12 Series Expansion of Gamma Function & the Reciprocal

12.1 Taylor Expansion around a

Higher Derivative of Gamma Function

The formula of the higher derivative of the gamma function & the reciprocal was discovered by Masayuki Ui in December 2016. (See [22 Higher Derivative of Composition Sec.3]) I reproduce it here as follows.

Formula 12.1.0 (Masayuki Ui)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function, and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expressions hold.

\[
\frac{d^n}{dz^n} \Gamma(z) = \Gamma(z) \sum_{k=1}^{n} B_{n,k}(\psi_0(z), \psi_1(z), \ldots, \psi_{n-1}(z))
\]

(0.1a)

\[
\frac{d^n}{dz^n} \frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z)} \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(z), \psi_1(z), \ldots, \psi_{n-1}(z))
\]

(0.1b)

Proof

When $f(z) = \log \Gamma(z)$,

\[
f_1 = \frac{d}{dz} \log \Gamma(z) = \psi_0(z), \quad f_2 = \frac{d}{dz} \psi_0(z) = \psi_1(z), \ldots
\]

\[
f_n = \frac{d}{dz} \psi_{n-2}(z) = \psi_{n-1}(z)
\]

Substituting these and $g_k = e^f$ for $k = 1, 2, 3, \ldots$ for the following Faà di Bruno’s Formula on a composite function

\[
\{g(f(x))\}^{(n)} = \sum_{r=1}^{n} g_k B_{n,k}(f_1, f_2, \ldots, f_n)
\]

we obtain

\[
\{e^{\log \Gamma(z)}\}^{(n)} = e^{\log \Gamma(z)} \sum_{k=1}^{n} B_{n,k}(\psi_0(z), \psi_1(z), \ldots, \psi_{n-1}(z))
\]

(0.1a)

When $f(z) = -\log \Gamma(z)$, in a similar way, we obtain

\[
\{e^{-\log \Gamma(z)}\}^{(n)} = e^{-\log \Gamma(z)} \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(z), \psi_1(z), \ldots, \psi_{n-1}(z))
\]

(0.1b)

Using this formula, we can perform the Taylor expansion of the gamma function $\Gamma(z)$ and the reciprocal $1/\Gamma(z)$. Where, we can not perform the Taylor expansion around $a = 0, -1, -2, -3, \ldots$ because, at these points, the differential coefficients are $\infty$ or 0.

Formula 12.1.1

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function, and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expression holds for $a \neq 0, -1, -2, -3, \ldots$.

\[
\Gamma(z) = \Gamma(a) + \sum_{n=1}^{\infty} \frac{c_n(a)}{n!} (z-a)^n
\]

(1.1)
where,

\[ c_n(a) = \Gamma(a) \sum_{k=1}^{\infty} B_{n,k}(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a)) \quad n=1, 2, 3, \ldots \]

Proof

\( \Gamma(z) \) can be expanded to Taylor series around \( a \neq 0, -1, -2, -3, \ldots \) as follows.

\[ \Gamma(z) = \Gamma(a) + \sum_{n=1}^{\infty} \frac{\Gamma^{(n)}(a)}{n!} (z-a)^n \]

Applying Formula 12.1.0 (0.1+) to this and replacing \( \Gamma^{(n)}(a) \) with \( c_n(a) \), we obtain the desired formula.

Example: Taylor expansion around 2 (symbolic calculation)

According to the formula, \( \Gamma(z) \) is expanded to Taylor series around 2. The polynomial \( B_{n,k}(f_1, f_2, \ldots) \) is generated using the function \texttt{BellY[]} of formula manipulation software \texttt{Mathematica}. The expansion until the 3rd term is as follows.

\[ \texttt{Tbl}\psi[n_+, z_] := \text{Table}[\psi_k[z], \{k, 0, n-1\}] \]
\[ \texttt{c[n_, z_] := Gamma[z] \sum_{k=1}^{n} \text{BellY}[n, k, \text{Tbl}\psi[n, z]]} \]
\[ \texttt{ft[z_, a_, m_] := Gamma[a] + \sum_{n=1}^{m} \frac{\text{c}[n, a]}{n!} (z - a)^n} \]

\[ 1 + (-2 + z) \psi_0[2] + \frac{1}{2} (-2 + z)^2 \left( \psi_0[2]^2 + \psi_1[2] \right) \]
\[ + \frac{1}{6} (-2 + z)^3 \left( \psi_0[2]^3 + 3 \psi_0[2] \psi_1[2] + \psi_2[2] \right) \]

On the other hand, when \( \Gamma(z) \) is expanded to series around 2 using the function \texttt{Series[]} of \texttt{Mathematica} it is as follows.

\[ \texttt{Series[Gamma[z], \{z, 2, 3\}]}; \]
\[ \texttt{ReplaceAll[\%}, \{\text{EulerGamma} \rightarrow y, \text{PolyGamma}[2, 2] \rightarrow \psi_2[2]\}]; \]
\[ \texttt{Collect[\%, \{z - 2\}, \text{Simplify}]} \]
\[ 1 + (-2 + z) (1 - y) + \frac{1}{12} (-2 + z)^2 \left( y^2 + 6 (-2 + y) y \right) \]
\[ + \frac{1}{12} (-2 + z)^3 \left( -4 - \pi^2 (-1 + y) + 6 y^2 - 2 y^3 + 2 \psi_2[2] \right) \]

Though they seem to be different, they are the same thing. Indeed, if \( \psi_0[2] = 1 - \gamma \), \( \psi_1[2] = \pi^2/6 - 1 \) are substituted for \( f_i'(z, 2, 3) \), it is as follows.

\[ \texttt{ReplaceAll[\%}, \{\psi_0[2] \rightarrow 1 - y, \psi_1[2] \rightarrow -1 + \frac{\pi^2}{6}\}]; \]
\[ \texttt{Collect[\%, \{z - 2\}, \text{Simplify}]} \]
\[ 1 + (-2 + z) (1 - y) + \frac{1}{12} (-2 + z)^2 \left( y^2 + 6 (-2 + y) y \right) \]
\[ + \frac{1}{12} (-2 + z)^3 \left( -4 - \pi^2 (-1 + y) + 6 y^2 - 2 y^3 + 2 \psi_2[2] \right) \]

-2-
Formula 12.1.2

When \( \Gamma(z) \) is the gamma function, \( \psi_n(z) \) is the polygamma function and \( B_{n,k}(f_1, f_2, \ldots) \) are Bell polynomials, the following expression holds for \( a \) s.t. \( a \neq 0, -1, -2, -3, \ldots \).

\[
\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(a)} + \sum_{n=1}^{\infty} \frac{c_n(a)}{n!} (z-a)^n
\]

where,

\[
c_n(a) = \frac{1}{\Gamma(a)} \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a))
\]

Proof

\( \frac{1}{\Gamma(z)} \) can be expanded to Taylor series around \( a \neq 0, -1, -2, -3, \ldots \) as follows.

\[
\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(a)} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\Gamma(z)} \right\}^{(n)}_{z=a} \frac{(z-a)^n}{n!}
\]

Applying the Formula 12.1.0 (0.1-), we obtain the desired formula.

Example: Taylor expansion around 2 (numeric calculation)

According to the formula, \( \frac{1}{\Gamma(z)} \) is expanded to Taylor series around 2. The polynomial \( B_{n,k}(f_1, f_2, \ldots) \) is generated using the function \( BellY[] \) of formula manipulation software Mathematica. If the right side is expanded until 20 terms and is illustrated with the left side, it is as follows. Both sides are exactly overlapped and the left side (blue) is invisible.

\[
\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(a)} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\Gamma(z)} \right\}^{(n)}_{z=a} \frac{(z-a)^n}{n!}
\]
12.2 Laurent Expansion of Gamma Function & the Reciprocal

We cannot perform the Maclaurin expansion of the gamma function $\Gamma(z)$ and the reciprocal $1/\Gamma(z)$. But, we can perform the Maclaurin expansion of the $\Gamma(1+z)$ and the reciprocal $1/\Gamma(1+z)$. Using this, we can perform the Laurent expansion of the $\Gamma(z)$ and the reciprocal $1/\Gamma(z)$ around 0.

**Formula 12.2.1 (Laurent expansion)**

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expression holds

$$\Gamma(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^{n-1}$$

where,

$$c_n = \sum_{k=1}^{n} B_{n,k}(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)) \quad n = 1, 2, 3, \ldots$$

**Proof**

$\Gamma(1+z)$ can be expanded to Maclaurin series as follows.

$$\Gamma(1+z) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma^{(n)}(1)}{n!} z^n$$

$$\Gamma^{(n)}(1) = \Gamma(1) \sum_{k=1}^{n} B_{n,k}(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1))$$

Replacing $\Gamma^{(n)}(1)$ with $c_n$, and dividing both sides by $z$,

$$\frac{\Gamma(1+z)}{z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^{n-1}$$

$$c_n = \Gamma(1) \sum_{k=1}^{n} B_{n,k}(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1))$$

Since $\Gamma(1+z) = z\Gamma(z)$, $\Gamma(1) = 1$, we obtain the desired expression.

**Numeric Calculation**

According the formula, $\Gamma(z)$ is expanded to Laurent series around 0. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function `BellY[]` of formula manipulation software **Mathematica**. The expansion until the 4th term is as follows.

```math
\text{Tbl\psi}[n_\ _, z_] := \text{Table}[\text{PolyGamma}[k, z], \{k, 0, n - 1\}];
\text{c}[n_\_] := \sum_{k=1}^{n} \text{BellY}[n, k, \text{Tbl\psi}[n, 1]]
\text{ft}[z_\_, m_\_] := \frac{1}{z} + \sum_{n=1}^{m} \frac{\text{c}[n]}{n!} z^{n-1}
\text{ft}[z_\_, 4];
\text{ReplaceAll}[%, \{\text{EulerGamma} \rightarrow \gamma, \text{PolyGamma}[2, 1] \rightarrow \psi_2[1]\}];
\frac{1}{z} - \gamma + \frac{1}{2} z \left(\frac{\pi^2}{6} + \gamma^2\right) + \frac{1}{6} z^2 \left(-\frac{\pi^2}{2} - \gamma^2 + \psi_2[1]\right) + \frac{1}{24} z^3 \left(\frac{3 \pi^4}{20} + \pi^2 \gamma^2 - \gamma^4 - 4 \gamma \psi_2[1]\right)
```
On the other hand, $\Gamma(z)$ is expanded to series around 0 using the function `Series[]` of Mathematica as follows. This is consistent with the above exactly.

\[
\text{Series}[\text{Gamma}[z], \{z, 0, 3\}];
\]

ReplaceAll[%, \{EulerGamma -> \[CapitalGamma], PolyGamma[2, 1] -> \[CapitalPsi][1]\}]

\[
\frac{1}{z} - \gamma + \frac{1}{12} \left( \pi^2 + 6 \gamma^2 \right) z + \frac{1}{6} \left( \frac{\pi^2}{2} - \gamma^3 + \psi'[1] \right) z^2
\]

\[
+ \frac{1}{24} \left( \frac{3 \pi^4}{20} + \frac{\pi^2}{2} \gamma^2 - \gamma^4 - 4 \gamma \psi'[1] \right) z^3 + O[z]^4
\]

In addition, if $f_i$ is expanded until 20 terms and is illustrated with $\Gamma$, it is as follows. Both are almost overlapped

Formula 12.2.2 (reciprocal Laurent expansion)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expression holds

\[
\frac{1}{\Gamma(z)} = z + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^{n+1}
\]

where,

\[
c_n = \sum_{k=1}^{n} (-1)^k B_{n,k} \left( \psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1) \right)
\]

$n = 1, 2, 3, \ldots$

Proof

$I(1+z)$ can be expanded to Maclaurin series as follows.

\[
\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \left\{ \frac{1}{\Gamma(1+z)} \right\}^{(n)} \frac{z^n}{n!}
\]

\[
\left\{ \frac{1}{\Gamma(1+z)} \right\}^{(n)} \bigg|_{z=0} = \frac{1}{\Gamma(1)} \sum_{k=1}^{n} (-1)^k B_{n,k} \left( \psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1) \right)
\]

Replacing the differential coefficient with $c_n$ and multiplying both sides by $z$,

\[
\frac{z}{\Gamma(1+z)} = z + \sum_{n=1}^{\infty} c_n \frac{z^{n+1}}{n!}
\]
\[ c_n = \frac{1}{\Gamma(1)} \sum_{k=1}^{n} (-1)^k B_{n,k} \left( \psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1) \right) \]

Since \( f(1+z) = z f(z) \), \( \Gamma(1) = 1 \), we obtain the desired expression.

**Symbolic Calculation**

According to the formula, \( 1/\Gamma(z) \) is expanded to series around 0. The polynomial \( B_{n,k}(f_1, f_2, \ldots) \) is generated using the function \( \text{BellY}[] \) of formula manipulation software \text{Mathematica}. The expansion until the 4th term is as follows.

\[
T_{n, k} := \text{Table}[\psi_k(z), \{k, 0, n - 1\}] \\
\sigma[n] := \sum_{k=1}^{n} (-1)^k \text{BellY}[n, k, T_{n, k}, 1] \\
f[z] := \frac{1}{\Gamma(z)} \\
f_t[z, m] := z + \sum_{n=1}^{m} \sigma[n] z^{n+1} \\
f_t[z, 4] \\
z - z^2 \psi_0[1] + \frac{1}{2} z^3 \left( \psi_1[1]^2 - \psi_1[1] \right) + \frac{1}{6} z^4 \left( -\psi_0[1]^3 + 3 \psi_0[1] \psi_1[1] - \psi_2[1] \right) \\
\quad + \frac{1}{24} z^5 \left( \psi_0[1]^4 - 6 \psi_0[1]^2 \psi_1[1] + 3 \psi_1[1]^2 + 4 \psi_0[1] \psi_2[1] \psi_3[1] \right)
\]

On the other hand, when \( 1/\Gamma(z) \) is expanded to series around 0 using the function \( \text{Series}[] \) of \text{Mathematica}, it is as follows.

\[
\text{Series}[f[z], \{z, 0, 5\}]; \\
\text{ReplaceAll}[\%\{\text{EulerGamma} \to \gamma, \text{PolyGamma}[2, 1] \to \psi_2[1]\}]; \\
\text{Collect}[\%, z, \text{Expand}] \\
z + z^2 \gamma + z^3 \left( -\frac{\pi^2}{12} + \frac{\gamma^2}{2} \right) + z^4 \left( -\frac{\pi^2 \gamma}{12} + \frac{\gamma^3}{6} - \frac{\psi_2[1]}{6} \right) \\
\quad + z^5 \left( \frac{\pi^4}{1440} - \frac{\pi^2 \gamma^2}{24} + \frac{\gamma^4}{24} - \frac{1}{6} \gamma \psi_2[1] \right)
\]

Though they seem to be different, they are the same thing. Indeed, if \( \psi_0[1] = -\gamma \), \( \psi_1[1] = \pi^2/6 \), \( \psi_3[1] = \pi^4/15 \) are substituted for \( f_t(z, 4) \), it is as follows.

\[
\text{ReplaceAll}[f_t[z, 4], \{\psi_0[1] \to -\gamma, \psi_1[1] \to \frac{\pi^2}{6}, \psi_3[1] \to \frac{\pi^4}{15}\}]; \\
\text{Collect}[\%, z, \text{Expand}] \\
z + z^2 \gamma + z^3 \left( -\frac{\pi^2}{12} + \frac{\gamma^2}{2} \right) + z^4 \left( -\frac{\pi^2 \gamma}{12} + \frac{\gamma^3}{6} - \frac{\psi_2[1]}{6} \right) \\
\quad + z^5 \left( \frac{\pi^4}{1440} - \frac{\pi^2 \gamma^2}{24} + \frac{\gamma^4}{24} - \frac{1}{6} \gamma \psi_2[1] \right)
\]
12.3 Maclaurin Expansion

Formula 12.3.1

When \( \Gamma(z) \) is the gamma function, \( \psi_n(z) \) is the polygamma function and \( B_{n,k}(f_1, f_2, \ldots) \) are Bell polynomials, the following expressions hold

\[
\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n
\]

(3.1-)

\[
\frac{1}{\Gamma(1-z)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{n!} z^n
\]

(3.1+)

\[
\frac{1}{\Gamma(1+z/2)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} z^n
\]

(3.2-)

\[
\frac{1}{\Gamma(1-z/2)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!} z^n
\]

(3.2+)

where,

\[
c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)) \quad n=1, 2, 3, \ldots
\]

**Proof**

\( 1/\Gamma(1+z) \) can be expanded to Maclaurin series as follows.

\[
\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \left\{ \frac{1}{\Gamma(1+z)} \right\}^{(n)} \frac{z^n}{n!}
\]

Replacing the differential coefficient with \( c_n \) we obtain (3.1-). And reversing the sign we obtain (3.1+).

Replacing \( z \) with \( z/2 \) in (3.1-) we obtain (3.2-). And reversing the sign we obtain (3.2+).

**Example: 1/\Gamma(1+z/2) (symbolic calculation)**

According the formula, \( 1/\Gamma(1+z/2) \) is expanded to series around 0. The polynomial \( B_{n,k}(f_1, f_2, \ldots) \) is generated using the function \( BellY[] \) of formula manipulation software Mathematica. The expansion until the 4th term is as follows.

\[
\text{Tbl}[z, z] := \text{Table}[\psi_k[z], \{k, 0, n-1\}]
\]

\[
\text{Bell}[n] := \sum_{k=1}^{n} (-1)^k \text{BellY}[n, k, \text{Tbl}[n, 1]]
\]

\[
f[z] := \frac{1}{\text{Gamma}[1 + z/2]} \quad \text{fm}[z, m] := 1 + \sum_{n=1}^{m} \frac{c[n]}{2^n n!} z^n
\]

\[
f[z, 4] = 1 - \frac{1}{2} z \psi_0[1] + \frac{1}{8} z^2 \left( \psi_0[1]^2 - \psi_1[1] \right) + \frac{1}{48} z^3 \left( -\psi_0[1]^3 + 3 \psi_0[1] \psi_1[1] - \psi_2[1] \right)
\]

\[
+ \frac{1}{384} z^4 \left( \psi_0[1]^4 - 6 \psi_0[1]^2 \psi_1[1] + 3 \psi_1[1]^2 + 4 \psi_0[1] \psi_2[1] - \psi_3[1] \right)
\]
On the other hand, when $f(z)$ is expanded to series around 0 using the function `Series[]` of Mathematica, it is as follows.

```
Series[f[z], {z, 0, 4}];
ReplaceAll[x, {EulerGamma -> γ, PolyGamma[2, 1] -> ψ_2[1]}];
Collect[x, z, Expand]
```

Though they seem to be different, they are the same thing. Indeed, if $ψ_0[1] = -γ$, $ψ_1[1] = π^2/6$, $ψ_3[1] = π^4/15$ are substituted for $f_m(z, 4)$, it is as follows.

```
ReplaceAll[f[m, z, 4], {ψ_0[1] -> -γ, ψ_1[1] -> π^2/6, ψ_3[1] -> π^4/15}];
Collect[x, z, Expand]
```

**Formula 12.3.2**

When $Γ(z)$ is the gamma function, $ψ_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expressions hold

\[
\sqrt{π} \frac{1}{Γ\left(\frac{1+z}{2}\right)} = 1 + \sum_{n=1}^{∞} \frac{c_n}{2^n n!} z^n
\]

(3.3)

\[
\sqrt{π} \frac{1}{Γ\left(\frac{1-z}{2}\right)} = 1 + \sum_{n=1}^{∞} (-1)^n \frac{c_n}{2^n n!} z^n
\]

(3.3)

where,

\[c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}(ψ_0\left(\frac{1}{2}\right), ψ_1\left(\frac{1}{2}\right), \ldots, ψ_{n-1}\left(\frac{1}{2}\right))\]

\[n = 1, 2, 3, \ldots\]

**Proof**

$1/Γ\left(\frac{1+z}{2}\right)$ can be expanded to Maclaurin series as follows.

\[
\frac{1}{Γ\left(\frac{1+z}{2}\right)} = \sum_{n=0}^{∞} \left(\frac{1}{Γ\left(\frac{1+z}{2}\right)}\right)^{(n)} z^n
\]

The first term is

\[
\left(\frac{1}{Γ\left(\frac{1+z}{2}\right)}\right)^{(0)}_{z=0} = \frac{1}{Γ\left(\frac{1+0}{2}\right)} = \frac{1}{\sqrt{π}}
\]

The second and the subsequent terms are obtained using [Formula 12.1.0]

\[
\frac{d^n}{dz^n} \frac{1}{Γ(z)} = \frac{1}{Γ(z)} \sum_{k=1}^{n} (-1)^k B_{n,k}(ψ_0(z), ψ_1(z), \ldots, ψ_{n-1}(z))
\]
as follows.

\[
\left\{ \frac{1}{\Gamma \left( (1+z)/2 \right)} \right\}^{(1)}_{z=0} = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^k B_{1,k} \left( \psi \left( \frac{1+0}{2} \right), \frac{1}{2} \right)^1
\]

\[
\left\{ \frac{1}{\Gamma \left( (1+z)/2 \right)} \right\}^{(2)}_{z=0} = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^k B_{1,k} \left( \psi \left( \frac{1+0}{2} \right), \psi \left( \frac{1+0}{2} \right), \frac{1}{2} \right)^2
\]

\[
\vdots
\]

\[
\left\{ \frac{1}{\Gamma \left( (1+z)/2 \right)} \right\}^{(n)}_{z=0} = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^k B_{n,k} \left( \psi_0 \left( \frac{1}{2} \right), \psi_1 \left( \frac{1}{2} \right), \ldots, \psi_{n-1} \left( \frac{1}{2} \right) \right) \frac{1}{2^n}
\]

\[n = 1, 2, 3, \ldots\]

i.e.

\[
\frac{1}{\Gamma \left( (1+z)/2 \right)} = \frac{1}{\sqrt{\pi}} \left[ 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^k B_{n,k} \left( \psi_0 \left( \frac{1}{2} \right), \psi_1 \left( \frac{1}{2} \right), \ldots, \psi_{n-1} \left( \frac{1}{2} \right) \right) \frac{z^n}{2^n n!} \right]
\]

Multiplying by \(\sqrt{\pi}\) both sides and replacing the inner \(\Sigma\) with \(c_n\), we obtain (3.3). In a similar way, (3.3+) is also obtained. (See the proof of formula 12.3.3)

Example: \(\sqrt{\pi} / \Gamma \left( (1-z)/2 \right)\) (numeric calculation)

According to the formula, \(\sqrt{\pi} / \Gamma \left( (1-z)/2 \right)\) is expanded to Maclaurin series. The Bell polynomial \(B_{n,k} \left( f_1, f_2, \ldots \right)\) is generated using the function \(BellY \left[ \right]\) of formula manipulation software Mathematica. If the right side is expanded until 15 terms and is illustrated with the left side, it is as follows. Both sides are exactly overlapped and the left side (blue) is invisible almost.

\(Tbl\left[ f, \right] := \text{Table}[\text{PolyGamma}[k, z], \{k, 0, n-1\}]\)

\(a[n_] := \sum_{k=1}^{n} (-1)^k \text{BellY}[n, k, Tbl \left[ f, \frac{1}{2} \right]]\)

\(f[z_] := \sqrt{\pi} / \Gamma \left( (1-z)/2 \right)\)

\(f[m, n] := 1 + \sum_{n=1}^{m} (-1)^n \frac{a[n]}{2^n n} z^n\)
Formula 12.3.3

When \( \Gamma(z) \) is the gamma function, \( \psi_n(z) \) is the polygamma function and \( B_{n,k}(f_1, f_2, \ldots) \) are Bell polynomials, the following expression holds

\[
\frac{\sqrt{\pi}}{2 \Gamma\left(\frac{3-z}{2}\right)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!} z^n
\]  

(3.4,)

where,

\[
c_n = \sum_{k=1}^{n} (-1)^k B_{n,k} \left( \psi_{\frac{3}{2}}, \psi_{\frac{3}{2}}, \ldots, \psi_{n-1}(\frac{3}{2}) \right)
\]

\( n = 1, 2, 3, \ldots \)

Proof

\( 1/\Gamma\left(\frac{3-z}{2}\right) \) can be expanded to Maclaurin series as follows.

\[
\frac{1}{\Gamma\left(\frac{3-z}{2}\right)} = \sum_{n=0}^{\infty} \left\{ \frac{1}{\Gamma\left(\frac{3-z}{2}\right)} \right\}^{(n)} \frac{z^n}{n!}
\]

The first term is

\[
\left\{ \frac{1}{\Gamma\left(\frac{3-z}{2}\right)} \right\}^{(0)} = \frac{1}{\Gamma\left(\frac{3-0}{2}\right)} = \frac{2}{\sqrt{\pi}}
\]

The second and the subsequent terms are obtained using

\[
\frac{d^n}{dz^n} \frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z)} \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(z), \psi_1(z), \ldots, \psi_{n-1}(z))
\]

as follows.

\[
\left\{ \frac{1}{\Gamma\left(\frac{3-z}{2}\right)} \right\}^{(1)} = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{1} (-1)^k B_{1,k}(\psi_{\frac{3-0}{2}}) \left( -\frac{1}{2} \right)^1
\]

\[
\left\{ \frac{1}{\Gamma\left(\frac{3-z}{2}\right)} \right\}^{(2)} = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{2} (-1)^k B_{2,k}(\psi_{\frac{3-0}{2}}, \psi_{\frac{3-0}{2}}) \left( -\frac{1}{2} \right)^2
\]

\[ \vdots \]

\[
\left\{ \frac{1}{\Gamma\left(\frac{3-z}{2}\right)} \right\}^{(n)} = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_{\frac{3}{2}}, \psi_{\frac{3}{2}}, \ldots, \psi_{n-1}(\frac{3}{2})) \left( -\frac{1}{2} \right)^n
\]

\[ n = 1, 2, 3, \ldots \]

i.e.

\[
\frac{1}{\Gamma\left(\frac{3-z}{2}\right)} = \frac{2}{\sqrt{\pi}} \left[ 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_{\frac{3}{2}}, \psi_{\frac{3}{2}}, \ldots, \psi_{n-1}(\frac{3}{2})) \left( -\frac{1}{2} \right)^n \right] \frac{2^n n!}{2^n n!}
\]

Multiplying \( \sqrt{\pi} / 2 \) both sides and replacing the inner \( \Sigma \) with \( c_n \), we obtain

(3.4.)

Symbolic Calculation

According the formula, \( \sqrt{\pi} / 2 \Gamma\left(\frac{3-z}{2}\right) \) is expanded to Maclaurin series. The Bell polynomial \( B_{n,k}(f_1, f_2, \ldots) \) is generated using the function \( BellY[] \) of formula manipulation software \( Mathematica \).

The expansion until the 4th term is as follows.
On the other hand, when \( f(z) \) is expanded to series around 0 using the function `Series[ ]` of Mathematica, it is as follows.

\[
\begin{align*}
1 + \frac{1}{2} z \psi_0 \left( \frac{3}{2} \right) &+ \frac{1}{8} z^2 \left( \psi_0 \left( \frac{3}{2} \right) - \psi_1 \left( \frac{3}{2} \right) \right) \\
- \frac{1}{48} z^3 &\left( -\psi_0 \left( \frac{3}{2} \right) \right)^3 + 3 \psi_0 \left( \frac{3}{2} \right) \psi_1 \left( \frac{3}{2} \right) - \psi_2 \left( \frac{3}{2} \right) \\
+ \frac{1}{384} z^4 &\left( \psi_0 \left( \frac{3}{2} \right)^4 - 6 \psi_0 \left( \frac{3}{2} \right)^2 \psi_1 \left( \frac{3}{2} \right) + 3 \psi_1 \left( \frac{3}{2} \right)^2 + 4 \psi_0 \left( \frac{3}{2} \right)^2 \psi_2 \left( \frac{3}{2} \right) - \psi_3 \left( \frac{3}{2} \right) \right)
\end{align*}
\]

Though they seem to be different, they are the same thing. Indeed, if \( \psi_1 \left( \frac{3}{2} \right) = \frac{\pi^2}{2} - 4 \) and \( \psi_3 \left( \frac{3}{2} \right) = \pi^4 - 96 \) are substituted for \( f_m(z, 4) \), it is as follows.

\[
\begin{align*}
1 &+ \frac{1}{2} z \psi_0 \left( \frac{3}{2} \right) + z^2 \left( \frac{1}{2} - \frac{\pi^2}{16} + \frac{1}{8} \psi_0 \left( \frac{3}{2} \right)^2 \right) \\
- \frac{1}{48} z^3 &\left( -\psi_0 \left( \frac{3}{2} \right) \right)^3 + \frac{1}{48} \psi_0 \left( \frac{3}{2} \right) \psi_1 \left( \frac{3}{2} \right) + \frac{1}{48} \psi_2 \left( \frac{3}{2} \right) \\
+ \frac{1}{384} z^4 &\left( \frac{3}{8} - \frac{\pi^2}{32} - \frac{\pi^4}{1536} + \frac{1}{16} \psi_0 \left( \frac{3}{2} \right)^2 - \frac{1}{128} \psi_0 \left( \frac{3}{2} \right)^2 + \frac{1}{384} \psi_0 \left( \frac{3}{2} \right)^4 + \frac{1}{96} \psi_0 \left( \frac{3}{2} \right) \psi_2 \left( \frac{3}{2} \right) \right)
\end{align*}
\]
12.4 Taylor Expansion around 1 (Part 1)

Formula 12.4.1

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expressions hold

$$\frac{1}{\Gamma(z)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!} (z-1)^n \quad (4.1)$$

$$\frac{1}{\Gamma(1-z)} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{n!} (z-1)^n \quad (4.1')$$

$$\frac{1}{\Gamma\left\{\left(\frac{1+z}{2}\right)\right\}} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} (z-1)^n \quad (4.2)$$

$$\frac{1}{\Gamma\left\{\left(\frac{3-z}{2}\right)\right\}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{2^n n!} (z-1)^n \quad (4.2')$$

$$\frac{1}{\Gamma(1-z)} = -(z-1) - \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{n!} (z-1)^{n+1} \quad (4.5)$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)) \quad n=1, 2, 3, \ldots$$

Proof

Replacing $z$ with $z-1$ in Formula 12.3.1 (3.1) ~ (3.2), we obtain (4.1) ~ (4.2). Multiplying both sides of (4.1) by $1-z$, we obtain (4.5). Strictly, (4.5) should be called reciprocal Laurent expansion.

Example: $1/\Gamma\left\{\left(\frac{1+z}{2}\right)\right\}$ (numeric calculation)

According to the formula, $1/\Gamma\left\{\left(\frac{1+z}{2}\right)\right\}$ is expanded to Taylor series around 1. The Bell polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\ ]$ of formula manipulation software Mathematica. If the right side is expanded until 15 terms and is illustrated with the left side, it is as follows. Both sides are exactly overlapped and the left side (blue) is invisible almost.
Formula 12.4.2

When $\Gamma(z)$ is the gamma function, $\psi_p(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expression holds

\[
\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!} (z-1)^n
\]

(4.5.)

where,

\[
c_n = \sum_{k=1}^{n} (-1)^k B_{n,k} \left( \psi_0(2), \psi_1(2) , \ldots , \psi_{n-1}(2) \right) \quad n=1, 2, 3, \ldots
\]

Proof

$1/\Gamma(1+z)$ can be expanded to Taylor series as follows.

\[
\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \left\{ \frac{1}{\Gamma(1+z)} \right\}^{(n)} \frac{(z-1)^n}{n!}
\]

Replacing the differential coefficient with $c_n$ we obtain (4.5.)

Symbolic Calculation

According the formula, $1/\Gamma(1+z)$ is expanded to Taylor series. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function BellY[] of formula manipulation software Mathematica. The expansion until the 3rd term is as follows.

```
Tbl\psi[n_, z_] := Table[\psi[k][z], {k, 0, n-1}]

c[n_] := \sum_{k=1}^{n} (-1)^k BellY[n, k, Tbl\psi[n, 2]]

ft[z_, m_] := 1 + \sum_{n=1}^{m} \frac{c[n]}{n!} (z-1)^n

ft[z, 3]
1 - (-1 + z) \psi_0[2] + \frac{1}{2} (-1 + z)^2 \left( \psi_0[2]^2 - \psi_1[2] \right)
+ \frac{1}{6} (-1 + z)^3 \left( -\psi_0[2]^3 + 3 \psi_0[2] \psi_1[2] - \psi_2[2] \right)
```

On the other hand, when $1/\Gamma(1+z)$ is expanded to series around 1 using the function Series[] of Mathematica, it is as follows.

```
Series[1 / Gamma[1 + z], {z, 1, 3}];
ReplaceAll[z, \{EulerGamma \rightarrow \gamma, PolyGamma[2, 2] \rightarrow \psi_2[2]\}];
Collect[%, \{z - 1\}, Expand]
1 + (-1 + z) (-1 + \gamma) + (-1 - z)^2 \left( 1 - \frac{\pi^2}{12} - \gamma + \frac{\gamma^2}{2} \right)
+ (-1 + z)^3 \left( -\frac{2}{3} + \frac{\pi^2}{12} + \gamma - \frac{\pi^2 \gamma}{12} - \frac{\gamma^2}{2} + \frac{\gamma^3}{6} - \psi_2[2] \right)
```

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Though they seem to be different, they are the same thing. Indeed, if $\psi_0[2] = 1 - \gamma$, $\psi_1[2] = \pi^2/6 - 1$ are substituted for $f(z, 3)$, it is as follows.

```
ReplaceAll[ft[z, 3], \{\psi_0[2] \to 1 - \gamma, \psi_1[2] \to \pi^2/6 - 1\}];
Collect[\%, (z - 1), Expand]
```

$$1 + (-1 + z) (-1 + \gamma) + (-1 - z)^2 \left( 1 - \frac{\pi^2}{12} - \gamma + \frac{\gamma^2}{2} \right)$$

$$+ (-1 + z)^3 \left( -\frac{2}{3} + \frac{\pi^2}{12} + \gamma - \frac{\gamma^2}{12} - \frac{\gamma^3}{2} + \frac{\gamma^3}{6} - \frac{\psi_2[2]}{6} \right)$$
12.5 Taylor Expansion around 1 (Part 2)

Formula 12.5.1

When \( \Gamma(z) \) is the gamma function, \( \psi_\nu(z) \) is the polygamma function and \( B_{n,k}(f_1, f_2, \ldots) \) are Bell polynomials, the following expressions hold

\[
\frac{\sqrt{\pi}}{\Gamma(z/2)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} (z-1)^n \quad (5.3)
\]

\[
\frac{\sqrt{\pi}}{\Gamma(1-z/2)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!} (z-1)^n \quad (5.3+)
\]

where,

\[
c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{n-1}\left(\frac{1}{2}\right)) \quad n = 1, 2, 3, \ldots
\]

Proof

\[
\frac{1}{\Gamma(z/2)} \text{ can be expanded to Taylor series as follows.}
\]

\[
\frac{1}{\Gamma(z/2)} = \sum_{n=0}^{\infty} \left\{ \frac{1}{\Gamma(z/2)} \right\}^{(n)} \frac{(z-1)^n}{n!}
\]

\[
\left\{ \frac{1}{\Gamma(z/2)} \right\}^{(0)}_{z=1} = \frac{1}{\Gamma(1/2)} = \frac{1}{\sqrt{\pi}}
\]

\[
\left\{ \frac{1}{\Gamma(z/2)} \right\}^{(n)}_{z=1} = \frac{1}{2^n \sqrt{\pi}} \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{n-1}\left(\frac{1}{2}\right)\right)
\]

\[
\text{i.e.}
\]

\[
\frac{1}{\Gamma(z/2)} = \frac{1}{\sqrt{\pi}} \left\{ 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{n-1}\left(\frac{1}{2}\right)\right) \right\} (z-1)^n
\]

Multiplying both sides by \( \sqrt{\pi} \) and replacing the inner \( \sum \) with \( c_n \), we obtain (5.3+). In a similar way, we obtain (5.3+).

Example: \( 1/\Gamma(1-z/2) \) (numeric calculation)

According the formula, \( \sqrt{\pi} / \Gamma(1-z/2) \) is expanded to Taylor series around 1. The Bell polynomial \( B_{n,k}(f_1, f_2, \ldots) \) is generated using the function \( \text{BellY}[] \) of formula manipulation software \text{Mathematica}. If the right side is expanded until 15 terms and is illustrated with the left side, it is as follows. Both sides are exactly overlapped and the left side (blue) is invisible almost.

\[
\text{Th1}[z_\_1, z_\_] := \text{Table}[\text{PolyGamma}[k, z], \{k, 0, n-1\}]
\]

\[
c[n_] := \sum_{k=1}^{n} (-1)^k \text{BellY}[n, k, \text{Th1}[n-1/2]]
\]

\[
\text{ft}[z_\_, m_] := 1 + \sum_{n=1}^{m} (-1)^n \frac{c[n]}{2^n n!} (z-1)^n
\]

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Formula 12.5.2

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expression holds

$$\frac{\sqrt{\pi}}{2 \Gamma(1+z/2)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} (z-1)^n$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k} \left( \psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{n-1}\left(\frac{3}{2}\right) \right)$$

$n = 1, 2, 3, \ldots$

Proof

$1/\Gamma(1+z/2)$ can be expanded to Taylor series as follows.

$$\frac{1}{\Gamma(1+z/2)} = \sum_{n=0}^{\infty} \frac{1}{(1+z/2)^n} \left( \frac{z-1}{n!} \right)$$

$$\left\{ \frac{1}{\Gamma(1+z/2)} \right\}^{(0)}_{z=1} = \frac{1}{\Gamma(3/2)} = \frac{2}{\sqrt{\pi}}$$

$$\left\{ \frac{1}{\Gamma(1+z/2)} \right\}^{(n)}_{z=1} = \frac{2}{2^n \sqrt{\pi}} \sum_{k=1}^{n} (-1)^k B_{n,k} \left( \psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{n-1}\left(\frac{3}{2}\right) \right)$$

$n = 1, 2, 3, \ldots$

i.e.

$$\frac{1}{\Gamma(1+z/2)} = \frac{2}{\sqrt{\pi}} \left\{ 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^k B_{n,k} \left( \psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{n-1}\left(\frac{3}{2}\right) \right) \left(\frac{z-1}{2^n n!}\right) \right\}$$

Multiplying both sides by $\sqrt{\pi}/2$ and replacing the inner $\Sigma$ with $c_n$, we obtain (5.6).

Symbolic Calculation

According the formula, $\sqrt{\pi}/2 \Gamma(1+z/2)$ is expanded to Taylor series. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[]$ of formula manipulation software Mathematica. The expansion until the 3rd term is as follows.
On the other hand, when $f(z)$ is expanded to series around 1 using the function `Series[]` of `Mathematica`, it is as follows.

$$
\text{Series}[f[z], \{z, 1, 3\}];
$$

ReplaceAll[$\%$, \{PolyGamma[0, $\frac{3}{2}$] $\rightarrow \psi_0[\frac{3}{2}]$, PolyGamma[2, $\frac{3}{2}$] $\rightarrow \psi_2[\frac{3}{2}]$\}];

Collect[$\%$, \{z - 1\}, simplify]

$$
1 - \frac{1}{2} \psi_0[\frac{3}{2}] (z - 1) + \frac{1}{16} \left(8 - \pi^2 + 2 \psi_0[\frac{3}{2}]^2\right) (z - 1)^2
+ \frac{1}{96} \left(3 (-8 + \pi^2) \psi_0[\frac{3}{2}] - 2 \psi_0[\frac{3}{2}]^3 - 2 \psi_2[\frac{3}{2}]\right) (z - 1)^3 + O[z - 1]^4
$$

Though they seem to be different, they are the same thing. Indeed, if $\psi_1[3/2] = \pi^2/2 - 4$ are substituted for $f(z, 3)$, it is as follows.

ReplaceAll[$\%$, $\psi_1[\frac{3}{2}]$ $\rightarrow \frac{\pi^2}{2} - 4$];

Collect[$\%$, \{z - 1\}, simplify]

$$
1 - \frac{1}{2} (-1 + z) \psi_0[\frac{3}{2}] - \frac{1}{8} (-1 + z)^2 \left(4 - \frac{\pi^2}{2} + \psi_0[\frac{3}{2}]^2\right)
+ \frac{1}{96} (-1 + z)^3 \left(3 (-8 + \pi^2) \psi_0[\frac{3}{2}] - 2 \psi_0[\frac{3}{2}]^3 - 2 \psi_2[\frac{3}{2}]\right)
$$