

11 Series Expansion of Reciprocal of Gamma Function

11.1 Functions with Integers as Roots

Function $f(z)$ with negative integers as roots can be described as follows.

$$f(z) = \left(1 + \frac{z}{1}\right) \left(1 + \frac{z}{2}\right) \left(1 + \frac{z}{3}\right) \cdots = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)$$

However, this infinite product diverges. That is, such a function cannot exist alone. So, let us consider the following function which added compensation to each factor.

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

Then, this infinite product uniformly converges for arbitrary z and results in function $e^{\gamma z} / \Gamma(1+z)$.

This was shown by **Weierstrass**. If the functions with various integers as roots are shown, it is as follows.

In addition, these left sides can be expanded to series of z . So, we call the formulas **factorization around 0**.

Formula 11.1.1 (factorization around 0)

When γ is Euler-Mascheroni constant and $\Gamma(z)$ is the Gamma function, the following expressions hold.

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} = \frac{e^{-\gamma z}}{\Gamma(1+z)} \quad (1,1.)$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} = \frac{e^{\gamma z}}{\Gamma(1-z)} \quad (1,1_+)$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{2n}\right) e^{-\frac{z}{2n}} = \frac{1}{\Gamma(1+z/2)} e^{-\frac{\gamma z}{2}} \quad (1,2.)$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{2n}\right) e^{\frac{z}{2n}} = \frac{1}{\Gamma(1-z/2)} e^{\frac{\gamma z}{2}} \quad (1,2_+)$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{2n-1}\right) e^{-\frac{z}{2n-1}} = \frac{\sqrt{\pi}}{\Gamma\{(1+z)/2\}} e^{-\left(\frac{\gamma}{2} + \log 2\right)z} \quad (1,3.)$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{2n-1}\right) e^{\frac{z}{2n-1}} = \frac{\sqrt{\pi}}{\Gamma\{(1-z)/2\}} e^{\left(\frac{\gamma}{2} + \log 2\right)z} \quad (1,3_+)$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{2n-1}\right) e^{-\frac{z}{2n}} = \frac{\sqrt{\pi}}{\Gamma\{(1+z)/2\}} e^{-\frac{\gamma z}{2}} \quad (1,4.)$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{2n-1}\right) e^{\frac{z}{2n}} = \frac{\sqrt{\pi}}{\Gamma\{(1-z)/2\}} e^{\frac{\gamma z}{2}} \quad (1,4_+)$$

Proof

Weierstrass expression on the gamma function was as follows.

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

From this, (1,1₋) is obtained immediately. Reversing the sign of z in (1,1₋), we obtain (1,1₊).

Replacing z with $z/2$ in (1,1₋) and (1,1₊), we obtain (1,2₋) and (1,2₊).

Next, dividing (1,1₋) by (1,2₋),

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{2n-1}\right) e^{-\frac{z}{2n-1}} = \frac{\Gamma(1+z/2)}{\Gamma(1+z)} \frac{e^{-\gamma z}}{e^{-\gamma z/2}} = \frac{\Gamma(z/2)}{2\Gamma(z)} e^{-\frac{\gamma z}{2}}$$

Here, using the Legendre duplication formula,

$$\frac{\Gamma(z/2)}{2\Gamma(z)} = \frac{\sqrt{\pi}}{2^z} \frac{1}{\Gamma\{(1+z)/2\}} = \frac{\sqrt{\pi} e^{-z \log 2}}{\Gamma\{(1+z)/2\}}$$

Substituting this for the above,

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{2n-1}\right) e^{-\frac{z}{2n-1}} = \frac{\sqrt{\pi}}{\Gamma\{(1+z)/2\}} e^{-\left(\frac{\gamma}{2} + \log 2\right)z} \quad (1.3.)$$

And, reversing the sign of z , we obtain (1,3₊).

Next,

$$\prod_{n=1}^{\infty} e^{\frac{z}{2n-1}} \cdot e^{-\frac{z}{2n}} = e^{z\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right)} = e^{z \log 2}$$

Multiplying each side of (1,3₊) by this,

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{2n-1}\right) e^{-\frac{z}{2n}} = \frac{\sqrt{\pi}}{\Gamma\{(1+z)/2\}} e^{-\frac{\gamma z}{2}} \quad (1.4.)$$

And, reversing the sign of z , we obtain (1,4₊).

Giving $z = 1$ to the formulas, we obtain the following special values.

Special Values

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-\frac{1}{n}} = e^{-\gamma} = 0.56145948\dots \quad (1.1^1)$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{2n}\right) e^{-\frac{1}{2n}} = \frac{2}{\sqrt{\pi}} e^{-\frac{\gamma}{2}} = 0.84550128\dots \quad (1.2^1)$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{2n}\right) e^{\frac{1}{2n}} = \frac{1}{\sqrt{\pi}} e^{\frac{\gamma}{2}} = 0.75294950\dots \quad (1.2^2)$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{2n-1}\right) e^{-\frac{1}{2n-1}} = \frac{\sqrt{\pi}}{2} e^{-\frac{\gamma}{2}} = 0.66405515\dots \quad (1.3^1)$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{2n-1}\right) e^{-\frac{1}{2n}} = \sqrt{\pi} e^{-\frac{\gamma}{2}} = 1.32811030\dots \quad (1.4^1)$$

From **Formula 11.1.1**, the following corollary is derived

Corollary 11.1.1

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \left(1 - \frac{z}{n}\right) = \frac{1}{\Gamma(1+z)\Gamma(1-z)} = \frac{\sin \pi z}{\pi z} \tag{1.1}$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{2n}\right) \left(1 - \frac{z}{2n}\right) = \frac{1}{\Gamma\left(1 + \frac{z}{2}\right)\Gamma\left(1 - \frac{z}{2}\right)} = \frac{2}{\pi z} \sin \frac{\pi z}{2} \tag{1.2}$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{2n-1}\right) \left(1 - \frac{z}{2n-1}\right) = \frac{\pi}{\Gamma\left(\frac{1+z}{2}\right)\Gamma\left(\frac{1-z}{2}\right)} = \cos \frac{\pi z}{2} \tag{1.3}$$

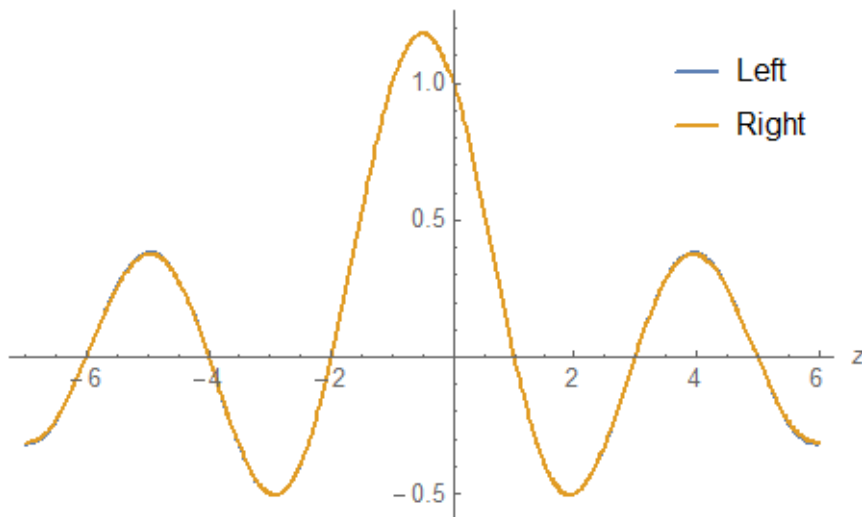
$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{2n-1}\right) \left(1 - \frac{z}{2n}\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1+z}{2}\right)\Gamma\left(1 - \frac{z}{2}\right)} \tag{1.2+}$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{2n-1}\right) \left(1 + \frac{z}{2n}\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-z}{2}\right)\Gamma\left(1 + \frac{z}{2}\right)} \tag{1.2-}$$

(1.1) is a function with all integers as roots. (1.2) is a function with all even numbers as roots, and (1.3) is a function with all odd numbers as roots. These are all expressed with an elementary function.

(1.2₊) is a function with negative odd numbers and positive even numbers as roots. (1.2₋) is a function with positive odd numbers and negative even numbers as roots. These cannot be expressed with an elementary function.

If both sides of (1.2₋) are illustrated, it is as follows. The product is calculated 500 terms. Since both sides are overlapped exactly, the left side (blue) is not visible.



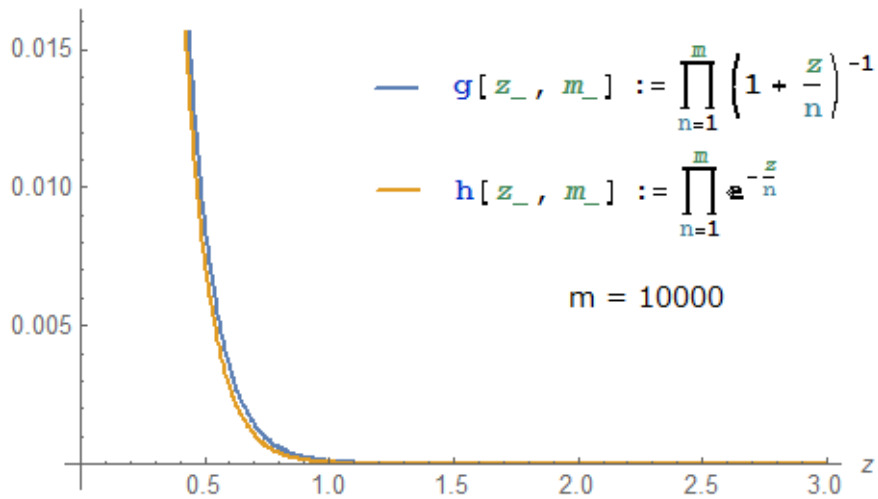
Compensation Term

We discuss (1,1₋) as an example.

$$\frac{e^{-\gamma z}}{\Gamma(1+z)} = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \tag{1,1-}$$

Although the compensation term is $e^{-z/n}$, the compensation term is not necessarily limited to this. First of all, the complete compensation term for the right side is $\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1}$. However, if such a compensation term is adopted, the function of the left side becomes a constant function and the utility value disappears. This is the reason why the compensation term like $e^{-z/n}$ is adopted.

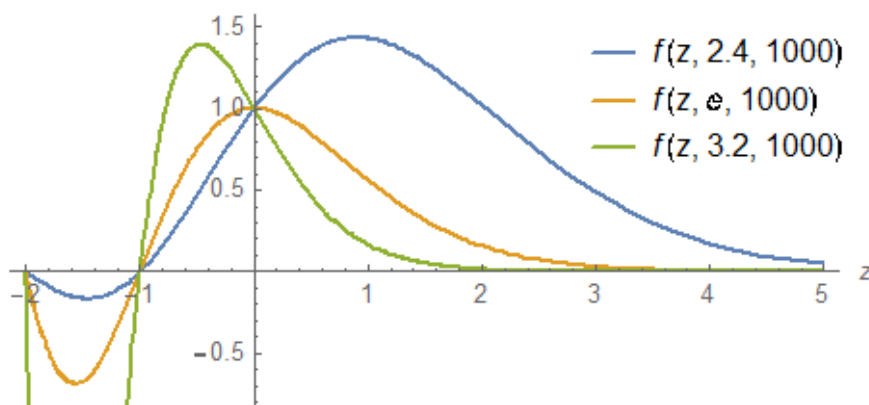
In fact, if the body and the compensation term are illustrated, it is as follows. The products are calculated 10000 terms respectively. Both are very close but they do not overlap.



Even from this figure, it is hard to consider compensation term other than an exponential function. On the other hand, it seems that it may be other exponential functions. So let us consider the following exponential function.

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) c^{-\frac{z}{n}} \quad c > 1$$

The necessity for the condition $c > 1$ is obvious. Anything is okay as long as this is satisfied. For example, if $f(z)$ is illustrated for $c = 2.4$, $c = 2.71828\dots$, $c = 3.2$, it is as follows.



We can see that the maximum value of $f(z)$ is minimum at $c = e$. Actually, if c deviates from e greatly, the maximum value of $f(z)$ becomes large surprising. Considering this, $e^{-z/n}$ seems to be optimal as a compensation term.

11.2 Maclaurin Expansion

Since the reciprocal of the gamma function is holomorphic over the whole complex plane, the Maclaurin expansion should be possible. In this section, treating the divergent product and divergent series as numbers or functions, we aim at the Maclaurin expansion of the reciprocal of gamma function.

Formula 11.1.2

When γ is Euler-Mascheroni constant, $\Gamma(z)$ is the Gamma function, H_n is a harmonic number and H is a harmonic series, the following expressions hold.

$$\begin{aligned} \frac{1}{\Gamma(1+z)} &= 1 + \gamma z + \left(\frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) z^2 \\ &\quad + \left(\frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} - \frac{\gamma-H}{n} \right) \right) z^3 + \dots \end{aligned} \quad (2.1)$$

$$\begin{aligned} \frac{1}{\Gamma(1-z)} &= 1 - \gamma z + \left(\frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) z^2 \\ &\quad - \left(\frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} - \frac{\gamma-H}{n} \right) \right) z^3 + \dots \end{aligned} \quad (2.1_+)$$

$$\begin{aligned} \frac{1}{\Gamma(1+z/2)} &= 1 + \frac{\gamma}{2} z + \frac{1}{2^2} \left(\frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) z^2 \\ &\quad + \frac{1}{2^3} \left(\frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} - \frac{\gamma-H}{n} \right) \right) z^3 + \dots \end{aligned} \quad (2.2)$$

$$\begin{aligned} \frac{1}{\Gamma(1-z/2)} &= 1 - \frac{\gamma}{2} z + \frac{1}{2^2} \left(\frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) z^2 \\ &\quad - \frac{1}{2^3} \left(\frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} - \frac{\gamma-H}{n} \right) \right) z^3 + \dots \end{aligned} \quad (2.2_+)$$

Proof

From the Weierstrass expression on the gamma function,

$$\frac{1}{\Gamma(1+z)} = e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \quad (2.1)$$

When H_n is a harmonic number and H is a harmonic series, the compensation term is

$$\prod_{n=1}^{\infty} e^{-\frac{z}{n}} = e^{-z \sum_{n=1}^{\infty} \frac{1}{n}} = e^{-Hz}$$

Then

$$e^{\gamma z} \prod_{n=1}^{\infty} e^{-\frac{z}{n}} = e^{(\gamma-H)z} = 1 + \frac{\gamma-H}{1!} z^1 + \frac{(\gamma-H)^2}{2!} z^2 + \frac{(\gamma-H)^3}{3!} z^3 + \dots \quad (2.1e)$$

Next, assume that the body is expanded as follows.

$$\begin{aligned} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) &= \left(1 + \frac{z}{1} \right) \left(1 + \frac{z}{2} \right) \left(1 + \frac{z}{3} \right) \left(1 + \frac{z}{4} \right) \dots \\ &= 1 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \dots \end{aligned}$$

Then, from Vieta's formulas,

$$\begin{aligned}
a_1 &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots = H \\
a_2 &= \frac{1}{1} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \right) + \frac{1}{2} \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \right) + \frac{1}{3} \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots \right) + \dots \\
&= \frac{H-H_1}{1} + \frac{H-H_2}{2} + \frac{H-H_3}{3} + \dots = \sum_{n=1}^{\infty} \frac{H-H_n}{n} \\
a_3 &= \frac{1}{1} \frac{1}{2} \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \right) + \frac{1}{1} \frac{1}{3} \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots \right) + \frac{1}{1} \frac{1}{4} \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots \right) + \dots \\
&\quad + \frac{1}{2} \frac{1}{3} \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots \right) + \frac{1}{2} \frac{1}{4} \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots \right) + \frac{1}{2} \frac{1}{5} \left(\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \right) + \dots \\
&\quad + \frac{1}{3} \frac{1}{4} \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots \right) + \frac{1}{3} \frac{1}{5} \left(\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \right) + \frac{1}{3} \frac{1}{6} \left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots \right) + \dots \\
&\quad + \\
&\quad \vdots \\
&= \frac{1}{1} \frac{1}{2} (H-H_2) + \frac{1}{1} \frac{1}{3} (H-H_3) + \frac{1}{1} \frac{1}{4} (H-H_4) + \dots \\
&\quad + \frac{1}{2} \frac{1}{3} (H-H_3) + \frac{1}{2} \frac{1}{4} (H-H_4) + \frac{1}{2} \frac{1}{5} (H-H_5) + \dots \\
&\quad + \frac{1}{3} \frac{1}{4} (H-H_4) + \frac{1}{3} \frac{1}{5} (H-H_5) + \frac{1}{3} \frac{1}{6} (H-H_6) + \dots \\
&\quad + \\
&\quad \vdots
\end{aligned}$$

Rearranging this along the diagonal line,

$$\begin{aligned}
a_3 &= \left(\frac{1}{1} \right) \frac{1}{2} (H-H_2) + \left(\frac{1}{1} + \frac{1}{2} \right) \frac{1}{3} (H-H_3) + \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right) \frac{1}{4} (H-H_4) + \dots \\
&= H_1 \frac{1}{2} (H-H_2) + H_2 \frac{1}{3} (H-H_3) + H_3 \frac{1}{4} (H-H_4) + \dots = \sum_{n=1}^{\infty} \frac{H_n (H-H_{n+1})}{n+1}
\end{aligned}$$

Thus,

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) = 1 + zH + z^2 \sum_{n=1}^{\infty} \frac{H-H_n}{n} + z^3 \sum_{n=1}^{\infty} \frac{H_n (H-H_{n+1})}{n+1} + \dots \quad (2.1p)$$

The right side of (2.1) is a product of (2.1e) and (2.1p). That is

$$e^{\gamma z} \prod_{n=1}^{\infty} e^{-\frac{z}{n}} = 1 + \frac{\gamma-H}{1!} z^1 + \frac{(\gamma-H)^2}{2!} z^2 + \frac{(\gamma-H)^3}{3!} z^3 + \dots \quad (2.1e)$$

$$\times \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) = 1 + zH + z^2 \sum_{n=1}^{\infty} \frac{H-H_n}{n} + z^3 \sum_{n=1}^{\infty} \frac{H_n (H-H_{n+1})}{n+1} + \dots \quad (2.1p)$$

Taking the Cauchy product,

The coefficient of z^1 is

$$H + \frac{\gamma - H}{1!} = \gamma$$

The coefficient of z^2 is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H - H_n}{n} + \frac{H(\gamma - H)}{1!} + \frac{(\gamma - H)^2}{2!} &= H^2 - \sum_{n=1}^{\infty} \frac{H_n}{n} + \frac{H\gamma}{1!} - \frac{H^2}{1!} + \frac{\gamma^2 - 2\gamma H + H^2}{2!} \\ &= \frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \end{aligned}$$

The coefficient of z^3 is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n(H - H_{n+1})}{n+1} + \frac{\gamma - H}{1!} \sum_{n=1}^{\infty} \frac{H - H_n}{n} + \frac{H(\gamma - H)^2}{2!} + \frac{(\gamma - H)^3}{3!} \\ = \sum_{n=1}^{\infty} \frac{H_n(H - H_{n+1})}{n+1} + \frac{\gamma - H}{1!} \sum_{n=1}^{\infty} \frac{H}{n} - \sum_{n=1}^{\infty} \frac{(\gamma - H)H_n}{n} + \frac{H(\gamma - H)^2}{2!} + \frac{(\gamma - H)^3}{3!} \\ = \frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H - H_{n+1}}{n+1} - \frac{\gamma - H}{n} \right) \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{\Gamma(1+z)} &= 1 + \gamma z + \left(\frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) z^2 \\ &\quad + \left(\frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H - H_{n+1}}{n+1} - \frac{\gamma - H}{n} \right) \right) z^3 + \dots \end{aligned} \quad (2.1.)$$

(2.1₊), (2.1) and (2.2₊) are easily derived from this.

Verification

The ordinary Maclaurin expansion of the left side of (2.1) is as follows. In addition, $\psi_n(z)$ is a polygamma function.

$$\begin{aligned} \frac{1}{\Gamma(1+z)} &= 1 + \gamma z + \left(\frac{\gamma^2}{2!} - \frac{\pi^2}{12} \right) z^2 + \left(\frac{\gamma^3}{3!} - \frac{\gamma\pi^2}{12} - \frac{\psi_2(1)}{6} \right) z^3 + \dots \\ &= 1 + 0.5772157 z - 0.6558781 z^2 - 0.04200264 z^3 + \dots \end{aligned}$$

Therefore, the following equations must hold.

$$\begin{aligned} \frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} &= \frac{\gamma^2}{2!} - \frac{\pi^2}{12} = -0.655878\dots \\ \frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H - H_{n+1}}{n+1} - \frac{\gamma - H}{n} \right) &= \frac{\gamma^3}{3!} - \frac{\gamma\pi^2}{12} - \frac{\psi_2(1)}{6} \\ &= -0.0420026\dots \end{aligned}$$

Since the harmonic series is infinite, we use the following limit value for the verification.

$$\lim_{m \rightarrow \infty} \left(\frac{H_m^2}{2} - \sum_{n=1}^m \frac{H_n}{n} \right)$$

Calculating these with the formula manipulation software Mathematica, we obtained the following result. The coefficients of each term are almost consistent with the above values, so, the correctness of this formula was numerically verified.

$$\begin{aligned}
 H[m_] &:= \text{HarmonicNumber}[m] & \gamma &:= \text{EulerGamma} \\
 c2[m_] &:= \frac{\gamma^2}{2!} + \frac{H[m]^2}{2} - \sum_{n=1}^m \frac{H[n]}{n} \\
 N[c2[20\,000]] & & \text{Limit}[c2[m], m \rightarrow \infty] & \\
 -0.655853 & & \frac{1}{12} (6 \text{EulerGamma}^2 - \pi^2) & \\
 c3[m_] &:= \frac{\gamma^3}{3!} + \frac{\gamma H[m]^2}{2} - \frac{2 H[m]^3}{3} + \sum_{n=1}^m H[n] \left\{ \frac{H[m] - H[n+1]}{n+1} - \frac{\gamma - H[m]}{n} \right\} \\
 N[c3[40\,000]] & & & \\
 -0.0419954 & & &
 \end{aligned}$$

Also, the following special values were obtained.

Special Values

$$H^2 - 2! \sum_{n=1}^{\infty} \frac{H_n}{n} = -\frac{\pi^2}{6} = -\zeta(2) \quad : \text{Riemann Zeta} \quad (2.3)$$

$$4H^3 - 3! \sum_{n=1}^{\infty} H_n \left(\frac{H - H_{n+1}}{n+1} + \frac{H}{n} \right) = \psi_2(1) = -2.404113\dots \quad (2.4)$$

If easier (2.3) is written down, it is as follows.

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \dots \right)^2 - 2! \left\{ \frac{1}{1} \binom{1}{1} + \frac{1}{2} \left(1 + \frac{1}{2} \right) + \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} \right) + \dots \right\} = -\zeta(2)$$

Note

A double series of H and H_n appears in the coefficient of z^4 , and a triple series appears in the coefficient of z^5 . These are far more complicated than notation by polygamma function. So, these are not calculated in this chapter..

11.3 Factorization around 1

Replacing z with $z-1$ in Formula 11.1.1, we obtain the following formula. These left sides can be expanded to power series of $z-1$. So, we call the formulas **factorization around 1**.

Formula 11.3.1

When γ is Euler-Mascheroni constant and $\Gamma(z)$ is the Gamma function, the following expressions hold.

$$\frac{1}{\Gamma(z)} = e^{\gamma(z-1)} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n}\right) e^{-\frac{z-1}{n}} \quad (3.1)$$

$$\frac{1}{\Gamma(2-z)} = e^{-\gamma(z-1)} \prod_{n=1}^{\infty} \left(1 - \frac{z-1}{n}\right) e^{\frac{z-1}{n}} \quad (3.1_+)$$

$$\frac{1}{\Gamma\{(1+z)/2\}} = e^{\frac{\gamma(z-1)}{2}} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n}\right) e^{-\frac{z-1}{2n}} \quad (3.2)$$

$$\frac{1}{\Gamma\{(3-z)/2\}} = e^{-\frac{\gamma(z-1)}{2}} \prod_{n=1}^{\infty} \left(1 - \frac{z-1}{2n}\right) e^{\frac{z-1}{2n}} \quad (3.2_+)$$

$$\frac{1}{\Gamma(z/2)} = \frac{1}{\sqrt{\pi}} e^{\left(\frac{\gamma}{2} + \log 2\right)(z-1)} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n-1}\right) e^{-\frac{z-1}{2n-1}} \quad (3.3)$$

$$\frac{1}{\Gamma(1-z/2)} = \frac{1}{\sqrt{\pi}} e^{-\left(\frac{\gamma}{2} + \log 2\right)(z-1)} \prod_{n=1}^{\infty} \left(1 - \frac{z-1}{2n-1}\right) e^{\frac{z-1}{2n-1}} \quad (3.2_+)$$

$$\frac{1}{\Gamma(z/2)} = \frac{1}{\sqrt{\pi}} e^{\frac{\gamma(z-1)}{2}} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n-1}\right) e^{-\frac{z-1}{2n}} \quad (3.4)$$

$$\frac{1}{\Gamma(1-z/2)} = \frac{1}{\sqrt{\pi}} e^{-\frac{\gamma(z-1)}{2}} \prod_{n=1}^{\infty} \left(1 - \frac{z-1}{2n-1}\right) e^{\frac{z-1}{2n}} \quad (3.4_+)$$

From this formula, the following formula is further derived.

Formula 11.3.2

When γ is Euler-Mascheroni constant and $\Gamma(z)$ is the Gamma function, the following expressions hold.

$$\frac{1}{\Gamma(1+z)} = e^{(\gamma-1)(z-1)} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n+1}\right) e^{-\frac{z-1}{n+1}} \quad (3.5)$$

$$\frac{1}{\Gamma(1-z)} = -(z-1) e^{-\gamma(z-1)} \prod_{n=1}^{\infty} \left(1 - \frac{z-1}{n}\right) e^{\frac{z-1}{n}} \quad (3.5_+)$$

$$\frac{1}{\Gamma(1+z/2)} = \frac{2}{\sqrt{\pi}} e^{\left(\frac{\gamma}{2} + \log 2 - 1\right)(z-1)} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n+1}\right) e^{-\frac{z-1}{2n+1}} \quad (3.6)$$

$$\frac{1}{\Gamma(1+z/2)} = \frac{2}{\sqrt{\pi}} e^{\frac{\gamma(z-1)}{2}} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n+1}\right) e^{-\frac{z-1}{2n}} \quad (3.7)$$

Proof

Dividing both sides of (3.1) by z ,

$$\frac{1}{z\Gamma(z)} = e^{\gamma(z-1)} \frac{1}{z} \left(1 + \frac{z-1}{1}\right) e^{-\frac{z-1}{1}} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n+1}\right) e^{-\frac{z-1}{n+1}}$$

i.e.

$$\frac{1}{\Gamma(1+z)} = e^{(\gamma-1)(z-1)} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n+1}\right) e^{-\frac{z-1}{n+1}} \quad (3.5)$$

Multiplying both sides of (3.1₊) by $1-z$,

$$\frac{1-z}{\Gamma(2-z)} = -(z-1) e^{-\gamma(z-1)} \prod_{n=1}^{\infty} \left(1 - \frac{z-1}{n}\right) e^{\frac{z-1}{n}}$$

i.e.

$$\frac{1}{\Gamma(1-z)} = -(z-1) e^{-\gamma(z-1)} \prod_{n=1}^{\infty} \left(1 - \frac{z-1}{n}\right) e^{\frac{z-1}{n}} \quad (3.5_+)$$

Dividing both sides of (3.3) by $z/2$,

$$\frac{1}{(z/2)\Gamma(z/2)} = \frac{2}{\sqrt{\pi}} e^{\left(\frac{\gamma}{2} + \log 2\right)(z-1)} \frac{1}{z} \left(1 + \frac{z-1}{2 \cdot 1 - 1}\right) e^{-\frac{z-1}{2 \cdot 1 - 1}} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n+1}\right) e^{-\frac{z-1}{2n+1}}$$

i.e.

$$\frac{1}{\Gamma(1+z/2)} = \frac{2}{\sqrt{\pi}} e^{\left(\frac{\gamma}{2} + \log 2 - 1\right)(z-1)} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n+1}\right) e^{-\frac{z-1}{2n+1}} \quad (3.6)$$

Dividing both sides of (3.4) by $z/2$,

$$\frac{1}{(z/2)\Gamma(z/2)} = \frac{2}{\sqrt{\pi}} e^{\frac{\gamma(z-1)}{2}} \frac{1}{z} \left(1 + \frac{z-1}{2 \cdot 1 - 1}\right) \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n+1}\right) e^{-\frac{z-1}{2n}}$$

i.e.

$$\frac{1}{\Gamma(1+z/2)} = \frac{2}{\sqrt{\pi}} e^{\frac{\gamma(z-1)}{2}} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n+1}\right) e^{-\frac{z-1}{2n}} \quad (3.7)$$

Giving $z=2$ to (3.5), (3.6), (3.7), we obtain the following special values.

Special Values

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n+1}\right) e^{-\frac{1}{n+1}} = \frac{e^{1-\gamma}}{2} = 0.763102\dots \quad (3.5^2)$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{2n+1}\right) e^{-\frac{1}{2n+1}} = \frac{\sqrt{\pi}}{4} e^{-\left(\frac{\gamma}{2}-1\right)} = 0.902544\dots \quad (3.6^2)$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{2n+1}\right) e^{-\frac{1}{2n}} = \frac{\sqrt{\pi}}{2} e^{-\frac{\gamma}{2}} = 0.664055\dots \quad (3.7^2)$$

11.4 Taylor Expansion (Part 1)

Formula 11.4.1

When γ is Euler-Mascheroni constant, $\Gamma(z)$ is the Gamma function, H_n is a harmonic number and H is a harmonic series, the following expressions hold.

$$\begin{aligned} \frac{1}{\Gamma(z)} &= 1 + \gamma(z-1) + \left(\frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) (z-1)^2 \\ &\quad + \left(\frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} - \frac{\gamma-H}{n} \right) \right) (z-1)^3 + \dots \quad (4.1) \end{aligned}$$

$$\begin{aligned} \frac{1}{\Gamma(2-z)} &= 1 - \gamma(z-1) + \left(\frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) (z-1)^2 \\ &\quad - \left(\frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} - \frac{\gamma-H}{n} \right) \right) (z-1)^3 + \dots \quad (4.1_+) \end{aligned}$$

$$\begin{aligned} \frac{1}{\Gamma\{(1+z)/2\}} &= 1 + \frac{\gamma}{2}(z-1) + \frac{1}{2^2} \left(\frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) (z-1)^2 \\ &\quad + \frac{1}{2^3} \left(\frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} - \frac{\gamma-H}{n} \right) \right) (z-1)^3 + \dots \quad (4.2) \end{aligned}$$

$$\begin{aligned} \frac{1}{\Gamma\{(3-z)/2\}} &= 1 - \frac{\gamma}{2}(z-1) + \frac{1}{2^2} \left(\frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) (z-1)^2 \\ &\quad - \frac{1}{2^3} \left(\frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} - \frac{\gamma-H}{n} \right) \right) (z-1)^3 + \dots \quad (4.2_+) \end{aligned}$$

$$\begin{aligned} \frac{1}{\Gamma(1-z)} &= -(z-1) + \gamma(z-1)^2 - \left(\frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) (z-1)^3 \\ &\quad + \left(\frac{\gamma^3}{3!} + \frac{\gamma H^2}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} - \frac{\gamma-H}{n} \right) \right) (z-1)^4 - \dots \quad (4.5_+) \end{aligned}$$

Proof

Replacing z with $z-1$ in Formula 11.2.1 (2.1) ~ (2.2₊), we obtain (4.1) ~ (4.2₊). Multiplying both sides of (4.1₊) by $1-z$, we obtain (4.5₊).

Formula 11.4.2

When γ is Euler-Mascheroni constant, $\Gamma(z)$ is the Gamma function, H_n is a harmonic number and H is a harmonic series, the following expression holds.

$$\begin{aligned} \frac{1}{\Gamma(1+z)} &= 1 - \left(1 - \frac{\gamma}{1!} \right) (z-1) + \left(1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} + \frac{H^2}{2!} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) (z-1)^2 \\ &\quad - \left(1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} - \frac{\gamma^3}{3!} + \frac{H^2(1-\gamma)}{2} + \frac{2H^3}{3} - \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} + \frac{1-\gamma+H}{n} \right) \right) (z-1)^3 + \dots \quad (4.5) \end{aligned}$$

Proof

Formula 11.3.2 (3.5) was as follows.

$$\frac{1}{\Gamma(1+z)} = e^{(\gamma-1)(z-1)} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n+1}\right) e^{-\frac{z-1}{n+1}} \quad (3.5)$$

At first, the compensation term is

$$\prod_{n=1}^{\infty} e^{-\frac{z-1}{n+1}} = e^{-(z-1) \sum_{n=1}^{\infty} \frac{1}{n+1}} = e^{(z-1) - (z-1) \sum_{n=1}^{\infty} \frac{1}{n}} = e^{-(H-1)(z-1)}$$

Then,

$$e^{(\gamma-1)(z-1)} \prod_{n=1}^{\infty} e^{-\frac{z-1}{n+1}} = e^{(\gamma-1)(z-1)} e^{-(H-1)(z-1)} = e^{(\gamma-H)(z-1)}$$

Expanding this,

$$e^{(\gamma-1)(z-1)} \prod_{n=1}^{\infty} e^{-\frac{z-1}{n+1}} = 1 + \frac{\gamma-H}{1!} (z-1)^1 + \frac{(\gamma-H)^2}{2!} (z-1)^2 + \frac{(\gamma-H)^3}{3!} (z-1)^3 + \dots \quad (4.1e)$$

Let the body be

$$\begin{aligned} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n+1}\right) &= \left(1 + \frac{z-1}{2}\right) \left(1 + \frac{z-1}{3}\right) \left(1 + \frac{z-1}{4}\right) \left(1 + \frac{z-1}{5}\right) \dots \\ &= 1 + a_1 (z-1)^1 + a_2 (z-1)^2 + a_3 (z-1)^3 + \dots \end{aligned}$$

Then, from Vieta's formulas,

$$\begin{aligned} a_1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = H-1 \\ a_2 &= \frac{1}{2} \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \right) + \frac{1}{3} \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots \right) + \frac{1}{4} \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots \right) + \dots \\ &= \frac{H-H_1}{1} + \frac{H-H_2}{2} + \frac{H-H_3}{3} + \dots - \frac{H-1}{1} = \sum_{n=1}^{\infty} \frac{H-H_n}{n} - \frac{H-1}{1} \\ a_3 &= \frac{1}{2} \frac{1}{3} (H-H_3) + \frac{1}{2} \frac{1}{4} (H-H_4) + \frac{1}{2} \frac{1}{5} (H-H_5) + \dots \\ &\quad + \frac{1}{3} \frac{1}{4} (H-H_4) + \frac{1}{3} \frac{1}{5} (H-H_5) + \frac{1}{3} \frac{1}{6} (H-H_6) + \dots \\ &\quad + \frac{1}{4} \frac{1}{5} (H-H_5) + \frac{1}{4} \frac{1}{6} (H-H_6) + \frac{1}{4} \frac{1}{7} (H-H_7) + \dots \\ &\quad + \\ &\quad \vdots \\ &= \frac{1}{1} \frac{1}{2} (H-H_2) + \frac{1}{1} \frac{1}{3} (H-H_3) + \frac{1}{1} \frac{1}{4} (H-H_4) + \dots \\ &\quad + \frac{1}{2} \frac{1}{3} (H-H_3) + \frac{1}{2} \frac{1}{4} (H-H_4) + \frac{1}{2} \frac{1}{5} (H-H_5) + \dots \\ &\quad + \frac{1}{3} \frac{1}{4} (H-H_4) + \frac{1}{3} \frac{1}{5} (H-H_5) + \frac{1}{3} \frac{1}{6} (H-H_6) + \dots \end{aligned}$$

$$\begin{aligned}
& + \\
& \vdots \\
& -\frac{1}{1} \frac{1}{2} (H-H_2) - \frac{1}{1} \frac{1}{3} (H-H_3) - \frac{1}{1} \frac{1}{4} (H-H_4) - \dots
\end{aligned}$$

Rearranging this along the diagonal line,

$$\begin{aligned}
a_3 &= \left(\frac{1}{1} \right) \frac{1}{2} (H-H_2) + \left(\frac{1}{1} + \frac{1}{2} \right) \frac{1}{3} (H-H_3) + \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right) \frac{1}{4} (H-H_4) + \dots \\
& - \frac{1}{1} \frac{1}{2} (H-H_2) - \frac{1}{1} \frac{1}{3} (H-H_3) - \frac{1}{1} \frac{1}{4} (H-H_4) - \dots \\
&= \sum_{n=1}^{\infty} \frac{H_n(H-H_{n+1})}{n+1} - \sum_{n=1}^{\infty} \frac{H-H_{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(H_n-1)(H-H_{n+1})}{n+1}
\end{aligned}$$

Thus,

$$\begin{aligned}
\prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n+1} \right) &= 1 + (z-1)(H-1) + (z-1)^2 \left(\sum_{n=1}^{\infty} \frac{H-H_n}{n} - \frac{H-1}{1} \right) \\
& + (z-1)^3 \sum_{n=1}^{\infty} \frac{(H_n-1)(H-H_{n+1})}{n+1} + \dots \quad (4.1p)
\end{aligned}$$

The right side of (3.5) is a product of (4.1e) and (4.1p). That is

$$\begin{aligned}
e^{(\gamma-1)(z-1)} \prod_{n=1}^{\infty} e^{-\frac{z-1}{n+1}} &= 1 + \frac{\gamma-H}{1!} (z-1)^1 + \frac{(\gamma-H)^2}{2!} (z-1)^2 + \frac{(\gamma-H)^3}{3!} (z-1)^3 + \dots \\
\times \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n+1} \right) &= 1 + (z-1)(H-1) + (z-1)^2 \left(\sum_{n=1}^{\infty} \frac{H-H_n}{n} - \frac{H-1}{1} \right) \\
& + (z-1)^3 \sum_{n=1}^{\infty} \frac{(H_n-1)(H-H_{n+1})}{n+1} + \dots
\end{aligned}$$

Taking the Cauchy product,

The coefficient of $(z-1)^1$ is

$$\frac{\gamma-H}{1!} + H-1 = - \left(1 - \frac{\gamma}{1!} \right)$$

The coefficient of $(z-1)^2$ is

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{H-H_n}{n} - \frac{H-1}{1} + \frac{(\gamma-H)(H-1)}{1!} + \frac{(\gamma-H)^2}{2!} \\
&= - \sum_{n=1}^{\infty} \frac{H_n}{n} + H^2 - \frac{H-1}{1} + \frac{\gamma(H-1)}{1!} - \frac{H(H-1)}{1!} + \frac{(\gamma-H)^2}{2!} \\
&= 1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n}
\end{aligned}$$

The coefficient of $(z-1)^3$ is as follows.

$$\sum_{n=1}^{\infty} \frac{(H_n-1)(H-H_{n+1})}{n+1} + \frac{\gamma-H}{1!} \left(\sum_{n=1}^{\infty} \frac{H-H_n}{n} - \frac{H-1}{1} \right) + \frac{(\gamma-H)^2(H-1)}{2!} + \frac{(\gamma-H)^3}{3!}$$

Here,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H-H_n}{n} &= H^2 - \sum_{n=1}^{\infty} \frac{H_n}{n} \\
\sum_{n=1}^{\infty} \frac{(H_n-1)(H-H_{n+1})}{n+1} &= \sum_{n=1}^{\infty} \frac{H_n(H-H_{n+1})}{n+1} - \sum_{n=1}^{\infty} \frac{H}{n+1} + \sum_{n=1}^{\infty} \frac{H_{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty} \frac{H_n(H-H_{n+1})}{n+1} - H \left(\sum_{n=1}^{\infty} \frac{1}{n} - 1 \right) + \sum_{n=1}^{\infty} \frac{H_n}{n} - \frac{H_1}{1} \\
&= \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} + \frac{1}{n} \right) - H^2 + H - 1
\end{aligned}$$

Then,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(H_n-1)(H-H_{n+1})}{n+1} + \frac{\gamma-H}{1!} \left(\sum_{n=1}^{\infty} \frac{H-H_n}{n} - \frac{H-1}{1} \right) + \frac{(\gamma-H)^2(H-1)}{2!} + \frac{(\gamma-H)^3}{3!} \\
&= \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} + \frac{1}{n} \right) - H^2 + H - 1 + \frac{\gamma-H}{1!} \left(H^2 - \sum_{n=1}^{\infty} \frac{H_n}{n} - \frac{H-1}{1} \right) \\
&\quad + \frac{(\gamma-H)^2(H-1)}{2!} + \frac{(\gamma-H)^3}{3!} \\
&= \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} + \frac{1-r+H}{n} \right) - H^2 + H - 1 + \frac{\gamma-H}{1!} \left(H^2 - \frac{H-1}{1} \right) \\
&\quad + \frac{(\gamma-H)^2(H-1)}{2!} + \frac{(\gamma-H)^3}{3!} \\
&= - \left(1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} - \frac{\gamma^3}{3!} \right) - \frac{H^2(1-\gamma)}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} + \frac{1-\gamma+H}{n} \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{\Gamma(1+z)} &= 1 - \left(1 - \frac{\gamma}{1!} \right) (z-1) + \left(1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} + \frac{H^2}{2!} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) (z-1)^2 \\
&\quad - \left(1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} - \frac{\gamma^3}{3!} + \frac{H^2(1-\gamma)}{2} + \frac{2H^3}{3} - \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} + \frac{1-\gamma+H}{n} \right) \right) (z-1)^3 + \dots
\end{aligned}$$

Verification

The ordinary Taylor expansion of the left side of (4.5) is as follows. Where, $\psi_n(z)$ is a polygamma function.

$$\begin{aligned}
\frac{1}{\Gamma(1+z)} &= 1 - \left(1 - \frac{\gamma}{1!} \right) (z-1) + \left(1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} - \frac{\pi^2}{12} \right) (z-1)^2 \\
&\quad - \left(1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} - \frac{\gamma^3}{3!} - \frac{(1-\gamma)\pi^2}{12} + \frac{\psi_2(1)}{6} \right) (z-1)^3 + \dots \\
&= 1 - 0.4227843(z-1) - 0.2330937(z-1)^2 + 0.1910911(z-1)^3 + \dots
\end{aligned}$$

Therefore, the following equations must hold.

$$1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} + \frac{H^2}{2} - \sum_{n=1}^{\infty} \frac{H_n}{n} = 1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} - \frac{\pi^2}{12} = -0.233093\dots$$

$$\begin{aligned}
& - \left(1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} - \frac{\gamma^3}{3!} \right) - \frac{H^2(1-\gamma)}{2} - \frac{2H^3}{3} + \sum_{n=1}^{\infty} H_n \left(\frac{H-H_{n+1}}{n+1} + \frac{1-\gamma+H}{n} \right) \\
& = - \left(1 - \frac{\gamma}{1!} + \frac{\gamma^2}{2!} - \frac{\gamma^3}{3!} \right) + \frac{(1-\gamma)\pi^2}{12} - \frac{\psi_2(1)}{6} = 0.191091\dots
\end{aligned}$$

Calculating these with the formula manipulation software Mathematica, we obtained the following result.

The coefficients of each term are almost consistent with the above values.

`H[m_] := HarmonicNumber[m] γ := EulerGamma`

`c2[m_] := 1 - $\frac{\gamma}{1!}$ + $\frac{\gamma^2}{2!}$ + $\frac{H[m]^2}{2}$ - $\sum_{n=1}^m \frac{H[n]}{n}$`

`N[c2[20000]] Limit[c2[m], m \rightarrow ∞]`
`-0.233069 1 - EulerGamma - $\frac{\text{EulerGamma}^2}{2}$ - $\frac{\pi^2}{12}$`

`c3[m_] := - $\left(1 - \frac{\gamma^1}{1!} + \frac{\gamma^2}{2!} - \frac{\gamma^3}{3!} \right)$ - $\frac{H[m]^2(1-\gamma)}{2}$ - $\frac{2H[m]^3}{3}$`
`+ $\sum_{n=1}^m H[n] \left(\frac{H[m] - H[n+1]}{n+1} + \frac{1-\gamma+H[m]}{n} \right)$`

`N[c3[20000]]`
`0.191081`

11.5 Taylor Expansion (Part 2)

Odd Harmonic Number & Odd Harmonic Series (definition)

We define odd harmonic number h_n and odd harmonic series h as follows.

$$h_n = 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} = H_{2n-1} - \frac{H_{n-1}}{2} \quad (5.h_n)$$

$$h = h_\infty = \frac{H}{2} + \log 2 \quad (5.h)$$

Formula 11.5.1

When γ is Euler-Mascheroni constant, $\Gamma(z)$ is the Gamma function, h_n is an odd harmonic number and h is an odd harmonic series, the following expressions hold.

$$\frac{\sqrt{\pi}}{\Gamma(z/2)} = 1 + c_1(z-1)^1 + c_2(z-1)^2 + c_3(z-1)^3 + \dots \quad (5.3.)$$

$$\frac{\sqrt{\pi}}{\Gamma(1-z/2)} = 1 - c_1(z-1)^1 + c_2(z-1)^2 - c_3(z-1)^3 + \dots \quad (5.3_+)$$

Where,

$$c_1 = \frac{\gamma}{2} + \log 2$$

$$c_2 = \frac{1}{2!} \left(\frac{\gamma}{2} + \log 2 \right)^2 + \frac{h^2}{2} - \sum_{n=1}^{\infty} \frac{h_n}{2n-1}$$

$$c_3 = \frac{1}{3!} \left(\frac{\gamma}{2} + \log 2 \right)^3 + \frac{h^2}{2} \left(\frac{\gamma}{2} + \log 2 \right) - \frac{2h^3}{3} + \sum_{n=1}^{\infty} h_n \left(\frac{h-h_{n+1}}{2n+1} - \frac{1}{2n-1} \left(\frac{\gamma}{2} + \log 2 - h \right) \right)$$

⋮

Proof

From Formula 11.3.1 (3.3.)

$$\frac{\sqrt{\pi}}{\Gamma(z/2)} = e^{\left(\frac{\gamma}{2} + \log 2\right)(z-1)} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n-1} \right) e^{-\frac{z-1}{2n-1}} \quad (5.3.)$$

At first, the compensation term is

$$\prod_{n=1}^{\infty} e^{-\frac{z-1}{2n-1}} = e^{-(z-1) \sum_{n=1}^{\infty} \frac{1}{2n-1}} = e^{-h(z-1)}$$

Then,

$$e^{\left(\frac{\gamma}{2} + \log 2\right)(z-1)} \prod_{n=1}^{\infty} e^{-\frac{z-1}{2n-1}} = e^{\left(\frac{\gamma}{2} + \log 2\right)(z-1)} e^{-h(z-1)} = e^{\left(\frac{\gamma}{2} + \log 2 - h\right)(z-1)}$$

Expanding this,

$$e^{\left(\frac{\gamma}{2} + \log 2\right)(z-1)} \prod_{n=1}^{\infty} e^{-\frac{z-1}{2n-1}} = 1 + \frac{1}{1!} \left(\frac{\gamma}{2} + \log 2 - h \right) (z-1)^1 + \frac{1}{2!} \left(\frac{\gamma}{2} + \log 2 - h \right)^2 (z-1)^2 + \frac{1}{3!} \left(\frac{\gamma}{2} + \log 2 - h \right)^3 (z-1)^3 + \dots \quad (5.1e)$$

Let the body be

$$\begin{aligned} \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n-1} \right) &= \left(1 + \frac{z-1}{1} \right) \left(1 + \frac{z-1}{3} \right) \left(1 + \frac{z-1}{5} \right) \left(1 + \frac{z-1}{7} \right) \dots \\ &= 1 + a_1(z-1)^1 + a_2(z-1)^2 + a_3(z-1)^3 + \dots \end{aligned}$$

Then, from Vieta's formulas,

$$\begin{aligned} a_1 &= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \dots = h \\ a_2 &= \frac{1}{1} \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots \right) + \frac{1}{3} \left(\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \dots \right) + \frac{1}{5} \left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \dots \right) + \dots \\ &= \frac{h-h_1}{1} + \frac{h-h_2}{3} + \frac{h-h_3}{5} + \dots = \sum_{n=1}^{\infty} \frac{h-h_n}{2n-1} \\ a_3 &= \frac{1}{1} \frac{1}{3} (h-h_2) + \frac{1}{1} \frac{1}{5} (h-h_3) + \frac{1}{1} \frac{1}{7} (h-h_4) + \dots \\ &\quad + \frac{1}{3} \frac{1}{5} (h-h_3) + \frac{1}{3} \frac{1}{7} (h-h_4) + \frac{1}{3} \frac{1}{9} (h-h_5) + \dots \\ &\quad + \frac{1}{5} \frac{1}{7} (h-h_4) + \frac{1}{5} \frac{1}{9} (h-h_5) + \frac{1}{5} \frac{1}{11} (h-h_6) + \dots \\ &\quad + \\ &\quad \vdots \end{aligned}$$

Rearranging this along the diagonal line,

$$\begin{aligned} a_3 &= \left(\frac{1}{1} \right) \frac{1}{3} (h-h_2) + \left(\frac{1}{1} + \frac{1}{3} \right) \frac{1}{5} (h-h_3) + \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} \right) \frac{1}{7} (h-h_4) + \dots \\ &= \sum_{n=1}^{\infty} \frac{h_n(h-h_{n+1})}{2n+1} \end{aligned}$$

Thus,

$$\prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n-1} \right) = 1 + (z-1)h + (z-1)^2 \sum_{n=1}^{\infty} \frac{h-h_n}{2n-1} + (z-1)^3 \sum_{n=1}^{\infty} \frac{h_n(h-h_{n+1})}{2n+1} + \dots \quad (5.1p)$$

The right side of (5.3) is a product of (5.1e) and (5.1p). That is

$$\begin{aligned} e^{\left(\frac{\gamma}{2} + \log 2 \right) (z-1)} \prod_{n=1}^{\infty} e^{-\frac{z-1}{2n-1}} &= 1 + \frac{1}{1!} \left(\frac{\gamma}{2} + \log 2 - h \right) (z-1)^1 + \frac{1}{2!} \left(\frac{\gamma}{2} + \log 2 - h \right)^2 (z-1)^2 \\ &\quad + \frac{1}{3!} \left(\frac{\gamma}{2} + \log 2 - h \right)^3 (z-1)^3 + \dots \\ \times \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{2n-1} \right) &= 1 + (z-1)h + (z-1)^2 \sum_{n=1}^{\infty} \frac{h-h_n}{2n-1} + (z-1)^3 \sum_{n=1}^{\infty} \frac{h_n(h-h_{n+1})}{2n+1} + \dots \end{aligned}$$

Taking the Cauchy product,

The coefficient of $(z-1)^1$ is

$$\frac{1}{1!} \left(\frac{\gamma}{2} + \log 2 - h \right) + h = \frac{\gamma}{2} + \log 2$$

The coefficient of $(z-1)^2$ is

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{h-h_n}{2n-1} + \frac{h}{1!} \left(\frac{\gamma}{2} + \log 2 - h \right) + \frac{1}{2!} \left(\frac{\gamma}{2} + \log 2 - h \right)^2 \\ &= - \sum_{n=1}^{\infty} \frac{h_n}{2n-1} + h^2 + \left(\frac{\gamma}{2} + \log 2 \right) h - h^2 + \frac{1}{2} \left(\frac{\gamma}{2} + \log 2 \right)^2 - \left(\frac{\gamma}{2} + \log 2 \right) h + \frac{h^2}{2} \\ &= \frac{1}{2!} \left(\frac{\gamma}{2} + \log 2 \right)^2 + \frac{h^2}{2} - \sum_{n=1}^{\infty} \frac{h_n}{2n-1} \end{aligned}$$

The coefficient of $(z-1)^3$ is as follows in a similar way.

$$\frac{1}{3!} \left(\frac{\gamma}{2} + \log 2 \right)^3 + \frac{h^2}{2} \left(\frac{\gamma}{2} + \log 2 \right) - \frac{2h^3}{3} + \sum_{n=1}^{\infty} h_n \left(\frac{h-h_{n+1}}{2n+1} - \frac{1}{2n-1} \left(\frac{\gamma}{2} + \log 2 - h \right) \right)$$

Then,

$$\begin{aligned} \frac{\sqrt{\pi}}{\Gamma(z/2)} &= 1 + \left(\frac{\gamma}{2} + \log 2 \right) (z-1) + \left\{ \frac{1}{2!} \left(\frac{\gamma}{2} + \log 2 \right)^2 + \frac{h^2}{2} - \sum_{n=1}^{\infty} \frac{h_n}{2n-1} \right\} (z-1)^2 \\ &+ \left\{ \frac{1}{3!} \left(\frac{\gamma}{2} + \log 2 \right)^3 + \frac{h^2}{2} \left(\frac{\gamma}{2} + \log 2 \right) - \frac{2h^3}{3} \right. \\ &\quad \left. + \sum_{n=1}^{\infty} h_n \left(\frac{h-h_{n+1}}{2n+1} - \frac{1}{2n-1} \left(\frac{\gamma}{2} + \log 2 - h \right) \right) \right\} (z-1)^2 + \dots \end{aligned}$$

And, changing this to alternating series, we obtain (5.3₊).

Verification

The ordinary Taylor expansion of the left side of (5.3) is as follows. Where, $\psi_n(z)$ is a polygamma function.

$$\begin{aligned} \frac{\sqrt{\pi}}{\Gamma(z/2)} &= 1 - \frac{1}{2} \psi_0 \left(\frac{1}{2} \right) (z-1) + \frac{1}{2!} \left(\frac{1}{4} \psi_0^2 \left(\frac{1}{2} \right) - \frac{\pi^2}{8} \right) (z-1)^2 \\ &+ \frac{1}{3!} \left(\frac{3}{16} \pi^2 \psi_0 \left(\frac{1}{2} \right) - \frac{1}{8} \psi_0^3 \left(\frac{1}{2} \right) - \frac{1}{8} \psi_2 \left(\frac{1}{2} \right) \right) (z-1)^3 + \dots \\ &= 1 + 0.981755(z-1) - 0.134928(z-1)^2 - 0.097286(z-1)^3 + \dots \end{aligned}$$

Therefore, the following equations must hold.

$$\begin{aligned} \frac{\gamma}{2} + \log 2 &= -\frac{1}{2} \psi_0 \left(\frac{1}{2} \right) = 0.981755 \\ \frac{1}{2!} \left(\frac{\gamma}{2} + \log 2 \right)^2 + \frac{h^2}{2} - \sum_{n=1}^{\infty} \frac{h_n}{2n-1} &= \frac{1}{2!} \left(\frac{1}{4} \psi_0^2 \left(\frac{1}{2} \right) - \frac{\pi^2}{8} \right) = -0.134928 \dots \\ \frac{1}{3!} \left(\frac{\gamma}{2} + \log 2 \right)^3 + \frac{h^2}{2} \left(\frac{\gamma}{2} + \log 2 \right) - \frac{2h^3}{3} + \sum_{n=1}^{\infty} h_n \left(\frac{h-h_{n+1}}{2n+1} - \frac{1}{2n-1} \left(\frac{\gamma}{2} + \log 2 - h \right) \right) &= \frac{1}{3!} \left(\frac{3}{16} \pi^2 \psi_0 \left(\frac{1}{2} \right) - \frac{1}{8} \psi_0^3 \left(\frac{1}{2} \right) - \frac{1}{8} \psi_2 \left(\frac{1}{2} \right) \right) = -0.0972863 \dots \end{aligned}$$

Calculating these with the formula manipulation software Mathematica, we obtained the following result.

The coefficients of each term are almost consistent with the above values.

$$\mathbf{H[m_]} := \mathbf{HarmonicNumber[m]} \quad \mathbf{\gamma} := \mathbf{EulerGamma}$$

$$c2[m_] := \frac{1}{2!} \left(\frac{\gamma}{2} + \text{Log}[2] \right)^2 + \frac{h[m]^2}{2} - \sum_{n=1}^m \frac{h[n]}{2n-1}$$

$N[c2[20000]]$

- 0.134923

$$c3[m_] := \frac{1}{3!} \left(\frac{\gamma}{2} + \text{Log}[2] \right)^3 + \frac{h[m]^2}{2} \left(\frac{\gamma}{2} + \text{Log}[2] \right) - \frac{2h[m]^3}{3} \\ + \sum_{n=1}^m h[n] \left(\frac{h[m] - h[n+1]}{2n+1} - \frac{1}{2n-1} \left(\frac{\gamma}{2} + \text{Log}[2] - h[m] \right) \right)$$

$N[c3[20000]]$

- 0.0972802

Also, the following special values were obtained.

Special Values

$$h^2 - 2! \sum_{n=1}^{\infty} \frac{h_n}{2n-1} = -\frac{\pi^2}{8} = -\lambda(2) \quad : \text{Dirichlet Lambda} \quad (5.4)$$

$$4h^3 - 3! \sum_{n=1}^{\infty} h_n \left(\frac{h-h_{n+1}}{2n+1} + \frac{h}{2n-1} \right) = \frac{1}{2^3} \psi_2 \left(\frac{1}{2} \right) = -2.10359958 \dots \quad (5.5)$$

If easier (5.4) is written down, it is as follows.

$$\left(1 + \frac{1}{3} + \frac{1}{5} + \dots \right)^2 - 2! \left\{ \frac{1}{1} \left(1 \right) + \frac{1}{3} \left(1 + \frac{1}{3} \right) + \frac{1}{5} \left(1 + \frac{1}{3} + \frac{1}{5} \right) + \dots \right\} = -\lambda(2)$$

Formula 11.5.2

When γ is Euler-Mascheroni constant, $\Gamma(z)$ is the Gamma function, h_n is an odd harmonic number and h is an odd harmonic series, the following expression holds.

$$\frac{\sqrt{\pi}}{2\Gamma(1+z/2)} = 1 - c_1(z-1)^1 + c_2(z-1)^2 - c_3(z-1)^3 + \dots \quad (5.6)$$

Where,

$$c_1 = 1 - \frac{\gamma}{2} - \log 2$$

$$c_2 = 1 - \frac{1}{1!} \left(\frac{\gamma}{2} + \log 2 \right) + \frac{1}{2!} \left(\frac{\gamma}{2} + \log 2 \right)^2 + \frac{h^2}{2} - \sum_{n=1}^{\infty} \frac{h_n}{2n-1}$$

$$c_3 = 1 - \frac{1}{1!} \left(\frac{\gamma}{2} + \log 2 \right) + \frac{1}{2!} \left(\frac{\gamma}{2} + \log 2 \right)^2 - \frac{1}{3!} \left(\frac{\gamma}{2} + \log 2 \right)^3 + \frac{h^2}{2} \left(1 - \frac{\gamma}{2} - \log 2 \right) \\ + \frac{2h^3}{3} - \sum_{n=1}^{\infty} h_n \left(\frac{h-h_{n+1}}{2n+1} + \frac{1}{2n-1} \left(1 - \frac{\gamma}{2} - \log 2 + h \right) \right)$$

:

Proof

In a similar way as the proof of Formula 11.4.2, we obtain the desired expression.

Verification

If the left side of (5.6.) is expanded to Taylor series with the formula manipulation software Mathematica, it is as follows.

$$\frac{\sqrt{\pi}}{2\Gamma(1+z/2)} = 1 - 0.01824(z-1)^1 - 0.11668(z-1)^2 + 0.01939(z-1)^3 + \dots$$

On the other hand, calculating c_1, c_2, c_3 according to the formula, we obtained the following results.

Both values are approximately equal.

```

H[n_] := HarmonicNumber[n]      h[n_] := H[2 n - 1] -  $\frac{H[n - 1]}{2}$ 
 $\gamma$  := EulerGamma

c1 := 1 -  $\frac{\gamma}{2}$  - Log[2]
N[-c1]
-0.018245

c2[m_] := 1 -  $\frac{1}{1!} \left( \frac{\gamma}{2} + \text{Log}[2] \right) + \frac{1}{2!} \left( \frac{\gamma}{2} + \text{Log}[2] \right)^2 + \frac{h[m]^2}{2} - \sum_{n=1}^m \frac{h[n]}{2 n - 1}$ 
N[c2[30 000]]
-0.116678

c3[m_] := 1 -  $\frac{1}{1!} \left( \frac{\gamma}{2} + \text{Log}[2] \right) + \frac{1}{2!} \left( \frac{\gamma}{2} + \text{Log}[2] \right)^2$ 
-  $\frac{1}{3!} \left( \frac{\gamma}{2} + \text{Log}[2] \right)^3 + \frac{h[m]^2}{2} \left( 1 - \frac{\gamma}{2} - \text{Log}[2] \right) + \frac{2 h[m]^3}{3}$ 
-  $\sum_{n=1}^m h[n] \left( \frac{h[m] - h[n + 1]}{2 n + 1} + \frac{1}{2 n - 1} \left( 1 - \frac{\gamma}{2} - \text{Log}[2] + h[m] \right) \right)$ 
N[-c3[5000]]
0.019397

```

11.6 Euler-Mascheroni Constant Function

Until the previous section, treating the divergent product and divergent series as numbers or functions, we performed a series expansion of the reciprocal of the gamma function. Although we succeeded in deriving the power series from the infinite product, we could not get the general formula.

However, there are not few merits of treating the divergent product and divergent series as numbers or functions. In this section, I will present one example.

Formula 11.6.1

When γ is Euler-Mascheroni constant and $\Gamma(z)$ is the Gamma function, the following expressions hold.

$$\gamma = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z} \log \left(1 + \frac{z}{n} \right) \right\} - \frac{\log \Gamma(1+z)}{z} \equiv g(z) \quad \text{Re}(z) > -1 \quad (6.1)$$

$$\gamma = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \log \left(1 + \frac{1}{n} \right) \right\} \quad (6.1')$$

$$\lim_{z \rightarrow 0} \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z} \log \left(1 + \frac{z}{n} \right) \right\} = 0 \quad (6.2)$$

$$\lim_{z \rightarrow 0} \frac{\log \Gamma(1+z)}{z} = -\gamma \quad (6.3)$$

Proof

Formula 11.1.1 (1.1.) was as follows.

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-\frac{z}{n}} = \frac{e^{-\gamma z}}{\Gamma(1+z)} \quad (1.1-)$$

The left side can be transformed as follows.

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-\frac{z}{n}} = e^{\log \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right)} \cdot e^{-\sum_{n=1}^{\infty} \frac{z}{n}} = e^{\log \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) - \sum_{n=1}^{\infty} \frac{z}{n}}$$

The right side can also be transformed as follows.

$$\frac{e^{-\gamma z}}{\Gamma(1+z)} = e^{-\log \Gamma(1+z)} e^{-\gamma z} = e^{-\log \Gamma(1+z) - \gamma z}$$

Then,

$$e^{\log \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) - \sum_{n=1}^{\infty} \frac{z}{n}} = e^{-\log \Gamma(1+z) - \gamma z}$$

Taking the logarithm of both sides and inverting the signs,

$$\gamma z + \log \Gamma(1+z) = \sum_{n=1}^{\infty} \frac{z}{n} - \log \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right)$$

From this,

$$\gamma = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z} \log \left(1 + \frac{z}{n} \right) \right\} - \frac{\log \Gamma(1+z)}{z} \quad \text{Re}(z) > -1 \quad (6.1)$$

Especially, when $z=1$,

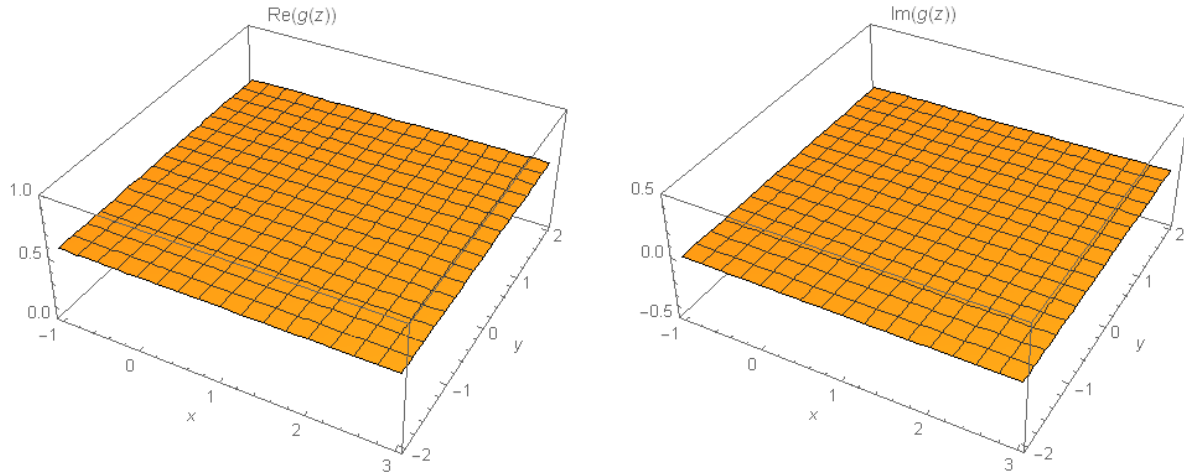
$$\gamma = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \log \left(1 + \frac{1}{n} \right) \right\} \quad (6.1')$$

Next, replacing z with Δz in the left side of (6.3),

$$\lim_{\Delta z \rightarrow 0} \frac{\log \Gamma(1 + \Delta z)}{\Delta z} = \frac{d}{dz} \log \Gamma(z) \Big|_{z=1} = \frac{\Gamma'(1)}{\Gamma(1)} = \psi(1) = -\gamma$$

Then, (6.3) holds. And (6.2) is derived from this and (6.1).

3D figure of (6.1) is as follows. The left figure is the real part and the right figure is the imaginary part.

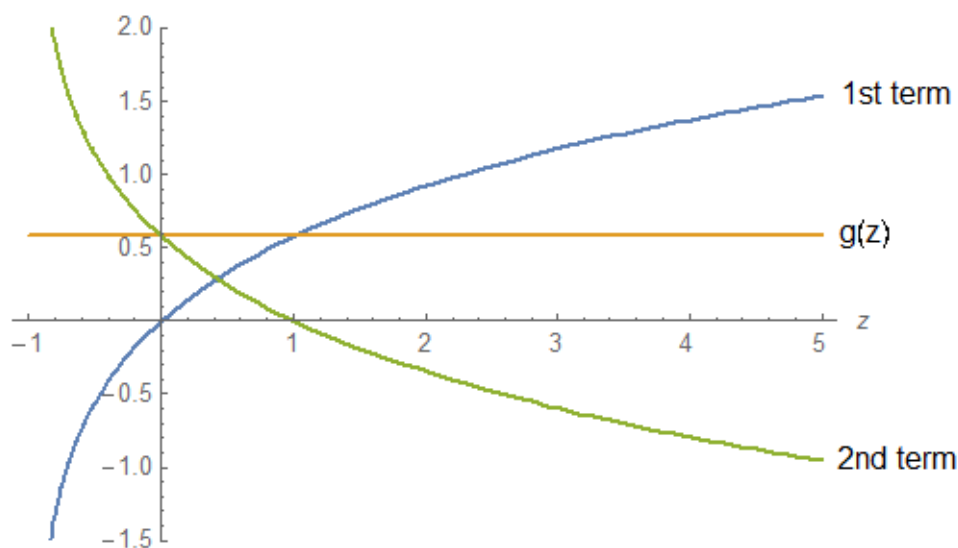


The domain is $Re(z) > -1$. Anywhere on this half plane, the real part is $0.57721566 \dots$ and the imaginary part is 0 . That is, this function $g(z)$ is a constant function which gives the Euler-Mascheroni constant γ .

In order to obtain γ , any value on this half plane may be chosen, and (6.1') is one of them. However, considering the convergence speed, (6.1') is not a good choice. The following is a study of the value of z and the number of terms necessary to obtain 5 significant digits.

```
N[{g[2, 70000], g[1, 35000], g[0.001, 34], g[0.0001, 3]}]
{0.577201, 0.577201, 0.577201, 0.577201}
```

According to this, in order to obtain effective 5 digits, 35,000 terms are required when $z=1$, and 3 terms are sufficient when $z=0.0001$. The reasons are (6.2) and (6.3). When this is illustrated clearly, it's as follows.



The first term of the function $g(z)$ is blue and the second term is yellow green. These are the line symmetry around $g = \gamma/2$. Therefore, the function value of $g(z)$ is γ regardless of the value of z . So, we call the function $g(z)$ **Euler-Mascheroni Constant Function**.

Although $g(z) = \gamma$ for any z s.t. $Re(z) > -1$, especially when $z = 0$, the first term is 0 and the second term is γ . That is, in the vicinity of 0, the role of the first term is small. That is why convergence of the first term is fast in the vicinity of 0.

By reference, giving $z = 0.000018$ to (6.3), we obtain -0.577201 with the same precision as the above.

Note

(6.1') can also be obtained directly from Euler's definition.

$$\begin{aligned} \gamma &= \lim_{m \rightarrow \infty} \left(\sum_{n=1}^m \frac{1}{n} - \log m \right) = \lim_{m \rightarrow \infty} \left\{ \sum_{n=1}^m \frac{1}{n} - \log \left(\frac{2}{1} \frac{3}{2} \frac{4}{3} \cdots \frac{m}{m-1} \right) \right\} \\ &= \lim_{m \rightarrow \infty} \left\{ \sum_{n=1}^m \frac{1}{n} - \sum_{n=1}^{m-1} \log \left(1 + \frac{1}{n} \right) \right\} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \log \left(1 + \frac{1}{n} \right) \right\} \end{aligned}$$

2016.12.20

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Alien's Mathematics