16 Split of Power Series

In this chapter, we divide the power series of a function into two or more series and investigate with what kind of function they are represented.

Although there are innumerable split series depending on the way of split, in this chapter, we will deal with a basic split series that is choosed at equal intervals from the original series.

16.1 Character for Split & Character for Alternating Split

Definition 16.1.1 (Character for Split)

When $n=2,3,4,\cdots$, $r=0,1,2,\cdots$, $\lambda_n=\{1+(-1)^n\}/2$, $\lfloor x \rfloor$ be floor function, we define Character for Split $c(0,n,r)$ as follows.

$$c(0,n,r)=\frac{1}{n}\left\{(-1)^{2r}-\lambda_n(-1)^r\right\}+\frac{1}{n}\sum_{s=1}^{\lfloor n/2 \rfloor}\left\{(-1)^{\frac{2s}{n}r}+(-1)^{-\frac{2s}{n}r}\right\}$$

The first few are written down as follows.

$$c(0,2,r)=\frac{(-1)^{2r}-(-1)^r}{2}+\frac{1}{2}\left\{(-1)^{\frac{2}{2}r}+(-1)^{-\frac{2}{2}r}\right\}$$

$$c(0,3,r)=\frac{(-1)^{2r}}{3}+\frac{1}{3}\left\{(-1)^{\frac{2}{3}r}+(-1)^{-\frac{2}{3}r}\right\}$$

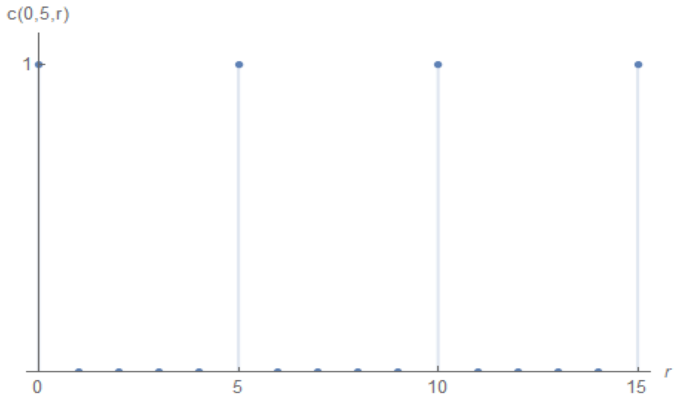
$$c(0,4,r)=\frac{(-1)^{2r}-(-1)^r}{4}+\frac{1}{4}\left\{(-1)^{\frac{2}{4}r}+(-1)^{-\frac{2}{4}r}\right\}+\frac{1}{4}\left\{(-1)^{\frac{4}{4}r}+(-1)^{-\frac{4}{4}r}\right\}$$

$$c(0,5,r)=\frac{(-1)^{2r}}{5}+\frac{1}{5}\left\{(-1)^{\frac{2}{5}r}+(-1)^{-\frac{2}{5}r}\right\}+\frac{1}{5}\left\{(-1)^{\frac{4}{5}r}+(-1)^{-\frac{4}{5}r}\right\}$$

$\vdots$

Example

$c(0,5,r)$ is drawn in 2D as follows.



Properties of the Character for Split

As seen from the above figure, this character has the following properties.

$$c(0,n,r)=\begin{cases}1 & r \bmod n=0\\ 0 & r \bmod n\neq 0\end{cases}$$

In the case of 2 -split

$(-1)^{2r}-(-1)^r$, $(-1)^{2r/2}+(-1)^{-2r/2}$ $r=0,1,2,\cdots$ are

$0,\ 2,\ 0,\ 2,\ 0,\ 2,\ \cdots$

$2,\ -2,\ 2,\ -2,\ 2,\ -2,\ \cdots$

So, $(-1)^{2r}-(-1)^{r}+(-1)^{2r/2}+(-1)^{-2r/2} \quad r=0,1,2,\cdots$ are

$$2,\ 0,\ 2,\ 0,\ 2,\ 0,\cdots$$

Dividing this by 2, $c(0,2,r) \quad r=0,1,2,\cdots$ become

$$1,\ 0,\ 1,\ 0,\ 1,\ 0,\cdots$$

In the case of 3-split

$(-1)^{2r}$, $(-1)^{2r/3}+(-1)^{-2r/3} \quad r=0,1,2,\cdots$ are

$$1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \cdots$$
$$2,\ -1,\ -1,\ 2,\ -1,\ -1,\ \cdots$$

So, $(-1)^{2r}+(-1)^{2r/3}+(-1)^{-2r/3} \quad r=0,1,2,\cdots$ are

$$3,\ 0,\ 0,\ 3,\ 0,\ 0,\cdots$$

Dividing this by 3, $c(0,3,r) \quad r=0,1,2,\cdots$ become

$$1,\ 0,\ 0,\ 1,\ 0,\ 0,\cdots$$

Definition 16.1.2 (Character for Alternating Split)

When $n=2,3,4,\cdots$, $r=0,1,2,\cdots$, $\underline{\lambda_n}=\{1-(-1)^n\}/2$, $\lfloor x \rfloor$ be floor function, we define Character for Alternating Split $\underline{c}(0,n,r)$ as follows.

$$\underline{c}(0,n,r)=\underline{\lambda_n}\cdot\frac{(-1)^r}{n}+\frac{1}{n}\sum_{s=1}^{\lfloor n/2 \rfloor}\left((-1)^{\frac{2s-1}{n}r}+(-1)^{-\frac{2s-1}{n}r}\right)$$

The first few are written down as follows.

$$\underline{c}(0,2,r)=\frac{1}{2}\left\{(-1)^{\frac{1}{2}r}+(-1)^{-\frac{1}{2}r}\right\}$$

$$\underline{c}(0,3,r)=\frac{(-1)^r}{3}+\frac{1}{3}\left\{(-1)^{\frac{1}{3}r}+(-1)^{-\frac{1}{3}r}\right\}$$

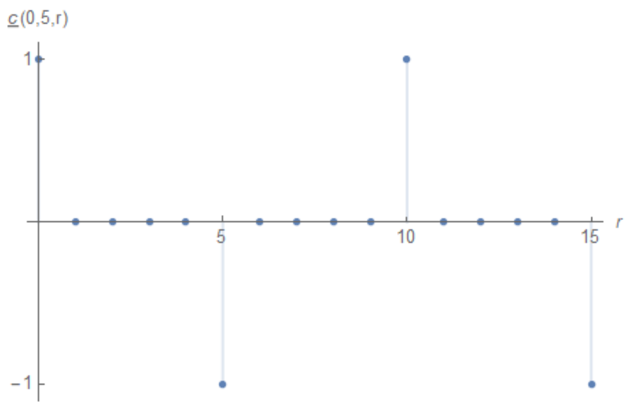
$$\underline{c}(0,4,r)=\frac{1}{4}\left\{(-1)^{\frac{1}{4}r}+(-1)^{-\frac{1}{4}r}\right\}+\frac{1}{4}\left\{(-1)^{\frac{3}{4}r}+(-1)^{-\frac{3}{4}r}\right\}$$

$$\underline{c}(0,5,r)=\frac{(-1)^r}{5}+\frac{1}{5}\left\{(-1)^{\frac{1}{5}r}+(-1)^{-\frac{1}{5}r}\right\}+\frac{1}{5}\left\{(-1)^{\frac{3}{5}r}+(-1)^{-\frac{3}{5}r}\right\}$$

$$\vdots$$

Example

$\underline{c}(0,5,r)$ is drawn in 2D as follows.



Properties of the Alternating Character for Split

As seen from the above figure, this character has the following properties.

$$\underline{c}(0,n,r) = \begin{cases} (-1)^{r} & r \bmod n = 0 \\ 0 & r \bmod n \neq 0 \end{cases}$$

In the case of 2 -split

$(-1)^{1r/2} + (-1)^{-1r/2}$ $r=0,1,2,\cdots$ are

$2,\ 0,\ -2,\ 0,\ 2,\ 0,\ -2,\ \cdots$

Dividing this by 2, $\underline{c}(0,2,r)$ $r=0,1,2,\cdots$ become

$1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ \cdots$

In the case of 3 -split

$(-1)^{r}$, $(-1)^{1r/3} + (-1)^{-1r/3}$ $r=0,1,2,\cdots$ are

$1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \cdots$

$2,\ 1,\ -1,\ -2,\ -1,\ 1,\ 2,\ 1,\ -1,\ -2,\ \cdots$

So, $(-1)^{r} + (-1)^{1r/3} + (-1)^{-1r/3}$ $r=0,1,2,\cdots$ are

$3,\ 0,\ 0,\ -3,\ 0,\ 0,\ 3,\ 0,\ 0,\ -3,\ \cdots$

Dividing this by 3, $\underline{c}(0,3,r)$ $r=0,1,2,\cdots$ become

$1,\ 0,\ 0,\ -1,\ 0,\ 0,\ 1,\ 0,\ 0,\ -1,\ \cdots$

16.2 n-split of Power Series

Formula 16.2.1 (n-split)

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \tag{2.0}$$

And let the n-split series $f(k,n,z)$ $k=0,1,2,\cdots,n-1$ are as follows.

$$f(0,n,z) = \sum_{r=0}^{\infty} a_{nr+0} z^{nr+0} = a_0 z^0 + a_n z^n + a_{2n} z^{2n} + a_{3n} z^{3n} + \cdots$$

$$f(1,n,z) = \sum_{r=0}^{\infty} a_{nr+1} z^{nr+1} = a_1 z^1 + a_{n+1} z^{n+1} + a_{2n+1} z^{2n+1} + a_{3n+1} z^{3n+1} + \cdots$$

$$f(2,n,z) = \sum_{r=0}^{\infty} a_{nr+2} z^{nr+2} = a_2 z^2 + a_{n+2} z^{n+2} + a_{2n+2} z^{2n+2} + a_{3n+2} z^{3n+2} + \cdots$$

$$\vdots$$

$$f(n-1,n,z) = \sum_{r=0}^{\infty} a_{nr+n-1} z^{nr+n-1} = a_{n-1} z^{n-1} + a_{2n-1} z^{2n-1} + a_{3n-1} z^{3n-1} + a_{4n-1} z^{4n-1} + \cdots$$

Then, the following expressions hold for $n = 2,3,4,\cdots$, $k = 0,1,2,\cdots,n-1$.

$$f(k,n,z) = \frac{f(z) - \lambda_n (-1)^k f(-z)}{n} + \frac{1}{n} \sum_{s=1}^{\lfloor n/2 \rfloor} \left[(-1)^{-\frac{2sk}{n}} f\left\{ (-1)^{\frac{2s}{n}} z \right\} + (-1)^{\frac{2sk}{n}} f\left\{ (-1)^{-\frac{2s}{n}} z \right\} \right]$$

Where, $\lambda_n = \{1+(-1)^n\}/2$, $\lfloor x \rfloor$ is the floor function.

Proof

Character for Split $c(0,n,r)$ in Definition 16.1.1 was as follows.

$$c(0,n,r) = \frac{1}{n}\left\{ (-1)^{2r} - \lambda_n (-1)^r \right\} + \frac{1}{n} \sum_{s=1}^{\lfloor n/2 \rfloor} \left\{ (-1)^{\frac{2s}{n}r} + (-1)^{-\frac{2s}{n}r} \right\}$$

Substituting $r=1$ for this,

$$c(0,n,1) = \frac{1}{n}\left\{ (-1)^{2} - \lambda_n (-1)^1 \right\} + \frac{1}{n} \sum_{s=1}^{\lfloor n/2 \rfloor} \left\{ (-1)^{\frac{2s}{n}} + (-1)^{-\frac{2s}{n}} \right\}$$

Multiply each term by z , substitute them into the function $f(z)$, and construct a polynomial using them, then

$$f(0,n,z) = \frac{f\{(-1)^2 z\} - \lambda_n f\{(-1)^1 z\}}{n} + \frac{1}{n} \sum_{s=1}^{\lfloor n/2 \rfloor} \left[f\left\{ (-1)^{\frac{2s}{n}} z \right\} + f\left\{ (-1)^{-\frac{2s}{n}} z \right\} \right]$$

Here,

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

So,

$$f(0,n,z) = \frac{1}{n}\left[\sum_{r=0}^{\infty} a_r \{(-1)^2 z\}^r - \lambda_n \sum_{r=0}^{\infty} a_r \{(-1)^1 z\}^r \right] + \frac{1}{n} \sum_{s=1}^{\lfloor n/2 \rfloor} \left[\sum_{r=0}^{\infty} a_r \left\{ (-1)^{\frac{2s}{n}} z \right\}^r + \sum_{r=0}^{\infty} a_r \left\{ (-1)^{-\frac{2s}{n}} z \right\}^r \right]$$

$$= \frac{1}{n}\left[\sum_{r=0}^{\infty} (-1)^{2r} a_r z^r - \lambda_n \sum_{r=0}^{\infty} (-1)^r a_r z^r \right] + \frac{1}{n} \sum_{s=1}^{\lfloor n/2 \rfloor} \left[\sum_{r=0}^{\infty} (-1)^{\frac{2s}{n}r} a_r z^r + \sum_{r=0}^{\infty} (-1)^{-\frac{2s}{n}r} a_r z^r \right]$$

$$= \frac{1}{n}\sum_{r=0}^{\infty}\left\{(-1)^{2r} - \lambda_n(-1)^r\right\}a_r z^r + \frac{1}{n}\sum_{r=0}^{\infty}\sum_{s=1}^{\lfloor n/2\rfloor}\left\{(-1)^{\frac{2s}{n}r} + (-1)^{-\frac{2s}{n}r}\right\}a_r z^r$$

i.e.

$$f(0,n,z) = \sum_{r=0}^{\infty}\left[\frac{1}{n}\left\{(-1)^{2r} - \lambda_n(-1)^r\right\} + \frac{1}{n}\sum_{s=1}^{\lfloor n/2\rfloor}\left\{(-1)^{\frac{2s}{n}r} + (-1)^{-\frac{2s}{n}r}\right\}\right]a_r z^r$$

Since the inside of [ ] is $c(0,n,r)$, due to the properties mentioned in the previous section,

$$f(0,n,z) = \sum_{r=0}^{\infty}c(0,n,r)a_r z^r = \sum_{r=0}^{\infty}a_{nr+0}z^{nr+0} = a_0 z^0 + a_n z^n + a_{2n}z^{2n} + a_{3n}z^{3n} + \cdots$$

Next, Character $c(1,n,r)$ for $f(1,n,z)$ is obtained by shifting $c(0,n,r)$ by 1 in the positive direction with respect to r . So,

$$\begin{aligned} c(1,n,r) &= c(0,n,r-1) \\ &= \frac{1}{n}\left\{(-1)^{2r}(-1)^{-2} - \lambda_n(-1)^r(-1)^{-1}\right\} \\ &\quad + \frac{1}{n}\sum_{s=1}^{\lfloor n/2\rfloor}\left\{(-1)^{\frac{2s}{n}r}(-1)^{-\frac{2s}{n}} + (-1)^{-\frac{2s}{n}r}(-1)^{-\frac{2s}{n}}\right\} \end{aligned}$$

Substituting $r=1$ for this,

$$\begin{aligned} c(1,n,1) &= \frac{1}{n}\left\{(-1)^2(-1)^{-2} - \lambda_n(-1)^1(-1)^{-1}\right\} \\ &\quad + \frac{1}{n}\sum_{s=1}^{\lfloor n/2\rfloor}\left\{(-1)^{\frac{2s}{n}}(-1)^{-\frac{2s}{n}} + (-1)^{-\frac{2s}{n}}(-1)^{-\frac{2s}{n}}\right\} \end{aligned}$$

Multiply each term by z , substitute them into the function $f(z)$, and construct a polynomial using them. Where, since the magenta parts were originally coefficients, these are left out of the function. Then,

$$\begin{aligned} f(1,n,z) &= \frac{(-1)^{-2}f(z) - \lambda_n(-1)^{-1}(-1)^1 f(-z)}{n} \\ &\quad + \frac{1}{n}\sum_{s=1}^{\lfloor n/2\rfloor}\left[(-1)^{-\frac{2s}{n}}f\left\{(-1)^{\frac{2s}{n}}z\right\} + (-1)^{\frac{2s}{n}}f\left\{(-1)^{-\frac{2s}{n}}z\right\}\right] \end{aligned}$$

For the same reasons as in $f(0,n,z)$,

$$\begin{aligned} f(1,n,z) &= \sum_{r=0}^{\infty}c(1,n,r)a_r z^r = \sum_{r=0}^{\infty}a_{nr+1}z^{nr+1} \\ &= a_1 z^1 + a_{n+1}z^{n+1} + a_{2n+1}z^{2n+1} + a_{3n+1}z^{3n+1} + \cdots \end{aligned}$$

Hereafter by induction, we obtain $f(k,n,z)$ $k=2,3,\cdots,n-1$.

Formula 16.2.2 (alternating n-split)

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty}a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

And let the alternating n-split series $\underline{f}(k,n,z)$ $k=0,1,2,\cdots,n-1$ are as follows.

$$\underline{f}(0,n,z) = \sum_{r=0}^{\infty}(-1)^r a_{nr+0}z^{nr+0} = a_0 z^0 - a_n z^n + a_{2n}z^{2n} - a_{3n}z^{3n} + - \cdots$$

$$\underline{f}(1,n,z) = \sum_{r=0}^{\infty}(-1)^r a_{nr+1}z^{nr+1} = a_1 z^1 - a_{n+1}z^{n+1} + a_{2n+1}z^{2n+1} - a_{3n+1}z^{3n+1} + - \cdots$$

$$\underline{f}(2,n,z) = \sum_{r=0}^{\infty}(-1)^r a_{nr+2}z^{nr+2} = a_2 z^2 - a_{n+2}z^{n+2} + a_{2n+2}z^{2n+2} - a_{3n+2}z^{3n+2} + - \cdots$$

$$\vdots$$

$$\underline{f}(n-1,n,z) = \sum_{r=0}^{\infty}(-1)^r a_{nr+n-1} z^{nr+n-1} = a_{n-1}z^{n-1} - a_{2n-1}z^{2n-1} + a_{3n-1}z^{3n-1} - a_{4n-1}z^{4n-1} + - \cdots$$

Then, following expressions hold for $n = 2,3,4,\cdots, k = 0,1,2,\cdots,n-1$.

$$\underline{f}(k,n,z) = \underline{\lambda_n}\frac{(-1)^k f(-z)}{n} + \frac{1}{n}\sum_{s=1}^{\lfloor n/2\rfloor}\left[(-1)^{-\frac{(2s-1)k}{n}} f\left\{(-1)^{\frac{2s-1}{n}}z\right\} + (-1)^{\frac{(2s-1)k}{n}} f\left\{(-1)^{-\frac{2s-1}{n}}z\right\}\right]$$

Where, $\underline{\lambda_n} = \{1-(-1)^n\}/2$, $\lfloor x\rfloor$ is the floor function.

Proof

It is proved in a similar way to Formula 16.2.1 using Definition 16.1.2 .

16.3 Two-split of Power Series

16.3.1 Two-split of Power Series

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \tag{2.0}$$

Two-split of the series are

$$f(0,2,z) = \sum_{r=0}^{\infty} a_{2r+0} z^{2r+0} = a_0 z^0 + a_2 z^2 + a_4 z^4 + a_6 z^6 + \cdots$$

$$f(1,2,z) = \sum_{r=0}^{\infty} a_{2r+1} z^{2r+1} = a_1 z^1 + a_3 z^3 + a_5 z^5 + a_7 z^7 + \cdots$$

And Formula 16.2.1 is written down as follows.

$$f(0,2,z) = \frac{f(z) - f(-z)}{2} + \frac{f\{(-1)^{2/2} z\} + f\{(-1)^{-2/2} z\}}{2}$$

$$f(1,2,z) = \frac{f(z) + f(-z)}{2} + \frac{(-1)^{-2/2} f\{(-1)^{2/2} z\} + (-1)^{2/2} f\{(-1)^{-2/2} z\}}{2}$$

Simplified

At first glance, they are reduced to

$$f(0,2,z) = \frac{f(z) + f(-z)}{2} \quad , \quad f(1,2,z) = \frac{f(z) - f(-z)}{2}$$

Example 1 Two-split of Exponential Series

$$f(z) = 1 + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \cdots = e^z \qquad \text{(Series to be splited)}$$

$$f(0,2,z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \frac{z^8}{8!} + \frac{z^{10}}{10!} + \cdots = \frac{e^z + e^{-z}}{2} = \cosh z$$

$$f(1,2,z) = \frac{z^1}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \frac{z^9}{9!} + \frac{z^{11}}{11!} + \cdots = \frac{e^z - e^{-z}}{2} = \sinh z$$

Example 2 Two-split of Exponential Generating Series

$$f(z) = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \frac{z^8}{1209600} + - \cdots = \frac{z}{e^z - 1} \qquad \text{(Series to be splited)}$$

$$f(0,2,z) = 1 + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \frac{z^8}{1209600} + - \cdots = \frac{1}{2}\left(\frac{z}{e^z - 1} + \frac{-z}{e^{-z} - 1}\right) = \frac{z}{2} \coth \frac{z}{2}$$

$$f(1,2,z) = -\frac{z}{2} \qquad\qquad = \frac{1}{2}\left(\frac{z}{e^z - 1} - \frac{-z}{e^{-z} - 1}\right) = -\frac{z}{2}$$

In fact, these right-hand sides are expanded into the Maclaurin series by the formula manipulation software *Mathematica* as follows.

$$\text{Normal}\left[\text{Series}\left[\frac{1}{2}\left(\frac{z}{e^z - 1} + \frac{-z}{e^{-z} - 1}\right), \{z, 0, 10\}\right]\right]$$

$$1 + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30\,240} - \frac{z^8}{1\,209\,600} + \frac{z^{10}}{47\,900\,160}$$

```
Normal[Series[1/2 (z/(e^z - 1) - (-z)/(e^(-z) - 1)), {z, 0, 11}]]
```

$$-\frac{z}{2}$$

16.3.2 Alternating two-split of Power Series

Alternating two-split of Power Series (2.0) are

$$\underline{f}(0,2,z) = \sum_{r=0}^{\infty}(-1)^r a_{2r+0} z^{2r+0} = a_0 z^0 - a_2 z^2 + a_4 z^4 - a_6 z^6 + - \cdots$$

$$\underline{f}(1,2,z) = \sum_{r=0}^{\infty}(-1)^r a_{2r+1} z^{2r+1} = a_1 z^1 - a_3 z^3 + a_5 z^5 - a_7 z^7 + - \cdots$$

And Formula 16.2.2 is written down as follows.

$$\underline{f}(0,2,z) = \frac{f\{(-1)^{1/2} z\} + f\{(-1)^{-1/2} z\}}{2}$$

$$\underline{f}(1,2,z) = \frac{(-1)^{-1/2} f\{(-1)^{1/2} z\} + (-1)^{1/2} f\{(-1)^{-1/2} z\}}{2}$$

Simplified

At first glance, they are reduced to

$$\underline{f}(0,2,z) = \frac{f(iz) + f(-iz)}{2} \quad , \quad \underline{f}(1,2,z) = \frac{f(iz) - f(-iz)}{2i}$$

Example 1 Alternating two-split of Exponential Series

$$f(z) = 1 + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \cdots = e^z \qquad \text{(Series to be splited)}$$

$$\underline{f}(0,2,z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + - \cdots = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

$$\underline{f}(1,2,z) = \frac{z^1}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + - \cdots = \frac{e^{iz} - e^{-iz}}{2i} = \sin z$$

Example 2 Alternating two-split of Exponential Generating Series

$$f(z) = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \frac{z^8}{1209600} + - \cdots = \frac{z}{e^z - 1} \qquad \text{(Series to be splited)}$$

$$\underline{f}(0,2,z) = 1 - \frac{z^2}{12} - \frac{z^4}{720} - \frac{z^6}{30240} - \frac{z^8}{1209600} - \cdots = \frac{1}{2}\left(\frac{iz}{e^{iz} - 1} + \frac{-iz}{e^{-iz} - 1}\right) = \frac{z}{2}\cot\frac{z}{2}$$

$$\underline{f}(1,2,z) = -\frac{z}{2} \qquad = \frac{1}{2i}\left(\frac{iz}{e^{iz} - 1} - \frac{-iz}{e^{-iz} - 1}\right) = -\frac{z}{2}$$

In fact, these right-hand sides are expanded into the Maclaurin series by the formula manipulation software *Mathematica* as follows.

```
Normal[Series[1/2 (i z/(e^(i z) - 1) + (-i z)/(e^(-i z) - 1)), {z, 0, 10}]]
```

$$1 - \frac{z^2}{12} - \frac{z^4}{720} - \frac{z^6}{30240} - \frac{z^8}{1209600} - \frac{z^{10}}{47900160}$$

$$\text{Normal}\left[\text{Series}\left[\frac{1}{2\,i}\left(\frac{i\,z}{e^{i\,z}-1}-\frac{-i\,z}{e^{-i\,z}-1}\right),\{z,0,11\}\right]\right]$$

$$-\frac{z}{2}$$

16.3.3 Two-split of Even and Odd Functions

When $f(z)$ is an even function or an odd function, it cannot be divided into two by 16.3.1 .
Because, the simplified type of 16.3.1 were

$$f(0,2,z) = \frac{f(z)+f(-z)}{2} \quad , \quad f(1,2,z) = \frac{f(z)-f(-z)}{2}$$

When $f(z)$ is an even function, since $f(z)=f(-z)$, $f(0,2,z) = f(z)$, $f(1,2,z) = 0$.
When $f(z)$ is an odd function, since $f(z)= -f(-z)$, $f(0,2,z) = 0$, $f(1,2,z) = f(z)$.

In such a case, it can be divided into two by the following formula. The proof is easy, so we will not mention it.

(1) When $f(z)$ is an even function,

$$f(0,2,z) = \frac{f(z)+f(iz)}{2} \quad , \quad f(1,2,z) = \frac{f(z)-f(iz)}{2}$$

(2) When $f(z)$ is an odd function,

$$f(0,2,z) = \frac{f(z)+i^{-1}f(iz)}{2} \quad , \quad f(1,2,z) = \frac{f(z)-i^{-1}f(iz)}{2}$$

Example 1 $f(z)=\cosh z$

$$1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\frac{z^8}{8!}+\frac{z^{10}}{10!}+\cdots = \cosh z \qquad \text{(Series to be splited)}$$

$$1+\frac{z^4}{4!}+\frac{z^8}{8!}+\frac{z^{12}}{12!}+\frac{z^{16}}{16!}+\frac{z^{20}}{20!}+\cdots = \frac{\cosh z+\cos z}{2}$$

$$\frac{z^2}{2!}+\frac{z^6}{6!}+\frac{z^{10}}{10!}+\frac{z^{14}}{14!}+\frac{z^{18}}{18!}+\cdots = \frac{\cosh z-\cos z}{2}$$

$(\because \ \cosh iz = \cos z)$

Especially when $z=1$,

$$1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\frac{1}{8!}+\frac{1}{10!}+\cdots = \cosh 1 = 1.54308063\cdots$$

$$1+\frac{1}{4!}+\frac{1}{8!}+\frac{1}{12!}+\frac{1}{16!}+\frac{1}{20!}+\cdots = \frac{\cosh 1+\cos 1}{2} = 1.04169147\cdots$$

$$\frac{1}{2!}+\frac{1}{6!}+\frac{1}{10!}+\frac{1}{14!}+\frac{1}{18!}+\cdots = \frac{\cosh 1-\cos 1}{2} = 0.50138916\cdots$$

Example 2 $f(z)=\sinh z$

$$\frac{z^1}{1!}+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\frac{z^9}{9!}+\frac{z^{11}}{11!}+\cdots = \sinh z \qquad \text{(Series to be splited)}$$

$$\frac{z^1}{1!}+\frac{z^5}{5!}+\frac{z^9}{9!}+\frac{z^{13}}{13!}+\frac{z^{17}}{17!}+\cdots = \frac{\sinh z+\sin z}{2}$$

$$\frac{z^3}{3!}+\frac{z^7}{7!}+\frac{z^{11}}{11!}+\frac{z^{15}}{15!}+\frac{z^{19}}{19!}+\cdots = \frac{\sinh z-\sin z}{2}$$

Especially when $z=1$,

$$\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\frac{1}{9!}+\frac{1}{11!}+\cdots = \sinh 1 = 1.17520119\cdots$$

$$\frac{1}{1!}+\frac{1}{5!}+\frac{1}{9!}+\frac{1}{13!}+\frac{1}{17!}+\cdots = \frac{\sinh 1 + \sin 1}{2} = 1.00833608\cdots$$

$$\frac{1}{3!}+\frac{1}{7!}+\frac{1}{11!}+\frac{1}{15!}+\frac{1}{19!}+\cdots = \frac{\sinh 1 - \sin 1}{2} = 0.16686510\cdots$$

16.4 Three-split of Power Series

16.4.1 Three-split of Power Series

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \tag{2.0}$$

Three-split of of the series are

$$f(0,3,z) = \sum_{r=0}^{\infty} a_{3r+0} z^{3r+0} = a_0 z^0 + a_3 z^3 + a_6 z^6 + a_9 z^9 + \cdots$$

$$f(1,3,z) = \sum_{r=0}^{\infty} a_{3r+1} z^{3r+1} = a_1 z^1 + a_4 z^4 + a_7 z^7 + a_{10} z^{10} + \cdots$$

$$f(2,3,z) = \sum_{r=0}^{\infty} a_{3r+2} z^{3r+2} = a_2 z^2 + a_5 z^5 + a_8 z^8 + a_{11} z^{11} + \cdots$$

And Formula 16.2.1 is written down as follows.

$$f(0,3,z) = \frac{f(z)}{3} + \frac{f\{(-1)^{2/3}z\} + f\{(-1)^{-2/3}z\}}{3}$$

$$f(1,3,z) = \frac{f(z)}{3} + \frac{(-1)^{-2/3} f\{(-1)^{2/3}z\} + (-1)^{2/3} f\{(-1)^{-2/3}z\}}{3}$$

$$f(2,3,z) = \frac{f(z)}{3} + \frac{(-1)^{-4/3} f\{(-1)^{2/3}z\} + (-1)^{4/3} f\{(-1)^{-2/3}z\}}{3}$$

Example 1 Three-split of Exponential Series

$$1 + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \cdots = e^z \qquad \text{(Series to be splited)}$$

$$1 + \frac{z^3}{3!} + \frac{z^6}{6!} + \frac{z^9}{9!} + \frac{z^{12}}{12!} + \frac{z^{15}}{15!} + \cdots = \frac{e^z}{3} + \frac{e^{(-1)^{2/3}z} + e^{(-1)^{-2/3}z}}{3} = \frac{e^z}{3} + \frac{2}{3\sqrt{e^z}} \cos\frac{\sqrt{3}\,z}{2}$$

$$\frac{z^1}{1!} + \frac{z^4}{4!} + \frac{z^7}{7!} + \frac{z^{10}}{10!} + \frac{z^{13}}{13!} + \frac{z^{16}}{16!} + \cdots = \frac{e^z}{3} + \frac{(-1)^{-2/3} e^{(-1)^{2/3}z} + (-1)^{2/3} e^{(-1)^{-2/3}z}}{3}$$

$$= \frac{e^z}{3} - \frac{1}{3\sqrt{e^z}} \cos\frac{\sqrt{3}\,z}{2} + \frac{1}{\sqrt{3e^z}} \sin\frac{\sqrt{3}\,z}{2}$$

$$\frac{z^2}{2!} + \frac{z^5}{5!} + \frac{z^8}{8!} + \frac{z^{11}}{11!} + \frac{z^{14}}{14!} + \frac{z^{17}}{17!} + \cdots = \frac{e^z}{3} + \frac{(-1)^{-4/3} e^{(-1)^{2/3}z} + (-1)^{4/3} e^{(-1)^{-2/3}z}}{3}$$

$$= \frac{e^z}{3} - \frac{1}{3\sqrt{e^z}} \cos\frac{\sqrt{3}\,z}{2} - \frac{1}{\sqrt{3e^z}} \sin\frac{\sqrt{3}\,z}{2}$$

Especially when $z=1$,

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots = e^1 = 2.71828182\cdots$$

$$1 + \frac{1}{3!} + \frac{1}{6!} + \frac{1}{9!} + \frac{1}{12!} + \frac{1}{15!} + \cdots = \frac{e}{3} + \frac{2\cos(\sqrt{3}/2)}{3\sqrt{e}} = 1.16805831\cdots$$

$$\frac{1}{1!} + \frac{1}{4!} + \frac{1}{7!} + \frac{1}{10!} + \frac{1}{13!} + \frac{z^{16}}{16!} + \cdots = \frac{e}{3} - \frac{\cos(\sqrt{3}/2)}{3\sqrt{e}} + \frac{\sin(\sqrt{3}/2)}{3\sqrt{e}} = 1.04186535\cdots$$

$$\frac{1}{2!} + \frac{1}{5!} + \frac{1}{8!} + \frac{1}{11!} + \frac{1}{14!} + \frac{1}{17!} + \cdots = \frac{e}{3} - \frac{\cos(\sqrt{3}/2)}{3\sqrt{e}} - \frac{\sin(\sqrt{3}/2)}{3\sqrt{e}} = 0.50835816\cdots$$

Example 2 Three-split of Logarithmic Series ($|z| < 1,\ z \ne 1$)

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\cdots = -\log(1-z) \qquad \text{(Series to be splited)}$$

$$\frac{z^3}{3}+\frac{z^6}{6}+\frac{z^9}{9}+\frac{z^{12}}{12}+\frac{z^{15}}{15}+\frac{z^{18}}{18}+\cdots = -\frac{\log(1-z)}{3} - \frac{\log\{1-(-1)^{2/3}z\}+\log\{1-(-1)^{-2/3}z\}}{3}$$

$$\frac{z^1}{1}+\frac{z^4}{4}+\frac{z^7}{7}+\frac{z^{10}}{10}+\frac{z^{13}}{13}+\frac{z^{16}}{16}+\cdots = -\frac{\log(1-z)}{3} - \frac{(-1)^{-2/3}\log\{1-(-1)^{2/3}z\}+(-1)^{2/3}\log\{1-(-1)^{-2/3}z\}}{3}$$

$$\frac{z^2}{2}+\frac{z^5}{5}+\frac{z^8}{8}+\frac{z^{11}}{11}+\frac{z^{14}}{14}+\frac{z^{17}}{17}+\cdots = -\frac{\log(1-z)}{3} - \frac{(-1)^{-4/3}\log\{1-(-1)^{2/3}z\}+(-1)^{4/3}\log\{1-(-1)^{-2/3}z\}}{3}$$

When $z = 1/3$,

$$\frac{1}{1\cdot 3}+\frac{1}{2\cdot 3^2}+\frac{1}{3\cdot 3^3}+\frac{1}{4\cdot 3^4}+\frac{1}{5\cdot 3^5}+\frac{1}{6\cdot 3^6}+\frac{1}{7\cdot 3^7}+\cdots = -\log\frac{2}{3} = 0.40546510\cdots$$

$$\frac{1}{3\cdot 3^3}+\frac{1}{6\cdot 3^6}+\frac{1}{9\cdot 3^9}+\frac{1}{12\cdot 3^{12}}+\frac{1}{15\cdot 3^{15}}+\frac{1}{18\cdot 3^{18}}+\cdots = \frac{1}{3}\log\frac{3}{2}-\frac{1}{3}\log\frac{13}{9} = 0.01258010\cdots$$

$$\frac{1}{1\cdot 3}+\frac{1}{4\cdot 3^4}+\frac{1}{7\cdot 3^7}+\frac{1}{10\cdot 3^{10}}+\frac{1}{13\cdot 3^{13}}+\frac{1}{16\cdot 3^{16}}+\cdots = \frac{1}{3}\log\frac{3}{2}+\frac{1}{6}\log\frac{13}{9}+\frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}}{7}$$
$$= 0.33648681\cdots$$

$$\frac{1}{2\cdot 3^2}+\frac{1}{5\cdot 3^5}+\frac{1}{8\cdot 3^8}+\frac{1}{11\cdot 3^{10}}+\frac{1}{14\cdot 3^{14}}+\frac{1}{17\cdot 3^{17}}+\cdots = \frac{1}{3}\log\frac{3}{2}+\frac{1}{6}\log\frac{13}{9}-\frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}}{7}$$
$$= 0.05639818\cdots$$

16.4.2 Alternating three-split of Power Series

Alternating three-split of Power Series (2.0) are

$$\underline{f}(0,3,z) = \sum_{r=0}^{\infty}(-1)^r a_{3r+0}z^{3r+0} = a_0z^0 - a_3z^3 + a_6z^6 - a_9z^9 + - \cdots$$

$$\underline{f}(1,3,z) = \sum_{r=0}^{\infty}(-1)^r a_{3r+1}z^{3r+1} = a_1z^1 - a_4z^4 + a_7z^7 - a_{10}z^{10} + - \cdots$$

$$\underline{f}(2,3,z) = \sum_{r=0}^{\infty}(-1)^r a_{3r+2}z^{3r+2} = a_2z^2 - a_5z^5 + a_8z^8 - a_{11}z^{11} + - \cdots$$

And Formula 16.2.2 is written down as follows.

$$\underline{f}(0,3,z) = \frac{f(-z)}{3} + \frac{f\{(-1)^{1/3}z\}+f\{(-1)^{-1/3}z\}}{3}$$

$$\underline{f}(1,3,z) = \frac{(-1)^1 f(-z)}{3} + \frac{(-1)^{-1/3}f\{(-1)^{1/3}z\}+(-1)^{1/3}f\{(-1)^{-1/3}z\}}{3}$$

$$\underline{f}(2,3,z) = \frac{(-1)^2 f(-z)}{3} + \frac{(-1)^{-2/3}f\{(-1)^{1/3}z\}+(-1)^{2/3}f\{(-1)^{-1/3}z\}}{3}$$

Example 1 Alternating three-split of Exponential Series

$$1-\frac{z^3}{3!}+\frac{z^6}{6!}-\frac{z^9}{9!}+\frac{z^{12}}{12!}-\frac{z^{15}}{15!}+\cdots=\frac{(-1)^0e^{-z}}{3}+\frac{e^{(-1)^{1/3}z}+e^{(-1)^{-1/3}z}}{3}$$

$$=\frac{1}{3e^z}+\frac{2\sqrt{e^z}}{3}\cos\frac{\sqrt{3}\,z}{2}$$

$$\frac{z^1}{1!}-\frac{z^4}{4!}+\frac{z^7}{7!}-\frac{z^{10}}{10!}+\frac{z^{13}}{13!}-\frac{z^{16}}{16!}+\cdots=\frac{(-1)^1e^{-z}}{3}+\frac{(-1)^{-1/3}e^{(-1)^{1/3}z}+(-1)^{1/3}e^{(-1)^{-1/3}z}}{3}$$

$$=-\frac{1}{3e^z}+\frac{\sqrt{e^z}}{3}\cos\frac{\sqrt{3}\,z}{2}+\sqrt{\frac{e^z}{3}}\sin\frac{\sqrt{3}\,z}{2}$$

$$\frac{z^2}{2!}-\frac{z^5}{5!}+\frac{z^8}{8!}-\frac{z^{11}}{11!}+\frac{z^{14}}{14!}-\frac{z^{17}}{17!}+\cdots=\frac{(-1)^2e^{-z}}{3}+\frac{(-1)^{-2/3}e^{(-1)^{1/3}z}+(-1)^{2/3}e^{(-1)^{-1/3}z}}{3}$$

$$=\frac{1}{3e^z}-\frac{\sqrt{e^z}}{3}\cos\frac{\sqrt{3}\,z}{2}+\sqrt{\frac{e^z}{3}}\sin\frac{\sqrt{3}\,z}{2}$$

Especially when $z=1$,

$$1-\frac{1}{3!}+\frac{1}{6!}-\frac{1}{9!}+\frac{1}{12!}-\frac{1}{15!}+-\cdots=\frac{1}{3e}+\frac{2}{3}\sqrt{e}\cos\frac{\sqrt{3}}{2}=0.83471946\cdots$$

$$\frac{1}{1!}-\frac{1}{4!}+\frac{1}{7!}-\frac{1}{10!}+\frac{1}{13!}-\frac{1}{16!}+-\cdots=\frac{1}{3e}+\frac{1}{3}\sqrt{e}\cos\frac{\sqrt{3}}{2}+\sqrt{\frac{e}{3}}\sin\frac{\sqrt{3}}{2}$$

$$=0.95853147\cdots$$

$$\frac{1}{2!}-\frac{1}{5!}+\frac{1}{8!}-\frac{1}{11!}+\frac{1}{14!}-\frac{1}{17!}+-\cdots=\frac{1}{3e}-\frac{1}{3}\sqrt{e}\cos\frac{\sqrt{3}}{2}+\sqrt{\frac{e}{3}}\sin\frac{\sqrt{3}}{2}$$

$$=0.49169144\cdots$$

Example 2 Three-split of Logarithmic Series ($|z|\le 1$)

$$-\frac{z^3}{3}+\frac{z^6}{6}-\frac{z^9}{9}+\frac{z^{12}}{12}-\frac{z^{15}}{15}+\frac{z^{18}}{18}-+\cdots=-\frac{\log(1+z)}{3}$$

$$-\frac{\log\{1-(-1)^{1/3}z\}+\log\{1-(-1)^{-1/3}z\}}{3}$$

$$\frac{z^1}{1}-\frac{z^4}{4}+\frac{z^7}{7}-\frac{z^{10}}{10}+\frac{z^{13}}{13}-\frac{z^{16}}{16}+-\cdots=-\frac{(-1)^1\log(1+z)}{3}$$

$$-\frac{(-1)^{-1/3}\log\{1-(-1)^{1/3}z\}+(-1)^{1/3}\log\{1-(-1)^{-1/3}z\}}{3}$$

$$\frac{z^2}{2}-\frac{z^5}{5}+\frac{z^8}{8}-\frac{z^{11}}{11}+\frac{z^{14}}{14}-\frac{z^{17}}{17}+-\cdots=-\frac{(-1)^2\log(1+z)}{3}$$

$$-\frac{(-1)^{-2/3}\log\{1-(-1)^{1/3}z\}+(-1)^{2/3}\log\{1-(-1)^{-1/3}z\}}{3}$$

Especially when $z=1$,

$$-\frac{1}{3}+\frac{1}{6}-\frac{1}{9}+\frac{1}{12}-\frac{1}{15}+\frac{1}{18}-+\cdots=-\frac{\log 2}{3}=-0.23104906\cdots$$

$$\frac{1}{1}-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\frac{1}{13}-\frac{1}{16}+-\cdots=\frac{1}{3}\left(\frac{\pi}{\sqrt{3}}+\log 2\right)=0.83564884\cdots$$

$$\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \frac{1}{14} - \frac{1}{17} +- \cdots = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} - log\,2 \right) = 0.37355072\cdots$$

cf. Mercator series

$$\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} +- \cdots = log(1+z)$$

16.5 Four-split of Power Series

16.5.1 Four-split of Power Series

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \tag{2.0}$$

Four-split of of the series are

$$f(0,4,z) = \sum_{r=0}^{\infty} a_{4r+0} z^{4r+0} = a_0 z^0 + a_4 z^4 + a_8 z^8 + a_{12} z^{12} + \cdots$$

$$f(1,4,z) = \sum_{r=0}^{\infty} a_{4r+1} z^{4r+1} = a_1 z^1 + a_5 z^5 + a_9 z^9 + a_{13} z^{13} + \cdots$$

$$f(2,4,z) = \sum_{r=0}^{\infty} a_{4r+2} z^{4r+2} = a_2 z^2 + a_6 z^6 + a_{10} z^{10} + a_{14} z^{14} + \cdots$$

$$f(3,4,z) = \sum_{r=0}^{\infty} a_{4r+3} z^{4r+3} = a_3 z^3 + a_7 z^7 + a_{11} z^{11} + a_{15} z^{15} + \cdots$$

And Formula 16.2.1 is written down as follows.

$$f(0,4,z) = \frac{f(z) - f(-z)}{4} + \frac{f\{(-1)^{2/4} z\} + f\{(-1)^{-2/4} z\}}{4} + \frac{f\{(-1)^{4/4} z\} + f\{(-1)^{-4/4} z\}}{4}$$

$$f(1,4,z) = \frac{f(z) + f(-z)}{4} + \frac{(-1)^{-2/4} f\{(-1)^{2/4} z\} + (-1)^{2/4} f\{(-1)^{-2/4} z\}}{4} + \frac{(-1)^{-4/4} f\{(-1)^{4/4} z\} + (-1)^{4/4} f\{(-1)^{-4/4} z\}}{4}$$

$$f(2,4,z) = \frac{f(z) - f(-z)}{4} + \frac{(-1)^{-4/4} f\{(-1)^{2/4} z\} + (-1)^{4/4} f\{(-1)^{-2/4} z\}}{4} + \frac{(-1)^{-8/4} f\{(-1)^{4/4} z\} + (-1)^{8/4} f\{(-1)^{-4/4} z\}}{4}$$

$$f(3,4,z) = \frac{f(z) + f(-z)}{4} + \frac{(-1)^{-6/4} f\{(-1)^{2/4} z\} + (-1)^{6/4} f\{(-1)^{-2/4} z\}}{4} + \frac{(-1)^{-12/4} f\{(-1)^{4/4} z\} + (-1)^{12/4} f\{(-1)^{-4/4} z\}}{4}$$

Example 1 Four-split of Binomial Series ($|z| < 1,\ z \neq 1$)

$$1 + \frac{1!!}{2!!} z^1 + \frac{3!!}{4!!} z^2 + \frac{5!!}{6!!} z^3 + \frac{7!!}{8!!} z^4 + \frac{9!!}{10!!} z^5 + \cdots = \frac{1}{\sqrt{1-z}} \quad \text{(Series to be splited)}$$

$$1 + \frac{7!!}{8!!} z^4 + \frac{15!!}{16!!} z^8 + \frac{23!!}{24!!} z^{12} + \frac{31!!}{32!!} z^{16} + \frac{39!!}{40!!} z^{20} + \cdots = \frac{1}{4}\left(\frac{1}{\sqrt{1-z}} - \frac{1}{\sqrt{1+z}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{1-(-1)^{2/4} z}} + \frac{1}{\sqrt{1-(-1)^{-2/4} z}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{1-(-1)^{4/4} z}} + \frac{1}{\sqrt{1-(-1)^{-4/4} z}}\right)$$

$$\frac{1!!}{2!!} z^1 + \frac{9!!}{10!!} z^5 + \frac{17!!}{18!!} z^9 + \frac{25!!}{26!!} z^{13} + \frac{33!!}{34!!} z^{17} + \frac{41!!}{42!!} z^{21} + \cdots = \frac{1}{4}\left(\frac{1}{\sqrt{1-z}} + \frac{1}{\sqrt{1+z}}\right)$$

$$+\frac{1}{4}\left(\frac{(-1)^{-2/4}}{\sqrt{1-(-1)^{2/4}z}}+\frac{(-1)^{2/4}}{\sqrt{1-(-1)^{-2/4}z}}\right)$$
$$+\frac{1}{4}\left(\frac{(-1)^{-4/4}}{\sqrt{1-(-1)^{4/4}z}}+\frac{(-1)^{4/4}}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

$$\frac{3!!}{4!!}z^2+\frac{11!!}{12!!}z^6+\frac{19!!}{20!!}z^{10}+\frac{27!!}{28!!}z^{14}+\frac{35!!}{36!!}z^{18}+\frac{43!!}{44!!}z^{22}+\cdots=\frac{1}{4}\left(\frac{1}{\sqrt{1-z}}-\frac{1}{\sqrt{1+z}}\right)$$
$$+\frac{1}{4}\left(\frac{(-1)^{-4/4}}{\sqrt{1-(-1)^{2/4}z}}+\frac{(-1)^{4/4}}{\sqrt{1-(-1)^{-2/4}z}}\right)$$
$$+\frac{1}{4}\left(\frac{(-1)^{-8/4}}{\sqrt{1-(-1)^{4/4}z}}+\frac{(-1)^{8/4}}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

$$\frac{5!!}{6!!}z^3+\frac{13!!}{14!!}z^7+\frac{21!!}{22!!}z^{11}+\frac{29!!}{30!!}z^{15}+\frac{37!!}{38!!}z^{19}+\frac{45!!}{46!!}z^{23}+\cdots=\frac{1}{4}\left(\frac{1}{\sqrt{1-z}}+\frac{1}{\sqrt{1+z}}\right)$$
$$+\frac{1}{4}\left(\frac{(-1)^{-6/4}}{\sqrt{1-(-1)^{2/4}z}}+\frac{(-1)^{6/4}}{\sqrt{1-(-1)^{-2/4}z}}\right)$$
$$+\frac{1}{4}\left(\frac{(-1)^{-12/4}}{\sqrt{1-(-1)^{4/4}z}}+\frac{(-1)^{12/4}}{\sqrt{1-(-1)^{-4/4}z}}\right)$$

If the right side of the last expression is expanded into the Maclaurin series by the formula manipulation software *Mathematica* , it is as follows.

$$\mathbf{f[3, 4, z\_] := \frac{1}{4}\left(\frac{1}{\sqrt{1-z}}+\frac{1}{\sqrt{1+z}}\right)+\frac{1}{4}\left(\frac{(-1)^{-6/4}}{\sqrt{1-(-1)^{2/4}z}}+\frac{(-1)^{6/4}}{\sqrt{1-(-1)^{-2/4}z}}\right)+\frac{1}{4}\left(\frac{(-1)^{-12/4}}{\sqrt{1-(-1)^{4/4}z}}+\frac{(-1)^{-12/4}}{\sqrt{1-(-1)^{-4/4}z}}\right)}$$

```
Normal[Simplify[Series[f[3, 4, z], {z, 0, 19}]]]
```

$$\frac{5z^3}{16}+\frac{429z^7}{2048}+\frac{88179z^{11}}{524288}+\frac{9694845z^{15}}{67108864}+\frac{4418157975z^{19}}{34359738368}$$

16.5.2 Alternating four-split of Power Series

Alternating four-split of Power Series (2.0) are

$$\underline{f}(0,4,z)=\sum_{r=0}^{\infty}(-1)^r a_{4r+0}z^{4r+0}=a_0z^0-a_4z^4+a_8z^8-a_{12}z^{12}+-\cdots$$
$$\underline{f}(1,4,z)=\sum_{r=0}^{\infty}(-1)^r a_{4r+1}z^{4r+1}=a_1z^1-a_5z^5+a_9z^9-a_{13}z^{13}+-\cdots$$
$$\underline{f}(2,4,z)=\sum_{r=0}^{\infty}(-1)^r a_{4r+2}z^{4r+2}=a_2z^2-a_6z^6+a_{10}z^{10}-a_{14}z^{14}+-\cdots$$
$$\underline{f}(3,4,z)=\sum_{r=0}^{\infty}(-1)^r a_{4r+3}z^{4r+3}=a_3z^3-a_7z^7+a_{11}z^{11}-a_{15}z^{15}+-\cdots$$

And Formula 16.2.2 is written down as follows.

$$\underline{f}(0,4,z)=\frac{f\{(-1)^{1/4}z\}+f\{(-1)^{-1/4}z\}}{4}+\frac{f\{(-1)^{3/4}z\}+f\{(-1)^{-3/4}z\}}{4}$$

$$\underline{f}(1,4,z) = \frac{(-1)^{-1/4} f\{(-1)^{1/4} z\} + (-1)^{1/4} f\{(-1)^{-1/4} z\}}{4} + \frac{(-1)^{-3/4} f\{(-1)^{3/4} z\} + (-1)^{3/4} f\{(-1)^{-3/4} z\}}{4}$$

$$\underline{f}(2,4,z) = \frac{(-1)^{-2/4} f\{(-1)^{1/4} z\} + (-1)^{2/4} f\{(-1)^{-1/4} z\}}{4} + \frac{(-1)^{-6/4} f\{(-1)^{3/4} z\} + (-1)^{6/4} f\{(-1)^{-3/4} z\}}{4}$$

$$\underline{f}(3,4,z) = \frac{(-1)^{-3/4} f\{(-1)^{1/4} z\} + (-1)^{3/4} f\{(-1)^{-1/4} z\}}{4} + \frac{(-1)^{-9/4} f\{(-1)^{3/4} z\} + (-1)^{9/4} f\{(-1)^{-3/4} z\}}{4}$$

Example 1 Alternating four-split of Logarithmic Series ($|z| \le 1$)

$$-\frac{z^4}{4} + \frac{z^8}{8} - \frac{z^{12}}{12} + \frac{z^{16}}{16} - \frac{z^{20}}{20} + \frac{z^{24}}{24} - + \cdots = -\frac{\log\{1-(-1)^{1/4} z\} + \log\{1-(-1)^{-1/4} z\}}{4} - \frac{\log\{1-(-1)^{3/4} z\} + \log\{1-(-1)^{-3/4} z\}}{4}$$

$$\frac{z^1}{1} - \frac{z^5}{5} + \frac{z^9}{9} - \frac{z^{13}}{13} + \frac{z^{17}}{17} - \frac{z^{21}}{21} + - \cdots = -\frac{(-1)^{-1/4}\log\{1-(-1)^{1/4} z\} + (-1)^{1/4}\log\{1-(-1)^{-1/4} z\}}{4} - \frac{(-1)^{-3/4}\log\{1-(-1)^{3/4} z\} + (-1)^{3/4}\log\{1-(-1)^{-3/4} z\}}{4}$$

$$\frac{z^2}{2} - \frac{z^6}{6} + \frac{z^{10}}{10} - \frac{z^{14}}{14} + \frac{z^{18}}{18} - \frac{z^{22}}{22} + - \cdots = -\frac{(-1)^{-2/4}\log\{1-(-1)^{1/4} z\} + (-1)^{2/4}\log\{1-(-1)^{-1/4} z\}}{4} - \frac{(-1)^{-6/4}\log\{1-(-1)^{3/4} z\} + (-1)^{6/4}\log\{1-(-1)^{-3/4} z\}}{4}$$

$$\frac{z^3}{3} - \frac{z^7}{7} + \frac{z^{11}}{11} - \frac{z^{15}}{15} + \frac{z^{19}}{19} - \frac{z^{23}}{23} + - \cdots = -\frac{(-1)^{-3/4}\log\{1-(-1)^{1/4} z\} + (-1)^{3/4}\log\{1-(-1)^{-1/4} z\}}{4} - \frac{(-1)^{-9/4}\log\{1-(-1)^{3/4} z\} + (-1)^{9/4}\log\{1-(-1)^{-3/4} z\}}{4}$$

Especially when $z=1$,

$$-\frac{1}{4} + \frac{1}{8} - \frac{1}{12} + \frac{1}{16} - \frac{1}{20} + \frac{1}{24} - + \cdots = -\frac{\log 2}{4} = -0.17328679\cdots$$

$$\frac{1}{1} - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \frac{1}{17} - \frac{1}{21} + - \cdots = \frac{\pi + 2\operatorname{arccoth}\sqrt{2}}{4\sqrt{2}} = 0.86697298\cdots$$

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \frac{1}{14} + \frac{1}{18} - \frac{1}{22} + - \cdots = \frac{\pi}{8} = 0.39269908\cdots$$

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{19}-\frac{1}{23}+-\cdots=\frac{\pi-2\operatorname{arccoth}\sqrt{2}}{4\sqrt{2}}=0.24374774\cdots$$

Example 2 Alternating four-split of Exponential Series

$$1-\frac{z^4}{4!}+\frac{z^8}{8!}-\frac{z^{12}}{12!}+\frac{z^{16}}{16!}-\frac{z^{20}}{20!}+-\cdots=\frac{e^{(-1)^{1/4}z}+e^{(-1)^{-1/4}z}}{4}+\frac{e^{(-1)^{3/4}z}+e^{(-1)^{-3/4}z}}{4}$$

$$=\cosh\frac{z}{\sqrt{2}}\cdot\cos\frac{z}{\sqrt{2}}$$

$$\frac{z^1}{1!}-\frac{z^5}{5!}+\frac{z^9}{9!}+\frac{z^{13}}{13!}+\frac{z^{17}}{17!}+\frac{z^{21}}{21!}+-\cdots=\frac{(-1)^{-1/4}e^{(-1)^{1/4}z}+(-1)^{1/4}e^{(-1)^{-1/4}z}}{4}$$

$$+\frac{(-1)^{-3/4}e^{(-1)^{3/4}z}+(-1)^{3/4}e^{(-1)^{-3/4}z}}{5}$$

$$=\frac{e^{z/\sqrt{2}}}{2\sqrt{2}}\left(\cos\frac{z}{\sqrt{2}}+\sin\frac{z}{\sqrt{2}}\right)-\frac{e^{-z/\sqrt{2}}}{2\sqrt{2}}\left(\cos\frac{z}{\sqrt{2}}-\sin\frac{z}{\sqrt{2}}\right)$$

$$\frac{z^2}{2!}-\frac{z^6}{6!}+\frac{z^{10}}{10!}-\frac{z^{14}}{14!}+\frac{z^{18}}{18!}-\frac{z^{22}}{22!}+-\cdots=\frac{(-1)^{-2/4}e^{(-1)^{1/4}z}+(-1)^{2/4}e^{(-1)^{-1/4}z}}{4}$$

$$+\frac{(-1)^{-6/4}e^{(-1)^{3/4}z}+(-1)^{6/4}e^{(-1)^{-3/4}z}}{4}$$

$$=\sinh\frac{z}{\sqrt{2}}\cdot\sin\frac{z}{\sqrt{2}}$$

$$\frac{z^3}{3!}-\frac{z^7}{7!}+\frac{z^{11}}{11!}-\frac{z^{15}}{15!}+\frac{z^{19}}{19}-\frac{z^{23}}{23!}+-\cdots=\frac{(-1)^{-3/4}e^{(-1)^{1/4}z}+(-1)^{3/4}e^{(-1)^{-1/4}z}}{4}$$

$$+\frac{(-1)^{-9/4}e^{(-1)^{3/4}z}+(-1)^{9/4}e^{(-1)^{-3/4}z}}{4}$$

$$=-\frac{e^{z/\sqrt{2}}}{2\sqrt{2}}\left(\cos\frac{z}{\sqrt{2}}-\sin\frac{z}{\sqrt{2}}\right)+\frac{e^{-z/\sqrt{2}}}{2\sqrt{2}}\left(\cos\frac{z}{\sqrt{2}}+\sin\frac{z}{\sqrt{2}}\right)$$

Especially when $z=1$,

$$1-\frac{1}{4!}+\frac{1}{8!}-\frac{1}{12!}+\frac{1}{16!}-\frac{1}{20!}+-\cdots=\cos\frac{1}{\sqrt{2}}\cosh\frac{1}{\sqrt{2}}=0.95835813\cdots$$

$$\frac{1}{1!}-\frac{1}{5!}+\frac{1}{9!}+\frac{1}{13!}+\frac{1}{17!}-\frac{1}{21!}+-\cdots=\frac{1-i}{2\sqrt{2}}\left(\sin\frac{1+i}{\sqrt{2}}+\sinh\frac{1+i}{\sqrt{2}}\right)=0.99166942\cdots$$

$$\frac{1}{2!}-\frac{1}{6!}+\frac{1}{10!}-\frac{1}{14!}+\frac{1}{18!}-\frac{1}{22!}+-\cdots=\sin\frac{1}{\sqrt{2}}\sinh\frac{1}{\sqrt{2}}=0.49861138\cdots$$

$$\frac{1}{3!}-\frac{1}{7!}+\frac{1}{11!}-\frac{1}{15!}+\frac{1}{19}-\frac{1}{23!}+-\cdots=\frac{1+i}{2\sqrt{2}}\left(\sin\frac{1+i}{\sqrt{2}}-\sinh\frac{1+i}{\sqrt{2}}\right)=0.16646827\cdots$$

16.6 Five-split of Power Series

16.6.1 Five-split of Power Series

Suppose that the function $f(z)$ is expanded into a power series on the domain D as follows.

$$f(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \tag{2.0}$$

Five-split of of the series are

$$f(0,5,z) = \sum_{r=0}^{\infty} a_{5r+0} z^{5r+0} = a_0 z^0 + a_5 z^5 + a_{10} z^{10} + a_{15} z^{15} + \cdots$$

$$f(1,5,z) = \sum_{r=0}^{\infty} a_{5r+1} z^{5r+1} = a_1 z^1 + a_6 z^6 + a_{11} z^{11} + a_{16} z^{16} + \cdots$$

$$f(2,5,z) = \sum_{r=0}^{\infty} a_{5r+2} z^{5r+2} = a_2 z^2 + a_7 z^7 + a_{12} z^{12} + a_{17} z^{17} + \cdots$$

$$f(3,5,z) = \sum_{r=0}^{\infty} a_{5r+3} z^{5r+3} = a_3 z^3 + a_8 z^8 + a_{13} z^{13} + a_{18} z^{18} + \cdots$$

$$f(4,5,z) = \sum_{r=0}^{\infty} a_{5r+4} z^{5r+4} = a_4 z^4 + a_9 z^9 + a_{14} z^{14} + a_{19} z^{19} + \cdots$$

And Formula 16.2.1 is written down as follows.

$$f(0,5,z) = \frac{f(z)}{5} + \frac{f\{(-1)^{2/5}z\} + f\{(-1)^{-2/5}z\}}{5} + \frac{f\{(-1)^{4/5}z\} + f\{(-1)^{-4/5}z\}}{5}$$

$$f(1,5,z) = \frac{f(z)}{5} + \frac{(-1)^{-2/5} f\{(-1)^{2/5}z\} + (-1)^{2/5} f\{(-1)^{-2/5}z\}}{5} + \frac{(-1)^{-4/5} f\{(-1)^{4/5}z\} + (-1)^{4/5} f\{(-1)^{-4/5}z\}}{5}$$

$$f(2,5,z) = \frac{f(z)}{5} + \frac{(-1)^{-4/5} f\{(-1)^{2/5}z\} + (-1)^{4/5} f\{(-1)^{-2/5}z\}}{5} + \frac{(-1)^{-8/5} f\{(-1)^{4/5}z\} + (-1)^{8/5} f\{(-1)^{-4/5}z\}}{5}$$

$$f(3,5,z) = \frac{f(z)}{5} + \frac{(-1)^{-6/5} f\{(-1)^{2/5}z\} + (-1)^{6/5} f\{(-1)^{-2/5}z\}}{5} + \frac{(-1)^{-12/5} f\{(-1)^{4/5}z\} + (-1)^{12/5} f\{(-1)^{-4/5}z\}}{5}$$

$$f(4,5,z) = \frac{f(z)}{5} + \frac{(-1)^{-8/5} f\{(-1)^{2/5}z\} + (-1)^{8/5} f\{(-1)^{-2/5}z\}}{5} + \frac{(-1)^{-16/5} f\{(-1)^{4/5}z\} + (-1)^{16/5} f\{(-1)^{-4/5}z\}}{5}$$

Example 1 Five-split of Exponential Series

$$1 + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \cdots = e^z \qquad \text{(Series to be splited)}$$

$$1 + \frac{z^5}{5!} + \frac{z^{10}}{10!} + \frac{z^{15}}{15!} + \frac{z^{20}}{20!} + \frac{z^{25}}{25!} + \cdots = \frac{e^z}{5} + \frac{e^{(-1)^{2/5}z} + e^{(-1)^{-2/5}z}}{5} + \frac{e^{(-1)^{4/5}z} + e^{(-1)^{-4/5}z}}{5}$$

$$\frac{z^1}{1!}+\frac{z^6}{6!}+\frac{z^{11}}{11!}+\frac{z^{16}}{16!}+\frac{z^{21}}{21!}+\frac{z^{26}}{26!}+\cdots = \frac{e^z}{5}+\frac{(-1)^{-2/5}e^{(-1)^{2/5}z}+(-1)^{2/5}e^{(-1)^{-2/5}z}}{5}$$
$$+\frac{(-1)^{-4/5}e^{(-1)^{4/5}z}+(-1)^{4/5}e^{(-1)^{-4/5}z}}{5}$$

$$\frac{z^2}{2!}+\frac{z^7}{7!}+\frac{z^{12}}{12!}+\frac{z^{17}}{17!}+\frac{z^{22}}{22!}+\frac{z^{27}}{27!}+\cdots = \frac{e^z}{5}+\frac{(-1)^{-4/5}e^{(-1)^{2/5}z}+(-1)^{4/5}e^{(-1)^{-2/5}z}}{5}$$
$$+\frac{(-1)^{-8/5}e^{(-1)^{4/5}z}+(-1)^{8/5}e^{(-1)^{-4/5}z}}{5}$$

$$\frac{z^3}{3!}+\frac{z^8}{8!}+\frac{z^{13}}{13!}+\frac{z^{18}}{18!}+\frac{z^{23}}{23!}+\frac{z^{28}}{28!}+\cdots = \frac{e^z}{5}+\frac{(-1)^{-6/5}e^{(-1)^{2/5}z}+(-1)^{6/5}e^{(-1)^{-2/5}z}}{5}$$
$$+\frac{(-1)^{-12/5}e^{(-1)^{4/5}z}+(-1)^{12/5}e^{(-1)^{-4/5}z}}{5}$$

$$\frac{z^4}{4!}+\frac{z^9}{9!}+\frac{z^{14}}{14!}+\frac{z^{19}}{19!}+\frac{z^{24}}{24!}+\frac{z^{29}}{29!}+\cdots = \frac{e^z}{5}+\frac{(-1)^{-8/5}e^{(-1)^{2/5}z}+(-1)^{8/5}e^{(-1)^{-2/5}z}}{5}$$
$$+\frac{(-1)^{-16/5}e^{(-1)^{4/5}z}+(-1)^{16/5}e^{(-1)^{-4/5}z}}{5}$$

Especially when $z=1$,

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots = 2.71828182\cdots$$
$$1+\frac{1}{5!}+\frac{1}{10!}+\frac{1}{15!}+\frac{1}{20!}+\frac{1}{25!}+\cdots = 1.00833360\cdots$$
$$\frac{1}{1!}+\frac{1}{6!}+\frac{1}{11!}+\frac{1}{16!}+\frac{1}{21!}+\frac{1}{26!}+\cdots = 1.00138891\cdots$$
$$\frac{1}{2!}+\frac{1}{7!}+\frac{1}{12!}+\frac{1}{17!}+\frac{1}{22!}+\frac{1}{27!}+\cdots = 0.50019841\cdots$$
$$\frac{1}{3!}+\frac{1}{8!}+\frac{1}{13!}+\frac{1}{18!}+\frac{1}{23!}+\frac{1}{28!}+\cdots = 0.16669146\cdots$$
$$\frac{1}{4!}+\frac{1}{9!}+\frac{1}{14!}+\frac{1}{19!}+\frac{1}{24!}+\frac{1}{29!}+\cdots = 0.04166942\cdots$$

Example 2 Five-split of Logarithmic Series ($|z| < 1,\ z \neq 1$)

$$\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\frac{z^6}{6}+\cdots = -\log(1-z)$$ (Series to be splited)

$$\frac{z^5}{5}+\frac{z^{10}}{10}+\frac{z^{15}}{15}+\frac{z^{20}}{20}+\frac{z^{25}}{25}+\frac{z^{30}}{30}+\cdots = -\frac{\log(1-z)}{5}$$
$$-\frac{\log\{1-(-1)^{2/5}z\}+\log\{1-(-1)^{-2/5}z\}}{5}$$
$$-\frac{\log\{1-(-1)^{4/5}z\}+\log\{1-(-1)^{-4/5}z\}}{5}$$

$$\frac{z^1}{1}+\frac{z^6}{6}+\frac{z^{11}}{11}+\frac{z^{16}}{16}+\frac{z^{21}}{21}+\frac{z^{26}}{26}+\cdots = -\frac{\log(1-z)}{5}$$
$$-\frac{(-1)^{-2/5}\log\{1-(-1)^{2/5}z\}+(-1)^{2/5}\log\{1-(-1)^{-2/5}z\}}{5}$$

$$-\frac{(-1)^{-4/5}log\{1-(-1)^{4/5}z\}+(-1)^{4/5}log\{1-(-1)^{-4/5}z\}}{5}$$

$$\frac{z^2}{2}+\frac{z^7}{7}+\frac{z^{12}}{12}+\frac{z^{17}}{17}+\frac{z^{22}}{22}+\frac{z^{27}}{27}+\cdots = -\frac{log(1-z)}{5}$$

$$-\frac{(-1)^{-4/5}log\{1-(-1)^{2/5}z\}+(-1)^{4/5}log\{1-(-1)^{-2/5}z\}}{5}$$

$$-\frac{(-1)^{-8/5}log\{1-(-1)^{4/5}z\}+(-1)^{8/5}log\{1-(-1)^{-4/5}z\}}{5}$$

$$\frac{z^3}{3}+\frac{z^8}{8}+\frac{z^{13}}{13}+\frac{z^{18}}{18}+\frac{z^{23}}{23}+\frac{z^{28}}{28}+\cdots = -\frac{log(1-z)}{5}$$

$$-\frac{(-1)^{-6/5}log\{1-(-1)^{2/5}z\}+(-1)^{6/5}log\{1-(-1)^{-2/5}z\}}{5}$$

$$-\frac{(-1)^{-12/5}log\{1-(-1)^{4/5}z\}+(-1)^{12/5}log\{1-(-1)^{-4/5}z\}}{5}$$

$$\frac{z^4}{4}+\frac{z^9}{9}+\frac{z^{14}}{14}+\frac{z^{19}}{19}+\frac{z^{24}}{24}+\frac{z^{29}}{29}+\cdots = -\frac{log(1-z)}{5}$$

$$-\frac{(-1)^{-8/5}log\{1-(-1)^{2/5}z\}+(-1)^{8/5}log\{1-(-1)^{-2/5}z\}}{5}$$

$$-\frac{(-1)^{-16/5}log\{1-(-1)^{4/5}z\}+(-1)^{16/5}log\{1-(-1)^{-4/5}z\}}{5}$$

When $z=1/2$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}+\frac{1}{6\cdot 2^6}+\frac{1}{7\cdot 2^7}+\cdots = 0.69314718\cdots = log\,2$$

$$\frac{1}{5\cdot 2^5}+\frac{1}{10\cdot 2^{10}}+\frac{1}{15\cdot 2^{15}}+\frac{1}{20\cdot 2^{20}}+\frac{1}{25\cdot 2^{25}}+\frac{1}{30\cdot 2^{30}}+\cdots = 0.00634973\cdots = log\,2-\frac{log\,31}{5}$$

$$\frac{1}{1\cdot 2}+\frac{1}{6\cdot 2^6}+\frac{1}{11\cdot 2^{11}}+\frac{1}{16\cdot 2^{16}}+\frac{1}{21\cdot 2^{21}}+\frac{1}{26\cdot 2^{26}}+\cdots = 0.50264953\cdots$$

$$\frac{1}{2\cdot 2^2}+\frac{1}{7\cdot 2^7}+\frac{1}{12\cdot 2^{12}}+\frac{1}{17\cdot 2^{17}}+\frac{1}{22\cdot 2^{22}}+\frac{1}{27\cdot 2^{27}}+\cdots = 0.12613687\cdots$$

$$\frac{1}{3\cdot 2^3}+\frac{1}{8\cdot 2^8}+\frac{1}{13\cdot 2^{13}}+\frac{1}{18\cdot 2^{18}}+\frac{1}{23\cdot 2^{23}}+\frac{1}{28\cdot 2^{28}}+\cdots = 0.04216455\cdots$$

$$\frac{1}{4\cdot 2^4}+\frac{1}{9\cdot 2^9}+\frac{1}{14\cdot 2^{14}}+\frac{1}{19\cdot 2^{19}}+\frac{1}{24\cdot 2^{24}}+\frac{1}{29\cdot 2^{29}}+\cdots = 0.01584647\cdots$$

16.6.2 Alternating five-split of Power Series

Alternating five-split of Power Series (2.0) are

$$\underline{f}(0,5,z) = \sum_{r=1}^{\infty}(-1)^r a_{5r+0}z^{5r+0} = a_0z^0 - a_5z^5 + a_{10}z^{10} - a_{15}z^{15} +-\cdots$$

$$\underline{f}(1,5,z) = \sum_{r=0}^{\infty}(-1)^r a_{5r+1}z^{5r+1} = a_1z^1 - a_6z^6 + a_{11}z^{11} - a_{16}z^{16} +-\cdots$$

$$\underline{f}(2,5,z) = \sum_{r=0}^{\infty}(-1)^r a_{5r+2}z^{5r+2} = a_2z^2 - a_7z^7 + a_{12}z^{12} - a_{17}z^{17} +-\cdots$$

$$\underline{f}(3,5,z) = \sum_{r=0}^{\infty}(-1)^r a_{5r+3}z^{5r+3} = a_3z^3 - a_8z^8 + a_{13}z^{13} - a_{18}z^{18} +-\cdots$$

$$\underline{f}(4,5,z) = \sum_{r=0}^{\infty}(-1)^r a_{5r+4} z^{5r+4} = a_4 z^4 - a_9 z^9 + a_{14} z^{14} - a_{19} z^{19} + - \cdots$$

And Formula 16.2.2 is written down as follows.

$$\underline{f}(0,5,z) = \frac{f(-z)}{5} + \frac{f\{(-1)^{1/5}z\} + f\{(-1)^{-1/5}z\}}{5} + \frac{f\{(-1)^{3/5}z\} + f\{(-1)^{-3/5}z\}}{5}$$

$$\underline{f}(1,5,z) = \frac{(-1)^1 f(-z)}{5} + \frac{(-1)^{-1/5} f\{(-1)^{1/5}z\} + (-1)^{1/5} f\{(-1)^{-1/5}z\}}{5} + \frac{(-1)^{-3/5} f\{(-1)^{3/5}z\} + (-1)^{3/5} f\{(-1)^{-3/5}z\}}{5}$$

$$\underline{f}(2,5,z) = \frac{(-1)^2 f(-z)}{5} + \frac{(-1)^{-2/5} f\{(-1)^{1/5}z\} + (-1)^{2/5} f\{(-1)^{-1/5}z\}}{5} + \frac{(-1)^{-6/5} f\{(-1)^{3/5}z\} + (-1)^{6/5} f\{(-1)^{-3/5}z\}}{5}$$

$$\underline{f}(3,5,z) = \frac{(-1)^3 f(-z)}{5} + \frac{(-1)^{-3/5} f\{(-1)^{1/5}z\} + (-1)^{3/5} f\{(-1)^{-1/5}z\}}{5} + \frac{(-1)^{-9/5} f\{(-1)^{3/5}z\} + (-1)^{9/5} f\{(-1)^{-3/5}z\}}{5}$$

$$\underline{f}(4,5,z) = \frac{(-1)^4 f(-z)}{5} + \frac{(-1)^{-4/5} f\{(-1)^{1/5}z\} + (-1)^{4/5} f\{(-1)^{-1/5}z\}}{5} + \frac{(-1)^{-12/5} f\{(-1)^{3/5}z\} + (-1)^{12/5} f\{(-1)^{-3/5}z\}}{5}$$

Example 1 Alternating five-split of Exponential Series

$$1 - \frac{z^5}{5!} + \frac{z^{10}}{10!} - \frac{z^{15}}{15!} + \frac{z^{20}}{20!} - \frac{z^{25}}{25!} + \cdots = \frac{e^z}{5} + \frac{e^{(-1)^{1/5}z} + e^{(-1)^{-1/5}z}}{5} + \frac{e^{(-1)^{3/5}z} + e^{(-1)^{-3/5}z}}{5}$$

$$\frac{z^1}{1!} - \frac{z^6}{6!} + \frac{z^{11}}{11!} - \frac{z^{16}}{16!} + \frac{z^{21}}{21!} - \frac{z^{26}}{26!} + \cdots = -\frac{e^z}{5} + \frac{(-1)^{-1/5} e^{(-1)^{1/5}z} + (-1)^{1/5} e^{(-1)^{-1/5}z}}{5} + \frac{(-1)^{-3/5} e^{(-1)^{3/5}z} + (-1)^{3/5} e^{(-1)^{-3/5}z}}{5}$$

$$\frac{z^2}{2!} - \frac{z^7}{7!} + \frac{z^{12}}{12!} - \frac{z^{17}}{17!} + \frac{z^{22}}{22!} - \frac{z^{27}}{27!} + \cdots = \frac{e^z}{5} + \frac{(-1)^{-2/5} e^{(-1)^{1/5}z} + (-1)^{2/5} e^{(-1)^{-1/5}z}}{5} + \frac{(-1)^{-6/5} e^{(-1)^{3/5}z} + (-1)^{6/5} e^{(-1)^{-3/5}z}}{5}$$

$$\frac{z^3}{3!} - \frac{z^8}{8!} + \frac{z^{13}}{13!} - \frac{z^{18}}{18!} + \frac{z^{23}}{23!} - \frac{z^{28}}{28!} + \cdots = -\frac{e^z}{5} + \frac{(-1)^{-3/5} e^{(-1)^{1/5}z} + (-1)^{3/5} e^{(-1)^{-1/5}z}}{5} + \frac{(-1)^{-9/5} e^{(-1)^{3/5}z} + (-1)^{9/5} e^{(-1)^{-3/5}z}}{5}$$

$$\frac{z^4}{4!} - \frac{z^9}{9!} + \frac{z^{14}}{14!} - \frac{z^{19}}{19!} + \frac{z^{24}}{24!} - \frac{z^{29}}{29!} + \cdots = \frac{e^z}{5} + \frac{(-1)^{-4/5} e^{(-1)^{1/5}z} + (-1)^{4/5} e^{(-1)^{-1/5}z}}{5}$$

$$+\frac{(-1)^{-12/5}e^{(-1)^{3/5}z}+(-1)^{12/5}e^{(-1)^{-3/5}z}}{5}$$

Especially when $z=1$,

$$1-\frac{1}{5!}+\frac{1}{10!}-\frac{1}{15!}+\frac{1}{20!}-\frac{1}{25!}+-\cdots=0.99166694\cdots$$

$$\frac{1}{1!}-\frac{1}{6!}+\frac{1}{11!}-\frac{1}{16!}+\frac{1}{21!}-\frac{1}{26!}+-\cdots=0.99861113\cdots$$

$$\frac{1}{2!}-\frac{1}{7!}+\frac{1}{12!}-\frac{1}{17!}+\frac{1}{22!}-\frac{1}{27!}+-\cdots=0.49980158\cdots$$

$$\frac{1}{3!}-\frac{1}{8!}+\frac{1}{13!}-\frac{1}{18!}+\frac{1}{23!}-\frac{1}{28!}+-\cdots=0.16664186\cdots$$

$$\frac{1}{4!}-\frac{1}{9!}+\frac{1}{14!}-\frac{1}{19!}+\frac{1}{24!}-\frac{1}{29!}+-\cdots=0.04166391\cdots$$

Example 2 Alternating five-split of Logarithmic Series ($|z|\le 1$)

$$-\frac{z^5}{5}+\frac{z^{10}}{10}-\frac{z^{15}}{15}+\frac{z^{20}}{20}-\frac{z^{25}}{25}+\frac{z^{30}}{30}-+\cdots=-\frac{\log(1+z)}{5}$$
$$-\frac{\log\{1-(-1)^{1/5}z\}+\log\{1-(-1)^{-1/5}z\}}{5}$$
$$-\frac{\log\{1-(-1)^{3/5}z\}+\log\{1-(-1)^{-3/5}z\}}{5}$$

$$\frac{z^1}{1}-\frac{z^6}{6}+\frac{z^{11}}{11}-\frac{z^{16}}{16}+\frac{z^{21}}{21}-\frac{z^{26}}{26}+-\cdots=-\frac{(-1)^{-1}\log(1+z)}{5}$$
$$-\frac{(-1)^{-1/5}\log\{1-(-1)^{1/5}z\}+(-1)^{1/5}\log\{1-(-1)^{-1/5}z\}}{5}$$
$$-\frac{(-1)^{-3/5}\log\{1-(-1)^{3/5}z\}+(-1)^{3/5}\log\{1-(-1)^{-3/5}z\}}{5}$$

$$\frac{z^2}{2}-\frac{z^7}{7}+\frac{z^{12}}{12}-\frac{z^{17}}{17}+\frac{z^{22}}{22}-\frac{z^{27}}{27}+-\cdots=-\frac{(-1)^{-2}\log(1+z)}{5}$$
$$-\frac{(-1)^{-2/5}\log\{1-(-1)^{1/5}z\}+(-1)^{2/5}\log\{1-(-1)^{-1/5}z\}}{5}$$
$$-\frac{(-1)^{-6/5}\log\{1-(-1)^{3/5}z\}+(-1)^{6/5}\log\{1-(-1)^{-3/5}z\}}{5}$$

$$\frac{z^3}{3}-\frac{z^8}{8}+\frac{z^{13}}{13}-\frac{z^{18}}{18}+\frac{z^{23}}{23}-\frac{z^{28}}{28}+-\cdots=-\frac{(-1)^{-3}\log(1+z)}{5}$$
$$-\frac{(-1)^{-3/5}\log\{1-(-1)^{1/5}z\}+(-1)^{3/5}\log\{1-(-1)^{-1/5}z\}}{5}$$
$$-\frac{(-1)^{-9/5}\log\{1-(-1)^{3/5}z\}+(-1)^{9/5}\log\{1-(-1)^{-3/5}z\}}{5}$$

$$\frac{z^4}{4}-\frac{z^9}{9}+\frac{z^{14}}{14}-\frac{z^{19}}{19}+\frac{z^{24}}{24}-\frac{z^{29}}{29}+-\cdots=-\frac{(-1)^{-4}\log(1+z)}{5}$$
$$-\frac{(-1)^{-4/5}\log\{1-(-1)^{1/5}z\}+(-1)^{4/5}\log\{1-(-1)^{-1/5}z\}}{5}$$

$$- \frac{(-1)^{-12/5} \log\{1-(-1)^{3/5} z\} + (-1)^{12/5} \log\{1-(-1)^{-3/5} z\}}{5}$$

Especially when $z=1$,

$$-\frac{1}{5}+\frac{1}{10}-\frac{1}{15}+\frac{1}{20}-\frac{1}{25}+\frac{1}{30}-+\cdots = -0.13862943\cdots = -\frac{\log 2}{5}$$

$$\frac{1}{1}-\frac{1}{6}+\frac{1}{11}-\frac{1}{16}+\frac{1}{21}-\frac{1}{26}+-\cdots = 0.88831357\cdots$$

$$\frac{1}{2}-\frac{1}{7}+\frac{1}{12}-\frac{1}{17}+\frac{1}{22}-\frac{1}{27}+-\cdots = 0.40690163\cdots$$

$$\frac{1}{3}-\frac{1}{8}+\frac{1}{13}-\frac{1}{18}+\frac{1}{23}-\frac{1}{28}+-\cdots = 0.25375156\cdots$$

$$\frac{1}{4}-\frac{1}{9}+\frac{1}{14}-\frac{1}{19}+\frac{1}{24}-\frac{1}{29}+-\cdots = 0.18064575\cdots$$

Note

It is possible to represent $(-1)^{m/n}$ $(m, n = 1, 2, 3, \cdots)$ in the formula with elementary transcendental functions or radicals. However, in the case of the 5th order or higher, it becomes very complicated.

2020.04.15
2021.03.20 Renewed

Kano Kono
Hiroshima, Japan

Alien's Mathematics