## 20 Series Expansion by Real & Imaginary Parts of Gamma Function

In this chapter, the gamma function is expanded into Taylor series, Laurent series and Maclaurin series by real and imaginary parts.

### Formulas to use

**1.** " **12 Series Expansion of Gamma Function & the Reciprocal** " Formula 12.1.1 , 12.1.2 , 12.2.1 , 12.2.2 .
**2.** " **14 Taylor Expansion by Real Part & Imaginary Part** " Formula 14.1.2 . This reprint is as follows.

### Formula 14.1.2 ( Reprint )

Suppose that a complex function $f(z)$ $(z=x+iy)$ is expanded around a real number $a$ into a Taylor series with real coefficients as follows.

$$f(z)=\sum_{s=0}^{\infty} f^{(s)}(a)\frac{(z-a)^s}{s!}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$ . Where, $0^0=1$ .

$$u(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

### 20.1 Taylor Expansion by Real & Imaginary Parts of Gamma Function & the Reciprocal

#### Formula 20.1.1

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , $z=x+iy$ and $u(x,y), v(x,y)$ are real part and imaginary part of $\Gamma(z)$ , the following expressions hold for $a \ne 0,-1,-2,-3,\cdots$ .

$$\Gamma(z)=\Gamma(a)+\sum_{s=1}^{\infty} c_s(a)\frac{(z-a)^s}{s!}$$

The radius of convergence is the distance from $a$ to the nearest singularity.

$$u(x,y)=\Gamma(a)+\sum_{s=1}^{\infty} c_s(a)\frac{(x-a)^s}{s!}+\sum_{r=1}^{\infty}\sum_{s=0}^{\infty} c_{2r+s}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} c_{2r+s+1}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$|z-a| \le$ ◇

where,

$$c_n(a)=\Gamma(a)\sum_{k=1}^{n} B_{n,k}(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a)) \qquad n=1,2,3,\cdots$$

$$0^0=1 .$$

### Proof

According to Formula 12.1.1, when $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expression holds for $a \ne 0,-1,-2,-3,\cdots$ .

$$\Gamma(z) = \Gamma(a) + \sum_{s=1}^{\infty} c_s(a) \frac{(z-a)^s}{s!} \tag{1.1}$$

where,

$$c_n(a) = \Gamma(a) \sum_{k=1}^{n} B_{n,k}\left(\psi_0(a), \psi_1(a), \dots, \psi_{n-1}(a)\right) \qquad n=1,2,3,\cdots$$

In (1.1) , changing the initial value of $s$ to $0$ ,

$$\Gamma(z) = \sum_{s=0}^{\infty} c_s(a) \frac{(z-a)^s}{s!} \qquad \begin{cases} c_s(a) = \Gamma(a) & s=0 \\ c_s(a) = c_s(a) & s>0 \end{cases}$$

Here, applying Formula 14.1.2 to the right side with $z = x+iy$ ,

$$f^{(s)}(a) = c_s(a)$$

$$f^{(2r+s)}(a) = c_{2r+s}(a)$$

$$f^{(2r+s+1)}(a) = c_{2r+s+1}(a)$$

So,

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} c_{2r+s}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!} \qquad \begin{cases} c_s(a) = \Gamma(a) & s=0 \\ c_s(a) = c_s(a) & s>0 \end{cases}$$

$$= c_{0+0} + \sum_{s=1}^{\infty} c_{0+s}(a) \frac{(x-a)^s}{s!} \frac{(-1)^0 y^0}{0!} + \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} c_{2r+s}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$= \Gamma(a) + \sum_{s=1}^{\infty} c_s(a) \frac{(x-a)^s}{s!} + \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} c_{2r+s}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} c_{2r+s+1}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

### Example: Taylor expansion around 2 ( numeric calculation )

According the formula, $\Gamma(z)$ is expanded to Taylor series around $2$ . The polynomial $B_{n,k}(f_1, f_2, \dots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The real and imaginary parts at $1+0.1i$ are calculated, and the function value and the series value are compared respectivery. the series are calculated up to $18$ terms. The results are as follows.

```
Unprotect[Power]; Power[0, 0] = 1;
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]
c[n_, a_] := Gamma[a] Sum[BellY[n, k, Tblψ[n, a]], {k, 1, n}]
Γ[z_, a_, m_] := Gamma[a] + Sum[c[s, a] (z - a)^s / s!, {s, 1, m}]
u[x_, y_, a_, m_] := Gamma[a] + Sum[c[s, a] (x - a)^s / s!, {s, 1, m}] + Sum[Sum[c[2 r + s, a] (x - a)^s / s! (-1)^r y^(2 r) / (2 r)!, {s, 0, m}], {r, 1, m}]
v[x_, y_, a_, m_] := Sum[Sum[c[2 r + s + 1, a] (x - a)^s / s! (-1)^r y^(2 r + 1) / (2 r + 1)!, {s, 0, m}], {r, 0, m}]
```

```
N[{Re[Gamma[1 + 0.1 i]], u[1, 0.1, 2, 18]}]
                {0.990207, 0.990206}
N[{Im[Gamma[1 + 0.1 i]], v[1, 0.1, 2, 18]}]
            {-0.0568238, -0.0568222}
```

In both the real and imaginary parts, the function value and the series value are almost the same.

## Formula 20.1.2

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, $z = x + iy$ and $u(x,y)$, $v(x,y)$ are real part and imaginary part of $1/\Gamma(z)$, the following expressions hold for $a \neq 0, -1, -2, -3, \cdots$.

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(a)} + \sum_{s=1}^{\infty} c_s(a) \frac{(z-a)^s}{s!} \qquad |z-a| < \infty \tag{1.2}$$

$$u(x,y) = \frac{1}{\Gamma(a)} + \sum_{s=1}^{\infty} c_s(a) \frac{(x-a)^s}{s!} + \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} c_{2r+s}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} c_{2r+s+1}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

where,

$$c_n(a) = \frac{1}{\Gamma(a)} \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a)\right)$$

$$0^0 = 1.$$

### Proof

(1.2) and $c_n(a)$ are according to Formula 12.1.2. Applying Formula 14.1.2 to the second term on the right side of (1.2), we obtain $u(x,y)$ and $v(x,y)$.

### Example: Taylor expansion around 2 ( numeric calculation )

According the formula, $1/\Gamma(z)$ is expanded to Taylor series around 2. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The real and imaginary parts at $1+0.1i$ are calculated, and the function value and the series value are compared respectivery. the series are calculated up to $15$ terms. The results are as follows.

```
Unprotect[Power]; Power[0, 0] = 1;
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]
```

$$\texttt{c[n\_, a\_] :=} \frac{1}{\texttt{Gamma[a]}} \sum_{k=1}^{n} (-1)^k\, \texttt{BellY[n, k, Tbl}\psi\texttt{[n, a]]}$$

$$\texttt{g[z\_, a\_, m\_] :=} \frac{1}{\texttt{Gamma[a]}} + \sum_{s=1}^{m} \texttt{c[s, a]} \frac{(z-a)^s}{s!}$$

$$\texttt{u[x\_, y\_, a\_, m\_] :=} \frac{1}{\texttt{Gamma[a]}} + \sum_{s=1}^{m} \texttt{c[s, a]} \frac{(x-a)^s}{s!} + \sum_{r=1}^{m}\sum_{s=0}^{m} \texttt{c[2 r + s, a]} \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$\texttt{v[x\_, y\_, a\_, m\_] :=} \sum_{r=0}^{m}\sum_{s=0}^{m} \texttt{c[2 r + s + 1, a]} \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$$N\left[\left\{Re\left[\frac{1}{Gamma[1+0.1 i]}\right], u[1, 0.1, 2, 15]\right\}\right]$$

{1.00658, 1.00658}

$$N\left[\left\{Im\left[\frac{1}{Gamma[1+0.1 i]}\right], v[1, 0.1, 2, 15]\right\}\right]$$

{0.0577631, 0.0577631}

The function value and the series value are exactly the same.

## 20.2 Laurent Expansion by Real & Imaginary Parts of Gamma Function & the Reciprocal

**Formula 20.2.1** **( Laurent expansion )**

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, $z = x + iy$ and $u(x,y), v(x,y)$ are real part and imaginary part of $\Gamma(z)$, the following expressions hold .

$$\Gamma(z) = \frac{1}{z} + \sum_{s=0}^{\infty} \frac{c_{s+1}}{s+1} \frac{z^s}{s!} \qquad |z| < 1$$

$$u(x,y) = \frac{x}{x^2+y^2} + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{c_{2r+s+1}}{2r+s+1} \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\frac{y}{x^2+y^2} + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{c_{2r+s+2}}{2r+s+2} \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad |z| < \Diamond$$

where,

$$c_n = \sum_{k=1}^{n} B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)\right) \qquad n = 1, 2, 3, \cdots$$

$$0^0 = 1 .$$

**Proof**

According to Formula 12.2.1, when $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expression holds.

$$\Gamma(z) = \frac{1}{z} + \sum_{s=1}^{\infty} \frac{c_s}{s!} z^{s-1} \tag{2.1}$$

where,

$$c_n = \sum_{k=1}^{n} B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)\right) \qquad n = 1, 2, 3, \cdots$$

In (2.1) , changing the initial value of $s$ to $0$ ,

$$\Gamma(z) = \frac{1}{z} + \sum_{s=0}^{\infty} \frac{c_{s+1}}{s+1} \frac{z^s}{s!}$$

Let $z = x + iy$ , separate the first term into the real part and the imaginary part, and apply Formula 14.1.2 to the second term. Then,

$$f^{(s)}(0) = \frac{c_{s+1}}{s+1}$$

$$f^{(2r+s)}(0) = \frac{c_{2r+s+1}}{2r+s+1}$$

$$f^{(2r+s+1)}(0) = \frac{c_{2r+s+2}}{2r+s+2}$$

Therefore,

$$u(x,y) = \frac{x}{x^2+y^2} + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{c_{2r+s+1}}{2r+s+1} \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\frac{y}{x^2+y^2} + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{c_{2r+s+2}}{2r+s+2} \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

**Example: ( numeric calculation )**

According the formula, $\Gamma(z)$ is expanded to Laurent series around $0$. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The real and imaginary parts at $0.5+0.1i$ are calculated, and the function value and the series value are compared respectivery. the series are calculated up to $20$ terms. The results are as follows.

```
Unprotect[Power]; Power[0, 0] = 1;
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]
```

$$c[n\_] := \sum_{k=1}^{n} \text{BellY}[n, k, \text{Tbl}\psi[n, 1]]$$

$$\Gamma[z\_, m\_] := \frac{1}{z} + \sum_{s=0}^{m} \frac{c[s+1]}{s+1} \frac{z^s}{s\,!}$$

$$u[x\_, y\_, m\_] := \frac{x}{x^2+y^2} + \sum_{r=0}^{m}\sum_{s=0}^{m} \frac{c[2r+s+1]}{2r+s+1} \frac{x^s}{s\,!} \frac{(-1)^r y^{2r}}{(2r)\,!}$$

$$v[x\_, y\_, m\_] := -\frac{y}{x^2+y^2} + \sum_{r=0}^{m}\sum_{s=0}^{m} \frac{c[2r+s+2]}{2r+s+2} \frac{x^s}{s\,!} \frac{(-1)^r y^{2r+1}}{(2r+1)\,!}$$

```
N[{Re[Gamma[0.5 + 0.1 i]], u[0.5, 0.1, 20]}]
        {1.69762, 1.69762}
N[{Im[Gamma[0.5 + 0.1 i]], v[0.5, 0.1, 20]}]
        {-0.332843, -0.332843}
```

In both the real and imaginary parts, the function value and the series value are exactly the same.

## Formula 20.2.2 ( reciprocal Laurent expansion )

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials , $z = x+iy$ and $u(x,y)$ ,$v(x,y)$ are real part and imaginary part of $1/\Gamma(z)$ , the following expressions hold .

$$\frac{1}{\Gamma(z)} = z + \sum_{s=2}^{\infty} s c_{s-1} \frac{z^s}{s\,!} \qquad |z| < \infty$$

$$u(x,y) = x + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (2r+s)\, c_{2r+s-1} \frac{x^s}{s\,!} \frac{(-1)^r y^{2r}}{(2r)\,!}$$

$$v(x,y) = y + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (2r+s+1)\, c_{2r+s} \frac{x^s}{s\,!} \frac{(-1)^r y^{2r+1}}{(2r+1)\,!}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)) \qquad n=1,2,3,\cdots$$

$$0^0 = 1$$

**Proof**

According to Formula 12.2.2, when $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, the following expression holds.

$$\frac{1}{\Gamma(z)} = z + \sum_{s=1}^{\infty} \frac{c_s}{s!} z^{s+1} \tag{2.2}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{n-1}(1)\right) \qquad n=1,2,3,\cdots$$

In (2.2) , changing the initial value of $s$ to $2$ ,

$$\frac{1}{\Gamma(z)} = z + \sum_{s=2}^{\infty} s c_{s-1} \frac{z^s}{s!}$$

Let $z = x + iy$ , separate the first term into the real part and the imaginary part, and apply Formula 14.1.2 to the second term. Then,

$$f^{(s)}(0) = s c_{s-1}$$

$$f^{(2r+s)}(0) = (2r+s)\, c_{2r+s-1}$$

$$f^{(2r+s+1)}(0) = (2r+s+1)\, c_{2r+s}$$

Therefore,

$$u(x,y) = x + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (2r+s)\, c_{2r+s-1} \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = y + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (2r+s+1)\, c_{2r+s} \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

**Example: ( numeric calculation )**

According the formula, $1/\Gamma(z)$ is expanded to reciprocal Laurent series. The polynomial $B_{n,k}\left(f_1, f_2, \dots\right)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. Real and imaginary parts at $0.5+0.1i$ are calculated, and the function value and the series value are compared respectivery. the series are calculated up to $10$ terms. The results are as follows.

```
Unprotect[Power]; Power[0, 0] = 1;
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]
c[n_] := Sum[(-1)^k BellY[n, k, Tblψ[n, 1]], {k, 1, n}]
g[z_, m_] := z + Sum[s c[s - 1] z^s/s!, {s, 2, m}]
u[x_, y_, m_] := x + Sum[(2 r + s) c[2 r + s - 1] x^s/s! (-1)^r y^(2 r)/(2 r)!, {r, 0, m}, {s, 0, m}]
v[x_, y_, m_] := y + Sum[(2 r + s + 1) c[2 r + s] x^s/s! (-1)^r y^(2 r + 1)/(2 r + 1)!, {r, 0, m}, {s, 0, m}]
N[{Re[1/Gamma[0.5 + 0.1 i]], u[0.5, 0.1, 10]}]
        {0.567255, 0.567255}
N[{Im[1/Gamma[0.5 + 0.1 i]], v[0.5, 0.1, 10]}]
        {0.111219, 0.111219}
```

In both the real and imaginary parts, the function value and the series value are exactly the same.

## 20.3 Maclaurin Expansion by Real & Imaginary Parts of Gamma Function & the Reciprocal

### Formula 20.3.1

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \dots)$ are Bell polynomials , $z = x + iy$ and $u(x,y), v(x,y)$ are real part and imaginary part of $\Gamma(1+z)$ , the following expressions hold .

$$\Gamma(1+z) = 1 + \sum_{s=1}^{\infty} c_s \frac{z^s}{s!} \qquad |z| < 1$$

$$u(x,y) = 1 + \sum_{s=1}^{\infty} c_s \frac{x^s}{s!} + \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} c_{2r+s} \frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} c_{2r+s+1} \frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad |z| < \Diamond$$

where,

$$c_n = \sum_{k=1}^{n} B_{n,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{n-1}(1)\right) \qquad n = 1, 2, 3, \cdots$$

$$0^0 = 1$$

### Proof

Giving $a = 1$ to Formula 20.1.1 and replacing $z$ with $z+1$ , we obtain the desired expressions.

### Example: ( numeric calculation )

According the formula, $\Gamma(1+z)$ is expanded to Maclaurin series. The polynomial $B_{n,k}(f_1, f_2, \dots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The real and imaginary parts at $0.4+0.3i$ are calculated, and the function value and the series value are compared respectivery. the series are calculated up to 18 terms. The results are as follows.

```
Unprotect[Power]; Power[0, 0] = 1;
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]
```

$$\texttt{c[n\_] :=} \sum_{k=1}^{n} \texttt{BellY[n, k, Tbl}\psi\texttt{[n, 1]]}$$

$$\texttt{f[z\_, m\_] := 1 +} \sum_{s=1}^{m} \texttt{c[s]} \frac{z^s}{s!}$$

$$\texttt{u[x\_, y\_, m\_] := 1 +} \sum_{s=1}^{m} \texttt{c[s]} \frac{x^s}{s!} + \sum_{r=1}^{m}\sum_{s=0}^{m} \texttt{c[2 r + s]} \frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$\texttt{v[x\_, y\_, m\_] :=} \sum_{r=0}^{m}\sum_{s=0}^{m} \texttt{c[2 r + s + 1]} \frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

```
N[{Re[Gamma[1 + 0.4 + 0.3 i]], u[0.4, 0.3, 18]}]
                    {0.847678, 0.847678}
N[{Im[Gamma[1 + 0.4 + 0.3 i]], v[0.4, 0.3, 18]}]
                  {-0.0119186, -0.0119186}
```

The function value and the series value are exactly the same.

## Formula 20.3.2

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials, $z = x + iy$ and $u(x,y), v(x,y)$ are real part and imaginary part of $1/\Gamma(1+z)$, the following expressions hold.

$$\frac{1}{\Gamma(1+z)} = 1 + \sum_{s=1}^{\infty} c_s \frac{z^s}{s!} \qquad |z| < \infty$$

$$u(x,y) = 1 + \sum_{s=1}^{\infty} c_s \frac{x^s}{s!} + \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} c_{2r+s} \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} c_{2r+s+1} \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

where,

$$c_n = \sum_{k=1}^{n} (-1)^k B_{n,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{n-1}(1)\right) \qquad n=1,2,3,\cdots$$

$$0^0 = 1$$

### Proof

Giving $a = 1$ to Formula 20.1.2 and replacing $z$ with $z+1$, we obtain the desired expressions.

### Example: ( numeric calculation )

According the formula, $1/\Gamma(1+z)$ is expanded to Maclaurin series. The polynomial $B_{n,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. The real and imaginary parts at $0.4+0.3i$ are calculated, and the function value and the series value are compared respectivery. the series are calculated up to $10$ terms. The results are as follows.

```
Unprotect[Power]; Power[0, 0] = 1;
Tblψ[n_, z_] := Table[PolyGamma[k, z], {k, 0, n - 1}]
c[n_] := Sum[(-1)^k BellY[n, k, Tblψ[n, 1]], {k, 1, n}]
g[z_, m_] := 1 + Sum[c[s] z^s/s!, {s, 1, m}]
u[x_, y_, m_] := 1 + Sum[c[s] x^s/s!, {s, 1, m}] + Sum[c[2 r + s] x^s/s! (-1)^r y^(2 r)/(2 r)!, {r, 1, m}, {s, 0, m}]
v[x_, y_, m_] := Sum[c[2 r + s + 1] x^s/s! (-1)^r y^(2 r+1)/(2 r + 1)!, {r, 0, m}, {s, 0, m}]
N[{Re[1/Gamma[1 + 0.4 + 0.3 i]], u[0.4, 0.3, 10]}]
{1.17946, 1.17946}
N[{Im[1/Gamma[1 + 0.4 + 0.3 i]], v[0.4, 0.3, 10]}]
{0.0165836, 0.0165836}
```

The function value and the series value are exactly the same.

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**Alien's Mathematics**