## 08 Series Expansion of Dirichlet Eta Function

### 8.1 Dirichlet Eta Series

Dirichlet eta function $\eta(z)$ is defined in $Re(z)>0$ by the following expression called Dirichlet eta series.

$$\eta(z) = \sum_{s=1}^{\infty}(-1)^{s-1}e^{-z\log s} = \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} +- \cdots \tag{1.0}$$

This is represented as follows for each real part and imaginary part.

### Formula 8.1.1

When the real and imaginary parts of Dirichlet eta function $\eta(x,y)$ are $\eta_r$ , $\eta_i$ respectively, the following expressions hold for $x>0$.

$$\eta_r(x,y) = \sum_{s=1}^{\infty}(-1)^{s-1}\frac{\cos(y\log s)}{s^x}$$

$$\eta_i(x,y) = -\sum_{s=1}^{\infty}(-1)^{s-1}\frac{\sin(y\log s)}{s^x}$$

### Proof

When $z=x+iy$ , from (1.0)

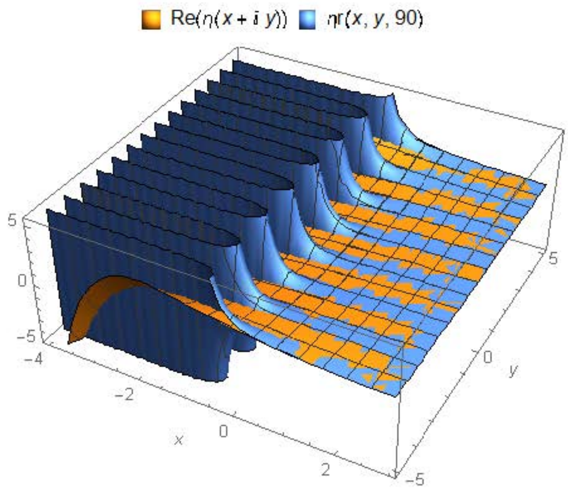
$$\eta(x,y) = \sum_{s=1}^{\infty}(-1)^{s-1}e^{-(x+iy)\log s} = \sum_{s=1}^{\infty}(-1)^{s-1}e^{-x\log s}\cdot e^{-iy\log s}$$

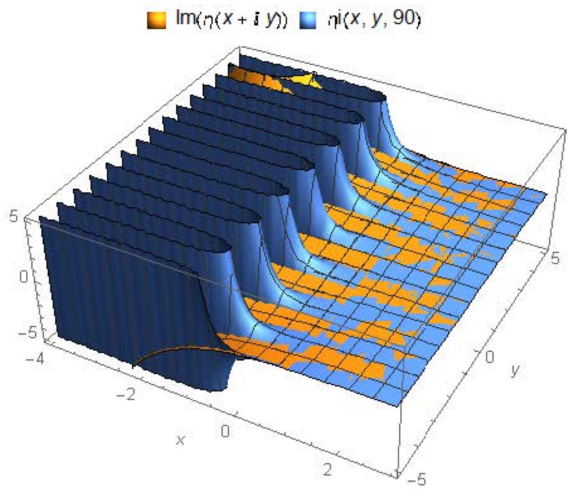
$$= \sum_{s=1}^{\infty}(-1)^{s-1}\frac{\cos(y\log s) - i\sin(y\log s)}{s^x}$$

$$= \sum_{s=1}^{\infty}(-1)^{s-1}\frac{\cos(y\log s)}{s^x} - i\sum_{s=1}^{\infty}(-1)^{s-1}\frac{\sin(y\log s)}{s^x}$$

From this, we obtain the desired expressions.

These are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. It is well known that $x=0$ is a line of convergence.



Formula 8.1.1 can be accelerated in convergence by applying the Knop transformation. Further, an analytic continuation can be expected. In addition, about Knopp transformation, see " **10 Convergence Acceleration & Summation Method by Double Series of Functions "** ( A la carte ).

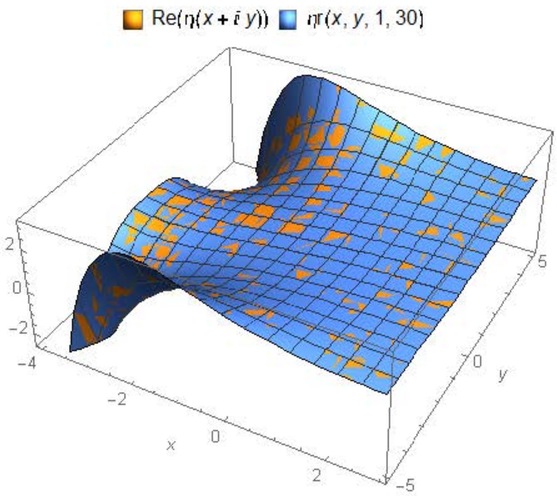
### Formula 8.1.1' ( Accelerated type )

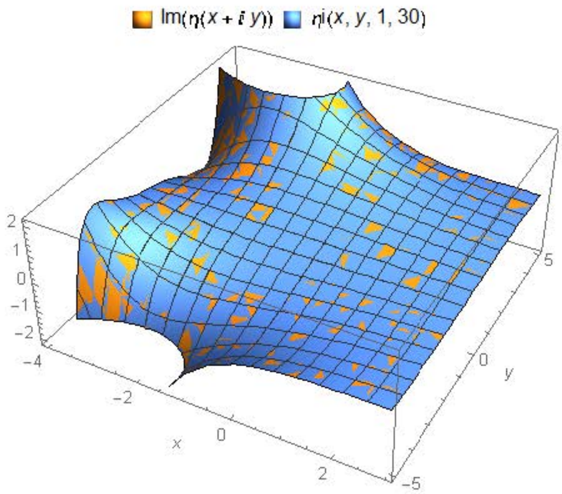
When the real and imaginary parts of Dirichlet eta function $\eta(x,y)$ are $\eta_r$ , $\eta_i$ respectively, the following expressions hold for arbitrary positive number $q$ .

$$\eta_r(x,y) = \sum_{k=1}^{\infty} \sum_{s=1}^{k} \frac{q^{k-s}}{(q+1)^{k+1}} \binom{k}{s} (-1)^{s-1} \frac{\cos(y \log s)}{s^x}$$

$$\eta_i(x,y) = -\sum_{k=1}^{\infty} \sum_{s=1}^{k} \frac{q^{k-s}}{(q+1)^{k+1}} \binom{k}{s} (-1)^{s-1} \frac{\sin(y \log s)}{s^x}$$

When $q=1$ , these are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. We can see that the line of convergence $x=0$ has disappeared due to the effect of the analytic continuation.




## Non-Trivial Zeros of Riemann Zeta

### Theorem 8.1.2

The non-trivial zeros of Riemann zeta function are given by the real solutions of the following simultaneous equations.

$$\begin{cases} \displaystyle\sum_{s=1}^{\infty} (-1)^{s-1} \frac{\cos(y \log s)}{s^x} = 0 \\ \displaystyle -\sum_{s=1}^{\infty} (-1)^{s-1} \frac{\sin(y \log s)}{s^x} = 0 \end{cases} \qquad \text{Where, } 0 < x < 1$$

And, Riemann hypothesis is that one of the solutions will always be $x=1/2$ . However, this simultaneous equation is a transcendental equation with respect to $x$ and $y$ , and the analysis is not easy.

## 8.2 Power Series with Stieltjes Constants

### 8.2.0 Power of Complex Number

### Lemma 8.2.0

When $x, y$ are real numbers, $r$ is non-negative integer, the following holds.

$$(x+iy)^r = \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} (-1)^s \binom{r}{2s} x^{r-2s} y^{2s} + i \sum_{s=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^s \binom{r}{2s+1} x^{r-2s-1} y^{2s+1} \tag{2.0}$$

Where, $0^0 = 1$ , $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

### Proof

$$(x+iy)^1 = \sum_{s=0}^{1} {}_1C_s\, x^{1-s} i^s y^s$$

$$= {}_1C_0\, x^{1-0} y^0 + i\left({}_1C_1\, x^{1-1} y^1\right)$$

$$= \sum_{s=0}^{0} (-1)^s {}_1C_{2s}\, x^{1-2s} y^{2s} + i \sum_{s=0}^{0} (-1)^s {}_1C_{2s+1}\, x^{1-2s-1} y^{2s+1} \quad \left( \begin{array}{l} 0 = \left\lceil \frac{1-1}{2} \right\rceil \\ 0 = \left\lfloor \frac{1-1}{2} \right\rfloor \end{array} \right)$$

$$(x+iy)^2 = \sum_{s=0}^{2} {}_2C_s\, x^{2-s} i^s y^s$$

$$= {}_2C_0\, x^{2-0} y^0 - {}_2C_2\, x^{2-2} y^2 + i\left({}_2C_1\, x^{2-1} y^1\right)$$

$$= \sum_{s=0}^{1} (-1)^s {}_2C_{2s}\, x^{2-2s} y^{2s} + i \sum_{s=0}^{0} (-1)^s {}_2C_{2s+1}\, x^{2-2s-1} y^{2s+1} \quad \left( \begin{array}{l} 1 = \left\lceil \frac{2-1}{2} \right\rceil \\ 0 = \left\lfloor \frac{2-1}{2} \right\rfloor \end{array} \right)$$

$$(x+iy)^3 = \sum_{s=0}^{3} {}_3C_s\, x^{3-s} i^s y^s$$

$$= {}_3C_0\, x^{3-0} y^0 - {}_3C_2\, x^{3-2} y^2 + i\left({}_3C_1\, x^{3-1} y^1 - {}_3C_3\, x^{3-3} y^3\right)$$

$$= \sum_{s=0}^{1} (-1)^s {}_3C_{2s}\, x^{3-2s} y^{2s} + i \sum_{s=0}^{1} (-1)^s {}_3C_{2s+1}\, x^{3-2s-1} y^{2s+1} \quad \left( \begin{array}{l} 1 = \left\lceil \frac{3-1}{2} \right\rceil \\ 1 = \left\lfloor \frac{3-1}{2} \right\rfloor \end{array} \right)$$

$$(x+iy)^4 = \sum_{s=0}^{4} {}_4C_s\, x^{4-s} i^s y^s$$

$$= {}_4C_0\, x^{4-0} y^0 - {}_4C_2\, x^{4-2} y^2 - {}_4C_4\, x^{4-4} y^4 + i\left({}_4C_1\, x^{4-1} y^1 - {}_4C_3\, x^{4-3} y^3\right)$$

$$= \sum_{s=0}^{2} (-1)^s {}_4C_{2s}\, x^{4-2s} y^{2s} + i \sum_{s=0}^{1} (-1)^s {}_4C_{2s+1}\, x^{4-2s-1} y^{2s+1} \quad \left( \begin{array}{l} 2 = \left\lceil \frac{4-1}{2} \right\rceil \\ 1 = \left\lfloor \frac{4-1}{2} \right\rfloor \end{array} \right)$$

Hereafter, by induction, we obtain the desired expression.

### 8.2.1 Power Series Expansion with Stieltjes Constants

" **03 Complementary Series of Dirichlet Series** " Formula 3.1.2 was as follows.

$$\eta(z) = \log 2 + \sum_{r=1}^{\infty} (-1)^r \left\{ \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s (\log 2)^{r-s} \right\} \frac{(z-1)^r}{r!}$$

Where, $\gamma_s = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} \frac{(\log k)^s}{k} - \frac{(\log n)^{s+1}}{s+1} \right\}$

This is represented as follows for each real part and imaginary part.

### Formula 8.2.1

When the real and imaginary parts of Dirichlet eta function $\eta(x,y)$ are $\eta_r$ , $\eta_i$ respectively, the following expressions hold on the whole complex plane.

$$\eta_r(x,y) = \log 2 + \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \left\{ \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s (\log 2)^{r-s} \right\} \times \sum_{t=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^t \binom{r}{2t} (x-1)^{r-2t} y^{2t}$$

$$\eta_i(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \left\{ \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s (\log 2)^{r-s} \right\} \times \sum_{t=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^t \binom{r}{2t+1} (x-1)^{r-2t-1} y^{2t+1}$$

Where, $0^0 = 1$ , $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

### Proof

" **03 Complementary Series of Dirichlet Series** " Formula 3.1.2 was as follows.

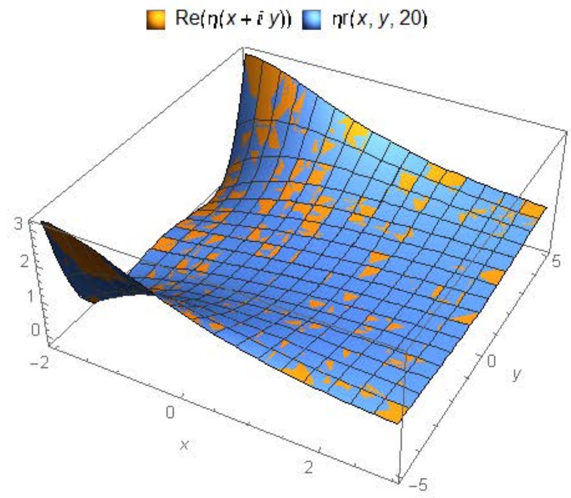
$$\eta(z) = \log 2 + \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \left\{ \frac{\log^{r+1} 2}{r+1} - \sum_{s=0}^{r-1} \binom{r}{s} \gamma_s (\log 2)^{r-s} \right\} (z-1)^r$$

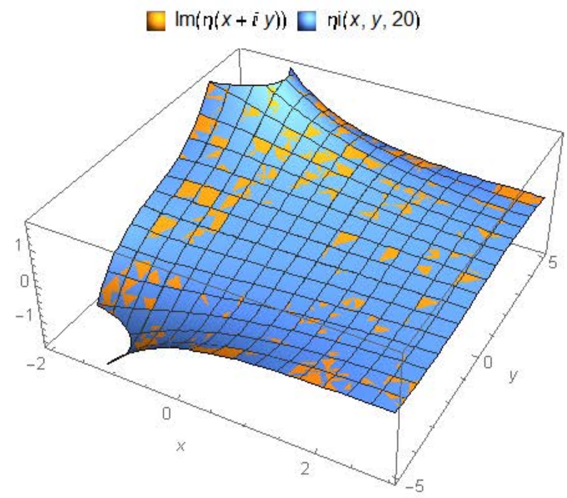
On the other hand, replacing $x$ with $x-1$ in Lemma 8.2.0 ,

$$(x-1+iy)^r = \sum_{t=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^t \binom{r}{2t} (x-1)^{r-2t} y^{2t} + i \sum_{t=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^t \binom{r}{2t+1} (x-1)^{r-2t-1} y^{2t+1}$$

Substituting this for $(z-1)^r$ and writing the real and imaginary parts as $\eta_r$ , $\eta_i$ , we obtain the formula.

These are drawn on the next page, The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. We can see that this formula holds on the whole complex plane.




The convergence of Formula 8.2.1 is very fast. However, it contains two binomial coefficients and is not beautiful. So applying Formula 14.1.2 in " 14 Taylor Expansion by Real Part & Imaginary Part " (A la carte) to (1.2) , we obtain the follows. It is a beautiful two-variable Taylor series, and the convergence is fast.

## Formula 8.2.2

When the real and imaginary parts of Dirichlet eta function $\eta(x,y)$ are $\eta_r$ , $\eta_i$ respectively, the following expressions hold on the whole complex plane.

$$\eta(z) = \log 2 + \sum_{s=1}^{\infty}(-1)^s\left\{\frac{(\log 2)^{s+1}}{s+1} - \sum_{t=0}^{s-1}\binom{s}{t}\gamma_t(\log 2)^{s-t}\right\}\frac{(z-1)^s}{s!}$$

$$\eta_r(x,y) = \log 2 + \sum_{s=1}^{\infty}\left\{\frac{(\log 2)^{s+1}}{s+1} - \sum_{t=0}^{s-1}\binom{s}{t}\gamma_t(\log 2)^{s-t}\right\}\frac{(-1)^s(x-1)^s}{s!}$$

$$+ \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\left\{\frac{(\log 2)^{2r+s+1}}{2r+s+1} - \sum_{t=0}^{2r+s-1}\binom{2r+s}{t}\gamma_t(\log 2)^{2r+s-t}\right\}\frac{(-1)^s(x-1)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$\eta_i(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left\{\frac{(\log 2)^{2r+s+2}}{2r+s+2} - \sum_{t=0}^{2r+s}\binom{2r+s+1}{t}\gamma_t(\log 2)^{2r+s+1-t}\right\}$$

$$\times\frac{(-1)^s(x-1)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $\gamma_t = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{(\log k)^t}{k} - \frac{(\log n)^{t+1}}{t+1}\right\}$ , $0^0 = 1$

The verification results by numerical calculation of the real and imaginary parts are as follows. The upper part is the real part and the lower part is the imaginary part. Both sides are exactly the same.

```
N[{Re[DirichletEta[-5 + i 1]] , ηr[-5, 1, 30]}]
{0.483565 , 0.483565}
N[{Im[DirichletEta[-5 + i 1]] , ηi[-5, 1, 30]}]
{-0.306893 , -0.306893}
```

## 8.3 Taylor Series & Complementary Series

### 8.3.1 Taylor Series

**Formula 8.3.1**

When the real and imaginary parts of Dirichlet eta function $\eta(x,y)$ are $\eta_r$ , $\eta_i$ respectively, the following expressions hold for $x>0$ and real constant $c$ other than zero of $\eta$ .

$$\eta_r(x,y) = \sum_{r=0}^{\infty}\left\{\sum_{s=0}^{\infty}\frac{(-1)^s \log^r(s+1)}{(s+1)^c}\right\}\frac{(-1)^r}{r!}\sum_{t=0}^{\left\lceil\frac{r-1}{2}\right\rceil}(-1)^t\binom{r}{2t}(x-c)^{r-2t}y^{2t}$$

$$\eta_i(x,y) = \sum_{r=0}^{\infty}\left\{\sum_{s=0}^{\infty}\frac{(-1)^s \log^r(s+1)}{(s+1)^c}\right\}\frac{(-1)^r}{r!}\sum_{t=0}^{\left\lfloor\frac{r-1}{2}\right\rfloor}(-1)^t\binom{r}{2t+1}(x-c)^{r-2t-1}y^{2t+1}$$

Where, $0^0=1$ , $\lceil x\rceil$ is the ceiling function , $\lfloor x\rfloor$ is the floor function.

**Proof**

" **03 Complementary Series of Dirichlet Series** " Formula 3.4.1 was as follows.

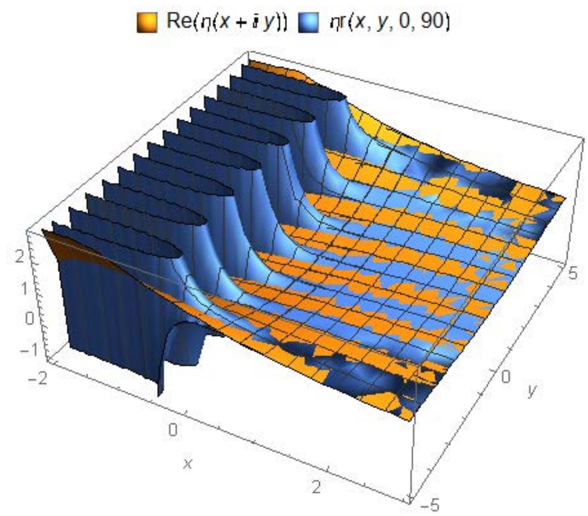
$$\eta(z) = \sum_{r=0}^{\infty}\left\{\sum_{s=0}^{\infty}\frac{(-1)^s \log^r(s+1)}{(s+1)^c}\right\}\frac{(-1)^r}{r!}(z-c)^r \quad \text{Where, } 0^0=1.$$

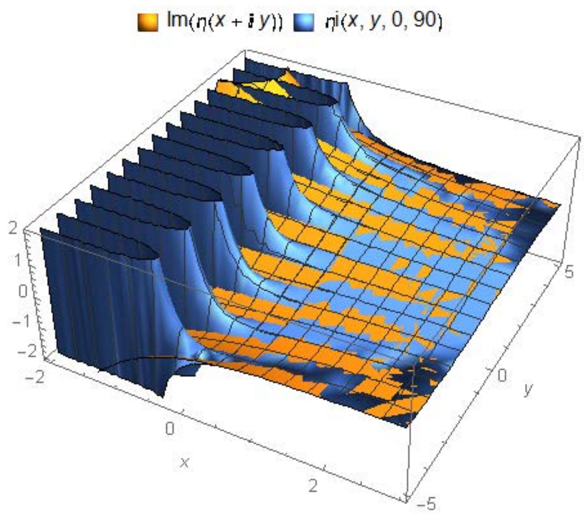
On the other hand, replacing $x$ with $x-c$ in Lemma 8.2.0 ,

$$(x-c+iy)^r = \sum_{t=0}^{\left\lceil\frac{r-1}{2}\right\rceil}(-1)^t\binom{r}{2t}(x-c)^{r-2t}y^{2t} + i\sum_{t=0}^{\left\lfloor\frac{r-1}{2}\right\rfloor}(-1)^t\binom{r}{2t+1}(x-c)^{r-2t-1}y^{2t+1}$$

Substituting this for $(z-c)^r$ and writing the real and imaginary parts as $\eta_r$ , $\eta_i$ , we obtain the formula.

When $c=0$ , these are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side.



Although this formula is a Taylor expansion, the convergence circle is not seen, but the line of convergence is seen instead. This formula is complicated and the convergence is slow.

So, we try to accelerate the convergence of Formula 8.3.1 . We use parallel acceleration method. In addition, about the method, see **" 13 Convergence Acceleration of Multiple Series "** ( A la carte ) .

### Formula 8.3.1' ( parallel acceleration )

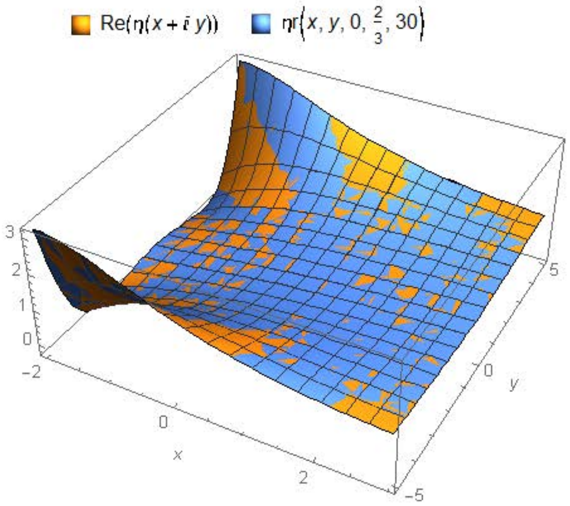
When the real and imaginary parts of Dirichlet eta function $\eta(x,y)$ are $\eta_r$ , $\eta_i$ respectively, the following expressions hold for arbitrary positive number $q$ and real constant $c$ other than zero of $\eta$ .

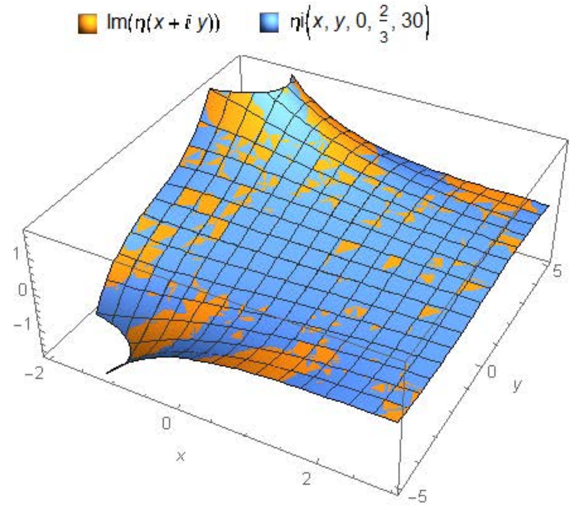
$$\eta_r(x,y) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k} \frac{q^{k-r-s}}{(q+1)^{k+1}}\binom{k}{r+s}\frac{(-1)^s \log^r(s+1)}{(s+1)^c}\frac{(-1)^r}{r!} \times \sum_{t=0}^{\lceil \frac{r-1}{2} \rceil}(-1)^t\binom{r}{2t}(x-c)^{r-2t}y^{2t}$$

$$\eta_i(x,y) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k} \frac{q^{k-r-s}}{(q+1)^{k+1}}\binom{k}{r+s}\frac{(-1)^s \log^r(s+1)}{(s+1)^c}\frac{(-1)^r}{r!} \times \sum_{t=0}^{\lfloor \frac{r-1}{2} \rfloor}(-1)^t\binom{r}{2t+1}(x-c)^{r-2t-1}y^{2t+1}$$

Where, $0^0=1$ , $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

When $c=0$ , $q=2/3$ , these are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. We can see that Formula 8.3.1 is analytically continued to the lefttward and the line of convergence has disappeared.




### 8.3.2 Complementary Series

Rearranging the double series of Formula 8.3.1 along the diagonal, we obtain a complementary series that displays the opposite side of the line of convergence.

**Formula 8.3.2**

When the real and imaginary parts of Dirichlet eta function $\eta(x,y)$ are $\eta_r$ , $\eta_i$ respectively, the following expressions hold for real constant $c$ other than zero of $\eta$ .

$$\eta_r(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{\log^s(r-s+1)}{(r-s+1)^c}\frac{(-1)^r}{s!}\sum_{t=0}^{\lceil\frac{s-1}{2}\rceil}(-1)^t\binom{s}{2t}(x-c)^{s-2t}y^{2t}$$

$$\eta_i(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{\log^s(r-s+1)}{(r-s+1)^c}\frac{(-1)^r}{s!}\sum_{t=0}^{\lfloor\frac{s-1}{2}\rfloor}(-1)^t\binom{s}{2t+1}(x-c)^{s-2t-1}y^{2t+1}$$

Where, $0^0 = 1$ , $\lceil x\rceil$ is the ceiling function , $\lfloor x\rfloor$ is the floor function.

**Proof**

" **03 Complementary Series of Dirichlet Series** " Formula 3.4.2 was as follows.

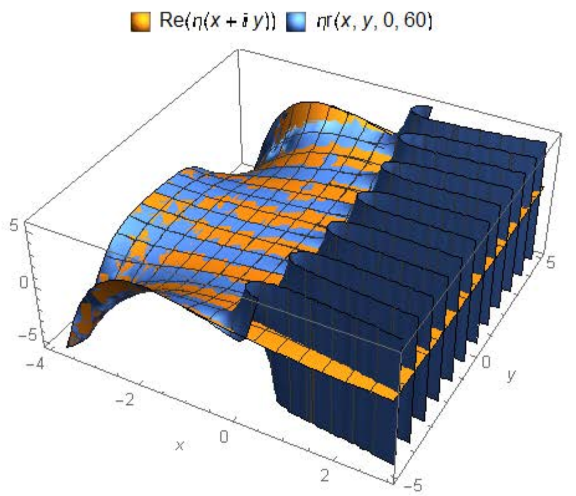
$$\eta(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^r\frac{\log^s(r-s+1)}{(r-s+1)^c}\frac{(z-c)^s}{s!} \qquad \text{Where, } 0^0 = 1 .$$

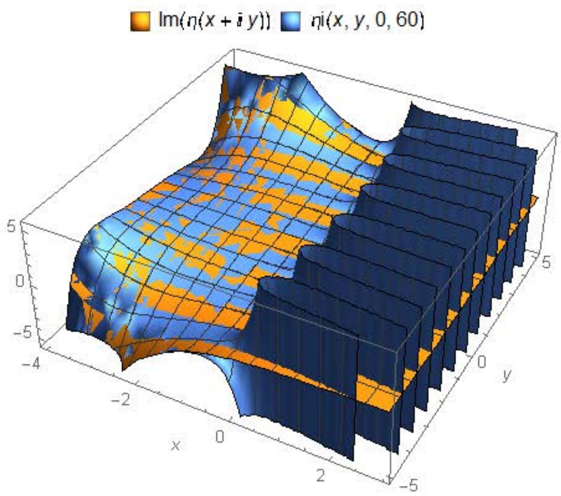
On the other hand, replacing $x$ with $x-c$ in Lemma 8.2.0 ,

$$(x-c+iy)^r = \sum_{t=0}^{\lceil\frac{r-1}{2}\rceil}(-1)^t\binom{r}{2t}(x-c)^{r-2t}y^{2t} + i\sum_{t=0}^{\lfloor\frac{r-1}{2}\rfloor}(-1)^t\binom{r}{2t+1}(x-c)^{r-2t-1}y^{2t+1}$$

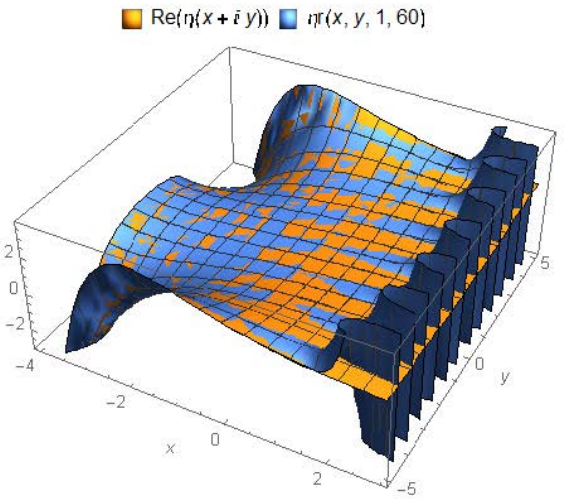
Substituting this for $(z-c)^r$ and writing the real and imaginary parts as $\eta_r$ , $\eta_i$ , we obtain the formula.

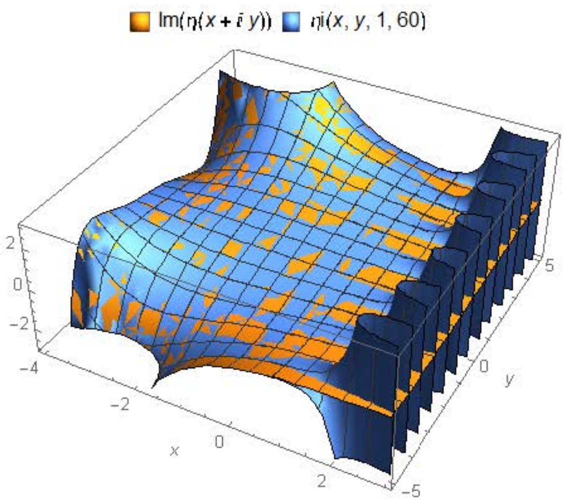
When $c=0$ , these are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. These are exactly the opposite sides of the figure in Formula 8.3.1 .




More surprisingly, the line of convergence is movable in this formula. For example, .When both sides are drawn with $c=1$ , the line of convergence moves from $x=0$ to $x=2$ .






As the result of drawing variously, I found that **the line of convergence in this formula is movable and the abscissa is $x = 2c$ .**

### 8.4 Dirichlet-Maclaurin Series

The formulas in the previous 2 sections were all complicated. The cause seems to have been in Lemma.8.2.0 So, in this section, we present a formula that does not depend on Lemma 8.2.0.

**Formula 8.4.1**

When the real and imaginary parts of Dirichlet eta function $\eta(x,y)$ are $\eta_r$, $\eta_i$ respectively, the following expressions hold for $x>0$.

$$\eta_r(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{\log^{2r}(s+1)}{(s+1)^x}\frac{y^{2r}}{(2r)!}$$

$$\eta_i(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{\log^{2r+1}(s+1)}{(s+1)^x}\frac{y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$.

**Proof**

$$\eta(x,y) = \sum_{s=1}^{\infty}(-1)^{s-1}e^{-(x+iy)\log s} = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^x}e^{-iy\log s}$$

i.e.

$$\eta(x,y) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}\cos(y\log s)}{s^x} - i\sum_{s=1}^{\infty}\frac{(-1)^{s-1}\sin(y\log s)}{s^x}$$

Here, expanding $\cos(y\log s), \sin(y\log s)$ into Maclaurin series with respect to $y$,

$$\cos(y\log s) = \sum_{r=0}^{\infty}(-1)^r\log^{2r}s\frac{y^{2r}}{(2r)!}$$

$$\sin(y\log s) = \sum_{r=0}^{\infty}(-1)^r\log^{2r+1}s\frac{y^{2r+1}}{(2r+1)!}$$

Substituting these for the above,

$$\eta(x,y) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^x}\sum_{r=0}^{\infty}(-1)^r\log^{2r}s\frac{y^{2r}}{(2r)!}$$

$$-i\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{s^x}\sum_{r=0}^{\infty}(-1)^r\log^{2r+1}s\frac{y^{2r+1}}{(2r+1)!}$$

Swapping rows and columns,

$$\eta(x,y) = \sum_{r=0}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s-1}\frac{\log^{2r}s}{s^x}\frac{y^{2r}}{(2r)!}$$

$$-i\sum_{r=0}^{\infty}\sum_{s=1}^{\infty}(-1)^{r+s-1}\frac{\log^{2r+1}s}{s^x}\frac{y^{2r+1}}{(2r+1)!}$$

Change the initial value of s from 1 to 0 ,

$$\eta(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{\log^{2r}(s+1)}{(s+1)^x}\frac{y^{2r}}{(2r)!}$$

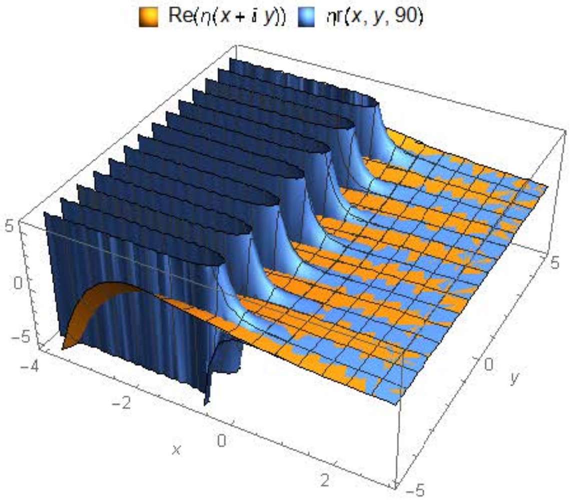
$$- i \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{r+s} \frac{\log^{2r+1}(s+1)}{(s+1)^x} \frac{y^{2r+1}}{(2r+1)!}$$

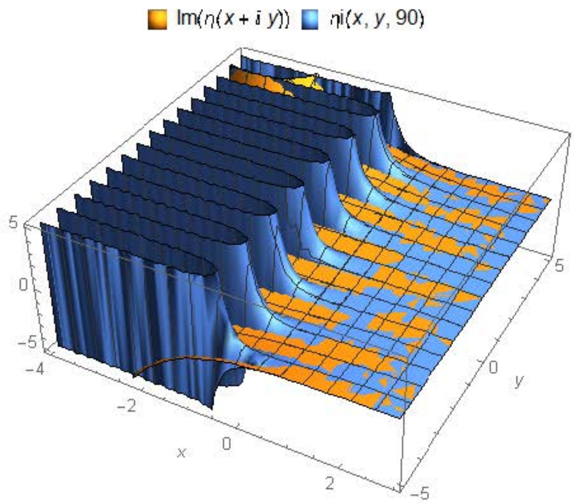
Writing the real and imaginary parts as $\eta_r$ , $\eta_i$ , we obtain the desired expressions.

### Note

This formula is a Dirichlet series with respect to the real part $x$ and a Maclaurin series with respect to the imaginary part $y$ . Therefore, we will call this **Dirichlet-Maclaurin series**. **This formula is the simplest so far.**

These are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. Although this formula is a Maclaurin expansion, the convergence circle is not seen, but the line of convergence $x = 0$ is seen instead.




### Disadvantages of Formula 8.4.1

This formula is a Dirichlet series with respect to $x$ and a Maclaurin series with respect to $y$ .
So, let us expand a few terms from the top of these series on the critical line $x = 1/2$ .
As first, expanding the left side using the function ***Series[ ]*** of the formula manipulation software ***Mathematica,***

```
Normal[N[Series[DirichletEta[1/2 + ⅈ y], {y, 0, 7}]]];
ComplexExpand[%]
```

$$0.604899 + 0.03368\, y^2 + 0.00134612\, y^4 - 0.000019988\, y^6$$
$$+ i\,(0. + 0.193289\, y + 0.000498958\, y^3 - 0.000271163\, y^5 - 2.27817 \times 10^{-6}\, y^7)$$

Next, expanding the first four terms on the right side with the upper limit of $\Sigma$ being 1000 respectively,

```
Unprotect[Power]; Power[0, 0] = 1;
Normal[N[Series[ηr[1/2, y, 1000], {y, 0, 7}]]]
```

$$0.620698 - 0.34348\,(0. + y)^2 + 1.50158\,(0. + y)^4 - 2.38728\,(0. + y)^6$$

```
Normal[N[Series[ηi[1/2, y, 1000], {y, 0, 7}]]]
```

```
0.0841258 (0.+ y) + 0.869033 (0.+ y)^3 - 2.07337 (0.+ y)^5 + 2.35631 (0.+ y)^7
```

Comparing both calculation results, we can find that only the 1st digit of the real part matches and the 2nd and subsequent terms do not match at all. In order to investigate this cause, each coefficient on the right side was calculated. Then,

```
Unprotect[Power]; Power[0, 0] = 1;
```

$$cr[r\_, x\_, m\_] := \sum_{s=0}^{m} (-1)^{r+s} \frac{(Log[s+1])^{2r}}{(s+1)^x} \frac{1}{(2r)!}$$

$$ci[r\_, x\_, m\_] := -\sum_{s=0}^{m} (-1)^{r+s} \frac{(Log[s+1])^{2r+1}}{(s+1)^x} \frac{1}{(2r+1)!}$$

$$N\left[cr\left[0, \frac{1}{2}, 350\,000\,000\,000\right]\right]$$

0.604899

$$N\left[ci\left[0, \frac{1}{2}, 310\,000\,000\,000\,000\right]\right]$$

0.193289 + 0. i

In order to obtain correct values for the 1st terms of the real part and imaginary part, this number of terms was necessary respectively. The 2nd and subsequent terms were not calculated correctly due to underflow. These are due to (1) the convergence speed is very slow and (2) the coefficients are too small. Formula 8.4.1 is not a good series numerically.

To make up for this shortcoming, this formula can be accelerated. So, we try to accelerate these double series using the series acceleration method by Euler transformation. In addition, Euler transformation is a kind of Knopp transformation. About the method, see **"13 Convergence Acceleration of Multiple Series"** (A la carte ) About Knopp transformation, see " **10 Convergence Acceleration & Summation Method by Double Series of Functions** " (A la carte ).

### Formula 8.4.1' ( series acceleration )

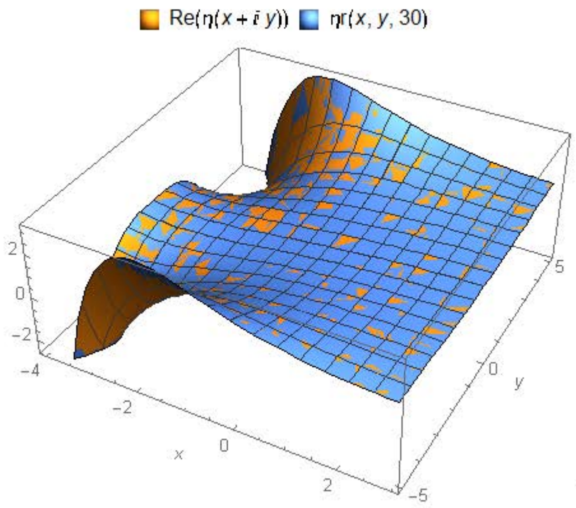
When the real and imaginary parts of Dirichlet eta function $\eta(x,y)$ are $\eta_r$ , $\eta_i$ respectively, the following expressions hold.

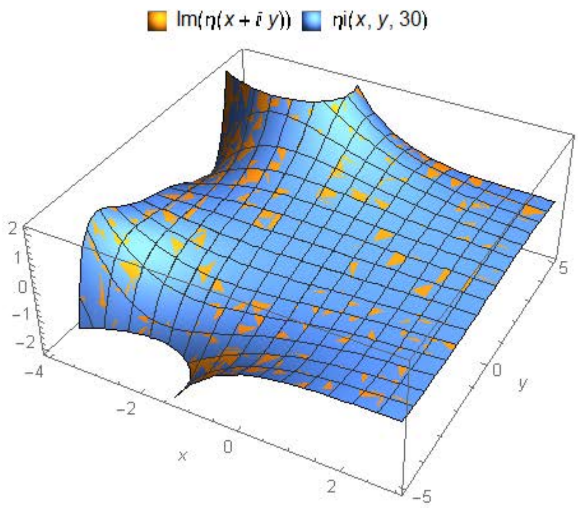
$$\eta_r(x,y) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{r} \frac{1}{2^{k+1}} \binom{k}{r} (-1)^r \frac{\log^{2r-2s}(s+1)}{(s+1)^x} \frac{y^{2r-2s}}{(2r-2s)!}$$

$$\eta_i(x,y) = -\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{r} \frac{1}{2^{k+1}} \binom{k}{r} (-1)^r \frac{\log^{2r-2s+1}(s+1)}{(s+1)^x} \frac{y^{2r-2s+1}}{(2r-2s+1)!}$$

Where, $0^0 = 1$.

Both sides of these are drawn as the next page. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. We can see that Formula 8.4.1 is analytically continued to the lefttward and the line of convergence has disappeared.




According to this formula, the first four terms of the right side on the critical line $x=1/2$ are expanded as follows. These coefficients are exactly consistent with the results of directly expanding the left side .

```
Normal[N[Series[ηr[1/2, y, 40], {y, 0, 7}]]]
0.604899 + 0.033638 (0 + y)^2 + 0.00134612 (0 + y)^4 - 0.000019988 (0 + y)^6
Normal[N[Series[ηi[1/2, y, 35], {y, 0, 7}]]]
0.193289(0.+y)+0.000498958(0.+y)^3-0.000271163(0.+y)^5-2.27817x10^-6(0.+y)^7
```

**Note**

The acceleration of only the coefficient of Formula 8.4.1 is as follows.

When the coefficients of $\eta_r$ , $\eta_i$ are $c_r(r,x)$ , $c_i(r,x)$ ,

$$c_r(r,x) = \sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{1}{2^{k+1}}\binom{k}{s}(-1)^{r+s}\frac{\log^{2r}(s+1)}{(s+1)^x}\frac{1}{(2r)!}$$

$$c_i(r,x) = -\sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{1}{2^{k+1}}\binom{k}{s}(-1)^{r+s}\frac{\log^{2r+1}(s+1)}{(s+1)^x}\frac{1}{(2r+1)!}$$

Where, $0^0 = 1$.

### 8.5 Non-Trivial Zeros of Riemann Zeta

Using Formula 8.4.1 , we obtain the following theorem about the non-trivial zeros of Riemann zeta function.

### Theorem 8.5.1

Non-trivial zeros of Riemann zeta function are given as real solutions of the following simultaneous equations.

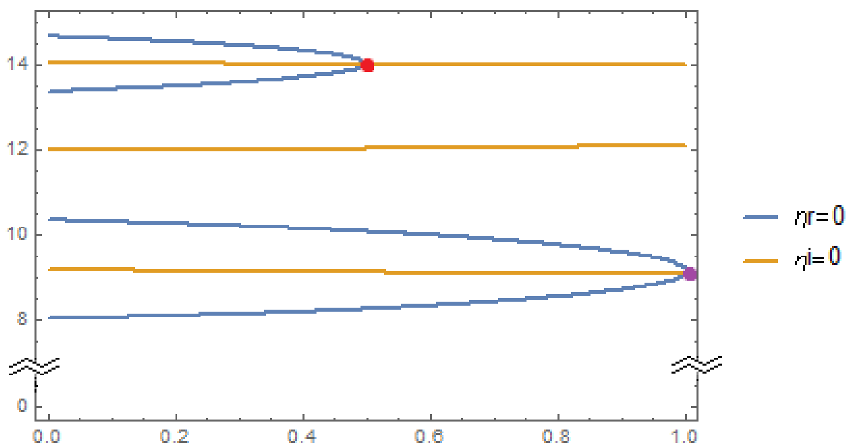
$$\begin{cases} \displaystyle\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{\infty}(-1)^{r+s}\frac{\log^{2r}(s+1)}{(s+1)^{x}(2r)!}\right\}y^{2r} = 0 \\ \displaystyle -\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{\infty}(-1)^{r+s}\frac{\log^{2r+1}(s+1)}{(s+1)^{x}(2r+1)!}\right\}y^{2r+1} = 0 \end{cases}$$

Where, $0^0 = 1$ , $0 < x < 1$ .

### Proof

It is known that zeros of the Riemann zeta function and the Dirichlet eta function are common at $0 < x < 1$ . Then, in Formula 8.4.1 , real solutions $(x, y)$ s.t. $\eta_r(x, y) = \eta_i(x, y) = 0$ , $0 < x < 1$ are zeros of Riemann zeta function.

A contour plot of $\eta_r(x, y) = \eta_i(x, y) = 0$ using Formula 8.4.1' is as follows. The red point is a non-trivial zero and the purple point is a $\eta$ specific zero.




This theorem is substantially the same as Theorem 8.1.2 in Section 1. However, while Theorem 8.1.2 is a transcendental equation with respect to both $x$ and $y$ , **Theorem 8.5.1 is an algebraic equation with respect to $y$** . If so, it seems that theories of algebraic equations are applicable.

### Hypothesis equivalent to the Riemann hypothesis

For example, we can propose the following hypothesis that is equivalent to the Riemann hypothesis, which seems to be applicable to theories of algebraic equations.

### Hypothesis 8.5.2

The simultaneous equations in Theorem 8.5.1 have no multiple roots with respect to $y$ .

## Proof of equivalence

Suppose Riemann hypothesis is false. Then, according to Theorem 7.4.1 in " **07 Completed Riemann Zeta** " ( Riemann Zeta ) , Riemann zeta function has at least one set of zeros as follows.

$$1/2+\alpha_1 \pm i\beta_1 \quad , \quad 1/2-\alpha_1 \pm i\beta_1 \qquad ( 0<\alpha_1<1/2 )$$

Since $\beta_1$ is a multiple root, this is contrary to this hypothesis. Therefore, Riemann hypothesis has to be true.

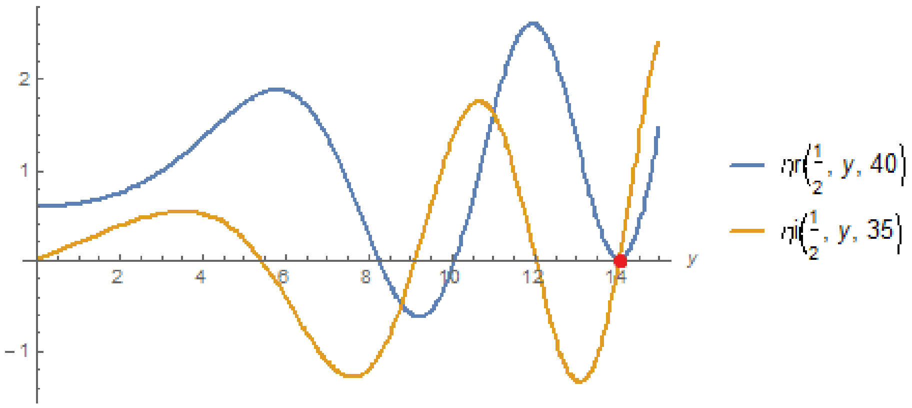
## Zeros on the critical line

Giving $x=1/2$ , Theorem 8.5.1 becomes as follows.

$$\begin{cases} \displaystyle\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{\infty}(-1)^{r+s}\frac{\log^{2r}(s+1)}{\sqrt{s+1}\,(2r)!}\right\}y^{2r} = 0 & (5.2r) \\ \displaystyle -\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{\infty}(-1)^{r+s}\frac{\log^{2r+1}(s+1)}{\sqrt{s+1}\,(2r+1)!}\right\}y^{2r+1} = 0 & (5.2i) \end{cases}$$

Where, $0^0=1$ .

Using Formula 8.4.1' , the real and imaginary parts on the critical line are drawn as follows: The red point is the zero on the critical line and is the first solution of this system.




Since { } is a constant in (5.2r), (5.2i) , these are algebraic equations. That is, **obtaining zeros on the critical line is reduce to solving the algebraic equations (5.2r) and (5.2i) respectively**. This seems to be easier than solving the above simultaneous equations.

2019.10.25

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**Alien's Mathematics**