05 Split of Dirichlet Series

5.1 Basic Split of Dirichlet Series

In this section, we will consider ways to split the Dirichlet series into two or more series.

Explicit series

As first, the series has to be explicitly expressed using Σ. Here, explicit means that the value of each term of the series is represented by a polynomial of the term number.
For examples,

$$\zeta(3) = \sum_{r=1}^{\infty}\frac{1}{r^3} \;,\quad \beta(2) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^2} \;,\quad s(4) = \sum_{r=1}^{\infty}\left(\frac{1}{r^4} - \frac{1}{r^3}\right)$$

On the contrary, the following series is not explicit series.

$$S(2) = \sum_{\rho:prime}\frac{1}{\rho^2}$$

Infinity of Split

When the series is absolute convergence, there are countless ways to split the series, even if it is only two-splits. For example,

$$\zeta(2) = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^2}+\frac{1}{10^2}+\cdots$$

Split 1

$$A_1 = 1+\frac{1}{3^2}+\frac{1}{6^2}+\frac{1}{10^2}+\frac{1}{15^2}+\frac{1}{21^2}+\cdots$$

$$A_2 = \frac{1}{2^2}+\left(\frac{1}{4^2}+\frac{1}{5^2}\right)+\left(\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^2}\right)+\cdots$$

Split 2

$$A_1 = 1+\left(\frac{1}{4^2}+\frac{1}{5^2}\right)+\left(\frac{1}{9^2}+\frac{1}{10^2}+\frac{1}{11^2}\right)+\left(\frac{1}{16^2}+\frac{1}{17^2}+\frac{1}{18^2}+\frac{1}{19^2}\right)+\cdots$$

$$A_2 = \left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}\right)+\left(\frac{1}{12^2}+\frac{1}{13^2}+\frac{1}{14^2}+\frac{1}{15^2}\right)+\cdots$$

Split 3

$$A_1 = 1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{8^2}+\frac{1}{16^2}+\frac{1}{32^2}+\frac{1}{64^2}+\cdots$$

$$A_2 = \frac{1}{3^2}+\left(\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}\right)+\left(\frac{1}{9^2}+\frac{1}{10^2}+\cdots+\frac{1}{15^2}\right)+\cdots$$

etc.

Although each of these split series is converged, most formulas of the sum are not known. So, we present a split method that the sum of each split series can be obtained as a formula.

Definition 5.1.0 (Basic Split)

When we can create positive or negative term series $A_k\ (k=1,2,\cdots m)$ by choosing the terms from the k th term $(k=1,2,\cdots m)$ with $(m-1)$ skipping in a series, We call this a **basic m-split** (or simply **m-split**).

Series in which basic split is impossible

According to **Definition 5.1.0** , in a series where the change in the sign of the terms is not cyclic, the basic split is impossible. For example, in the following series S , the basic split is impossible.

$$S = 1-\frac{1}{2^n}+\left(\frac{1}{3^n}+\frac{1}{4^n}\right)-\left(\frac{1}{5^n}+\frac{1}{6^{2n}}+\frac{1}{7^n}\right)+\left(\frac{1}{8^n}+\cdots+\frac{1}{11^n}\right)-\left(\frac{1}{12^n}+\cdots+\frac{1}{16^n}\right)+\cdots$$

Series in which basic split is possible

According to **Definition 5.1.0** ,

(1) In a positive or negative term series, **two or more basic splits are uniquely possible**.

(2) In a series where the change in the sign of the terms is cyclic, the basic split is possible. However, split with any number is impossible . For example, Dirichlet Eta series

$$\eta(n) = 1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\frac{1}{7^n}-\frac{1}{8^n}+\frac{1}{9^n}-\frac{1}{10^n}+\frac{1}{11^n}-\frac{1}{12^n}+-\cdots$$

is **uniquely split** respectively as follows.

2-split

$$A_1 = 1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{9^n}+\cdots$$

$$A_2 = -\left(\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{6^n}+\frac{1}{8^n}+\frac{1}{10^n}+\cdots\right)$$

3-split

Impossible

4-split

$$A_1 = 1+\frac{1}{5^n}+\frac{1}{9^n}+\frac{1}{13^n}+\frac{1}{17^n}+\cdots$$

$$A_2 = -\left(\frac{1}{2^n}+\frac{1}{6^n}+\frac{1}{10^n}+\frac{1}{14^n}+\frac{1}{18^n}+\cdots\right)$$

$$A_3 = \frac{1}{3^n}+\frac{1}{7^n}+\frac{1}{11^n}+\frac{1}{15^n}+\frac{1}{19^n}+\cdots$$

$$A_4 = -\left(\frac{1}{4^n}+\frac{1}{8^n}+\frac{1}{12^n}+\frac{1}{16^n}+\frac{1}{20^n}+\cdots\right)$$

5-split

Impossible

⋮

If Dirichlet series is split in this way, the sums of each split series are obtained as formula. Below, we show that.

Formula 5.1.1

Let $\zeta(n,z)$ be Hurwitz zeta function, $\psi_n(z)$ be polygamma function, and Riemann zeta series $\zeta(n)$ and the m -split series A_k are as follows respectively.

$$\zeta(n) = \sum_{r=1}^{\infty} \frac{1}{r^n}$$

$$A_k = \sum_{r=0}^{\infty} \frac{1}{(mr+k)^n} \qquad k=1,2, \cdots, m$$

Then, the following expressions hold for $k=1,2, \cdots, m$.

$$\sum_{r=0}^{\infty} \frac{1}{(mr+k)^n} = \frac{1}{m^n}\zeta\left(n, \frac{k}{m}\right) \tag{1.1}$$

$$= \frac{(-1)^n}{m^n(n-1)!}\psi_{n-1}\left(\frac{k}{m}\right) \tag{1.1'}$$

$$= \frac{(-1)^n}{m^n(n-1)!}\int_0^{\infty} \frac{t^{n-1}e^{-\frac{k}{m}t}}{1-e^{-t}}dt \tag{1.1''}$$

Proof

$$\sum_{r=0}^{\infty}\left(\frac{1}{mr+k}\right)^n = \frac{1}{m^n}\sum_{r=0}^{\infty}\frac{1}{(r+k/m)^n} = \frac{1}{m^n}\zeta\left(n, \frac{k}{m}\right) \tag{1.1}$$

There is the following relation between Hurwitz zeta function $\zeta(n,z)$ and polygamma function $\psi_n(z)$.

$$\psi_n(z) = (-1)^{n+1} n!\,\zeta(n+1,z)$$

From this,

$$\zeta\left(n, \frac{k}{m}\right) = \frac{(-1)^n}{(n-1)!}\psi_{n-1}\left(\frac{k}{m}\right)$$

Substituting this for (1.1) ,

$$\sum_{r=0}^{\infty} \frac{1}{(mr+k)^n} = \frac{(-1)^n}{m^n(n-1)!}\psi_{n-1}\left(\frac{k}{m}\right)$$

From *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Milton Abramowitz and Irene A. Stegun)

$$\psi_n(z) = (-1)^{n+1}\int_0^{\infty} \frac{t^n e^{-zt}}{1-e^{-t}}dt$$

So,

$$\psi_{n-1}\left(\frac{k}{m}\right) = (-1)^n\int_0^{\infty} \frac{t^{n-1}e^{-\frac{k}{m}t}}{1-e^{-t}}dt$$

Substituting this for (1.1') , weobtain (1.1") .

Formula 5.1.2

Let $\zeta(n,z)$ be Hurwitz zeta function, $\psi_n(z)$ be polygamma function, and Dirichlet lambda series $\lambda(n)$ and the m -split series A_k are as follows respectively.

$$\lambda(n) = \sum_{r=0}^{\infty} \frac{1}{(2r+1)^n}$$

$$A_k = \sum_{r=0}^{\infty} \frac{1}{(2mr+2k-1)^n} \qquad k=1,2, \cdots ,m$$

Then, the following expressions hold for $k=1,2, \cdots , m$.

$$\sum_{r=0}^{\infty} \frac{1}{(2mr+2k-1)^n} = \frac{1}{(2m)^n} \zeta\left(n, \frac{2k-1}{2m}\right) \tag{1.2}$$

$$= \frac{(-1)^n}{(2m)^n (n-1)!} \psi_{n-1}\left(\frac{2k-1}{2m}\right) \tag{1.2'}$$

$$= \frac{1}{(2m)^n (n-1)!} \int_0^{\infty} \frac{t^{n-1} e^{-\frac{2k-1}{2m}t}}{1-e^{-t}} dt \tag{1.2''}$$

Proof

By the similar way to the previous proof.

Note

The advantage of **Definition 5.1.0** is that the sum of each split series is expressed by the Hurwitz zeta function. This is because the definition of the basic split is consistent with the definition of the Hurwitz zeta.

5.2 Two-split of Dirichlet Series

5.2.1 Two-split of $\zeta(4)$

Riemann zeta series $\zeta(4)$ is

$$\zeta(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \frac{1}{9^4} + \cdots$$

The two-split is

$$A_1 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(2r+1)^4}$$

$$A_2 = \frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(2r+2)^4}$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$\text{A1}[m\_] := \sum_{r=0}^{m} \frac{1}{(2r+1)^4} \qquad \text{A2}[m\_] := \sum_{r=0}^{m} \frac{1}{(2r+2)^4}$$

```
N[{A1[10000], A2[10000]}, 10]
{1.014678032, 0.06764520211}
```

```
N[A1[10000] + A2[10000], 10]        N[Zeta[4], 10]
1.082323234                         1.082323234
```

Expression by Hurwitz Zeta function

According to Formula 5.1.1 (1.1), the sums of the series A_1, A_2 are given by the following expressions respectively.

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^4} = \frac{1}{2^4}\zeta\left(4, \frac{1}{2}\right) =: B_1$$

$$\sum_{r=0}^{\infty} \frac{1}{(2r+2)^4} = \frac{1}{2^2}\zeta\left(4, \frac{2}{2}\right) =: B_2$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

```
ζ[n_, z_] := HurwitzZeta[n, z]
```

$$\text{B1} := \frac{1}{2^4}\zeta\left[4, \frac{1}{2}\right] \qquad \text{B2} := \frac{1}{2^4}\zeta\left[4, \frac{2}{2}\right]$$

```
N[{B1, B2}, 10]                     N[B1 + B2, 10]
{1.014678032, 0.06764520211}        1.082323234
```

Expression by $\zeta(4)$

There is the following relation between Hurwitz zeta function $\zeta(n,z)$ and Riemann zeta function $\zeta(z)$.

$$\zeta\left(4, \frac{1}{2}\right) = \left(2^4 - 1\right)\zeta(4) , \qquad \zeta\left(4, \frac{2}{2}\right) = \zeta(4)$$

Then,

$$\sum_{r=0}^{\infty}\frac{1}{(2r+1)^4}=\frac{(2^4-1)\zeta(4)}{2^4}=:C_1$$

$$\sum_{r=0}^{\infty}\frac{1}{(2r+2)^4}=\frac{\zeta(4)}{2^4}=:C_2$$

Clearly $C_1+C_2=\zeta(4)$.

5.2.2 Two-split of $\lambda(4)$

Dirichlet Lambda series $\lambda(4)$ is

$$\lambda(4)=1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{11^4}+\frac{1}{13^4}+\frac{1}{15^4}+\frac{1}{17^4}+\cdots$$

The two-split is

$$A_1=1+\frac{1}{5^4}+\frac{1}{9^4}+\frac{1}{13^4}+\frac{1}{17^4}+\cdots=\sum_{r=0}^{\infty}\frac{1}{(4r+1)^4}$$

$$A_2=\frac{1}{3^4}+\frac{1}{7^4}+\frac{1}{11^4}+\frac{1}{15^4}+\cdots=\sum_{r=0}^{\infty}\frac{1}{(4r+3)^4}$$

$$\lambda(4)=A_1+A_2$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$\text{A1}[m\_]:=\sum_{r=0}^{m}\frac{1}{(4r+1)^4}\qquad \text{A2}[m\_]:=\sum_{r=0}^{m}\frac{1}{(4r+3)^4}$$

```
N[{A1[10000], A2[10000]}, 10]
{1.001811292 , 0.01286673993}

N[A1[10000] + A2[10000], 10]
1.014678032

N[DirichletLambda[4], 10]
1.014678032
```

Expression by polygamma function

According to Formula 5.1.2 (1.2') , the sums of the series A_1 , A_2 are given by the following expressions respectively.

$$\sum_{r=0}^{\infty}\frac{1}{(4r+1)^4}=\frac{1}{4^4\,3!}\psi_3\left(\frac{1}{4}\right)=:B_1$$

$$\sum_{r=0}^{\infty}\frac{1}{(4r+3)^4}=\frac{1}{4^4\,3!}\psi_3\left(\frac{3}{4}\right)=:B_2$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$\psi_{n\_}[z\_]:=\text{PolyGamma}[n,z]$$

$$\text{B1}:=\frac{1}{4^4\,3!}\psi_3\left[\frac{1}{4}\right]\qquad \text{B2}:=\frac{1}{4^4\,3!}\psi_3\left[\frac{3}{4}\right]$$

```
N[{B1, B2}, 10]
{1.001811292 , 0.01286673993}

N[B1 + B2, 10]
1.014678032
```

Expression by π and $\beta(4)$

According to ***MathWorld*** (http://mathworld.wolfram.com/PolygammaFunction.html) ,

$$\psi_3\left(\frac{1}{4}\right) = 8\pi^4 + 768\beta(4) \quad , \quad \psi_3\left(\frac{3}{4}\right) = 8\pi^4 - 768\beta(4)$$

Where, $\beta(z)$ is Dirichlet Beta function. Substituting these for the above ,

$$\sum_{r=0}^{\infty} \frac{1}{(4r+1)^4} = \frac{8\pi^4 + 768\beta(4)}{4^4\, 3!} =: C_1$$

$$\sum_{r=0}^{\infty} \frac{1}{(4r+3)^4} = \frac{8\pi^4 - 768\beta(4)}{4^4\, 3!} =: C_2$$

$$C_1 + C_2 = \frac{\pi^4}{96} = \lambda(4)$$

5.2.3 Two-split of $\beta(4)$

Dirichlet Beta series $\beta(4)$ is

$$\beta(4) = 1 - \frac{1}{3^4} + \frac{1}{5^4} - \frac{1}{7^4} + \frac{1}{9^4} - \frac{1}{11^4} + \frac{1}{13^4} - \frac{1}{15^4} + \frac{1}{17^4} - + \cdots$$

Two-split of this is

$$\beta(4) = A_1 - A_2$$

A_1 , A_2 are the same as **5.2.2** . Then,

$$A_1 = 1.00181129167\cdots = \frac{1}{4^4\, 3!}\psi_3\left(\frac{1}{4}\right) = \frac{8\pi^4 + 768\beta(4)}{4^4\, 3!}$$

$$A_2 = 0.01286673993\cdots = \frac{1}{4^4\, 3!}\psi_3\left(\frac{3}{4}\right) = \frac{8\pi^4 - 768\beta(4)}{4^4\, 3!}$$

$$A_1 - A_2 = 0.9889445517\cdots = \frac{2\times 768\beta(4)}{3!\, 4^4} = \beta(4)$$

5.3 Three-split of Dirichlet Series

5.3.1 Three-split of $\zeta(3)$

Riemann zeta series $\zeta(3)$ is

$$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{9^3} + \cdots$$

The three-split is

$$A_1 = 1 + \frac{1}{4^3} + \frac{1}{7^3} + \frac{1}{10^3} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(3r+1)^3}$$

$$A_2 = \frac{1}{2^3} + \frac{1}{5^3} + \frac{1}{8^3} + \frac{1}{11^3} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(3r+2)^3}$$

$$A_3 = \frac{1}{3^3} + \frac{1}{6^3} + \frac{1}{9^3} + \frac{1}{12^3} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(3r+3)^3}$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$\text{A1}[m\_] := \sum_{r=0}^{m} \frac{1}{(3r+1)^3} \qquad \text{A2}[m\_] := \sum_{r=0}^{m} \frac{1}{(3r+2)^3} \qquad \text{A3}[m\_] := \sum_{r=0}^{m} \frac{1}{(3r+3)^3}$$

```
N[{A1[50000], A2[50000], A3[50000]}, 10]
{1.020780044, 0.1367562327, 0.04452062604}

N[A1[50000] + A2[50000] + A3[50000], 10] 10]
1.202056903

N[Zeta[3], 10]
1.202056903
```

Expression by polygamma function

According to Formula 5.1.1 (1.1') , the sums of the series A_1, A_2, A_3 are given by the following expressions respectively.

$$\sum_{r=0}^{\infty} \frac{1}{(3r+1)^3} = \frac{-1}{3^3\, 2!}\psi_2\left(\frac{1}{3}\right) =: B_1$$

$$\sum_{r=0}^{\infty} \frac{1}{(3r+2)^3} = \frac{-1}{3^3\, 2!}\psi_2\left(\frac{2}{3}\right) =: B_2$$

$$\sum_{r=0}^{\infty} \frac{1}{(3r+3)^3} = \frac{-1}{3^3\, 2!}\psi_2\left(\frac{3}{3}\right) =: B_3$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$\psi_{n\_}[z\_] := \text{PolyGamma}[n, z]$$

$$\text{B1} := \frac{-1}{3^3\, 2!}\psi_2\left[\frac{1}{3}\right] \qquad \text{B2} := \frac{-1}{3^3\, 2!}\psi_2\left[\frac{2}{3}\right] \qquad \text{B3} := \frac{-1}{3^3\, 2!}\psi_2\left[\frac{3}{3}\right]$$

```
N[{B1, B2, B3}, 10]
{1.020780044, 0.1367562327, 0.04452062604}

N[B1 + B2 + B3, 10]
1.202056903
```

5.3.2 Three-split of $\lambda(3)$

Dirichlet lambda series $\lambda(3)$ and the three-split are as follows.

$$\lambda(3) = 1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{15^3} + \frac{1}{17^3} + \frac{1}{19^3} + \cdots$$

$$A_1 = 1 + \frac{1}{7^3} + \frac{1}{13^3} + \frac{1}{19^3} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(6r+1)^3}$$

$$A_2 = \frac{1}{3^3} + \frac{1}{9^3} + \frac{1}{15^3} + \frac{1}{21^3} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(6r+3)^3}$$

$$A_3 = \frac{1}{5^3} + \frac{1}{11^3} + \frac{1}{17^3} + \frac{1}{23^3} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(6r+5)^3}$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$\mathrm{A1}[m\_] := \sum_{r=0}^{m} \frac{1}{(6r+1)^3} \qquad \mathrm{A2}[m\_] := \sum_{r=0}^{m} \frac{1}{(6r+3)^3} \qquad \mathrm{A3}[m\_] := \sum_{r=0}^{m} \frac{1}{(6r+5)^3}$$

```
N[{A1[70000], A2[70000], A3[70000]}, 10]
{1.003685515, 0.03895554779, 0.009158727129}

N[A1[70000] + A2[70000] + A3[70000], 10]        1.051799790
N[DirichletLambda[3], 10]                        1.051799790
```

Expression by polygamma function

According to Formula 5.1.2 (1.2') , the sums of the series A_1, A_2, A_3 are given by the following expressions respectively.

$$\sum_{r=0}^{\infty} \frac{1}{(6r+1)^3} = \frac{-1}{6^3 2!} \psi_1\left(\frac{1}{6}\right) =: B_1$$

$$\sum_{r=0}^{\infty} \frac{1}{(6r+3)^3} = \frac{-1}{6^3 2!} \psi_1\left(\frac{3}{6}\right) =: B_2$$

$$\sum_{r=0}^{\infty} \frac{1}{(6r+5)^3} = \frac{-1}{6^3 2!} \psi_1\left(\frac{5}{6}\right) =: B_3$$

Expression by π and $\zeta(3)$

According to ***MathWorld*** (http://mathworld.wolfram.com/PolygammaFunction.html) ,

$$\psi_2\left(\frac{1}{6}\right) = -182\zeta(3) - 4\sqrt{3}\pi^3 , \quad \psi_2\left(\frac{3}{6}\right) = -14\zeta(3) , \quad \psi_2\left(\frac{5}{6}\right) = -182\zeta(3) + 4\sqrt{3}\pi^3$$

Substituting these for the above ,

$$\sum_{r=0}^{\infty} \frac{1}{(6r+1)^3} = \frac{182\zeta(3) + 4\sqrt{3}\pi^3}{6^3 2!} =: C_1$$

$$\sum_{r=0}^{\infty} \frac{1}{(6r+3)^3} = \frac{14\zeta(3)}{6^3 2!} =: C_2$$

$$\sum_{r=0}^{\infty} \frac{1}{(6r+5)^3} = \frac{182\zeta(3) - 4\sqrt{3}\pi^3}{6^3\, 2!} \quad =: C_3$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$\zeta 3 := \text{Zeta}[3]$$

$$C1 := \frac{182\,\zeta 3 + 4\sqrt{3}\,\pi^3}{6^3\, 2!} \qquad C2 := \frac{14\,\zeta 3}{6^3\, 2!} \qquad C3 := \frac{182\,\zeta 3 - 4\sqrt{3}\,\pi^3}{6^3\, 2!}$$

```
N[{C1, C2, C3}, 10]
{ 1.003685515, 0.03895554779, 0.009158727129 }
N[C1 + C2 + C3, 10]     1.051799790
```

5.3.3 Three-split of $\beta(3)$

Since Dirichlet beta series $\beta(3)$ is alternating series, the basic 3-split is impossible. In the case of such a series, we must make composite three series by combining four or more basic strip of the series. This should also be called "composite split". (See **Sec.5**)

5.4 Four-split of Dirichlet Series

5.4.1 Four-split of $\zeta(2)$

Riemann zeta series $\zeta(2)$ is

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \cdots$$

The four-split is

$$A_1 = 1 + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(4r+1)^2}$$

$$A_2 = \frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \frac{1}{14^2} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(4r+2)^2}$$

$$A_3 = \frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{15^2} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(4r+3)^2}$$

$$A_4 = \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{12^2} + \frac{1}{16^2} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(4r+4)^2}$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$\text{A1}[m\_] := \sum_{r=0}^{m} \frac{1}{(4r+1)^2} \qquad \text{A2}[m\_] := \sum_{r=0}^{m} \frac{1}{(4r+2)^2} \qquad \text{A3}[m\_] := \sum_{r=0}^{m} \frac{1}{(4r+3)^2} \qquad \text{A4}[m\_] := \sum_{r=0}^{m} \frac{1}{(4r+4)^2}$$

```
N[{A1[100 000], A2[100 000], A3[100 000], A4[100 000]},10]
{ 1.074832447, 0.3084245125, 0.1588668530, 0.1028077542 }

N[Zeta[2], 10]
1.644931567

N[A1[100 000] + A2[100 000] + A3[100 000] + A4[100 000], 10]
1.644934067
```

Expression by polygamma function

According to Formula 5.1.1 (1.1') , the sums of the series A_1, A_2, A_3, A_4 are given by the following expressions respectively.

$$\sum_{r=0}^{\infty} \frac{1}{(4r+1)^2} = \frac{1}{4^2\, 1!} \psi_1\left(\frac{1}{4}\right) =: B_1$$

$$\sum_{r=0}^{\infty} \frac{1}{(4r+2)^2} = \frac{1}{4^2\, 1!} \psi_1\left(\frac{2}{4}\right) =: B_2$$

$$\sum_{r=0}^{\infty} \frac{1}{(4r+3)^2} = \frac{1}{4^2\, 1!} \psi_1\left(\frac{3}{4}\right) =: B_3$$

$$\sum_{r=0}^{\infty} \frac{1}{(4r+4)^2} = \frac{1}{4^2\, 1!} \psi_1\left(\frac{4}{4}\right) =: B_4$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$\psi_{n\_}[z\_] := \text{PolyGamma}[n, z]$$

$$\text{B1} := \frac{1}{4^2\, 1!} \psi_1\left[\frac{1}{4}\right] \qquad \text{B2} := \frac{1}{4^2\, 1!} \psi_1\left[\frac{2}{4}\right] \qquad \text{B3} := \frac{1}{4^2\, 1!} \psi_1\left[\frac{3}{4}\right] \qquad \text{B4} := \frac{1}{4^2\, 1!} \psi_1\left[\frac{4}{4}\right]$$

```
N[{B1, B2, B3, B4}, 10]
{1.074833072, 0.3084251375, 0.1588674780, 0.1028083792 }

N[B1 + B2 + B3 + B4, 10]
1.644934067
```

Expression by π and $\beta(2)$

According to *MathWorld* (http://mathworld.wolfram.com/PolygammaFunction.html) ,

$$\psi_1\left(\frac{1}{4}\right)=\pi^4+8\beta(2)\ ,\ \psi_1\left(\frac{2}{4}\right)=\frac{\pi^2}{2}\ ,\ \psi_1\left(\frac{3}{4}\right)=\pi^4-8\beta(2)\ ,\ \psi_1\left(\frac{4}{4}\right)=\frac{\pi^2}{6}$$

Where, $\beta(2)$ is the Dirichlet beta function. Substituting these for the above,

$$\sum_{r=0}^{\infty}\frac{1}{(4r+1)^2}=\frac{\pi^4+8\beta(2)}{4^2\,1!}=:C_1$$

$$\sum_{r=0}^{\infty}\frac{1}{(4r+2)^2}=\frac{\pi^2}{2\times4^2\,1!}=:C_2$$

$$\sum_{r=0}^{\infty}\frac{1}{(4r+3)^2}=\frac{\pi^4-8\beta(2)}{4^2\,1!}=:C_3$$

$$\sum_{r=0}^{\infty}\frac{1}{(4r+4)^2}=\frac{\pi^2}{6\times4^2\,1!}=:C_4$$

$$C_1+C_1+C_1+C_1=\frac{\pi^4}{6}\left(\frac{6}{4^2}+\frac{3}{4^2}+\frac{6}{4^2}+\frac{1}{4^2}\right)=\zeta(2)$$

5.4.2 Four-split of $\lambda(2)$

Dirichlet lambda series $\lambda(2)$ is

$$\lambda(2)=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{15^2}+\frac{1}{17^2}+\cdots$$

The four-split is

$$A_1=1+\frac{1}{9^2}+\frac{1}{17^2}+\frac{1}{25^2}+\cdots=\sum_{r=0}^{\infty}\frac{1}{(8r+1)^2}$$

$$A_2=\frac{1}{3^2}+\frac{1}{11^2}+\frac{1}{19^2}+\frac{1}{27^2}+\cdots=\sum_{r=0}^{\infty}\frac{1}{(8r+3)^2}$$

$$A_3=\frac{1}{5^2}+\frac{1}{13^2}+\frac{1}{21^2}+\frac{1}{29^2}+\cdots=\sum_{r=0}^{\infty}\frac{1}{(8r+5)^2}$$

$$A_4=\frac{1}{7^2}+\frac{1}{15^2}+\frac{1}{23^2}+\frac{1}{31^2}+\cdots=\sum_{r=0}^{\infty}\frac{1}{(8r+7)^2}$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$\text{A1}[m\_]:=\sum_{r=0}^{m}\frac{1}{(8r+1)^2}\qquad \text{A2}[m\_]:=\sum_{r=0}^{m}\frac{1}{(8r+3)^2}\qquad \text{A3}[m\_]:=\sum_{r=0}^{m}\frac{1}{(8r+5)^2}\qquad \text{4}[m\_]:=\sum_{r=0}^{m}\frac{1}{(8r+7)^2}$$

```
N[{A1[200000], A2[200000], A3[200000], A4[200000]}, 10]
{1.021689507, 0.1275276974, 0.05314340895, 0.03133962434}
N[A1[200000] + A2[200000] + A3[200000] + A4[200000], 10]     1.233700238
                              N[DirichletLambda[2], 10]     1.233700550
```

Expression by integral

According to Formula 5.1.2 (1.2") , the sums of the series A_1, A_2, A_3, A_4 are given by the following expressions respectively.

$$\sum_{r=0}^{\infty}\frac{1}{(8r+1)^2} = \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-\frac{1}{8}t}}{1-e^{-t}}dt =: B_1$$

$$\sum_{r=0}^{\infty}\frac{1}{(8r+3)^2} = \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-\frac{3}{8}t}}{1-e^{-t}}dt =: B_2$$

$$\sum_{r=0}^{\infty}\frac{1}{(8r+5)^2} = \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-\frac{5}{8}t}}{1-e^{-t}}dt =: B_3$$

$$\sum_{r=0}^{\infty}\frac{1}{(8r+7)^2} = \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-\frac{7}{8}t}}{1-e^{-t}}dt =: B_4$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

$$B1 := \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-\frac{1}{8}t}}{1-e^{-t}}dt \qquad B2 := \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-\frac{3}{8}t}}{1-e^{-t}}dt$$

$$B3 := \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-\frac{5}{8}t}}{1-e^{-t}}dt \qquad B4 := \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-\frac{7}{8}t}}{1-e^{-t}}dt$$

```
N[{B1, B2, B3, B4}, 10]
{1.021689585,0.1275277755,0.05314348708,0.03133970247}

N[B1 + B2 + B3 + B4, 10]
1.233700550
```

5.4.3 Four-split of $\beta(2)$

Dirichlet beta series $\beta(2)$ ($= 0.9159655941\cdots$) is expanded as follows.

$$\beta(2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \frac{1}{13^2} - \frac{1}{15^2} + \frac{1}{17^2} -+\cdots$$

The four-split is

$$\beta(2) = A_1 - A_2 + A_3 - A_4$$

Here, A_1, A_2, A_3, A_4 are same as **5.4.2** . Therefore,

$$A_1 = 1 + \frac{1}{9^2} + \frac{1}{17^2} + \frac{1}{25^2} + \cdots = \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-1t/8}}{1-e^{-t}}dt$$

$$A_2 = \frac{1}{3^2} + \frac{1}{11^2} + \frac{1}{19^2} + \frac{1}{27^2} + \cdots = \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-3t/8}}{1-e^{-t}}dt$$

$$A_3 = \frac{1}{5^2} + \frac{1}{13^2} + \frac{1}{21^2} + \frac{1}{29^2} + \cdots = \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-5t/8}}{1-e^{-t}}dt$$

$$A_4 = \frac{1}{7^2} + \frac{1}{15^2} + \frac{1}{23^2} + \frac{1}{31^2} + \cdots = \frac{1}{8^2\,1!}\int_0^{\infty}\frac{t\,e^{-7t/8}}{1-e^{-t}}dt$$

5.5 Composite Split of Dirichlet Series

Definition 5.5.0 (Composite Split)

Composing fewer m series from the basic n-split series is called **composite m-split** .

5.5.1 Composite 2-split of $\lambda(3)$

According to **5.3.2** , Dirichlet lambda series $\lambda(3)$ split into basic 3 series.

$$A_1 = 1 + \frac{1}{7^3} + \frac{1}{13^3} + \frac{1}{19^3} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(6r+1)^3}$$

$$A_2 = \frac{1}{3^3} + \frac{1}{9^3} + \frac{1}{15^3} + \frac{1}{21^3} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(6r+3)^3}$$

$$A_3 = \frac{1}{5^3} + \frac{1}{11^3} + \frac{1}{17^3} + \frac{1}{23^3} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(6r+5)^3}$$

$$\lambda(3) = A_1 + A_2 + A_3$$

There are the following 3 combinations in composing the series A_1, A_2, A_3 into 2 series.

$$A_1 + (A_2 + A_3) \quad , \quad A_2 + (A_1 + A_3) \quad , \quad A_3 + (A_1 + A_2)$$

When $\lambda(3) = a_1 + a_2$, these are expanded as follows.

$A_1 + (A_2 + A_3)$

$$a_1 = 1 + \frac{1}{7^3} + \frac{1}{13^3} + \frac{1}{19^3} + \frac{1}{25^3} + \frac{1}{31^3} + \cdots$$

$$a_2 = \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{9^3} + \frac{1}{11^3} + \frac{1}{15^3} + \frac{1}{17^3} + \cdots$$

$A_2 + (A_1 + A_3)$

$$a_1 = \frac{1}{3^3} + \frac{1}{9^3} + \frac{1}{15^3} + \frac{1}{21^3} + \frac{1}{27^3} + \cdots$$

$$a_2 = 1 + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{17^3} + \cdots$$

$A_3 + (A_1 + A_2)$

$$a_1 = \frac{1}{5^3} + \frac{1}{11^3} + \frac{1}{17^3} + \frac{1}{23^3} + \frac{1}{29^3} + \cdots$$

$$a_2 = 1 + \frac{1}{3^3} + \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{13^3} + \frac{1}{15^3} + \cdots$$

From **5.3.2** , these sums are described as follows.

$A_1 + (A_2 + A_3)$

$$a_1 = 1 + \frac{1}{7^3} + \frac{1}{13^3} + \frac{1}{19^3} + \frac{1}{25^3} + \frac{1}{31^3} + \cdots = \frac{182\zeta(3) + 4\sqrt{3}\pi^3}{6^3\, 2!}$$

$$a_2 = \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{9^3} + \frac{1}{11^3} + \frac{1}{15^3} + \frac{1}{17^3} + \cdots = \frac{196\zeta(3) - 4\sqrt{3}\pi^3}{6^3\, 2!}$$

$A_2 + (A_1 + A_3)$

$$a_1 = \frac{1}{3^3} + \frac{1}{9^3} + \frac{1}{15^3} + \frac{1}{21^3} + \frac{1}{27^3} + \cdots = \frac{14\zeta(3)}{6^3\,2!}$$

$$a_2 = 1 + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{17^3} + \cdots = \frac{2\times 182\zeta(3)}{6^3\,2!}$$

$A_3 + (A_1 + A_2)$

$$a_1 = \frac{1}{5^3} + \frac{1}{11^3} + \frac{1}{17^3} + \frac{1}{23^3} + \frac{1}{29^3} + \cdots = \frac{182\zeta(3) - 4\sqrt{3}\pi^3}{6^3\,2!}$$

$$a_2 = 1 + \frac{1}{3^3} + \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{13^3} + \frac{1}{15^3} + \cdots = \frac{196\zeta(3) + 4\sqrt{3}\pi^3}{6^3\,2!}$$

When the third pair is calculated with ***Mathematica***, it is as follows.

```
N[DirichletLambda[3], 10]        N[ζ3 = Zeta[3]]
1.051799790                      1.202056903
```

A3+(A1+A2)

```
a1[m_] := Sum[1/(6 r + 5)^3, {r, 0, m}]      a2[m_] := Sum[1/(6 r + 1)^3 + 1/(6 r + 3)^3, {r, 0, m}]

N[{a1[70000], a2[70000]}, 10]       N[a1[70000] + a2[70000], 10]
{0.009158727129, 1.042641063}       1.051799790
```

$$c1 := \frac{182\,\zeta3 - 4\sqrt{3}\,\pi^3}{6^3\,2!} \qquad c2 := \frac{196\,\zeta3 + 4\sqrt{3}\,\pi^3}{6^3\,2!}$$

```
N[{c1, c2}, 10]                      N[c1 + c2, 10]
{0.009158727129, 1.042641063}        1.051799790
```

5.5.2 Composite 2-split of $\beta(2)$

According to **5.4.3** , Dirichlet beta series $\beta(2)$ split into basic 4 series.

$$A_1 = 1 + \frac{1}{9^2} + \frac{1}{17^2} + \frac{1}{25^2} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+1)^2}$$

$$A_2 = \frac{1}{3^2} + \frac{1}{11^2} + \frac{1}{19^2} + \frac{1}{27^2} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+3)^2}$$

$$A_3 = \frac{1}{5^2} + \frac{1}{13^2} + \frac{1}{21^2} + \frac{1}{29^2} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+5)^2}$$

$$A_4 = \frac{1}{7^2} + \frac{1}{15^2} + \frac{1}{23^2} + \frac{1}{31^2} + \cdots = \sum_{r=0}^{\infty} \frac{1}{(8r+7)^2}$$

$$\beta(2) = A_1 - A_2 + A_3 - A_4$$

There are the following 7 combinations in composing the series A_1, A_2, A_3, A_4 into 2 series.

$$A_1+(-A_2+A_3-A_4) \quad , \quad -A_2+(A_1+A_3-A_4)$$
$$A_3+(A_1-A_2-A_4) \quad , \quad -A_4+(A_1-A_2+A_3)$$
$$(A_1-A_2)+(A_3-A_4) \ , \ (A_1+A_3)-(A_2+A_4) \ , \ (A_1-A_4)-(A_2-A_3)$$

When $\beta(2)=a_1+a_2$, if these are expanded and each sum is described by **5.4.3** , it is as follows.

$A_1+(-A_2+A_3-A_4)$

$$a_1= 1+\frac{1}{9^2}+\frac{1}{17^2}+\frac{1}{25^2}+\frac{1}{33^2}+\frac{1}{41^2}+\cdots = \frac{1}{8^2 1!}\int_0^\infty \frac{t\,e^{-\frac{1}{8}t}}{1-e^{-t}}dt$$

$$a_2= -\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}-\frac{1}{11^2}+\frac{1}{13^2}-\frac{1}{15^2}-+-\cdots = \frac{1}{8^2 1!}\int_0^\infty \frac{t\left(-e^{-\frac{3}{8}t}+e^{-\frac{5}{8}t}-e^{-\frac{7}{8}t}\right)}{1-e^{-t}}dt$$

$-A_2+(A_1+A_3-A_4)$

$$a_1= -\left(\frac{1}{3^2}+\frac{1}{11^2}+\frac{1}{19^2}+\frac{1}{27^2}+\frac{1}{35^2}+\cdots\right) = \frac{1}{8^2 1!}\int_0^\infty \frac{-t\,e^{-\frac{3}{8}t}}{1-e^{-t}}dt$$

$$a_2= 1+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{13^2}-\frac{1}{15^2}++-\cdots = \frac{1}{8^2 1!}\int_0^\infty \frac{t\left(e^{-\frac{1}{8}t}+e^{-\frac{5}{8}t}-e^{-\frac{7}{8}t}\right)}{1-e^{-t}}dt$$

$A_3+(A_1-A_2-A_4)$

$$a_1= \frac{1}{5^2}+\frac{1}{13^2}+\frac{1}{21^2}+\frac{1}{29^2}+\frac{1}{21^2}+\cdots = \frac{1}{8^2 1!}\int_0^\infty \frac{t\,e^{-\frac{5}{8}t}}{1-e^{-t}}dt$$

$$a_2= 1-\frac{1}{3^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}-\frac{1}{15^2}+--\cdots = \frac{1}{8^2 1!}\int_0^\infty \frac{t\left(e^{-\frac{1}{8}t}-e^{-\frac{3}{8}t}-e^{-\frac{7}{8}t}\right)}{1-e^{-t}}dt$$

$-A_4+(A_1-A_2+A_3)$

$$a_1= -\left(\frac{1}{7^2}+\frac{1}{15^2}+\frac{1}{23^2}+\frac{1}{31^2}+\frac{1}{37^2}+\cdots\right) = \frac{1}{8^2 1!}\int_0^\infty \frac{-t\,e^{-\frac{7}{8}t}}{1-e^{-t}}dt$$

$$a_2= 1-\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{9^2}-\frac{1}{11^2}+\frac{1}{13^2}+-+\cdots = \frac{1}{8^2 1!}\int_0^\infty \frac{t\left(e^{-\frac{1}{8}t}-e^{-\frac{3}{8}t}+e^{-\frac{5}{8}t}\right)}{1-e^{-t}}dt$$

$(A_1-A_2)+(A_3-A_4)$

$$1-\frac{1}{3^2}+\frac{1}{9^2}-\frac{1}{11^2}+\frac{1}{17^2}-\frac{1}{19^2}+-\cdots = \frac{1}{8^2 1!}\int_0^\infty \frac{t\left(e^{-\frac{1}{8}t}-e^{-\frac{3}{8}t}\right)}{1-e^{-t}}dt$$

$$\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{13^2}-\frac{1}{15^2}+\frac{1}{21^2}-\frac{1}{23^2}+-\cdots = \frac{1}{8^2 1!}\int_0^\infty \frac{t\left(e^{-\frac{5}{8}t}-e^{-\frac{7}{8}t}\right)}{1-e^{-t}}dt$$

$(A_1 + A_3) + (-A_2 - A_4)$

$$1+\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\frac{1}{17^2}+\frac{1}{21^2}+\cdots=\frac{1}{8^2 1!}\int_0^\infty \frac{t\left(e^{-\frac{1}{8}t}+e^{-\frac{5}{8}t}\right)}{1-e^{-t}}dt$$

$$-\frac{1}{3^2}-\frac{1}{7^2}-\frac{1}{11^2}-\frac{1}{15^2}-\frac{1}{19^2}-\cdots=\frac{1}{8^2 1!}\int_0^\infty \frac{t\left(-e^{-\frac{3}{8}t}-e^{-\frac{7}{8}t}\right)}{1-e^{-t}}dt$$

$(A_1 - A_4) + (-A_2 + A_3)$

$$1-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{15^2}+\frac{1}{17^2}-\frac{1}{23^2}+-\cdots=\frac{1}{8^2 1!}\int_0^\infty \frac{t\left(e^{-\frac{1}{8}t}-e^{-\frac{7}{8}t}\right)}{1-e^{-t}}dt$$

$$-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{11^2}+\frac{1}{13^2}-\frac{1}{19^2}+\frac{1}{21^2}-+\cdots=\frac{1}{8^2 1!}\int_0^\infty \frac{t\left(-e^{-\frac{3}{8}t}+e^{-\frac{5}{8}t}\right)}{1-e^{-t}}dt$$

When the 1st pair and the 7th pair are calculated with ***Mathematica***, it is as follows.

```
N[DirichletBeta[2], 10]     0.9159655942
```

A1 + (-A2 + A3 - A4)

$$a1[m\_] := \sum_{r=0}^{m} \frac{1}{(8r+1)^2} \qquad a2[m\_] := \sum_{r=0}^{m}\left(-\frac{1}{(8r+3)^2}+\frac{1}{(8r+5)^2}-\frac{1}{(8r+7)^2}\right)$$

```
N[{a1[200 000], a2[200 000]}, 10]     N[a1[200 000] + a2[200 000], 10]
{1.021689507, -0.1057239128}          0.9159655942
```

$$b1 := \frac{1}{8^2\, 1!}\int_0^\infty \frac{t\, e^{-\frac{1}{8}t}}{1-e^{-t}}\,dt \qquad b2 := \frac{1}{8^2\, 1!}\int_0^\infty \frac{t\left(-e^{-\frac{3}{8}t}+e^{-\frac{5}{8}t}-e^{-\frac{7}{8}t}\right)}{1-e^{-t}}\,dt$$

```
N[{b1, b2}, 10]                       N[b1 + b2, 10]
{1.021689585, -0.1057239909}          0.9159655942
```

(A1 - A4) + (-A2 + A3)

$$a1[m\_] := \sum_{r=0}^{m}\left(\frac{1}{(8r+1)^2}-\frac{1}{(8r+7)^2}\right) \qquad a2[m\_] := \sum_{r=0}^{m}\left(-\frac{1}{(8r+3)^2}+\frac{1}{(8r+5)^2}\right)$$

```
N[{a1[200 000], a2[200 000]}, 10]     N[a1[200 000] + a2[200 000], 10]
{ 0.9903498826, -0.07438428843 }      0.9159655942
```

$$b1 := \frac{1}{8^2\, 1!}\int_0^\infty \frac{t\left(e^{-\frac{1}{8}t}-e^{-\frac{7}{8}t}\right)}{1-e^{-t}}\,dt \qquad b2 := \frac{1}{8^2\, 1!}\int_0^\infty \frac{t\left(-e^{-\frac{3}{8}t}+e^{-\frac{5}{8}t}\right)}{1-e^{-t}}\,dt$$

```
N[{b1, b2}, 10]                       N[b1 + b2, 10]
{ 0.9903498826, -0.07438428843 }      0.9159655942
```

Note

Basic n-split exists innumerably. So, **composite m-split ($m < n$) which is these composition also exists innumerably.**

5.6 Composite 2-split of Madhava–Leibniz series

Madhava–Leibniz series $\beta(1)$ $(=\pi/4)$ is as follows.

$$\beta(1)=1-\frac{1}{3^1}+\frac{1}{5^1}-\frac{1}{7^1}+\frac{1}{9^1}-\frac{1}{11^1}+\frac{1}{13^1}-\frac{1}{15^1}+\frac{1}{17^1}-+\cdots$$

The four-split is

$$A_1=1+\frac{1}{9^1}+\frac{1}{17^1}+\frac{1}{25^1}+\cdots=\sum_{r=0}^{\infty}\frac{1}{(8r+1)^1}$$

$$A_2=\frac{1}{3^1}+\frac{1}{11^1}+\frac{1}{19^1}+\frac{1}{27^1}+\cdots=\sum_{r=0}^{\infty}\frac{1}{(8r+3)^1}$$

$$A_3=\frac{1}{5^1}+\frac{1}{13^1}+\frac{1}{21^1}+\frac{1}{29^1}+\cdots=\sum_{r=0}^{\infty}\frac{1}{(8r+5)^1}$$

$$A_4=\frac{1}{7^1}+\frac{1}{15^1}+\frac{1}{23^1}+\frac{1}{31^1}+\cdots=\sum_{r=0}^{\infty}\frac{1}{(8r+7)^1}$$

$$\beta(1)=A_1-A_2+A_3-A_4$$

There are the following 7 combinations in composing the series A_1, A_2, A_3, A_4 into 2 series.

$$A_1+(-A_2+A_3-A_4)\quad,\quad -A_2+(A_1+A_3-A_4)$$

$$A_3+(A_1-A_2-A_4)\quad,\quad -A_4+(A_1-A_2+A_3)$$

$$(A_1-A_2)+(A_3-A_4)\quad,\quad(A_1+A_3)-(A_2+A_4)\quad,\quad(A_1-A_4)-(A_2-A_3)$$

However, among these, one and three pairs are impossible. Because, A_1, A_2, A_3, A_4 are divergence series. And $(A_1+A_3)-(A_2+A_4)$ is also impossible for the same reason. Thus, possible combinations are the following two pairs. Since these split series are alternating series, they converge conditionally.

Split 1

$$\begin{cases} a_{11}=1-\frac{1}{7^1}+\frac{1}{9^1}-\frac{1}{15^1}+\frac{1}{17^1}-\frac{1}{23^1}+-\cdots=A_1-A_4 \\ a_{12}=\frac{1}{3^1}-\frac{1}{5^1}+\frac{1}{11^1}-\frac{1}{13^1}+\frac{1}{19^1}-\frac{1}{21^1}+-\cdots=A_2-A_3 \end{cases}$$

$$\beta(1)=a_{11}-a_{12}$$

Split 2

$$\begin{cases} a_{21}=1-\frac{1}{3^1}+\frac{1}{9^1}-\frac{1}{11^1}+\frac{1}{17^1}-\frac{1}{19^1}+-\cdots=A_1-A_2 \\ a_{22}=\frac{1}{5^1}-\frac{1}{7^1}+\frac{1}{13^1}-\frac{1}{15^1}+\frac{1}{21^1}-\frac{1}{23^1}+-\cdots=A_3-A_4 \end{cases}$$

$$\beta(1)=a_{21}+a_{22}$$

Expression by polygamma function

Left sides are

Split 1

$$\begin{cases} 1-\frac{1}{7^1}+\frac{1}{9^1}-\frac{1}{15^1}+\frac{1}{17^1}-\frac{1}{23^1}+-\cdots=\sum_{r=0}^{\infty}\left\{\frac{1}{(8r+1)^1}-\frac{1}{(8r+7)^1}\right\} \\ \frac{1}{3^1}-\frac{1}{5^1}+\frac{1}{11^1}-\frac{1}{13^1}+\frac{1}{19^1}-\frac{1}{21^1}+-\cdots=\sum_{r=0}^{\infty}\left\{\frac{1}{(8r+3)^1}-\frac{1}{(8r+5)^1}\right\} \end{cases}$$

Split 2

$$\begin{cases} 1-\frac{1}{3^1}+\frac{1}{9^1}-\frac{1}{11^1}+\frac{1}{17^1}-\frac{1}{19^1}+-\cdots=\sum_{r=0}^{\infty}\left\{\frac{1}{(8r+1)^1}-\frac{1}{(8r+3)^1}\right\} \\ \frac{1}{5^1}-\frac{1}{7^1}+\frac{1}{13^1}-\frac{1}{15^1}+\frac{1}{21^1}-\frac{1}{23^1}+-\cdots=\sum_{r=0}^{\infty}\left\{\frac{1}{(8r+5)^1}-\frac{1}{(8r+7)^1}\right\} \end{cases}$$

If Formula 5.1.2 (1.2') is applied to these by force, it is as follows.

Split 1
$$\begin{cases} a_{11} = -\dfrac{1}{8^1 0!}\left\{\psi_0\left(\dfrac{1}{8}\right) - \psi_0\left(\dfrac{7}{8}\right)\right\} \\ a_{12} = -\dfrac{1}{8^1 0!}\left\{\psi_0\left(\dfrac{3}{8}\right) - \psi_0\left(\dfrac{5}{8}\right)\right\} \end{cases}$$

$$\beta(1) = a_{11} - a_{12}$$

Split 2
$$\begin{cases} a_{21} = -\dfrac{1}{8^1 0!}\left\{\psi_0\left(\dfrac{1}{8}\right) - \psi_0\left(\dfrac{3}{8}\right)\right\} \\ a_{22} = -\dfrac{1}{8^1 0!}\left\{\psi_0\left(\dfrac{5}{8}\right) - \psi_0\left(\dfrac{7}{8}\right)\right\} \end{cases}$$

$$\beta(1) = a_{21} + a_{22}$$

Expression by *Gauss's Digamma Theorem*

Amoung the polygamma function $\psi_n(z)$, the special value of $\psi_0(z)$ is given by the following equation by **Gauss**.
(See http://mathworld.wolfram.com/GausssDigammaTheorem.html)

$$\psi_0\left(\frac{k}{m}\right) = -\gamma - \ln(2m) - \frac{\pi}{2}cot\left(\frac{k\pi}{m}\right) + 2\sum_{r=1}^{\lceil m/2\rceil -1} cos\left(\frac{2rk\pi}{m}\right)\ln\left[sin\frac{r\pi}{m}\right]$$

Where, $0 < k < m$, γ : Euler-Mascheroni constant

From this,

$$\psi_0\left(\frac{1}{8}\right) = -\gamma - 4\ln 2 - \frac{\pi}{2}cot\frac{\pi}{8} + \sqrt{2}\ln tan\frac{\pi}{8}$$

$$\psi_0\left(\frac{3}{8}\right) = -\gamma - 4\ln 2 - \frac{\pi}{2}tan\frac{\pi}{8} - \sqrt{2}\ln tan\frac{\pi}{8}$$

$$\psi_0\left(\frac{5}{8}\right) = -\gamma - 4\ln 2 + \frac{\pi}{2}tan\frac{\pi}{8} - \sqrt{2}\ln tan\frac{\pi}{8}$$

$$\psi_0\left(\frac{7}{8}\right) = -\gamma - 4\ln 2 + \frac{\pi}{2}cot\frac{\pi}{8} + \sqrt{2}\ln tan\frac{\pi}{8}$$

Substituting these for the above,

Split 1
$$\begin{cases} a_{11} = \dfrac{\pi}{8}cot\dfrac{\pi}{8} \\ a_{12} = \dfrac{\pi}{8}tan\dfrac{\pi}{8} \end{cases}$$

Split 2
$$\begin{cases} a_{21} = \dfrac{\pi}{16}\left(cot\dfrac{\pi}{8} - tan\dfrac{\pi}{8}\right) - \dfrac{\sqrt{2}}{4}\ln tan\dfrac{\pi}{8} \\ a_{22} = \dfrac{\pi}{16}\left(cot\dfrac{\pi}{8} - tan\dfrac{\pi}{8}\right) + \dfrac{\sqrt{2}}{4}\ln tan\dfrac{\pi}{8} \end{cases}$$

Further,

$$cot\frac{\pi}{8} = \sqrt{2}+1 \quad , \quad tan\frac{\pi}{8} = \sqrt{2}-1$$

Using these, we obtain

Split 1
$$\begin{cases} a_{11} = 1 - \dfrac{1}{7^1} + \dfrac{1}{9^1} - \dfrac{1}{15^1} + \dfrac{1}{17^1} - \dfrac{1}{23^1} +- \cdots = \dfrac{\pi}{8}(\sqrt{2}+1) \\ a_{12} = \dfrac{1}{3^1} - \dfrac{1}{5^1} + \dfrac{1}{11^1} - \dfrac{1}{13^1} + \dfrac{1}{19^1} - \dfrac{1}{21^1} +- \cdots = \dfrac{\pi}{8}(\sqrt{2}-1) \end{cases}$$

$$\beta(1) = a_{11} - a_{12}$$

Split 2

$$\begin{cases} a_{21} = 1 - \frac{1}{3^1} + \frac{1}{9^1} - \frac{1}{11^1} + \frac{1}{17^1} - \frac{1}{19^1} + - \cdots = \frac{\pi}{8} - \frac{\sqrt{2}}{4} \ln\left(\sqrt{2}-1\right) \\ a_{22} = \frac{1}{5^1} - \frac{1}{7^1} + \frac{1}{13^1} - \frac{1}{15^1} + \frac{1}{21^1} - \frac{1}{23^1} + - \cdots = \frac{\pi}{8} + \frac{\sqrt{2}}{4} \ln\left(\sqrt{2}-1\right) \end{cases}$$

$$\beta(1) = a_{21} + a_{22}$$

The calculation result by the formula manipulation software ***Mathematica*** is as follows.

```
N[DirichletBeta[2], 10]      0.7853981634      ψn_[z_] := PolyGamma[n, z]
```

(A1 - A4) - (A2 - A3)

$$a1[m\_] := \sum_{r=0}^{m} \left(\frac{1}{(8r+1)^1} - \frac{1}{(8r+7)^1} \right) \qquad a2[m\_] := \sum_{r=0}^{m} \left(\frac{1}{(8r+3)^1} - \frac{1}{(8r+5)^1} \right)$$

```
N[{a1[1000000], a2[1000000]}, 10]      N[a1[1000000] - a2[1000000], 10]
{0.948059324, 0.1626611606}            0.7853981634
```

$$c1 := \frac{\pi}{8}\left(\sqrt{2}+1\right) \qquad c2 := \frac{\pi}{8}\left(\sqrt{2}-1\right)$$

```
N[{c1, c2}, 10]                        N[c1 - c2, 10]
{0.948059449, 0.1626612856}            0.7853981634
```

(A1 - A2) + (A3 - A4)

$$a1[m\_] := \sum_{r=0}^{m} \left(\frac{1}{(8r+1)^1} - \frac{1}{(8r+3)^1} \right) \qquad a2[m\_] := \sum_{r=0}^{m} \left(\frac{1}{(8r+5)^1} - \frac{1}{(8r+7)^1} \right)$$

```
N[{a1[1000000], a2[1000000]}, 10]      N[a1[1000000] + a2[1000000], 10]
{0.7043117018, 0.08108646163}          0.7853981634
```

$$c1 := \frac{\pi}{8} - \frac{\sqrt{2}}{4} \mathrm{Log}\left[\sqrt{2}-1\right] \qquad c2 := \frac{\pi}{8} + \frac{\sqrt{2}}{4} \mathrm{Log}\left[\sqrt{2}-1\right]$$

```
N[{c1, c2}, 10]                        N[c1 + c2, 10]
{0.7043117018, 0.08108646163}          0.7853981634
```

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2019.02.15 Updated Sec.1

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Alien's Mathematics