12 Series Expansion of Gamma Function & the Reciprocal

12.1 Taylor Expansion around a

Higher Derivative of Gamma Function

The formula of the higher derivative of the gamma function & the reciprocal was discovered by ***Masayuki Ui*** in December 2016. (See **22 Higher Derivative of Composition Sec.3**) I reproduce it here as follows.

Formula 12.1.0 (Masayuki Ui)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \dots)$ are Bell polynomials , the following expressions hold.

$$\frac{d^n}{dz^n}\Gamma(z) = \Gamma(z)\sum_{k=1}^{n} B_{n,k}(\psi_0(z), \psi_1(z), \dots, \psi_{n-1}(z)) \tag{0.1$_+$}$$

$$\frac{d^n}{dz^n}\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z)}\sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(z), \psi_1(z), \dots, \psi_{n-1}(z)) \tag{0.1$_-$}$$

Proof

When $f(z) = \log\Gamma(z)$,

$$f_1 = \frac{d}{dz}\log\Gamma(z) = \psi_0(z) \quad , \quad f_2 = \frac{d}{dz}\psi_0(z) = \psi_1(z) \quad , \quad \cdots$$

$$f_n = \frac{d}{dz}\psi_{n-2}(z) = \psi_{n-1}(z)$$

Substituting these and $g_k = e^f \;\; k=1,2,3,\cdots$ for the following ***Faà di Bruno's Formula*** on a composite function

$$\{g\{f(x)\}\}^{(n)} = \sum_{r=1}^{n} g_k B_{n,k}(f_1, f_2, \dots, f_n)$$

we obtain

$$\left\{e^{\log\Gamma(z)}\right\}^{(n)} = e^{\log\Gamma(z)}\sum_{k=1}^{n} B_{n,k}(\psi_0(z), \psi_1(z), \dots, \psi_{n-1}(z)) \tag{0.1$_+$}$$

When $f(z) = -\log\Gamma(z)$, in a similar way, we obtain

$$\left\{e^{-\log\Gamma(z)}\right\}^{(n)} = e^{-\log\Gamma(z)}\sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(z), \psi_1(z), \dots, \psi_{n-1}(z)) \tag{0.1$_-$}$$

Using this formula, we can perform the Taylor expansion of the gamma function $\Gamma(z)$ and the reciprocal $1/\Gamma(z)$. Where, we can not perform the Taylor expansion around $a = 0, -1, -2, -3, \cdots$. Because, at these points, the differential coefficients are ∞ or 0 .

Formula 12.1.1

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \dots)$ are Bell polynomials , the following expression holds for $a \;\; s.t. \;\; a \ne 0, -1, -2, -3, \cdots$.

$$\Gamma(z) = 1 + \sum_{n=1}^{\infty}\frac{c_n(a)}{n!}(z-a)^n \tag{1.1}$$

where,

$$c_n(a) = \Gamma(a) \sum_{k=1}^{n} B_{n,k}\left(\psi_0(a), \psi_1(a), \dots, \psi_{n-1}(a)\right) \qquad n=1,2,3,\cdots$$

Proof

$\Gamma(z)$ can be expanded to Taylor series around $a \neq 0,-1,-2,-3,\cdots$ as follows.

$$\Gamma(z) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma^{(n)}(a)}{n!}(z-a)^n$$

Applying Formula 12.1.0 (0.1_+) to this and replacing $\Gamma^{(n)}(a)$ with $c_n(a)$, we obtain the desired formula.

Example: Taylor expansion around 2 (symbolic calculation)

According the formula, $\Gamma(z)$ is expanded to Taylor series around 2. The polynomial $B_{n,k}(f_1, f_2, \dots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The expansion until the 3rd term is as follows.

```
Tblψ[n_, z_] := Table[ψk[z], {k, 0, n - 1}]
c[n_, z_] := Gamma[z] Sum[BellY[n, k, Tblψ[n, z]], {k, 1, n}]
ft[z_, a_, m_] := 1 + Sum[c[n, a]/n! (z - a)^n, {n, 1, m}]
ft[z, 2, 3]
```

$$1 + (-2+z)\,\psi_0[2] + \frac{1}{2}(-2+z)^2\left(\psi_0[2]^2 + \psi_1[2]\right) + \frac{1}{6}(-2+z)^3\left(\psi_0[2]^3 + 3\,\psi_0[2]\,\psi_1[2] + \psi_2[2]\right)$$

On the other hand, when $\Gamma(z)$ is expanded to series around 2 using the function $Series[\,]$ of ***Mathematica*** it is as follows.

```
Series[Gamma[z], {z, 2, 3}];
ReplaceAll[%, {EulerGamma → γ, PolyGamma[2, 2] -> ψ2[2]}];
Collect[%, (z - 2), Simplify]
```

$$1 + (-2+z)(1-\gamma) + \frac{1}{12}(-2+z)^2\left(\pi^2 + 6(-2+\gamma)\gamma\right) + \frac{1}{12}(-2+z)^3\left(-4 - \pi^2(-1+\gamma) + 6\gamma^2 - 2\gamma^3 + 2\,\psi_2[2]\right)$$

Though they seem to be different, they are the same thing. Indeed, if $\psi_0[2] = 1-\gamma$, $\psi_1[2] = \pi^2/6 - 1$ are substituted for $f_t(z,2,3)$, it is as follows.

```
ReplaceAll[ft[z, 2, 3], {ψ0[2] → 1 - γ, ψ1[2] -> -1 + π^2/6}];
Collect[%, (z - 2), Simplify]
```

$$1 + (-2+z)(1-\gamma) + \frac{1}{12}(-2+z)^2\left(\pi^2 + 6(-2+\gamma)\gamma\right) + \frac{1}{12}(-2+z)^3\left(-4 - \pi^2(-1+\gamma) + 6\gamma^2 - 2\gamma^3 + 2\,\psi_2[2]\right)$$

Formula 12.1.2

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , the following expression holds for a s.t. $a \neq 0,-1,-2,-3,\cdots$.

$$\frac{1}{\Gamma(z)} = 1 + \sum_{n=1}^{\infty} \frac{c_n(a)}{n!}(z-a)^n \tag{1.2}$$

where,

$$c_n(a) = \frac{1}{\Gamma(a)} \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a)\right)$$

Proof

$1/\Gamma(z)$ can be expanded to Taylor series around $a \neq 0,-1,-2,-3,\cdots$ as follows.

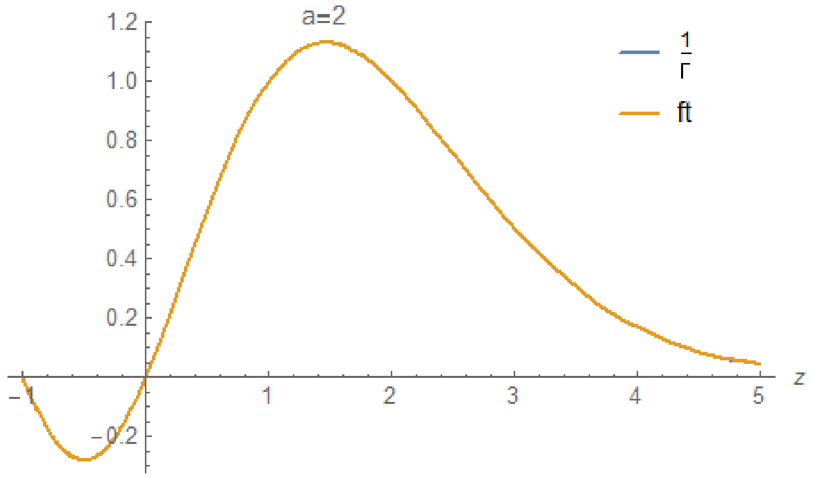
$$\frac{1}{\Gamma(z)} = 1 + \sum_{n=1}^{\infty} \left\{\frac{1}{\Gamma(z)}\right\}_{z=a}^{(n)} \frac{(z-a)^n}{n!}$$

Applying Formula 12.1.0 (0.1.) to this and replacing the differential coefficient with $c_n(a)$, we obtain the desired formula.

Example: Taylor expansion around 2 (numeric calculation)

According the formula, $1/\Gamma(z)$ is expanded to Taylor series around 2. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. If the right side is expanded until 20 terms and is illustrated with the left side, it is as follows. Both sides are exactly overlapped and the left side (blue) is invisible.

```
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]
c[n_, z_] := 1/Gamma[z] Sum[(-1)^k BellY[n, k, Tblψ[n, z]], {k, 1, n}]
ft[z_, a_, m_] := 1 + Sum[c[n, a]/n! (z - a)^n, {n, 1, m}]
```



12.2 Laurent Expansion of Gamma Function & the Reciprocal

We can not perform the Maclaurin expansion of the gamma function $\Gamma(z)$ and the reciprocal $1/\Gamma(z)$. But, we can perform the Maclaurin expansion of the $\Gamma(1+z)$ and the reciprocal $1/\Gamma(1+z)$. Using this, we can perform the Laurent expansion of the $\Gamma(z)$ and the reciprocal $1/\Gamma(z)$ around 0.

Formula 12.2.1 (Laurent expansion)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , the following expression holds

$$\Gamma(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^{n-1} \tag{2.1}$$

where,

$$c_n = \sum_{k=1}^{n} B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)\right) \qquad n=1,2,3,\cdots$$

Proof

$\Gamma(1+z)$ can be expanded to Maclaurin series as follows.

$$\Gamma(1+z) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma^{(n)}(1)}{n!} z^n$$

$$\Gamma^{(n)}(1) = \Gamma(1) \sum_{k=1}^{n} B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)\right)$$

Replacing $\Gamma^{(n)}(1)$ with c_n and dividing both sides by z ,

$$\frac{\Gamma(1+z)}{z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^{n-1}$$

$$c_n = \Gamma(1) \sum_{k=1}^{n} B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)\right)$$

Since $\Gamma(1+z) = z\Gamma(z)$, $\Gamma(1) = 1$, we obtain the desired expression.

Numeric Calculation

According the formula, $\Gamma(z)$ is expanded to Laurent series around 0. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The expansion until the 4th term is as follows.

```
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]

c[n_] := Sum[BellY[n, k, Tblψ[n, 1]], {k, 1, n}]

ft[z_, m_] := 1/z + Sum[c[n]/n! z^(n-1), {n, 1, m}]

ft[z, 4];
ReplaceAll[%, {EulerGamma → γ, PolyGamma[2, 1] -> ψ2[1]}]
```

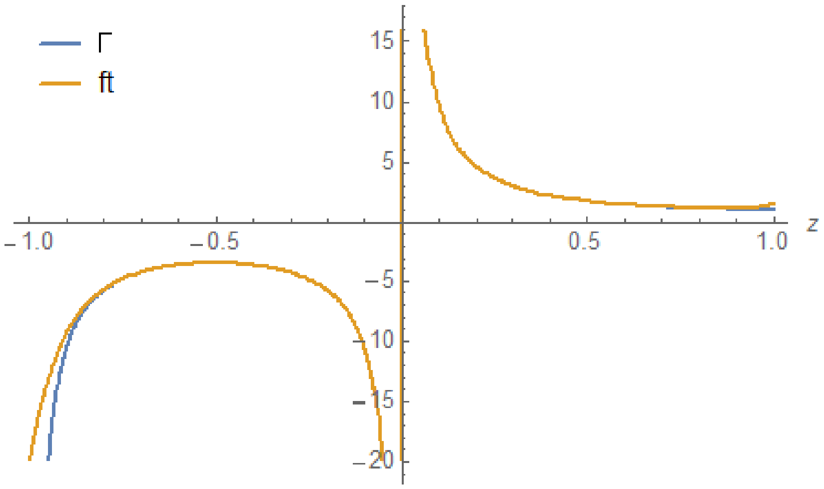
$$\frac{1}{z} - \gamma + \frac{1}{2} z \left(\frac{\pi^2}{6} + \gamma^2\right) + \frac{1}{6} z^2 \left(-\frac{\pi^2 \gamma}{2} - \gamma^3 + \psi_2[1]\right) + \frac{1}{24} z^3 \left(\frac{3\pi^4}{20} + \pi^2 \gamma^2 + \gamma^4 - 4\gamma\psi_2[1]\right)$$

On the other hand, $\Gamma(z)$ is expanded to series around 0 using the function $Series[\,]$ of ***Mathematica*** as follows. This is consistent with the above exactly.

```
Series[Gamma[z], {z, 0, 3}];
ReplaceAll[%, {EulerGamma → γ, PolyGamma[2, 1] -> ψ2[1]}]
```

$$\frac{1}{z} - \gamma + \frac{1}{12}\left(\pi^2 + 6\gamma^2\right) z + \frac{1}{6}\left(-\frac{\pi^2\gamma}{2} - \gamma^3 + \psi_2[1]\right) z^2 + \frac{1}{24}\left(\frac{3\pi^4}{20} + \pi^2\gamma^2 + \gamma^4 - 4\gamma\psi_2[1]\right) z^3 + O[z]^4$$

In addition, if f_t is expanded until 20 terms and is illustrated with Γ, it is as follows. Both are almost overlapped



Formula 12.2.2 (reciprocal Laurent expansion)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , the following expression holds

$$\frac{1}{\Gamma(z)} = z + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^{n+1} \tag{2.2}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)\right) \qquad n=1,2,3,\cdots$$

Proof

$\Gamma(1+z)$ can be expanded to Maclaurin series as follows.

$$\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \left\{\frac{1}{\Gamma(1+z)}\right\}_{z=0}^{(n)} \frac{z^n}{n!}$$

$$\left\{\frac{1}{\Gamma(1+z)}\right\}_{z=0}^{(n)} = \frac{1}{\Gamma(1)} \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)\right)$$

Replacing the differential coefficient with c_n and multiplying both sides by z ,

$$\frac{z}{\Gamma(1+z)} = z + \sum_{n=1}^{\infty} c_n \frac{z^{n+1}}{n!}$$

$$c_n = \frac{1}{\Gamma(1)} \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(1) , \psi_1(1) , \dots , \psi_{n-1}(1) \right)$$

Since $\Gamma(1+z) = z\Gamma(z)$, $\Gamma(1) = 1$, we obtain the desired expression.

Symbolic Calculation

According the formula, $1/\Gamma(z)$ is expanded to series around 0. The polynomial $B_{n,k}\left(f_1 , f_2 , \dots\right)$ is generated using the function $BellY[\ ]$ of formula manipulation software ***Mathematica***. The expansion until the 4th term is as follows.

```
Tblψ[n_, z_] := Table[ψ_k[z], {k, 0, n - 1}]
```

$$c[n\_] := \sum_{k=1}^{n} (-1)^k \, \mathrm{BellY}[n, k, \mathrm{Tbl}\psi[n, 1]]$$

$$f[z\_] := \frac{1}{\mathrm{Gamma}[z]} \qquad \mathrm{ft}[z\_, m\_] := z + \sum_{n=1}^{m} \frac{c[n]}{n!} z^{n+1}$$

```
ft[z, 4]
```

$$z - z^2 \psi_0[1] + \frac{1}{2} z^3 \left(\psi_0[1]^2 - \psi_1[1]\right) + \frac{1}{6} z^4 \left(-\psi_0[1]^3 + 3 \psi_0[1] \psi_1[1] - \psi_2[1]\right) + \frac{1}{24} z^5 \left(\psi_0[1]^4 - 6 \psi_0[1]^2 \psi_1[1] + 3 \psi_1[1]^2 + 4 \psi_0[1] \psi_2[1] - \psi_3[1]\right)$$

On the other hand, when $1/\Gamma(z)$ is expanded to series around 0 using the function $Series[\ ]$ of ***Mathematica,*** it is as follows.

```
Series[f[z], {z, 0, 5}];
ReplaceAll[%, {EulerGamma → γ, PolyGamma[2, 1] -> ψ_2[1]}];
Collect[%, z, Expand]
```

$$z + z^2 \gamma + z^3 \left(-\frac{\pi^2}{12} + \frac{\gamma^2}{2}\right) + z^4 \left(-\frac{\pi^2 \gamma}{12} + \frac{\gamma^3}{6} - \frac{\psi_2[1]}{6}\right) + z^5 \left(\frac{\pi^4}{1440} - \frac{\pi^2 \gamma^2}{24} + \frac{\gamma^4}{24} - \frac{1}{6} \gamma \psi_2[1]\right)$$

Though they seem to be different, they are the same thing. Indeed, if $\psi_0[1] = -\gamma$, $\psi_1[1] = \pi^2/6$, $\psi_3[1] = \pi^4/15$ are substituted for $f_t(z,4)$, it is as follows.

$$\mathrm{ReplaceAll}\left[\mathrm{ft}[z, 4], \left\{\psi_0[1] \to -\gamma, \psi_1[1] \text{ -> } \frac{\pi^2}{6}, \psi_3[1] \text{ -> } \frac{\pi^4}{15}\right\}\right];$$

```
Collect[%, z, Expand]
```

$$z + z^2 \gamma + z^3 \left(-\frac{\pi^2}{12} + \frac{\gamma^2}{2}\right) + z^4 \left(-\frac{\pi^2 \gamma}{12} + \frac{\gamma^3}{6} - \frac{\psi_2[1]}{6}\right) + z^5 \left(\frac{\pi^4}{1440} - \frac{\pi^2 \gamma^2}{24} + \frac{\gamma^4}{24} - \frac{1}{6} \gamma \psi_2[1]\right)$$

12.3 Maclaurin Expansion

Formula 12.3.1

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , the following expressions hold

$$\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n \tag{3.1-}$$

$$\frac{1}{\Gamma(1-z)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{n!} z^n \tag{3.1+}$$

$$\frac{1}{\Gamma(1+z/2)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} z^n \tag{3.2-}$$

$$\frac{1}{\Gamma(1-z/2)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!} z^n \tag{3.1+}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)) \qquad n=1,2,3,\cdots$$

Proof

$1/\Gamma(1+z)$ can be expanded to Maclaurin series as follows.

$$\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \left\{ \frac{1}{\Gamma(1+z)} \right\}_{z=0}^{(n)} \frac{z^n}{n!}$$

$$\left\{ \frac{1}{\Gamma(1+z)} \right\}_{z=0}^{(n)} = \frac{1}{\Gamma(1)} \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1))$$

Replacing the differential coefficient with c_n we obtain (3.1-) . And reversing the sign we obtain (3.1+) .

Replacing z with $z/2$ in (3.1-) we obtain (3.2-) . And reversing the sign we obtain (3.2+) .

Example: $1/\Gamma(1+z/2)$ (symbolic calculation)

According the formula, $1/\Gamma(1+z/2)$ is expanded to series around 0. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The expansion until the 4th term is as follows.

```
Tblψ[n_, z_] := Table[ψ_k[z], {k, 0, n - 1}]
c[n_] := Sum[(-1)^k BellY[n, k, Tblψ[n, 1]], {k, 1, n}]
f[z_] := 1 / Gamma[1 + z / 2]          fm[z_, m_] := 1 + Sum[c[n] / (2^n n!) z^n, {n, 1, m}]
fm[z, 4]
```

$$1 - \frac{1}{2} z\,\psi_0[1] + \frac{1}{8} z^2 \left(\psi_0[1]^2 - \psi_1[1]\right) + \frac{1}{48} z^3 \left(-\psi_0[1]^3 + 3\,\psi_0[1]\,\psi_1[1] - \psi_2[1]\right)$$
$$+ \frac{1}{384} z^4 \left(\psi_0[1]^4 - 6\,\psi_0[1]^2\,\psi_1[1] + 3\,\psi_1[1]^2 + 4\,\psi_0[1]\,\psi_2[1] - \psi_3[1]\right)$$

On the other hand, when $f(z)$ is expanded to series around 0 using the function $Series[\,]$ of ***Mathematica,*** it is as follows.

```
Series[f[z], {z, 0, 4}];
ReplaceAll[%, {EulerGamma → γ, PolyGamma[2, 1] -> ψ2[1]}];
Collect[%, z, Expand]
```

$$1+\frac{z\gamma}{2}+z^2\left(-\frac{\pi^2}{48}+\frac{\gamma^2}{8}\right)+z^3\left(-\frac{\pi^2\gamma}{96}+\frac{\gamma^3}{48}-\frac{\psi_2[1]}{48}\right)+z^4\left(\frac{\pi^4}{23040}-\frac{\pi^2\gamma^2}{384}+\frac{\gamma^4}{384}-\frac{1}{96}\gamma\,\psi_2[1]\right)$$

Though they seem to be different, they are the same thing. Indeed, if $\psi_0[1]=-\gamma$, $\psi_1[1]=\pi^2/6$, $\psi_3[1]=\pi^4/15$ are substituted for $f_m(z,4)$, it is as follows.

```
ReplaceAll[fm[z, 4], {ψ0[1] → -γ, ψ1[1] -> π^2/6, ψ3[1] -> π^4/15}];
Collect[%, z, Expand]
```

$$1+\frac{z\gamma}{2}+z^2\left(-\frac{\pi^2}{48}+\frac{\gamma^2}{8}\right)+z^3\left(-\frac{\pi^2\gamma}{96}+\frac{\gamma^3}{48}-\frac{\psi_2[1]}{48}\right)+z^4\left(\frac{\pi^4}{23040}-\frac{\pi^2\gamma^2}{384}+\frac{\gamma^4}{384}-\frac{1}{96}\gamma\,\psi_2[1]\right)$$

12.4 Taylor Expansion around 1 (Part 1)

Formula 12.4.1

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , the following expressions hold

$$\frac{1}{\Gamma(z)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!}(z-1)^n \tag{4.1$_-$}$$

$$\frac{1}{\Gamma(2-z)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{n!}(z-1)^n \tag{4.1$_+$}$$

$$\frac{1}{\Gamma\{(1+z)/2\}} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!}(z-1)^n \tag{4.2$_-$}$$

$$\frac{1}{\Gamma\{(3-z)/2\}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!}(z-1)^n \tag{4.2$_+$}$$

$$\frac{1}{\Gamma(1-z)} = -(z-1) - \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{n!}(z-1)^{n+1} \tag{4.5$_+$}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\big(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)\big) \qquad n=1,2,3,\cdots$$

Proof

Replacing z with $z-1$ in Formula 12.3.1 (3.1$_-$) ~ (3.2$_+$) , we obtain (4.1$_-$) ~ (4.2$_+$) . Multiplying both sides of (4.1$_+$) by $1-z$, we obtain (4.5$_+$) . Strictly, (4.5$_+$) should be called reciprocal Laurent expansion.

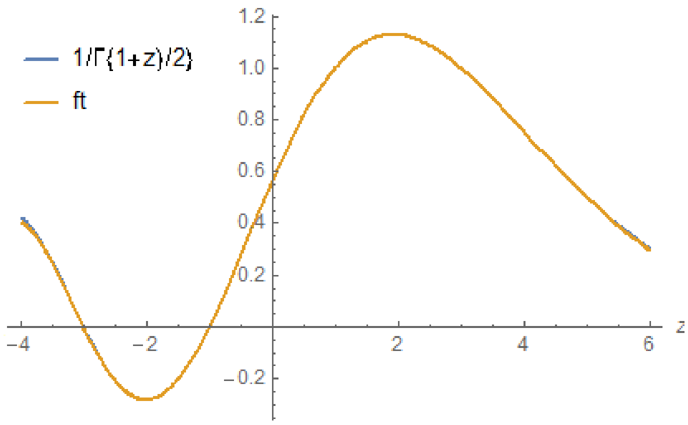
Example: $1/\Gamma\{(1+z)/2\}$ (numeric calculation)

According the formula, $1/\Gamma\{(1+z)/2\}$ is expanded to Taylor series around 1. The Bell polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\ ]$ of formula manipulation software ***Mathematica*** .
If the right side is expanded until 15 terms and is illustrated with the left side, it is as follows. Both sides are exactly overlapped and the left side (blue) is invisible almost.

```
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]
```

$$\texttt{c[n\_]} := \sum_{k=1}^{n} (-1)^k \texttt{BellY[n, k, Tbl}\psi\texttt{[n, 1]]}$$

$$\texttt{ft[z\_, m\_]} := 1 + \sum_{n=1}^{m} \frac{\texttt{c[n]}}{2^n\, n!} (z-1)^n$$



Formula 12.4.2

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , the following expression holds

$$\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!}(z-1)^n \tag{4.5.}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(2), \psi_1(2), \ldots, \psi_{n-1}(2)) \qquad n=1,2,3,\cdots$$

Proof

$1/\Gamma(1+z)$ can be expanded to Taylor series as follows.

$$\frac{1}{\Gamma(1+z)} = 1 + \sum_{n=1}^{\infty} \left\{ \frac{1}{\Gamma(1+z)} \right\}_{z=1}^{(n)} \frac{(z-1)^n}{n!}$$

$$\left\{ \frac{1}{\Gamma(1+z)} \right\}_{z=1}^{(n)} = \frac{1}{\Gamma(2)} \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(2), \psi_1(2), \ldots, \psi_{n-1}(2))$$

Replacing the differential coefficient with c_n we obtain (4.5.) .

Symbolic Calculation

According the formula, $1/\Gamma(1+z)$ is expanded to Taylor series. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The expansion until the 3rd term is as follows.

```
Tblψ[n_, z_] := Table[ψk[z], {k, 0, n - 1}]
```

$$\texttt{c[n\_] :=} \sum_{k=1}^{n} (-1)^k \texttt{ BellY[n, k, Tbl}\psi\texttt{[n, 2]]}$$

$$\texttt{ft[z\_, m\_] := 1 +} \sum_{n=1}^{m} \frac{\texttt{c[n]}}{\texttt{n!}} (z-1)^n$$

```
ft[z, 3]
```

$$1 - (-1+z)\,\psi_0[2] + \frac{1}{2}(-1+z)^2 \left(\psi_0[2]^2 - \psi_1[2]\right) + \frac{1}{6}(-1+z)^3 \left(-\psi_0[2]^3 + 3\,\psi_0[2]\,\psi_1[2] - \psi_2[2]\right)$$

On the other hand, when $1/\Gamma(1+z)$ is expanded to series around 1 using the function $Series[\,]$ of ***Mathematica,*** it is as follows.

```
Series[1 / Gamma[1 + z], {z, 1, 3}];
ReplaceAll[%, {EulerGamma → γ, PolyGamma[2, 2] -> ψ2[2]}];
Collect[%, (z - 1), Expand]
```

$$1 + (-1+z)(-1+\gamma) + (-1-z)^2 \left(1 - \frac{\pi^2}{12} - \gamma + \frac{\gamma^2}{2}\right) + (-1+z)^3 \left(-\frac{2}{3} + \frac{\pi^2}{12} + \gamma - \frac{\pi^2\gamma}{12} - \frac{\gamma^2}{2} + \frac{\gamma^3}{6} - \frac{\psi_2[2]}{6}\right)$$

Though they seem to be different, they are the same thing. Indeed, if $\psi_0[2] = 1 - \gamma$, $\psi_1[2] = \pi^2/6 - 1$ are substituted for $f_t(z, 3)$, it is as follows.

```
ReplaceAll[ft[z, 3], {ψ0[2] → 1 - γ, ψ1[2] -> π^2/6 - 1}];
Collect[%, (z - 1), Expand]
```

$$1 + (-1 + z)(-1 + \gamma) + (-1 - z)^2 \left(1 - \frac{\pi^2}{12} - \gamma + \frac{\gamma^2}{2}\right) + (-1 + z)^3 \left(-\frac{2}{3} + \frac{\pi^2}{12} + \gamma - \frac{\pi^2 \gamma}{12} - \frac{\gamma^2}{2} + \frac{\gamma^3}{6} - \frac{\psi_2[2]}{6}\right)$$

12.5 Taylor Expansion around 1 (Part 2)

Formula 12.5.1

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , the following expressions hold

$$\frac{\sqrt{\pi}}{\Gamma(z/2)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!}(z-1)^n \tag{5.3.}$$

$$\frac{\sqrt{\pi}}{\Gamma(1-z/2)} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_n}{2^n n!}(z-1)^n \tag{5.3+}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{n-1}\left(\frac{1}{2}\right)\right) \qquad n=1,2,3,\cdots$$

Proof

$1/\Gamma(z/2)$ can be expanded to Taylor series as follows.

$$\frac{1}{\Gamma(z/2)} = \sum_{n=0}^{\infty} \left\{ \frac{1}{\Gamma(z/2)} \right\}_{z=1}^{(n)} \frac{(z-1)^n}{n!}$$

$$\left\{ \frac{1}{\Gamma(z/2)} \right\}_{z=1}^{(0)} = \frac{1}{\Gamma(1/2)} = \frac{1}{\sqrt{\pi}}$$

$$\left\{ \frac{1}{\Gamma(z/2)} \right\}_{z=1}^{(n)} = \frac{1}{2^n \sqrt{\pi}} \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{n-1}\left(\frac{1}{2}\right)\right)$$

$$n = 1,2,3,\cdots$$

i.e.

$$\frac{1}{\Gamma(z/2)} = \frac{1}{\sqrt{\pi}} \left\{ 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{n-1}\left(\frac{1}{2}\right)\right) \frac{(z-1)^n}{2^n n!} \right\}$$

Multiplying both sides by $\sqrt{\pi}$ and replacing the inner Σ with c_n, we obtain (5.3.) . In a similar way, we obtain (5.3+) .

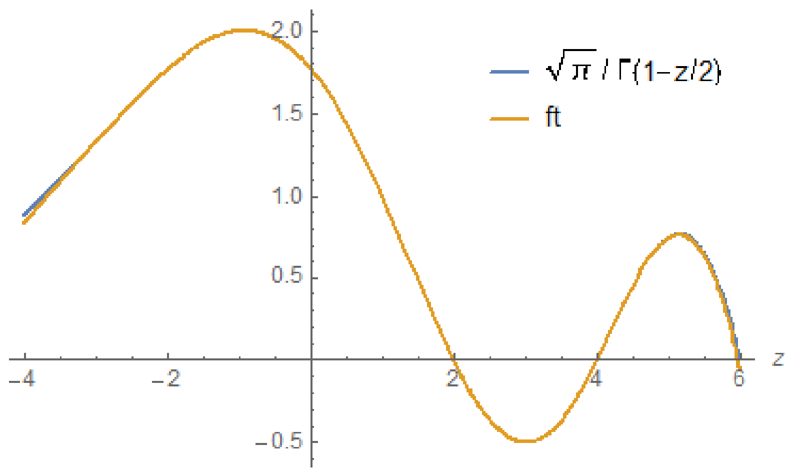
Example: $1/\Gamma(1-z/2)$ (numeric calculation)

According the formula, $\sqrt{\pi}/\Gamma(1-z/2)$ is expanded to Taylor series around 1. The Bell polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. If the right side is expanded until 15 terms and is illustrated with the left side, it is as follows. Both sides are exactly overlapped and the left side (blue) is invisible almost.

```
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]

c[n_] := Sum[(-1)^k BellY[n, k, Tblψ[n, 1/2]], {k, 1, n}]

ft[z_, m_] := 1 + Sum[(-1)^n c[n]/(2^n n!) (z - 1)^n, {n, 1, m}]
```



Formula 12.5.2

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , the following expression holds

$$\frac{\sqrt{\pi}}{2\Gamma(1+z/2)} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{2^n n!} (z-1)^n \tag{5.6.}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{n-1}\left(\frac{3}{2}\right)\right) \qquad n=1,2,3,\cdots$$

Proof

$1/\Gamma(1+z/2)$ can be expanded to Taylor series as follows.

$$\frac{1}{\Gamma(1+z/2)} = \sum_{n=0}^{\infty} \left\{\frac{1}{(1+z/2)}\right\}_{z=1}^{(n)} \frac{(z-1)^n}{n!}$$

$$\left\{\frac{1}{\Gamma(1+z/2)}\right\}_{z=1}^{(0)} = \frac{1}{\Gamma(3/2)} = \frac{2}{\sqrt{\pi}}$$

$$\left\{\frac{1}{\Gamma(1+z/2)}\right\}_{z=1}^{(n)} = \frac{2}{2^n\sqrt{\pi}} \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{n-1}\left(\frac{3}{2}\right)\right)$$

$$n = 1,2,3,\cdots$$

i.e.

$$\frac{1}{\Gamma(1+z/2)} = \frac{2}{\sqrt{\pi}}\left\{1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^k B_{n,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{n-1}\left(\frac{3}{2}\right)\right)\frac{(z-1)^n}{2^n n!}\right\}$$

Multiplying both sides by $\sqrt{\pi}/2$ and replacing the inner Σ with c_n, we obtain (5.6.) .

Symbolic Calculation

According the formula, $\sqrt{\pi}/2\Gamma(1+z/2)$ is expanded to Taylor series. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The expansion until the 3rd term is as follows.

```
Tblψ[n_, z_] := Table[ψ_k[z], {k, 0, n - 1}]
```

$$c[n\_] := \sum_{k=1}^{n} (-1)^k \text{BellY}\left[n, k, \text{Tbl}\psi\left[n, \frac{3}{2}\right]\right]$$

$$f[z\_] := \frac{\sqrt{\pi}}{2\,\text{Gamma}[1 + z/2]} \qquad ft[z\_, m\_] := 1 + \sum_{n=1}^{m} \frac{c[n]}{2^n\, n!} (z-1)^n$$

```
ft[z, 3]
```

$$1 - \frac{1}{2}(-1+z)\,\psi_0\left[\frac{3}{2}\right] + \frac{1}{8}(-1+z)^2\left(\psi_0\left[\frac{3}{2}\right]^2 - \psi_1\left[\frac{3}{2}\right]\right) + \frac{1}{48}(-1+z)^3\left(-\psi_0\left[\frac{3}{2}\right]^3 + 3\,\psi_0\left[\frac{3}{2}\right]\psi_1\left[\frac{3}{2}\right] - \psi_2\left[\frac{3}{2}\right]\right)$$

On the other hand, when $f(z)$ is expanded to series around 1 using the function $Series[\,]$ of ***Mathematica,*** it is as follows.

$$\text{Series}[f[z], \{z, 1, 3\}];$$
$$\text{ReplaceAll}\left[\%, \left\{\text{PolyGamma}\left[0, \frac{3}{2}\right] \to \psi_0\left[\frac{3}{2}\right], \text{PolyGamma}\left[2, \frac{3}{2}\right] \to \psi_2\left[\frac{3}{2}\right]\right\}\right];$$
$$\text{Collect}[\%, (z-1), \text{Simplify}]$$

$$1 - \frac{1}{2}\psi_0\left[\frac{3}{2}\right](z-1) + \frac{1}{16}\left(8 - \pi^2 + 2\,\psi_0\left[\frac{3}{2}\right]^2\right)(z-1)^2 + \frac{1}{96}\left(3\,(-8+\pi^2)\,\psi_0\left[\frac{3}{2}\right] - 2\,\psi_0\left[\frac{3}{2}\right]^3 - 2\,\psi_2\left[\frac{3}{2}\right]\right)(z-1)^3 + O[z-1]^4$$

Though they seem to be different, they are the same thing. Indeed, if $\psi_1[3/2] = \pi^2/2 - 4$ are substituted for $f_t(z,3)$, it is as follows.

$$\text{ReplaceAll}\left[ft[z, 3], \psi_1\left[\frac{3}{2}\right] \to \frac{\pi^2}{2} - 4\right];$$
$$\text{Collect}[\%, (z-1), \text{Simplify}]$$

$$1 - \frac{1}{2}(-1+z)\,\psi_0\left[\frac{3}{2}\right] + \frac{1}{8}(-1+z)^2\left(4 - \frac{\pi^2}{2} + \psi_0\left[\frac{3}{2}\right]^2\right) + \frac{1}{96}(-1+z)^3\left(3\,(-8+\pi^2)\,\psi_0\left[\frac{3}{2}\right] - 2\,\psi_0\left[\frac{3}{2}\right]^3 - 2\,\psi_2\left[\frac{3}{2}\right]\right)$$

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Kano. Kono

Alien's Mathematics