frac{1}{55^3}+\frac{1}{57^3}-\frac{1}{83^3}+-\cdots=-\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{1}{28}}-(-1)^{\frac{27}{28}}\right\}}{\left\{(-1)^{\frac{1}{28}}+(-1)^{\frac{27}{28}}\right\}^3}$$

$$-\left(\frac{1}{3^3}-\frac{1}{25^3}+\frac{1}{31^3}-\frac{1}{53^3}+\frac{1}{59^3}-\frac{1}{81^3}+-\cdots\right)=\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{3}{28}}-(-1)^{\frac{25}{28}}\right\}}{\left\{(-1)^{\frac{3}{28}}+(-1)^{\frac{25}{28}}\right\}^3}$$

$$\frac{1}{5^3}-\frac{1}{23^3}+\frac{1}{33^3}-\frac{1}{51^3}+\frac{1}{61^3}-\frac{1}{79^3}+-\cdots=-\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{5}{28}}-(-1)^{\frac{23}{28}}\right\}}{\left\{(-1)^{\frac{5}{28}}+(-1)^{\frac{23}{28}}\right\}^3}$$

$$-\left(\frac{1}{7^3}-\frac{1}{21^3}+\frac{1}{35^3}-\frac{1}{49^3}+\frac{1}{63^3}-\frac{1}{77^3}+-\cdots\right)=\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{7}{28}}-(-1)^{\frac{21}{28}}\right\}}{\left\{(-1)^{\frac{7}{28}}+(-1)^{\frac{21}{28}}\right\}^3}$$

$$\frac{1}{9^3}-\frac{1}{19^3}+\frac{1}{37^3}-\frac{1}{47^3}+\frac{1}{65^3}-\frac{1}{75^3}+-\cdots=-\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{9}{28}}-(-1)^{\frac{19}{28}}\right\}}{\left\{(-1)^{\frac{9}{28}}+(-1)^{\frac{19}{28}}\right\}^3}$$

$$-\left(\frac{1}{11^3}-\frac{1}{17^3}+\frac{1}{39^3}-\frac{1}{45^3}+\frac{1}{67^3}-\frac{1}{73^3}+-\cdots\right)=\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{11}{28}}-(-1)^{\frac{17}{28}}\right\}}{\left\{(-1)^{\frac{11}{28}}+(-1)^{\frac{17}{28}}\right\}^3}$$

$$\frac{1}{13^3}-\frac{1}{15^3}+\frac{1}{41^3}-\frac{1}{43^3}+\frac{1}{69^3}-\frac{1}{71^3}+-\cdots=-\frac{2^2\pi^3}{28^3}\frac{i\left\{(-1)^{\frac{13}{28}}-(-1)^{\frac{15}{28}}\right\}}{\left\{(-1)^{\frac{13}{28}}+(-1)^{\frac{15}{28}}\right\}^3}$$

The calculation results of the 1st and the 2nd series by *Mathematica* are as follows.

$$a1[m\_]:=\sum_{r=0}^{m}\left(\frac{1}{(28r+1)^3}-\frac{1}{(28r+27)^3}\right)$$

N[a1[10000], 10]

0.9999893925

$$N\left[-\frac{2^2\pi^3}{28^3}\frac{i\left((-1)^{\frac{1}{28}}-(-1)^{\frac{27}{28}}\right)}{\left((-1)^{\frac{1}{28}}+(-1)^{\frac{27}{28}}\right)^3},10\right]$$

$0.9999893925+0\times 10^{-12}\,i$

$$a2[m\_]:=-\sum_{r=0}^{m}\left(\frac{1}{(28r+3)^3}-\frac{1}{(28r+25)^3}\right)$$

N[a2[10000], 10]

-0.03700417387

$$N\left[\frac{2^2\pi^3}{28^3}\frac{i\left((-1)^{\frac{3}{28}}-(-1)^{\frac{25}{28}}\right)}{\left((-1)^{\frac{3}{28}}+(-1)^{\frac{25}{28}}\right)^3},10\right]$$

$-0.03700417387+0\times 10^{-14}\,i$

6.5 Reflection $m2^n$ -split of Dirichlet Series

Amoung the reflection split, in particular, the sum of the $m2^n$ -split series is expressed by the nested radical of m or 2 and their addition and multiplication. In this section, we prove and exemplify it.

Formula 6.5.1

When T_n is Chebyshev Polynomial of the 1st kind, the following expressions hold for natural numbers m,k.

$$\cos\frac{(2k-1)\pi}{m} = T_{2k-1}\left(\cos\frac{\pi}{m}\right) \tag{5.1c}$$

$$\sin\frac{(2k-1)\pi}{m} = (-1)^{k-1}T_{2k-1}\left(\sin\frac{\pi}{m}\right) \tag{5.1s}$$

Proof

According *http://mathworld.wolfram.com/Multiple-AngleFormulas.html*, the multiple-angle formulas for trigonometric functions are represented using Chebyshev Polynomial of the 1st kind as follows.

$$\cos nx = T_n(\cos x)$$

$$\sin nx = (-1)^{(n-1)/2}\,T_n(\sin x) \qquad \text{for } n \text{ odd}$$

Substituting $n=2k-1$, $x=\pi/m$ for these, we obtain the desired expressions.

Example

$$\sin\frac{5\pi}{32} = (-1)^{3-1}T_{2\cdot3-1}\left(\sin\frac{\pi}{32}\right) = 5\sin\frac{\pi}{32} - 20\sin^3\frac{\pi}{32} + 16\sin^5\frac{\pi}{32}$$

Formula 6.5.2

The following expressions hold for natural number n .

$$\begin{cases} \cos\dfrac{\pi}{2^n} = \dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \\ \sin\dfrac{\pi}{2^n} = \dfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \end{cases} \quad (n-1)\text{-nests} \tag{5.2$_2$}$$

$$\begin{cases} \cos\dfrac{\pi}{3\cdot2^n} = \dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{3}}}} \\ \sin\dfrac{\pi}{3\cdot2^n} = \dfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{3}}}} \end{cases} \quad n\text{-nests} \tag{5.2$_3$}$$

$$\begin{cases} \cos\dfrac{\pi}{5\cdot2^n} = \dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\dfrac{1+\sqrt{5}}{2}}}}} \\ \sin\dfrac{\pi}{5\cdot2^n} = \dfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\dfrac{1+\sqrt{5}}{2}}}}} \end{cases} \quad n\text{-nests} \tag{5.2$_3$}$$

Proof

$$\cos\frac{\theta}{2^1} = \sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}$$

Putting $\theta = \pi/2$,

$$\cos\frac{\pi}{2^2} = \sqrt{\frac{1}{2}+\frac{1}{2}\cos\frac{\pi}{2}} = \sqrt{\frac{1}{2}+0} = \frac{1}{2}\sqrt{2}$$

$$\cos\frac{\pi}{2^3} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\frac{\pi}{2}}} = \frac{2}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\frac{1}{2}\sqrt{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

$$\cos\frac{\pi}{2^4} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\frac{\pi}{2}}}} = \frac{2}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\frac{1}{2}\sqrt{2+\sqrt{2}}}$$

$$= \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$$

$\vdots$

Hereafter, by induction, we obtain $\cos(\pi/2^n)$. $\sin(\pi/2^n)$ is obtained using $\sqrt{1-\cos^2 z}$.
(5.2_3) and (5.2_5) are also derived in a similar way.

Theorem 6.5.3

The sum of reflection split series $a_k \quad k=1,2,\cdots,2^n$ in Definition 6.1.3' is expressed with addition, multiplication and the nested radical of 2 , except circle ratio π .

Proof

According to Formula 6.1.4 , the sum a_k of the 2^n -split series is given by the following expression.

(1) For Dirichlet lambda series $\lambda(2h)$,

$$a_k = -\frac{\pi}{(4\cdot 2^n)^{2h}(2h-1)!}\frac{d^{2h-1}}{dz^{2h-1}}\cot(\pi z)\Bigg|_{\frac{2k-1}{4\cdot 2^n}} \qquad k=1,2,\cdots,2^n$$

As seen in the calculation example of Formula 6.1.2, this higher order difference quotient is a polynomial of $\cot\frac{(2k-1)\pi}{2^{n+2}}$ and $\csc\frac{(2k-1)\pi}{2^{n+2}}$. Since $\cot z = \cos z/\sin z$, $\csc z = 1/\sin z$, this is expressed with a polynomial of $\cos\frac{\pi}{2^{n+2}}$ and $\sin\frac{\pi}{2^{n+2}}$ from Formula 6.5.1 . And, from Formula 6.5.2 (5.2_2) , this higher order difference quotient results in a polynomial with the following elements.

$$\frac{1}{2}\sqrt{2\pm\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \qquad (n+1)\text{-nests}$$

Corollary 6.5.3

(1) The sum of reflection split series $a_k \quad k=1,2,\cdots,3\cdot 2^n$ is expressed with addition, multiplication and the nested radical of 2 and 3 , except circle ratio π .

(2) The sum of reflection split series $a_k \quad k=1,2,\cdots,5\cdot 2^n$ is expressed with addition, multiplication and the nested radical of 2 and 5 , except circle ratio π .

Example 1: Reflection 8-split of $\lambda(2)$

Dirichlet lambda series $\lambda(2)$ is

$$\lambda(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots$$

According to Definition 6.1.3' (1) ,

$$a_k = \sum_{r=0}^{\infty} \left\{ \frac{1}{(4mr+2k-1)^{2n}} + \frac{1}{(4mr+4m-2k+1)^{2n}} \right\} \qquad k=1,2,\cdots,m$$

$$a_1 + a_2 + \cdots + a_m = \lambda(2n)$$

Putting $n=1, m=8$,

$$a_k = \sum_{r=0}^{\infty} \left\{ \frac{1}{(32r+2k-1)^2} + \frac{1}{(32r+32-2k+1)^2} \right\} \qquad k=1,2,\cdots,8$$

Expanding these,

$$a_1 = 1 + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{63^2} + \frac{1}{65^2} + \frac{1}{95^2} + \cdots$$

$$a_2 = \frac{1}{3^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{61^2} + \frac{1}{67^2} + \frac{1}{93^2} + \cdots$$

$$\vdots$$

$$a_8 = \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{79^2} + \frac{1}{81^2} + \cdots$$

$$\lambda(2) = a_1 + a_2 + a_3 + \cdots + a_8$$

From Formula 6.1.4 (1) ,

$$a_k = -\frac{\pi}{32^2 1!} \frac{d}{dz} \cot(\pi z) \Big|_{\frac{2k-1}{32}} \qquad k=1,2,\cdots,8$$

Here,

$$\frac{d}{dz} \cot(\pi z) = -\pi \csc^2(\pi z)$$

So,

$$a_k = \frac{\pi^2}{32^2 1!} \csc^2 \frac{(2k-1)\pi}{32} = \frac{\pi^2}{32^2 1!} \left\{ \sin \frac{(2k-1)\pi}{32} \right\}^{-2} \qquad k=1,2,\cdots,8$$

From Formula 6.5.1 , the following expression holds for $k=1,2,\cdots,8$.

$$\sin \frac{(2k-1)\pi}{32} = (-1)^{k-1} T_{2k-1}(\beta) \quad , \quad \beta = \sin \frac{\pi}{32}$$

Using Formula 6.5.2 (5.2_2) , let

$$s_k = (-1)^{k-1} T_{2k-1}(\beta) \ , \quad \beta = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

Then,

$$a_k = \frac{\pi^2}{32^2 1!} \frac{1}{s_k^2} =: b_k \qquad k=1,2,\cdots,8$$

After calculating and simplifying b_k using *Mathematica*, if the results are written together with a_k , it is as follows. Of course, both sides are numerically consistent.

$$a_1 = 1 + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{63^2} + \frac{1}{65^2} + \frac{1}{95^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{61^2} + \frac{1}{67^2} + \frac{1}{93^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2+\sqrt{2-\sqrt{2}}}\right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{59^2} + \frac{1}{69^2} + \frac{1}{91^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2-\sqrt{2-\sqrt{2}}}\right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{57^2} + \frac{1}{71^2} + \frac{1}{89^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2-\sqrt{2+\sqrt{2}}}\right)}$$

$$a_5 = \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{41^2} + \frac{1}{55^2} + \frac{1}{73^2} + \frac{1}{87^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2-\sqrt{2+\sqrt{2}}}\right)}$$

$$a_6 = \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{43^2} + \frac{1}{53^2} + \frac{1}{75^2} + \frac{1}{85^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2-\sqrt{2-\sqrt{2}}}\right)}$$

$$a_7 = \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{45^2} + \frac{1}{51^2} + \frac{1}{77^2} + \frac{1}{83^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2+\sqrt{2-\sqrt{2}}}\right)}$$

$$a_8 = \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{79^2} + \frac{1}{81^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}$$

$$\lambda(2) = a_1 + a_2 + a_3 + \cdots + a_8$$

Note

When $\lambda(2)$ is split in 2^n series, the denominator on the right side is the n -nested square root of 2 .

$$a_k = \sum_{r=0}^{\infty}\left\{\frac{1}{(4\cdot 2^n r+2k-1)^2} + \frac{1}{(4\cdot 2^n r+4\cdot 2^n-2k+1)^2}\right\} = \frac{\pi^2}{2^{2n+2}\left(2\pm\sqrt{2\pm\sqrt{2\pm\cdots\pm\sqrt{2}}}\right)}$$

where, the sign differs according to k

Example 2: Reflection 6-split of $\beta(3)$

Dirichlet beta series $\beta(3)$ is

$$\beta(3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + - \cdots$$

According to Definition 6.1.3' (2) ,

$$a_k = (-1)^{k-1}\sum_{r=0}^{\infty}\left\{\frac{1}{(4mr+2k-1)^{2n-1}} - \frac{1}{(4mr+4m-2k+1)^{2n-1}}\right\} \qquad k=1,2,\cdots,m$$

$$a_1 + a_2 + \cdots + a_m = \beta(2n-1)$$

Putting $n=2, m=6$,

$$a_k = (-1)^{k-1}\sum_{r=0}^{\infty}\left\{\frac{1}{(24r+2k-1)^3} - \frac{1}{(24r+24-2k+1)^3}\right\} \qquad k=1,2,\cdots,6$$

Expanding these,

$$a_1 = 1 - \frac{1}{23^3} + \frac{1}{25^3} - \frac{1}{47^3} + \frac{1}{49^3} - \frac{1}{71^3} + - \cdots$$

$$a_2 = -\left(\frac{1}{3^3} - \frac{1}{21^3} + \frac{1}{27^3} - \frac{1}{45^3} + \frac{1}{51^3} - \frac{1}{69^3} + - \cdots \right)$$

$$\vdots$$

$$a_5 = \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{33^3} - \frac{1}{39^3} + \frac{1}{57^3} - \frac{1}{63^3} + - \cdots$$

$$a_6 = -\left(\frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{35^3} - \frac{1}{37^3} + \frac{1}{59^3} - \frac{1}{61^3} + - \cdots \right)$$

$$a_1 + a_2 + \cdots + a_6 = \beta(3)$$

From Formula 6.1.4 (2) ,

$$a_k = \frac{(-1)^{k-1}\pi}{24^3 2!} \frac{d^2}{dz^2} cot(\pi z) \Big|_{\frac{2k-1}{24}} \qquad k=1,2,\cdots,6$$

Here,

$$\frac{d^2}{dz^2} cot(\pi z) = 2\pi^2 cot(\pi z) csc^2(\pi z)$$

So,

$$a_k = \frac{(-1)^{k-1}\pi^3}{24^3} cot\frac{(2k-1)\pi}{24} csc^2\frac{(2k-1)\pi}{24}$$

From Formula 6.5.1 , the following expressions hold for $k=1,2,\cdots,6$.

$$cos\frac{(2k-1)\pi}{24} = T_{2k-1}(\alpha) \quad , \quad \alpha = cos\frac{\pi}{24}$$

$$sin\frac{(2k-1)\pi}{24} = (-1)^{k-1} T_{2k-1}(\beta) \quad , \quad \beta = sin\frac{\pi}{24}$$

From Formula 6.5.2 (5.2_3) ,

$$cos\frac{\pi}{24} = cos\frac{\pi}{3 \cdot 2^3} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}$$

$$sin\frac{\pi}{24} = sin\frac{\pi}{3 \cdot 2^3} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}$$

So, let

$$c_k = T_{2k-1}(\alpha) \quad , \quad \alpha = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}$$

$$s_k = (-1)^{k-1} T_{2k-1}(\beta) \quad , \quad \beta = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}$$

Substituting these for the above ,

$$a_k = (-1)^{k-1}\frac{\pi^3}{24^3}\frac{c_k}{s_k^3} =: b_k \qquad k=1,2,\cdots,6$$

After calculating and simplifying b_k using *Mathematica*, if the results are written together with a_k , it is as follows. Where, b_6 is drawn using $cos(\pi - \theta) = -cos\theta$, $sin(\pi - \theta) = sin\theta$ because rationalization of the denominator is difficult. Of course, both sides are numerically consistent.

$$a_1 = 1 - \frac{1}{23^3} + \frac{1}{25^3} - \frac{1}{47^3} + \frac{1}{49^3} - \frac{1}{71^3} + - \cdots$$
$$= \frac{\left(56+39\sqrt{2}+32\sqrt{3}+23\sqrt{6}\right)\pi^3}{6912}$$

$$a_2 = -\left(\frac{1}{3^3} - \frac{1}{21^3} + \frac{1}{27^3} - \frac{1}{45^3} + \frac{1}{51^3} - \frac{1}{69^3} + - \cdots\right) = \frac{\left(-4-3\sqrt{2}\right)\pi^3}{6912}$$

$$a_3 = \frac{1}{5^3} - \frac{1}{19^3} + \frac{1}{29^3} - \frac{1}{43^3} + \frac{1}{53^3} - \frac{1}{67^3} + - \cdots$$
$$= \frac{\left(56-39\sqrt{2}-32\sqrt{3}+23\sqrt{6}\right)\pi^3}{6912}$$

$$a_4 = -\left(\frac{1}{7^3} - \frac{1}{17^3} + \frac{1}{31^3} - \frac{1}{41^3} + \frac{1}{55^3} - \frac{1}{65^3} + - \cdots\right)$$
$$= \frac{\left(56+39\sqrt{2}-32\sqrt{3}-23\sqrt{6}\right)\pi^3}{6912}$$

$$a_5 = \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{33^3} - \frac{1}{39^3} + \frac{1}{57^3} - \frac{1}{63^3} + - \cdots = \frac{\left(-4+3\sqrt{2}\right)\pi^3}{6912}$$

$$a_6 = -\left(\frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{35^3} - \frac{1}{37^3} + \frac{1}{59^3} - \frac{1}{61^3} + - \cdots\right)$$
$$= \frac{\left(56-39\sqrt{2}+32\sqrt{3}-23\sqrt{6}\right)\pi^3}{6912}$$

$$a_1 + a_2 + \cdots + a_6 = \beta(3)$$

6.6 Reflection $m2^n$-split of $\lambda(2)$

In this section, we deal with reflection $m2^n$-split of Dirichlet lambda series $\lambda(2)$ chiefly, which is as follows.

$$\lambda(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots$$

If 2-split, 4-split, 6-split, 8-split and 10-split of this are calculated by the method mentioned in the previous section, it is as follows.

2-split

$$a_1 = 1 + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{23^2} + \cdots = \frac{\pi^2}{16\left(2-\sqrt{2}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{21^2} + \cdots = \frac{\pi^2}{16\left(2+\sqrt{2}\right)}$$

4-split

$$a_1 = 1 + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{47^2} + \cdots = \frac{\pi^2}{64\left(2-\sqrt{2+\sqrt{2}}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{45^2} + \cdots = \frac{\pi^2}{64\left(2-\sqrt{2-\sqrt{2}}\right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{43^2} + \cdots = \frac{\pi^2}{64\left(2+\sqrt{2-\sqrt{2}}\right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{41^2} + \cdots = \frac{\pi^2}{64\left(2+\sqrt{2+\sqrt{2}}\right)}$$

6-split

$$a_1 = 1 + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{71^2} + \cdots = \frac{\pi^2}{144\left(2-\sqrt{2+\sqrt{3}}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{45^2} + \frac{1}{51^2} + \frac{1}{69^2} + \cdots = \frac{\pi^2}{3^2\,16\left(2-\sqrt{2}\right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{43^2} + \frac{1}{53^2} + \frac{1}{67^2} + \cdots = \frac{\pi^2}{144\left(2-\sqrt{2-\sqrt{3}}\right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{41^2} + \frac{1}{55^2} + \frac{1}{65^2} + \cdots = \frac{\pi^2}{144\left(2+\sqrt{2-\sqrt{3}}\right)}$$

$$a_5 = \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{33^2} + \frac{1}{39^2} + \frac{1}{57^2} + \frac{1}{63^2} + \cdots = \frac{\pi^2}{3^2\,16\left(2+\sqrt{2}\right)}$$

$$a_6 = \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{35^2} + \frac{1}{37^2} + \frac{1}{59^2} + \frac{1}{61^2} + \cdots = \frac{\pi^2}{144\left(2+\sqrt{2+\sqrt{3}}\right)}$$

Where, a_2 , a_5 are $1/3^2$ times the reflection 2-split series of $\lambda(2)$.

8-split

$$a_1 = 1 + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{63^2} + \frac{1}{65^2} + \frac{1}{95^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{61^2} + \frac{1}{67^2} + \frac{1}{93^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2+\sqrt{2-\sqrt{2}}}\right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{59^2} + \frac{1}{69^2} + \frac{1}{91^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2-\sqrt{2-\sqrt{2}}}\right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{57^2} + \frac{1}{71^2} + \frac{1}{89^2} + \cdots = \frac{\pi^2}{256\left(2-\sqrt{2-\sqrt{2+\sqrt{2}}}\right)}$$

$$a_5 = \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{41^2} + \frac{1}{55^2} + \frac{1}{73^2} + \frac{1}{87^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2-\sqrt{2+\sqrt{2}}}\right)}$$

$$a_6 = \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{43^2} + \frac{1}{53^2} + \frac{1}{75^2} + \frac{1}{85^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2-\sqrt{2-\sqrt{2}}}\right)}$$

$$a_7 = \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{45^2} + \frac{1}{51^2} + \frac{1}{77^2} + \frac{1}{83^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2+\sqrt{2-\sqrt{2}}}\right)}$$

$$a_8 = \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{47^2} + \frac{1}{49^2} + \frac{1}{79^2} + \frac{1}{81^2} + \cdots = \frac{\pi^2}{256\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}$$

10-split

$$a_1 = 1 + \frac{1}{39^2} + \frac{1}{41^2} + \frac{1}{79^2} + \frac{1}{81^2} + \frac{1}{119^2} + \cdots = \frac{\pi^2}{400\left(2-\sqrt{2+\sqrt{2+\frac{1+\sqrt{5}}{2}}}\right)}$$

$$a_2 = \frac{1}{3^2} + \frac{1}{37^2} + \frac{1}{43^2} + \frac{1}{77^2} + \frac{1}{83^2} + \frac{1}{117^2} + \cdots = \frac{\pi^2}{400\left(2-\sqrt{2+\sqrt{2+\frac{1-\sqrt{5}}{2}}}\right)}$$

$$a_3 = \frac{1}{5^2} + \frac{1}{35^2} + \frac{1}{45^2} + \frac{1}{75^2} + \frac{1}{85^2} + \frac{1}{115^2} + \cdots = \frac{\pi^2}{5^2\,16\left(2-\sqrt{2}\right)}$$

$$a_4 = \frac{1}{7^2} + \frac{1}{33^2} + \frac{1}{47^2} + \frac{1}{73^2} + \frac{1}{87^2} + \frac{1}{113^2} + \cdots = \frac{\pi^2}{400\left(2-\sqrt{2-\sqrt{2+\frac{1-\sqrt{5}}{2}}}\right)}$$

$$a_5 = \frac{1}{9^2} + \frac{1}{31^2} + \frac{1}{49^2} + \frac{1}{71^2} + \frac{1}{89^2} + \frac{1}{111^2} + \cdots = \frac{\pi^2}{400\left(2-\sqrt{2-\sqrt{2+\frac{1+\sqrt{5}}{2}}}\right)}$$

$$a_6 = \frac{1}{11^2} + \frac{1}{29^2} + \frac{1}{51^2} + \frac{1}{69^2} + \frac{1}{91^2} + \frac{1}{109^2} + \cdots = \frac{\pi^2}{400\left(2+\sqrt{2-\sqrt{2+\frac{1+\sqrt{5}}{2}}}\right)}$$

$$a_7 = \frac{1}{13^2} + \frac{1}{27^2} + \frac{1}{53^2} + \frac{1}{67^2} + \frac{1}{93^2} + \frac{1}{107^2} + \cdots = \frac{\pi^2}{400\left(2+\sqrt{2-\sqrt{2+\frac{1-\sqrt{5}}{2}}}\right)}$$

$$a_8 = \frac{1}{15^2} + \frac{1}{25^2} + \frac{1}{55^2} + \frac{1}{65^2} + \frac{1}{95^2} + \frac{1}{105^2} + \cdots = \frac{\pi^2}{5^2 16(2+\sqrt{2})}$$

$$a_9 = \frac{1}{17^2} + \frac{1}{23^2} + \frac{1}{57^2} + \frac{1}{63^2} + \frac{1}{97^2} + \frac{1}{103^2} + \cdots = \frac{\pi^2}{400\left(2+\sqrt{2+\sqrt{2+\frac{1-\sqrt{5}}{2}}}\right)}$$

$$a_{10} = \frac{1}{19^2} + \frac{1}{21^2} + \frac{1}{59^2} + \frac{1}{61^2} + \frac{1}{99^2} + \frac{1}{101^2} + \cdots = \frac{\pi^2}{400\left(2+\sqrt{2+\sqrt{2+\frac{1+\sqrt{5}}{2}}}\right)}$$

Where, a_2, a_5 are $1/5^2$ times the reflection 2-split series of $\lambda(2)$.

Remark

The denominators are not rationalized purposely. If this expression is used, the sum of each split series is determined by the combination of + and - in the nested square roots. However, in 6-split and 10-split, it is impossible to represent all the sums with only these combinations. Interestingly, the shortags are filled up by reflection 2-split series of $\lambda(2)$. So, the following thorem holds.

Theorem 6.6.1

Let p be a prime number greater than 2 and reflection $2p$ -split series a_k of $\lambda(2)$ are as follows.

$$a_k = \sum_{r=0}^{\infty}\left\{\frac{1}{(8pr+2k-1)^2} + \frac{1}{(8pr+8p-2k+1)^2}\right\} \qquad k=1,2,\cdots,2p$$

$$a_1 + a_2 + \cdots + a_{2p} = \lambda(2)$$

Then, following expressions hold.

$$a_{\frac{p+1}{2}} = \frac{\pi^2}{p^2 16(2-\sqrt{2})} \quad , \quad a_{\frac{3p+1}{2}} = \frac{\pi^2}{p^2 16(2+\sqrt{2})}$$

Proof

According to Formula 6.1.4 (1) , the sum b_k of the reflection 2 -split series and the sum a_k of the reflection $2p$ -split series are given as follows respectively.

$$b_1 = \frac{\pi^2}{8^2} csc^2 \frac{1\pi}{8} = \frac{\pi^2}{16(2-\sqrt{2})} \quad , \quad b_2 = \frac{\pi^2}{8^2} csc^2 \frac{3\pi}{8} = \frac{\pi^2}{16(2+\sqrt{2})}$$

$$a_k = \frac{\pi^2}{(8p)^2} csc^2 \frac{(2k-1)\pi}{8p} \qquad k=1,2,\cdots,2p$$

When $2k-1 = p$ $i.e.$ $k = (p+1)/2$,

$$a_{\frac{p+1}{2}} = \frac{\pi^2}{(8p)^2} csc^2 \frac{p\pi}{8p} = \frac{\pi^2}{p^2 8^2} csc^2 \frac{\pi}{8} = \frac{\pi^2}{p^2 16(2-\sqrt{2})}$$

When $2k-1 = 3p$ $i.e.$ $k = (3p+1)/2$,

$$a_{\frac{3p+1}{2}} = \frac{\pi^2}{(8p)^2} csc^2 \frac{3p\pi}{8p} = \frac{\pi^2}{p^2 8^2} csc^2 \frac{3\pi}{8} = \frac{\pi^2}{p^2 16(2+\sqrt{2})}$$

Example: 14-split

For the purpose, expression by the nested radicals of $\cos(\pi/7)$ is required first. According to Wikipedia (https://en.wikipedia.org/wiki/Trigonometric_constants_expressed_in_real_radicals) , it is as follows.

$$\cos\frac{\pi}{7} = \frac{1}{24}\sqrt{3\left(80 + \sqrt[3]{14336+\sqrt{-5549064193}} + \sqrt[3]{14336-\sqrt{-5549064193}}\right)}$$

We have to apply this half-angle formula 3 times to find $\sin(\pi/56)$. Then, we must calculate the sum $a_k \quad (k=1,2,3,\cdots,2p)$ of each split series. However, they become very long formulas and it is difficult to describe here.

Nevertheless, if placed with $p=7$ in the theorem, only a_4 , a_{11} are expressed simply exceptionally. That is

$$a_4 = \frac{1}{7^2} + \frac{1}{49^2} + \frac{1}{63^2} + \frac{1}{105^2} + \frac{1}{119^2} + \frac{1}{161^2} + \cdots = \frac{\pi^2}{7^2\,16\left(2-\sqrt{2}\right)}$$

$$a_{11} = \frac{1}{21^2} + \frac{1}{35^2} + \frac{1}{77^2} + \frac{1}{91^2} + \frac{1}{133^2} + \frac{1}{147^2} + \cdots = \frac{\pi^2}{7^2\,16\left(2+\sqrt{2}\right)}$$

In fact, the calculation results of both sides by *Mathematica* are as follows.

$$a_{k\_} := \sum_{r=0}^{200\,000}\left(\frac{1}{(56\,r+2\,k-1)^2} + \frac{1}{(56\,r+56-2\,k+1)^2}\right)$$

$$N\left[\left\{a_4,\ \frac{\pi^2}{7^2\,16\left(2-\sqrt{2}\right)}\right\}\right] \qquad N\left[\left\{a_{11},\ \frac{\pi^2}{7^2\,16\left(2+\sqrt{2}\right)}\right\}\right]$$

{0.0214904 , 0.0214904}   {0.00368717 , 0.00368717}

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Kano Kono

Alien's Mathematics