24 Sugioka's Theorem on the Series of Higher (Repeated) Integrals

Mikio Sugioka discovered that the series of the higher integral of arbitrary functions results in one integral in March, 2003. (http://www5b.biglobe.ne.jp/~sugi_m/page030.htm). I was introducing these theorems in the one section in my work " 16 Higher Integral of the product of two functions ". Since Mr. Sugioka discovered many theorems after that, they can no longer be housed in one section. Then, I decided to prepare a chapter independent here and to introduce his wonderful theorems.

Although Mr. Sugioka named the thorem "An operator's theorem", in this paper, I named like the title so that the contents might be understood well.

24.1 Series of the n-th order Integrals

In this section, we ask for the sum of the following series of higher integrals.

$$\int_a^x f(x)\,dx \pm \int_a^x\!\!\int_a^x f(x)\,dx^2 + \int_a^x\!\!\int_a^x\!\!\int_a^x f(x)\,dx^3 \pm \int_a^x\!\cdots\!\int_a^x f(x)\,dx^4 + \pm \cdots$$

Theorem 24.1.1 (Sugioka's Theorem 3)

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $a, x \in I$.

$$\sum_{r=1}^{m} \int_a^x\!\cdots\!\int_a^x f(x)\,dx^r = e^x \int_a^x f(x)e^{-x}dx - e^x \int_a^x \left\{ \int_a^x\!\cdots\!\int_a^x f(x)\,dx^m \right\} e^{-x}dx \tag{1.1}$$

$$\sum_{r=1}^{\infty} \int_a^x\!\cdots\!\int_a^x f(x)\,dx^r = e^x \int_a^x f(x)e^{-x}dx \tag{1.2}$$

Proof

Let

$$f^{<r>}(x) = \int_a^x\!\cdots\!\int_a^x f(x)\,dx^r \qquad r=1,2,\cdots,m \tag{1.r}$$

Then $f^{<r>}(a) = 0 \;\; (r=1,2,\cdots,m)$. So,

$$\int_a^x f(x)e^{-x}dx = \left[f^{<1>}(x)e^{-x}\right]_a^x + \int_a^x f^{<1>}(x)e^{-x} = f^{<1>}(x)e^{-x} + \int_a^x f^{<1>}(x)e^{-x}$$

$$= f^{<1>}(x)e^{-x} + \left[f^{<2>}(x)e^{-x}\right]_a^x + \int_a^x f^{<2>}(x)e^{-x}$$

$$= f^{<1>}(x)e^{-x} + f^{<2>}(x)e^{-x} + \int_a^x f^{<2>}(x)e^{-x}$$

$$\vdots$$

$$= e^{-x}\sum_{r=1}^{m} f^{<r>}(x) + \int_a^x f^{<m>}e^{-x}dx$$

Substituting (1.r) for this,

$$\int_a^x f(x)e^{-x}dx = e^{-x}\sum_{r=1}^{m} \int_a^x\!\cdots\!\int_a^x f(x)\,dx^r + \int_a^x \left\{ \int_a^x\!\cdots\!\int_a^x f(x)\,dx^m \right\} e^{-x}dx$$

Multiplying by e^x the both sides, we obtain (1.1) .

Next, according to Theorem 4.1.3 (**4.1**), the following expression holds.

$$f^{<m>}(x) = \sum_{r=0}^{m-1} f^{<m-r>}(a)\frac{(x-a)^r}{r\,!} + \int_a^x\!\cdots\!\int_a^x f(x)\,dx^m \tag{1.m}$$

On the other hand, since $f(x)$ is analytic function on I, this can be expanded to Taylor series as follows.

$$f(x) = \sum_{r=0}^{\infty} f^{(r)}(a)\frac{(x-a)^r}{r!} = \sum_{r=0}^{\infty} f^{<-r>}(a)\frac{(x-a)^r}{r!}$$

Shifting by m the index of the integration operator of $f(x)$

$$f^{<m>}(x) = \sum_{r=0}^{\infty} f^{<m-r>}(a)\frac{(x-a)^r}{r!} \tag{1.t}$$

Comparing the right-hand sides of (1.m) and (1.t), we obtain

$$\lim_{m\to\infty}\int_a^x\cdots\int_a^x f(x)\,dx^m = 0$$

Thus, (1.2) holds.

Relation with a differential equation (Sugioka's Theorem 2)

The right side of the theorem is transformed as follows.

$$e^x\int_a^x f(x)e^{-x}dx = e^x\left[\int f(x)e^{-x}dx\right]_a^x$$

$$= e^x\left[e^{-x}y_1(x)\right]_a^x \qquad \left\{y_1(x)\equiv e^x\int f(x)e^{-x}dx\right\}$$

$$= y_1(x) - y_1(a)e^{x-a}$$

And $y_1(x)$ is transformed as follows.

$$y_1(x) = e^x\int f(x)e^{-x}dx = e^{-\int(-1)dx}\left\{\int f(x)\,e^{\int(-1)dx}dx\right\}$$

Then, we notice that this is a particular solution of the 1st order differential equation $y'(x) - y(x) = f(x)$.

Example 1 $\int_a^x dx + \int_a^x\int_a^x dx^2 + \int_a^x\int_a^x\int_a^x dx^3 + \cdots$

$$f(x)=1\,, \qquad \int_a^x\cdots\int_a^x dx^m = \frac{(x-a)^m}{m!}$$

Substituting these for (1.1) and using the incomplete gamma function $\Gamma(x,y)$,

$$\sum_{r=1}^{m}\int_a^x\cdots\int_a^x dx^r = e^x\int_a^x e^{-x}dx - e^x\int_a^x\frac{(x-a)^m}{m!}e^{-x}dx$$

$$= e^x\left[-e^{-x}\right]_a^x - \frac{e^x}{m!}\left[-e^{-a}\Gamma(m+1,x-a)\right]_a^x$$

That is

$$\sum_{r=1}^{m}\int_a^x\cdots\int_a^x dx^r = e^{x-a} - 1 + \frac{e^{x-a}}{m!}\{\Gamma(m+1,x-a) - \Gamma(m+1,0)\}$$

And, since $\lim_{m\to\infty}\frac{e^{x-a}}{m!}\{\Gamma(m+1,x-a) - \Gamma(m+1,0)\} = 0$,

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x dx^r = e^{x-a} - 1$$

Example 2 $\int_a^x e^x dx + \int_a^x\int_a^x e^x dx^2 + \int_a^x\int_a^x\int_a^x e^x dx^3 + \cdots$

$$f(x) = e^x \quad , \quad \int_a^x \cdots \int_a^x e^x dx^m = e^x - e^a \sum_{r=0}^{m-1} \frac{(x-a)^r}{r!}$$

Substituting these for (1.1) and using the incomplete gamma function $\Gamma(x, y)$,

$$\sum_{r=1}^{m} \int_a^x \cdots \int_a^x e^x dx^r = e^x \int_a^x e^x e^{-x} dx - e^x \int_a^x \left\{ \int_a^x \cdots \int_a^x e^x dx^m \right\} e^{-x} dx$$

$$= e^x \int_a^x dx - e^x \int_a^x \left\{ e^x - e^a \sum_{r=0}^{m-1} \frac{(x-a)^r}{r!} \right\} e^{-x} dx$$

$$= e^x (x-a) - e^x \int_a^x e^x e^{-x} dx + e^{x+a} \sum_{r=0}^{m-1} \frac{1}{r!} \int_a^x (x-a)^r e^{-x} dx$$

$$= e^x (x-a) - e^x \int_a^x e^x e^{-x} dx - e^{x+a} \sum_{r=0}^{m-1} \frac{e^{-a}}{r!} \{ \Gamma(r+1, x-a) - \Gamma(r+1, 0) \}$$

$$= e^x (x-a) - e^x (x-a) - e^x \sum_{r=0}^{m-1} \frac{\Gamma(r+1, x-a) - \Gamma(r+1)}{r!}$$

i.e.

$$\sum_{r=1}^{m} \int_a^x \cdots \int_a^x e^x dx^r = e^x (x-a) - e^x (x-a) - e^x \sum_{r=0}^{m-1} \left\{ \frac{\Gamma(r+1, x-a)}{r!} - 1 \right\}$$

Next,

$$\Gamma(n, x) = (n-1)!\, e^{-x} \sum_{s=0}^{n-1} \frac{x^s}{s!}$$

From this,

$$\frac{\Gamma(r+1, x-a)}{r!} = e^{-(x-a)} \sum_{s=0}^{r} \frac{(x-a)^s}{s!}$$

Using this,

$$\sum_{r=1}^{m} \int_a^x \cdots \int_a^x e^x dx^r = -e^x \sum_{r=0}^{m-1} \left\{ \frac{\Gamma(r+1, x-a)}{r!} - 1 \right\}$$

$$= -e^x e^{-(x-a)} \sum_{r=0}^{m-1} \left\{ \sum_{s=0}^{r} \frac{(x-a)^s}{s!} - e^{(x-a)} \right\}$$

Here,

$$\sum_{r=0}^{m-1} \left\{ \sum_{s=0}^{r} \frac{(x-a)^s}{s!} - e^{(x-a)} \right\} = \sum_{r=0}^{m-1} \left\{ \frac{(m-r)(x-a)^r}{r!} - e^{(x-a)} \right\}$$

$$= -\sum_{r=0}^{m-1} \frac{r(x-a)^r}{r!} + \sum_{r=0}^{m-1} \frac{m(x-a)^r}{r!} - me^{(x-a)}$$

$$= -(x-a) \sum_{r=1}^{m-1} \frac{(x-a)^{r-1}}{(r-1)!} + m \sum_{r=0}^{m-1} \frac{(x-a)^r}{r!} - me^{(x-a)}$$

Substituting this for the above,

$$\sum_{r=1}^{m} \int_a^x \cdots \int_a^x e^x dx^r = -e^x \sum_{r=0}^{m-1} \left\{ \frac{\Gamma(r+1, x-a)}{r!} - 1 \right\}$$

$$= e^x e^{-(x-a)} \left\{ (x-a) \sum_{r=1}^{m-1} \frac{(x-a)^{r-1}}{(r-1)!} - m \sum_{r=0}^{m-1} \frac{(x-a)^r}{r!} + me^{(x-a)} \right\}$$

When $m \to \infty$,

$$\sum_{r=1}^{\infty}\int_a^x \cdots \int_a^x e^x dx^r = e^x e^{-(x-a)}\left\{ (x-a)e^{(x-a)} - me^{(x-a)} + me^{(x-a)}\right\}$$

That is,

$$\sum_{r=1}^{\infty}\int_a^x \cdots \int_a^x e^x dx^r = e^x (x-a)$$

Note

If we adopt the lineal higher integral $\int_{-\infty}^x \cdots \int_{-\infty}^x e^x dx^m = e^x$ then $\sum_{r=1}^{\infty}\int_{-\infty}^x \cdots \int_{-\infty}^x e^x dx^m = \infty$.

Therefore, it is for a collateral higher integral that this theorem is significant.

A by-product

$$\lim_{m\to\infty}\sum_{r=0}^{m-1}\left\{ \sum_{s=0}^{r}\frac{(x-a)^s}{s!} - e^{(x-a)}\right\} = \lim_{m\to\infty}\sum_{r=0}^{m-1}\left\{ \frac{(m-r)(x-a)^r}{r!} - e^{(x-a)}\right\}$$

$$= -e^{(x-a)}(x-a)$$

Example 3 $\int_0^x \log x dx + \int_0^x\int_0^x \log x dx^2 + \int_0^x\int_0^x\int_0^x \log x dx^3 + \cdots$

Substituting $f(x) = \log x$, $a=0$ for (1.2),

$$\sum_{r=1}^{\infty}\int_0^x \cdots \int_0^x \log x dx^r = e^x \int_0^x \log x e^{-x} dx$$

The integral of the right side is as follows according to Formula 16.6.2 (**16.6**)

$$\int_0^x \log x e^{-x} dx = \left[-e^{-x}\log|x| + Ei(-x)\right]_0^x$$

$$= -e^{-x}\log|x| + Ei(-x) - \gamma$$

Where, $Ei(x) = \int_{-\infty}^x \frac{e^t}{t}dt$, $\gamma = 0.5772\cdots$ (Euler-Mascheroni Constant).

This integral is not lineal as understood from this constant. Multiplying by e^x the both sides, we obtain

$$\sum_{r=1}^{\infty}\int_0^x \cdots \int_0^x \log x dx^r = -\log|x| + e^x (Ei(-x) - \gamma)$$

The higher integral of the left side is the following lineal integral.

$$\int_0^x \cdots \int_0^x \log x dx^n = \frac{x^n}{n!}\left(\log|x| - \sum_{s=1}^{n}\frac{1}{s}\right)$$

Then, the left side is

$$\sum_{r=1}^{\infty}\frac{x^r}{r!}\left(\log|x| - \sum_{s=1}^{r}\frac{1}{s}\right)$$

When the first 10 terms of $\sum$ are calculated and both sides are illustrated, it is as follows.
Both sides overlap exactly and blue (left) can not be seen.

```
m = 10;
fl[x_] := Sum[x^r/r! (Log[Abs[x]] - Sum[1/s, {s, 1, r}]), {r, 1, m}]
fr[x_] := -Log[Abs[x]] + e^x (ExpIntegralEi[-x] - EulerGamma)
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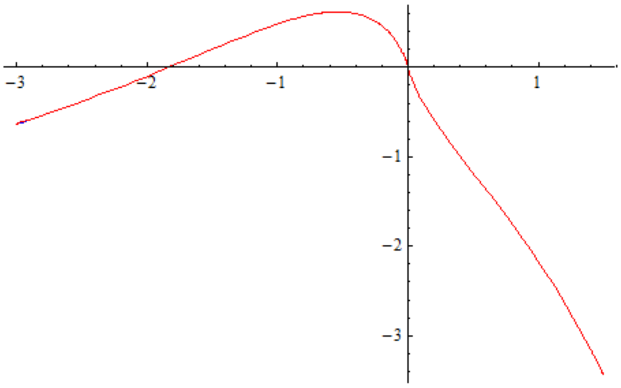

Theorem 24.1.2 (Sugioka's Theorem 4)

When $f(x)$ is analytic function defined on a closed interval I , the following expressions hold for $a,x\in I$.

$$\sum_{r=1}^{m}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^r = e^{-x}\int_a^x f(x)e^x dx - (-1)^m e^{-x}\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}e^x dx \tag{2.1}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^r = e^{-x}\int_a^x f(x)e^x dx \tag{2.2}$$

Proof

In a similar way to Theorem 24.1.1, we obtain the desired expressions.

Example $\int_a^x \sin x\,dx - \int_a^x\int_a^x \sin x\,dx^2 + \int_a^x\int_a^x\int_a^x \sin x\,dx^3 - \int_a^x\cdots\int_a^x \sin x\,dx^4 + - \cdots$

Substituting $f(x)=\sin x$ for (2.2) ,

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x \sin x\,dx^r = e^{-x}\int_a^x \sin x\, e^x dx$$

The right side is as follows according to Formula 16.5.3 (**16.5**).

$$e^{-x}\int_a^x \sin x\cdot e^x\,dx = e^{-x}\left[\left(\sin\frac{\pi}{4}\right)e^x\sin\left(x-\frac{\pi}{4}\right)\right]_a^x$$

$$= \frac{\sqrt{2}e^{-x}}{2}\left\{e^x\sin\left(x-\frac{\pi}{4}\right) - e^a\sin\left(a-\frac{\pi}{4}\right)\right\}$$

Here, $\sqrt{2}\sin\left(x-\frac{\pi}{4}\right) = \sin x - \cos x$. Then

$$e^{-x}\int_a^x \sin x\cdot e^x\,dx = \frac{\sin x-\cos x}{2} - \frac{\sin a-\cos a}{2}e^{a-x}$$

That is

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x \sin x\,dx^r = \frac{\sin x-\cos x}{2} - \frac{\sin a-\cos a}{2}e^{a-x}$$

The higher integral of the left side is as follows according to Theorem 4.1.3 (**4.1**)

$$\int_a^x \cdots \int_a^x \sin x \, dx^n = \sin\left(x - \frac{\pi n}{2}\right) - \sum_{s=0}^{n-1} \frac{(x-a)^s}{s!} \sin\left\{a - \frac{\pi(n-s)}{2}\right\}$$

Then, the left side is

$$\sum_{r=1}^{\infty} (-1)^{r-1} \left\{ \sin\left(x - \frac{\pi r}{2}\right) - \sum_{s=0}^{r-1} \frac{(x-a)^s}{s!} \sin\left\{a - \frac{\pi(r-s)}{2}\right\} \right\}$$

When $a = 3/2$ and the first 25 terms of $\sum$ are calculated and both sides are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.

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m = 25;
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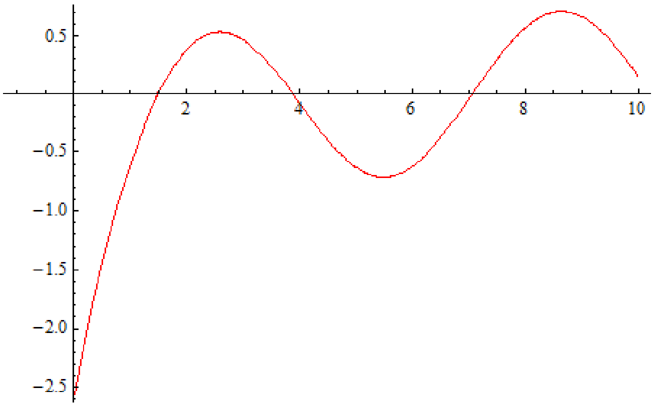
$$\texttt{fl[a\_] :=} \sum_{r=1}^{m} (-1)^{r-1} \left(\texttt{Sin}\left[x - \frac{\pi r}{2}\right] - \sum_{s=0}^{r-1} \frac{(x-a)^s}{s!} \texttt{Sin}\left[a - \frac{\pi (r-s)}{2}\right] \right)$$

$$\texttt{fr[a\_] :=} \frac{\texttt{Sin[x] - Cos[x]}}{2} - \frac{\texttt{Sin[a] - Cos[a]}}{2} e^{a-x}$$



24.2 Series of the odd-th order Integrals

In this section, we ask for the sum of the following series of higher integrals.

$$\int_a^x f(x)dx \pm \int_a^x\int_a^x\int_a^x f(x)dx^3 + \int_a^x\cdots\int_a^x f(x)dx^5 \pm \int_a^x\cdots\int_a^x f(x)dx^7 +\pm \cdots$$

Theorem 24.2.1

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $a, x \in I$.

$$\sum_{r=1}^{m}\int_a^x\cdots\int_a^x f(x)dx^{2r-1} = \frac{1}{2}\left\{e^x\int_a^x f(x)e^{-x}dx + e^{-x}\int_a^x f(x)e^x dx\right\}$$

$$-(-1)^m \cosh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\frac{e^x+(-1)^{-m}e^{-x}}{2}dx$$

$$+(-1)^m \sinh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\frac{e^x-(-1)^{-m}e^{-x}}{2}dx \tag{1.1}$$

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x f(x)dx^{2r-1} = \frac{1}{2}\left\{e^x\int_a^x f(x)e^{-x}dx + e^{-x}\int_a^x f(x)e^x dx\right\} \tag{1.2}$$

Proof

Formula of repeated integration by parts was as follows (" 0**1 Generalized Taylor's Theorem** ") .

$$\int_a^x f(x)g(x)dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)g^{(r-1)}(x)\right]_a^x + (-1)^m\int_a^x f^{<m>}(x)g^{(m)}(x)dx$$

When $g(x) = \cosh x, \sinh x$,

$$(\cosh x)^{(r-1)} = \frac{e^x+(-1)^{-r+1}e^{-x}}{2}, \quad (\cosh x)^{(m)} = \frac{e^x+(-1)^{-m}e^{-x}}{2}$$

$$(\sinh x)^{(r-1)} = \frac{e^x-(-1)^{-r+1}e^{-x}}{2}, \quad (\sinh x)^{(m)} = \frac{e^x-(-1)^{-m}e^{-x}}{2}$$

Substituting these for the above,

$$\int_a^x f(x)\cosh x dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)\frac{e^x+(-1)^{-r+1}e^{-x}}{2}\right]_a^x,$$

$$+(-1)^m\int_a^x f^{<m>}(x)\frac{e^x+(-1)^{-m}e^{-x}}{2}dx,$$

$$\int_a^x f(x)\sinh x dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)\frac{e^x-(-1)^{-r+1}e^{-x}}{2}\right]_a^x,$$

$$+(-1)^m\int_a^x f^{<m>}(x)\frac{e^x-(-1)^{-m}e^{-x}}{2}dx,$$

Here, let

$$f^{<r>}(x) = \int_a^x\cdots\int_a^x f(x)dx^r \qquad r=1,2,\cdots,m$$

Since $f^{<r>}(a) = 0 \ (r=1,2,\cdots,m)$,

$$\int_a^x f(x)\cosh x\,dx = \sum_{r=1}^{m}(-1)^{r-1}\frac{e^x+(-1)^{-r+1}e^{-x}}{2}\int_a^x\cdots\int_a^x f(x)\,dx^r$$

$$+(-1)^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}\frac{e^x+(-1)^{-m}e^{-x}}{2}dx$$

$$\int_a^x f(x)\cosh x\,dx = \sum_{r=1}^{m}(-1)^{r-1}\frac{e^x+(-1)^{-r+1}e^{-x}}{2}\int_a^x\cdots\int_a^x f(x)\,dx^r$$

$$+(-1)^m\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}\frac{e^x-(-1)^{-m}e^{-x}}{2}dx$$

Expanding a part of the 1st term of the right side,

$$\sum_{r=1}^{m}(-1)^{r-1}\frac{e^x+(-1)^{-r+1}e^{-x}}{2} = \cosh x - \sinh x + \cosh x - \sinh x +-\cdots$$

$$\sum_{r=1}^{m}(-1)^{r-1}\frac{e^x-(-1)^{-r+1}e^{-x}}{2} = \sinh x - \cosh x + \sinh x - \cosh x +-\cdots$$

Multiplying both sides by $\cosh x, \sinh x$ respectively,

$$\cosh x\sum_{r=1}^{m}(-1)^{r-1}\frac{e^x+(-1)^{-r+1}e^{-x}}{2} = \cosh^2 x - \cosh x\sinh x + \cosh^2 x - \cosh x\sinh x +-\cdots$$

$$\sinh x\sum_{r=1}^{m}(-1)^{r-1}\frac{e^x-(-1)^{-r+1}e^{-x}}{2} = \sinh^2 x - \sinh x\cosh x + \sinh^2 x - \sinh x\cosh x +-\cdots$$

The 2nd terms of the right side are as follows.

$$(-1)^m\cosh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}\frac{e^x+(-1)^{-m}e^{-x}}{2}dx$$

$$(-1)^m\sinh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}\frac{e^x-(-1)^{-m}e^{-x}}{2}dx$$

Then

$$\cosh x\int_a^x f(x)\cosh x\,dx - \sinh x\int_a^x f(x)\sinh x\,dx = \sum_{r=1}^{m}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1}$$

$$+(-1)^m\cosh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}\frac{e^x+(-1)^{-m}e^{-x}}{2}dx$$

$$-(-1)^m\sinh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}\frac{e^x-(-1)^{-m}e^{-x}}{2}dx$$

Furthermore,

$$\cosh x\int_a^x f(x)\cosh x\,dx - \sinh x\int_a^x f(x)\sinh x\,dx$$

$$= \frac{e^x+e^{-x}}{2}\int_a^x f(x)\frac{e^x+e^{-x}}{2}dx - \frac{e^x-e^{-x}}{2}\int_a^x f(x)\frac{e^x-e^{-x}}{2}dx$$

$$= \frac{1}{2}\left\{e^x\int_a^x f(x)e^{-x}dx + e^{-x}\int_a^x f(x)e^x dx\right\}$$

Using this, we obtain

$$\sum_{r=1}^{m}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \frac{1}{2}\left\{e^x\int_a^x f(x)e^{-x}dx + e^{-x}\int_a^x f(x)e^{x}dx\right\}$$

$$- (-1)^m \cosh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}\frac{e^x+(-1)^{-m}e^{-x}}{2}dx$$

$$+ (-1)^m \sinh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}\frac{e^x-(-1)^{-m}e^{-x}}{2}dx$$

And since $\lim_{m\to\infty}\int_a^x\cdots\int_a^x f(x)\,dx^m = 0$ holds by assumption (See proof of Theorem 24.1.1), we obtain (1.2).

Theorem 24.2.2 (Sugioka's Theorem 5)

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $a, x \in I$.

$$\sum_{r=1}^{m}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \cos x\int_a^x f(x)\cos x\,dx + \sin x\int_a^x f(x)\sin x\,dx$$

$$+ (-1)^m \cos x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}\cos\left(x+\frac{m\pi}{2}\right)dx$$

$$+ (-1)^m \sin x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)\,dx^m\right\}\sin\left(x+\frac{m\pi}{2}\right)dx \tag{2.1}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \cos x\int_a^x f(x)\cos x\,dx + \sin x\int_a^x f(x)\sin x\,dx \tag{2.2}$$

Proof

Formula of repeated integration by parts was as follows (" **1 Generalized Taylor's Theorem** ") .

$$\int_a^x f(x)g(x)\,dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)g^{(r-1)}(x)\right]_a^x + (-1)^m\int_a^x f^{<m>}(x)g^{(m)}(x)\,dx$$

When $g(x) = \cos x\ ,\ \sin x$,

$$(\cos x)^{(r-1)} = \cos\left\{x+\frac{(r-1)\pi}{2}\right\} \quad , \quad (\cos x)^{(m)} = \cos\left(x+\frac{m\pi}{2}\right)$$

$$(\sin x)^{(r-1)} = \sin\left\{x+\frac{(r-1)\pi}{2}\right\} \quad , \quad (\sin x)^{(m)} = \sin\left(x+\frac{m\pi}{2}\right)$$

Substituting these for the above,

$$\int_a^x f(x)\cos x\,dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)\cos\left\{x+\frac{(r-1)\pi}{2}\right\}\right]_a^x$$

$$+ (-1)^m\int_a^x f^{<m>}(x)\cos\left(x+\frac{m\pi}{2}\right)dx$$

$$\int_a^x f(x)\sin x\,dx = \sum_{r=1}^{m}(-1)^{r-1}\left[f^{<r>}(x)\sin\left\{x+\frac{(r-1)\pi}{2}\right\}\right]_a^x$$

$$+ (-1)^m\int_a^x f^{<m>}(x)\sin\left(x+\frac{m\pi}{2}\right)dx$$

Here, let

$$f^{<r>}(x) = \int_a^x \cdots \int_a^x f(x)\,dx^r \qquad r=1,2,\cdots,m$$

Since $f^{<r>}(a)=0 \ (r=1,2,\cdots,m)$,

$$\int_a^x f(x)\cos x\,dx = \sum_{r=1}^{m}(-1)^{r-1}\cos\left\{x+\frac{(r-1)\pi}{2}\right\}\int_a^x \cdots \int_a^x f(x)\,dx^r$$
$$+(-1)^m\int_a^x\left\{\int_a^x \cdots \int_a^x f(x)\,dx^m\right\}\cos\left(x+\frac{m\pi}{2}\right)dx$$

$$\int_a^x f(x)\sin x\,dx = \sum_{r=1}^{m}(-1)^{r-1}\sin\left\{x+\frac{(r-1)\pi}{2}\right\}\int_a^x \cdots \int_a^x f(x)\,dx^r$$
$$+(-1)^m\int_a^x\left\{\int_a^x \cdots \int_a^x f(x)\,dx^m\right\}\sin\left(x+\frac{m\pi}{2}\right)dx$$

Expanding a part of the 1st term of the right side,

$$\sum_{r=1}^{m}(-1)^{r-1}\cos\left\{x+\frac{(r-1)\pi}{2}\right\} = \cos x+\sin x-\cos x-\sin x ++--\cdots$$

$$\sum_{r=1}^{m}(-1)^{r-1}\sin\left\{x+\frac{(r-1)\pi}{2}\right\} = \sin x-\cos x-\sin x+\cos x ++--\cdots$$

Multiplying both sides by $\cos x, \sin x$ respectively,

$$\cos x\sum_{r=1}^{m}(-1)^{r-1}\cos\left\{x+\frac{(r-1)\pi}{2}\right\} = \cos^2 x+\sin x\cos x-\cos^2 x-\sin x\cos x ++--\cdots$$

$$\sin x\sum_{r=1}^{m}(-1)^{r-1}\sin\left\{x+\frac{(r-1)\pi}{2}\right\} = \sin^2 x-\sin x\cos x-\sin^2 x+\sin x\cos x ++--\cdots$$

The 2nd terms of the right side are as follows.

$$(-1)^m\cos x\int_a^x\left\{\int_a^x \cdots \int_a^x f(x)\,dx^m\right\}\cos\left(x+\frac{m\pi}{2}\right)dx$$

$$(-1)^m\sin x\int_a^x\left\{\int_a^x \cdots \int_a^x f(x)\,dx^m\right\}\sin\left(x+\frac{m\pi}{2}\right)dx$$

Then

$$\cos x\int_a^x f(x)\cos x\,dx+\sin x\int_a^x f(x)\sin x\,dx = \sum_{r=1}^{m}(-1)^{r-1}\int_a^x \cdots \int_a^x f(x)\,dx^{2r-1}$$
$$+(-1)^m\cos x\int_a^x\left\{\int_a^x \cdots \int_a^x f(x)\,dx^m\right\}\cos\left(x+\frac{m\pi}{2}\right)dx$$
$$+(-1)^m\sin x\int_a^x\left\{\int_a^x \cdots \int_a^x f(x)\,dx^m\right\}\sin\left(x+\frac{m\pi}{2}\right)dx$$

Transposing the remainder terms , we obtain (2.1) .

And since $\lim_{m\to\infty}\int_a^x \cdots \int_a^x f(x)\,dx^m = 0$ holds by assumption (See proof of Theorem 24.1.1), we obtain (2.2).

Example1 $\int_a^x dx + \int_a^x\int_a^x\int_a^x dx^3 + \int_a^x \cdots \int_a^x dx^5 + \cdots$

Substituting $f(x)=1$ for (1.2) ,

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x dx^{2r-1} = \frac{1}{2}\left\{e^x\int_a^x e^{-x}dx + e^{-x}\int_a^x e^x dx\right\}$$

$$= \frac{1}{2}\left\{e^x\left[-e^{-x}\right]_a^x + e^{-x}\left[e^x\right]_a^x\right\} = \frac{e^{x-a} - e^{a-x}}{2}$$

i.e.

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x dx^{2r-1} = sech(x-a)$$

Example2 $\int_a^x e^x dx - \int_a^x\int_a^x\int_a^x e^x dx^3 + \int_a^x\cdots\int_a^x e^x dx^5 - \int_a^x\cdots\int_a^x e^x dx^7 + - \cdots$

Substituting $f(x)=e^x$ for (2.2) ,

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x e^x dx^{2r-1} = \cos x\int_a^x e^x \cos x dx + \sin x\int_a^x e^x \sin x dx$$

Right side is as follow.

$$\int_a^x e^x \cos x dx = \frac{1}{2}\{e^x(\cos x+\sin x) - e^a(\cos a+\sin a)\}$$

$$\int_a^x e^x \sin x dx = -\frac{1}{2}\{e^x(\cos x-\sin x) - e^a(\cos a-\sin a)\}$$

From these,

$$\cos x\int_a^x e^x \cos x dx = \frac{1}{2}\{e^x(\cos^2 x+\cos x\sin x) - e^a\cos x(\cos a+\sin a)\}$$

$$\sin x\int_a^x e^x \sin x dx = -\frac{1}{2}\{e^x(\sin x\cos x-\sin^2 x) - e^a\sin x(\cos a-\sin a)\}$$

Then

$$\cos x\int_a^x e^x \cos x dx + \sin x\int_a^x e^x \sin x dx = \frac{e^x}{2}$$

$$-\frac{e^a}{2}\{\cos x(\cos a+\sin a) - \sin x(\cos a-\sin a)\}$$

i.e.

$$\cos x\int_a^x e^x \cos x dx + \sin x\int_a^x e^x \sin x dx = \frac{e^x}{2} - \frac{e^a}{2}\{\cos(x-a) - \sin(x-a)\}$$

Therefore,

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x e^x dx^{2r-1} = \frac{e^x}{2} - \frac{e^a}{2}\{\cos(x-a) - \sin(x-a)\}$$

The higher integral of the left side is as follows according to Theorem 4.1.3 (**4.1**)

$$\int_a^x\cdots\int_a^x e^x dx^n = e^x - \sum_{s=0}^{n-1} e^a\frac{(x-a)^s}{s!}$$

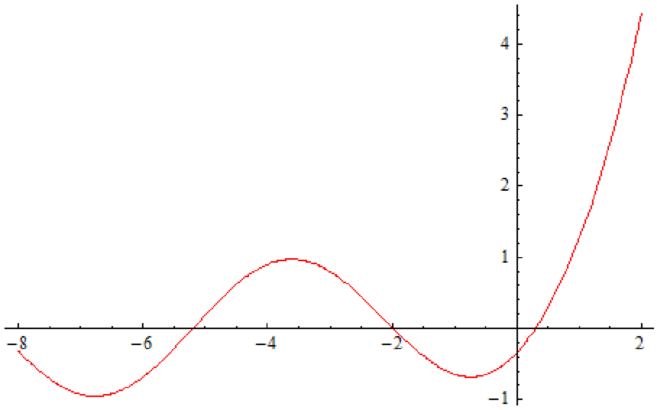
Then, the left side is

$$\sum_{r=1}^{\infty}(-1)^{r-1}\left\{e^x - e^a\sum_{s=0}^{2r-2}\frac{(x-a)^s}{s!}\right\}$$

When $a=0.3$ and the first 20 terms of ∑ are calculated and both sides are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.

```
m = 20;
```

$$fl[a\_] := \sum_{r=1}^{m} (-1)^{r-1} \left(e^{x} - e^{a} \sum_{s=0}^{2r-2} \frac{(x-a)^{s}}{s!} \right) \qquad fr[a\_] := \frac{e^{x}}{2} - \frac{e^{a}}{2} (Cos[x-a] - Sin[x-a])$$



24.3 Series of the even-th order Integrals

In this section, we ask for the sum of the following series of higher integrals.

$$\int_a^x\int_a^x f(x)dx^2 \pm \int_a^x\cdots\int_a^x f(x)dx^4 + \int_a^x\cdots\int_a^x f(x)dx^6 \pm \int_a^x\cdots\int_a^x f(x)dx^8 +\pm \cdots$$

Theorem 24.3.1

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $a, x \in I$.

$$\sum_{r=1}^{m}\int_a^x\cdots\int_a^x f(x)dx^{2r} = \frac{1}{2}\left\{e^x\int_a^x f(x)e^{-x}dx - e^{-x}\int_a^x f(x)e^x dx\right\}$$
$$- (-1)^m \sinh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\frac{e^x+(-1)^{-m}e^{-x}}{2}dx$$
$$+ (-1)^m \cosh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\frac{e^x-(-1)^{-m}e^{-x}}{2}dx \tag{1.1}$$

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x f(x)dx^{2r} = \frac{1}{2}\left\{e^x\int_a^x f(x)e^{-x}dx - e^{-x}\int_a^x f(x)e^x dx\right\} \tag{1.2}$$

Proof

The following expressions were obtained during the proof of Theorem 24.2.1.

$$\sum_{r=1}^{m}(-1)^{r-1}\frac{e^x+(-1)^{-r+1}e^{-x}}{2} = \cosh x - \sinh x + \cosh x - \sinh x +-\cdots$$

$$\sum_{r=1}^{m}(-1)^{r-1}\frac{e^x-(-1)^{-r+1}e^{-x}}{2} = \sinh x - \cosh x + \sinh x - \cosh x +-\cdots$$

Multiplying both sides by $\sinh x, \cosh x$ respectively,

$$\sinh x\sum_{r=1}^{m}(-1)^{r-1}\frac{e^x+(-1)^{-r+1}e^{-x}}{2} = \sinh x\cosh x - \sinh^2 x + \sinh x\cosh x - \sinh^2 x +-\cdots$$

$$\cosh x\sum_{r=1}^{m}(-1)^{r-1}\frac{e^x-(-1)^{-r+1}e^{-x}}{2} = \cosh x\sinh x - \cosh^2 x + \cosh x\sinh x - \cosh^2 x +-\cdots$$

The 2nd terms of the right side are as follows.

$$(-1)^m \sinh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\frac{e^x+(-1)^{-m}e^{-x}}{2}dx$$

$$(-1)^m \cosh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\frac{e^x-(-1)^{-m}e^{-x}}{2}dx$$

Then

$$\sinh x\int_a^x f(x)\cosh x\,dx - \cosh x\int_a^x f(x)\sinh x\,dx = \sum_{r=1}^{m}\int_a^x\cdots\int_a^x f(x)dx^{2r}$$
$$+ (-1)^m \sinh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\frac{e^x+(-1)^{-m}e^{-x}}{2}dx$$
$$- (-1)^m \cosh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\frac{e^x-(-1)^{-m}e^{-x}}{2}dx$$

Furthermore,

$$\sinh x\int_a^x f(x)\cosh x\,dx-\cosh x\int_a^x f(x)\sinh x\,dx$$

$$=\frac{e^x-e^{-x}}{2}\int_a^x f(x)\frac{e^x+e^{-x}}{2}dx-\frac{e^x+e^{-x}}{2}\int_a^x f(x)\frac{e^x-e^{-x}}{2}dx$$

$$=\frac{1}{2}\left\{e^x\int_a^x f(x)e^{-x}dx-e^{-x}\int_a^x f(x)e^x dx\right\}$$

Using this, we obtain

$$\sum_{r=1}^{m}\int_a^x\cdots\int_a^x f(x)dx^{2r}=\frac{1}{2}\left\{e^x\int_a^x f(x)e^{-x}dx-e^{-x}\int_a^x f(x)e^x dx\right\}$$

$$-(-1)^m\sinh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\frac{e^x+(-1)^{-m}e^{-x}}{2}dx$$

$$+(-1)^m\cosh x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\frac{e^x-(-1)^{-m}e^{-x}}{2}dx$$

And since $\lim_{m\to\infty}\int_a^x\cdots\int_a^x f(x)dx^m=0$ holds by assumption (See proof of Theorem 24.1.1), we obtain (1.2).

Theorem 24.3.2

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $a, x\in I$.

$$\sum_{r=1}^{m}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)dx^{2r}=\sin x\int_a^x f(x)\cos x\,dx-\cos x\int_a^x f(x)\sin x\,dx$$

$$-(-1)^m\sin x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\cos\left(x+\frac{m\pi}{2}\right)dx$$

$$+(-1)^m\cos x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\sin\left(x+\frac{m\pi}{2}\right)dx \tag{2.1}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)dx^{2r}=\sin x\int_a^x f(x)\cos x\,dx-\cos x\int_a^x f(x)\sin x\,dx \tag{2.2}$$

Proof

The following expressions were obtained during the proof of Theorem 24.2.2.

$$\sum_{r=1}^{m}(-1)^{r-1}\cos\left\{x+\frac{(r-1)\pi}{2}\right\}=\cos x+\sin x-\cos x-\sin x++--\cdots$$

$$\sum_{r=1}^{m}(-1)^{r-1}\sin\left\{x+\frac{(r-1)\pi}{2}\right\}=\sin x-\cos x-\sin x+\cos x++--\cdots$$

Multiplying both sides by $\sin x, \cos x$ respectively,

$$\sin x\sum_{r=1}^{m}(-1)^{r-1}\cos\left\{x+\frac{(r-1)\pi}{2}\right\}=\sin x\cos x+\sin^2 x-\sin x\cos x-\sin^2 x++--\cdots$$

$$\cos x\sum_{r=1}^{m}(-1)^{r-1}\sin\left\{x+\frac{(r-1)\pi}{2}\right\}=\cos x\sin x-\cos^2 x-\cos x\sin x+\cos^2 x++--\cdots$$

The 2nd terms of the right side are as follows.

$$(-1)^m\sin x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}\cos\left(x+\frac{m\pi}{2}\right)dx$$

$$(-1)^m \cos x \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^m \right\} \sin\left(x + \frac{m\pi}{2} \right) dx$$

Then

$$\sin x \int_a^x f(x) \cos x\, dx - \cos x \int_a^x f(x) \sin x\, dx = \sum_{r=1}^{m} (-1)^{r-1} \int_a^x \cdots \int_a^x f(x)\, dx^{2r}$$

$$+ (-1)^m \sin x \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^m \right\} \cos\left(x + \frac{m\pi}{2} \right) dx$$

$$- (-1)^m \cos x \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x)\, dx^m \right\} \sin\left(x + \frac{m\pi}{2} \right) dx$$

Transposing the remainder terms , we obtain (2.1) .

And since $\lim_{m \to \infty} \int_a^x \cdots \int_a^x f(x)\, dx^m = 0$ holds by assumption (See proof of Theorem 24.1.1), we obtain (2.2).

Example1 $\int_a^x \int_a^x dx^2 + \int_a^x \cdots \int_a^x dx^4 + \int_a^x \cdots \int_a^x dx^6 + \cdots$

Substituting $f(x) = 1$ for (1.2) ,

$$\sum_{r=1}^{\infty} \int_a^x \cdots \int_a^x dx^{2r} = \frac{1}{2} \left\{ e^x \int_a^x e^{-x} dx - e^{-x} \int_a^x e^x dx \right\} = \frac{1}{2} \left\{ e^x \left[-e^{-x} \right]_a^x - e^{-x} \left[e^x \right]_a^x \right\}$$

$$= \frac{1}{2} \left\{ -1 + e^{x-a} - \left(1 - e^{a-x} \right) \right\} = \frac{e^{x-a} + e^{a-x}}{2} - 1$$

i.e.

$$\sum_{r=1}^{\infty} \int_a^x \cdots \int_a^x dx^{2r} = \cosh(x-a) - 1$$

Example2 $\int_0^x \int_0^x x^2 dx^2 - \int_0^x \cdots \int_0^x x^2 dx^4 + \int_0^x \cdots \int_0^x x^2 dx^6 - \int_0^x \cdots \int_0^x x^2 dx^8 +- \cdots$

Substituting $f(x) = x^2$ for (2.2) ,

$$\sum_{r=1}^{\infty} (-1)^{r-1} \int_0^x \cdots \int_0^x x^2 dx^{2r-1} = \sin x \int_0^x x^2 \cos x\, dx - \cos x \int_0^x x^2 \sin x\, dx$$

Then, the left side is as follows.

$$\int_0^x x^2 \cos x\, dx = 2x \cos x + \left(x^2 - 2 \right) \sin x \ , \quad \int_0^x x^2 \sin x\, dx = 2x \sin x - \left(x^2 - 2 \right) \cos x - 2$$

From these,

$$\sin x \int_0^x x^2 \cos x\, dx - \cos x \int_0^x x^2 \sin x\, dx = x^2 - 2 + 2\cos x$$

Therefore,

$$\sum_{r=1}^{\infty} (-1)^{r-1} \int_0^x \cdots \int_0^x x^2 dx^{2r} = x^2 - 2 + 2\cos x$$

The higher integral of the left side is as follows according to Theorem 4.1.3 (**4.1**)

$$\int_0^x \cdots \int_0^x x^2\, dx^n = \frac{2!}{(2+n)!} x^{2+n}$$

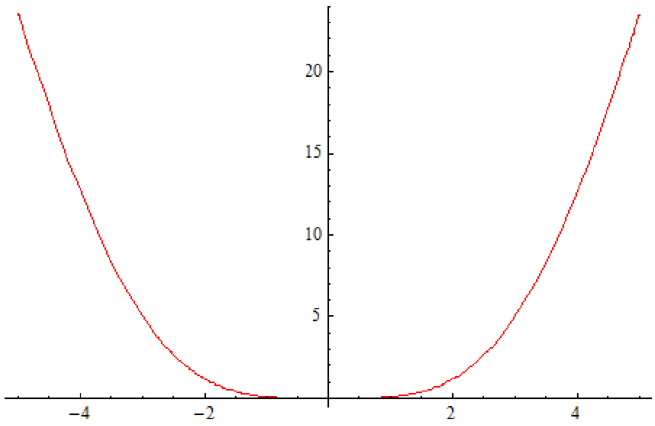
Then, the left side is

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{2!}{(2+2r)!}x^{2+2r}$$

If the first 10 terms of ∑ are calculated and both sides are illustrated, it is as follows.
Both sides overlap exactly and blue (left) can not be seen.

$$m = 10;$$

$$fl[x\_] := \sum_{r=1}^{m}(-1)^{r-1}\frac{2\,!}{(2+2\,r)\,!}\,x^{2+2\,r} \qquad fr[x\_] := x^2 - 2 + 2\,Cos[x]$$



24.4 Integrals Series Expansion (exp x)

Mikio Sugioka discovered that the series of the higher integral of arbitrary functions is expanded into the series of the higher integrals of exponential functions or trigonometric functions in March, 2012. (http://www5b.biglobe.ne.jp/~sugi_m/page216.htm). The expressions by the higher integral series are very beautiful. In addition, the expressions by a double series are useful for numerical calculations. Then I introduce them in the following three sections.

Theorem 24.4.1

When $f(x)$ is analytic function defined on a closed interval I , the following expressions hold for $a, x \in I$.

$$\sum_{r=0}^{\infty} \int_a^x \cdots \int_a^x f(x)\,dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=0}^{\infty} \frac{(x-a)^{s+r}}{(s+r)!} \tag{1.1}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a) \int_a^x \cdots \int_a^x e^{x-a}\,dx^r \tag{1.1'}$$

$$\sum_{r=0}^{\infty} (-1)^r \int_a^x \cdots \int_a^x f(x)\,dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=0}^{\infty} (-1)^s \frac{(x-a)^{s+r}}{(s+r)!} \tag{1.2}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a) \int_a^x \cdots \int_a^x e^{-(x-a)}\,dx^r \tag{1.2'}$$

Proof

Let $f^{(r)}(a) = f_a^r$ and expand $f(x)$ into Taylor series around a . Then

$$f(x) = f_a^0 \cdot \frac{(x-a)^0}{0!} + f_a^1 \cdot \frac{(x-a)^1}{1!} + f_a^2 \cdot \frac{(x-a)^2}{2!} + f_a^3 \cdot \frac{(x-a)^3}{3!} + \cdots \tag{0}$$

Integrating both sides of this with respect to x from a to x one by one,

$$\int_a^x f(x)\,dx = f_a^0 \cdot \frac{(x-a)^1}{1!} + f_a^1 \cdot \frac{(x-a)^2}{2!} + f_a^2 \cdot \frac{(x-a)^3}{3!} + f_a^3 \cdot \frac{(x-a)^4}{4!} + \cdots$$

$$\int_a^x\int_a^x f(x)\,dx^2 = f_a^0 \cdot \frac{(x-a)^2}{2!} + f_a^1 \cdot \frac{(x-a)^3}{3!} + f_a^2 \cdot \frac{(x-a)^4}{4!} + f_a^3 \cdot \frac{(x-a)^5}{5!} + \cdots$$

$$\int_a^x\int_a^x\int_a^x f(x)\,dx^3 = f_a^0 \cdot \frac{(x-a)^3}{3!} + f_a^1 \cdot \frac{(x-a)^4}{4!} + f_a^2 \cdot \frac{(x-a)^5}{5!} + f_a^3 \cdot \frac{(x-a)^6}{6!} + \cdots$$

$$\vdots$$

Adding these also including (0) perpendicularly.

$$f(x) + \int_a^x f(x)\,dx + \int_a^x\int_a^x f(x)\,dx^2 + \int_a^x\int_a^x\int_a^x f(x)\,dx^3 + \cdots$$

$$= f_a^0 \left\{ \frac{(x-a)^0}{0!} + \frac{(x-a)^1}{1!} + \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} + \cdots \right\}$$

$$+ f_a^1 \left\{ \frac{(x-a)^1}{1!} + \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} + \frac{(x-a)^4}{4!} + \cdots \right\}$$

$$+ f_a^2 \left\{ \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} + \frac{(x-a)^4}{4!} + \frac{(x-a)^5}{5!} + \cdots \right\}$$

$$\vdots$$

$$= \sum_{r=0}^{\infty} f_a^r \sum_{s=0}^{\infty} \frac{(x-a)^{s+r}}{(s+r)!}$$

i.e.

$$\sum_{r=0}^{\infty} \int_a^x \cdots \int_a^x f(x) dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=0}^{\infty} \frac{(x-a)^{s+r}}{(s+r)!} \tag{1.1}$$

Furthermore,

$$\sum_{s=0}^{\infty} \frac{(x-a)^{s+r}}{(s+r)!} = \int_a^x \cdots \int_a^x e^{x-a} dx^r$$

Then

$$\sum_{r=0}^{\infty} \int_a^x \cdots \int_a^x f(x) dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \int_a^x \cdots \int_a^x e^{x-a} dx^r \tag{1.1'}$$

Next,

$$f(x) = f_a^0 \cdot \frac{(x-a)^0}{0!} + f_a^1 \cdot \frac{(x-a)^1}{1!} + f_a^2 \cdot \frac{(x-a)^2}{2!} + f_a^3 \cdot \frac{(x-a)^3}{3!} + \cdots$$

$$-\int_a^x f(x) dx = -f_a^0 \cdot \frac{(x-a)^1}{1!} - f_a^1 \cdot \frac{(x-a)^2}{2!} - f_a^2 \cdot \frac{(x-a)^3}{3!} - f_a^3 \cdot \frac{(x-a)^4}{4!} - \cdots$$

$$\int_a^x \int_a^x f(x) dx^2 = f_a^0 \cdot \frac{(x-a)^2}{2!} + f_a^1 \cdot \frac{(x-a)^3}{3!} + f_a^2 \cdot \frac{(x-a)^4}{4!} + f_a^3 \cdot \frac{(x-a)^5}{5!} + \cdots$$

$$-\int_a^x \int_a^x \int_a^x f(x) dx^3 = -f_a^0 \cdot \frac{(x-a)^3}{3!} - f_a^1 \cdot \frac{(x-a)^4}{4!} - f_a^2 \cdot \frac{(x-a)^5}{5!} - f_a^3 \cdot \frac{(x-a)^6}{6!} - \cdots$$

Adding these perpendicularly.

$$f(x) - \int_a^x f(x) dx + \int_a^x \int_a^x f(x) dx^2 - \int_a^x \int_a^x \int_a^x f(x) dx^3 + - \cdots$$

$$= f_a^0 \left\{ \frac{(x-a)^0}{0!} - \frac{(x-a)^1}{1!} + \frac{(x-a)^2}{2!} - \frac{(x-a)^3}{3!} + - \cdots \right\}$$

$$+ f_a^1 \left\{ \frac{(x-a)^1}{1!} - \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} - \frac{(x-a)^4}{4!} + - \cdots \right\}$$

$$+ f_a^2 \left\{ \frac{(x-a)^2}{2!} - \frac{(x-a)^3}{3!} + \frac{(x-a)^4}{4!} - \frac{(x-a)^5}{5!} + - \cdots \right\}$$

$$\vdots$$

$$= \sum_{r=0}^{\infty} (-1)^r f_a^r \sum_{s=0}^{\infty} (-1)^s \frac{(x-a)^{s+r}}{(s+r)!}$$

i.e.

$$\sum_{r=0}^{\infty} (-1)^r \int_a^x \cdots \int_a^x f(x) dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=0}^{\infty} (-1)^s \frac{(x-a)^{s+r}}{(s+r)!} \tag{1.2}$$

Furthermore,

$$\sum_{s=0}^{\infty} (-1)^s \frac{(x-a)^{s+r}}{(s+r)!} = \int_a^x \cdots \int_a^x e^{-(x-a)} dx^r$$

Then

$$\sum_{r=0}^{\infty}(-1)^r\int_a^x\cdots\int_a^x f(x)dx^r = \sum_{r=0}^{\infty}f^{(r)}(a)\int_a^x\cdots\int_a^x e^{-(x-a)}dx^r \tag{1.2'}$$

Example1 $\log x+\int_1^x \log x dx+\int_1^x\int_1^x \log x dx^2+\int_1^x\int_1^x\int_1^x \log x dx^3+\cdots$

$$f(x)=\log x\,,\quad (\log x)^{(n)}=(-1)^{n-1}(n-1)!\ x^{-n}\qquad n=1,2,3,\cdots$$

Substituting these for (1.1), (1.1') ,

$$\sum_{r=0}^{\infty}\int_1^x\cdots\int_1^x \log x dx^r = \sum_{r=1}^{\infty}(-1)^{r-1}(r-1)!\sum_{s=0}^{\infty}\frac{(x-1)^{s+r}}{(s+r)!}$$

$$= \sum_{r=1}^{\infty}(-1)^{r-1}(r-1)!\int_1^x\cdots\int_1^x e^{x-1}dx^r$$

$$= 0!\int_1^x e^{x-1}dx-1!\int_1^x\int_1^x e^{x-1}dx^2+2!\int_1^x\int_1^x\int_1^x e^{x-1}dx^3-3!\int_1^x\cdots\int_1^x e^{x-1}dx^4+-\cdots$$

When higher integrals are replaced with Riemann-Liouville Integral and one arbitrary point $x=0.8$ is given, it is as follows.

```
m = 10;
f1[x_] := Log[x] + Sum[1/Gamma[r] Integrate[(x - t)^(r-1) Log[t], {t, 1, x}], {r, 1, m}]
fm[x_] := Sum[(-1)^(r-1) (r - 1)! Sum[(x - 1)^(s+r)/(s + r)!, {s, 0, ∞}], {r, 1, m}]
fr[x_] := Sum[(-1)^(r-1) (r - 1)! 1/Gamma[r] Integrate[(x - t)^(r-1) e^(t-1), {t, 1, x}], {r, 1, m}]
```

$$f1[x\_] := \mathrm{Log}[x]+\sum_{r=1}^{m}\frac{1}{\mathrm{Gamma}[r]}\int_1^x (x-t)^{r-1}\mathrm{Log}[t]\,dt$$

$$fm[x\_] := \sum_{r=1}^{m}(-1)^{r-1}(r-1)!\sum_{s=0}^{\infty}\frac{(x-1)^{s+r}}{(s+r)!}$$

$$fr[x\_] := \sum_{r=1}^{m}(-1)^{r-1}(r-1)!\frac{1}{\mathrm{Gamma}[r]}\int_1^x (x-t)^{r-1}e^{t-1}dt$$

```
N[f1[0.8]]        N[fm[0.8]]        N[fr[0.8]]
-0.202997         -0.202997         -0.202997
```

Example2 $\sqrt{x}-\int_1^x\sqrt{x}dx+\int_1^x\int_1^x\sqrt{x}dx^2-\int_1^x\int_1^x\int_1^x\sqrt{x}dx^3+-\ \cdots$

$$f(x)=\sqrt{x}$$

$$f^{(n)}(1)=\frac{\Gamma(1+1/2)}{\Gamma(1+1/2-n)}1^{\frac{1}{2}-n}=(-1)^{n-1}\frac{(2n-3)!!}{2^n}$$

Substituting these for (1.2), (1.2') ,

$$\sum_{r=0}^{\infty}(-1)^r\int_1^x\cdots\int_1^x\sqrt{x}dx^r=e^{-(x-1)}+\sum_{r=1}^{\infty}(-1)^{r-1}\frac{(2r-3)!!}{2^r}\sum_{s=0}^{\infty}(-1)^s\frac{(x-1)^{s+r}}{(s+r)!}$$

$$=e^{-(x-1)}+\sum_{r=1}^{\infty}(-1)^{r-1}\frac{(2r-3)!!}{2^r}\int_1^x\cdots\int_1^x e^{-(x-1)}dx^r$$

$$=e^{-(x-1)}+\frac{(-1)!!}{2^1}\int_1^x e^{x-1}dx-\frac{1!!}{2^2}\int_1^x\int_1^x e^{x-1}dx^2+\frac{3!!}{2^3}\int_1^x\int_1^x\int_1^x e^{x-1}dx^3-+\cdots$$

During the proof of the previous theorem, if the calculation is done without including (0) , we obtain the following theorem.

Theorem 24.4.1'

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $a, x \in I$.

$$\sum_{r=1}^{\infty} \int_a^x \cdots \int_a^x f(x)\,dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=1}^{\infty} \frac{(x-a)^{s+r}}{(s+r)!}$$

$$\sum_{r=1}^{\infty} (-1)^{r-1} \int_a^x \cdots \int_a^x f(x)\,dx^r = \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=1}^{\infty} (-1)^{s-1} \frac{(x-a)^{s+r}}{(s+r)!}$$

Combining Theorem 24.4.1' and **24.1** , we obtain the following formula which gives the collateral integral of the product of an exponential function and arbitrary functions.

Formula 24.4.2

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $a, x \in I$.

$$\int_a^x e^x f(x)\,dx = e^x \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=1}^{\infty} (-1)^{s-1} \frac{(x-a)^{s+r}}{(s+r)!} \tag{2.1}$$

$$\int_a^x e^{-x} f(x)\,dx = e^{-x} \sum_{r=0}^{\infty} f^{(r)}(a) \sum_{s=1}^{\infty} \frac{(x-a)^{s+r}}{(s+r)!} \tag{2.2}$$

Example $\int_a^x e^{-x} \cos x\,dx$

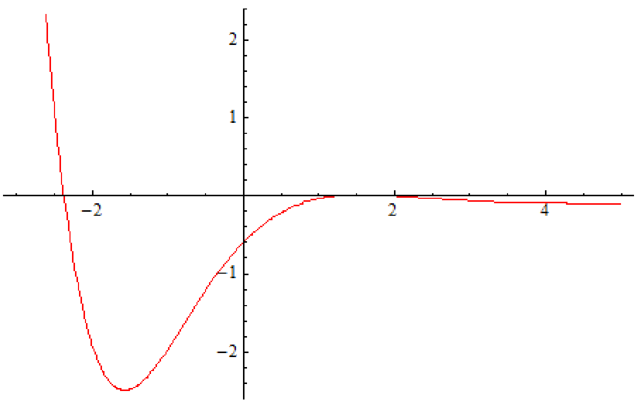
Since $(\cos x)^{(n)} = \cos(x+n\pi/2)$ $n=1,2,3,\cdots$,

$$\int_a^x e^{-x} \cos x\,dx = e^{-x} \sum_{r=0}^{\infty} \cos\left(a+\frac{r\pi}{2}\right) \sum_{s=r}^{\infty} \frac{(x-a)^{s+1}}{(s+1)!}$$

When $a=2$ and the first 50 terms of $\sum$ are calculated and both sides are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.

```
m = 50;
```

$$\mathtt{f1[a\_]} := \int_a^x e^{-t} \operatorname{Cos}[t]\,dt \qquad \mathtt{fr[a\_]} := e^{-x} \sum_{r=0}^{m} \operatorname{Cos}\left[a+\frac{r\pi}{2}\right] \sum_{s=1}^{m} \frac{(x-a)^{s+r}}{(s+r)!}$$



24.5 Integrals Series Expansion (sin x)

Theorem 24.5.1

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $a, x \in I$.

$$\sum_{r=1}^{\infty}\int_a^x \cdots \int_a^x f(x)\,dx^{2r-1} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}\frac{(x-a)^{2s+1+r}}{(2s+1+r)!} \tag{1.1}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x \cdots \int_a^x \sinh(x-a)\,dx^r \tag{1.1'}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x \cdots \int_a^x f(x)\,dx^{2r-1} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}(-1)^s\frac{(x-a)^{2s+1+r}}{(2s+1+r)!} \tag{1.2}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x \cdots \int_a^x \sin(x-a)\,dx^r \tag{1.2'}$$

Proof

Let $f^{(r)}(a) = f_a^r$ and expand $f(x)$ into Taylor series around a. Then

$$f(x) = f_a^0\cdot\frac{(x-a)^0}{0!} + f_a^1\cdot\frac{(x-a)^1}{1!} + f_a^2\cdot\frac{(x-a)^2}{2!} + f_a^3\cdot\frac{(x-a)^3}{3!} + \cdots$$

Integrating both sides of this with respect to x from a to x alternately,

$$\int_a^x f(x)\,dx = f_a^0\cdot\frac{(x-a)^1}{1!} + f_a^1\cdot\frac{(x-a)^2}{2!} + f_a^2\cdot\frac{(x-a)^3}{3!} + f_a^3\cdot\frac{(x-a)^4}{4!} + \cdots$$

$$\int_a^x\int_a^x\int_a^x f(x)\,dx^3 = f_a^0\cdot\frac{(x-a)^3}{3!} + f_a^1\cdot\frac{(x-a)^4}{4!} + f_a^2\cdot\frac{(x-a)^5}{5!} + f_a^3\cdot\frac{(x-a)^6}{6!} + \cdots$$

$$\int_a^x \cdots \int_a^x f(x)\,dx^5 = f_a^0\cdot\frac{(x-a)^5}{5!} + f_a^1\cdot\frac{(x-a)^6}{6!} + f_a^2\cdot\frac{(x-a)^7}{7!} + f_a^3\cdot\frac{(x-a)^8}{8!} + \cdots$$

$$\vdots$$

Adding these perpendicularly.

$$\int_a^x f(x)\,dx + \int_a^x\int_a^x\int_a^x f(x)\,dx^3 + \int_a^x \cdots \int_a^x f(x)\,dx^5 + \cdots$$

$$= f_a^0\left\{\frac{(x-a)^1}{1!} + \frac{(x-a)^3}{3!} + \frac{(x-a)^5}{5!} + \frac{(x-a)^7}{7!} + \cdots\right\}$$

$$+ f_a^1\left\{\frac{(x-a)^2}{2!} + \frac{(x-a)^4}{4!} + \frac{(x-a)^6}{6!} + \frac{(x-a)^8}{8!} + \cdots\right\}$$

$$+ f_a^2\left\{\frac{(x-a)^3}{3!} + \frac{(x-a)^5}{5!} + \frac{(x-a)^7}{7!} + \frac{(x-a)^9}{9!} + \cdots\right\}$$

$$\vdots$$

$$= \sum_{r=0}^{\infty} f_a^r \sum_{s=0}^{\infty}\frac{(x-a)^{2s+1+r}}{(2s+1+r)!}$$

i.e.

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}\frac{(x-a)^{2s+1+r}}{(2s+1+r)!} \tag{1.1}$$

Furthermore,

$$\sum_{s=0}^{\infty}\frac{(x-a)^{2s+1+r}}{(2s+1+r)!} = \int_a^x\cdots\int_a^x sinh(x-a)\,dx^r$$

Then,

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x sinh(x-a)\,dx^r \tag{1.1'}$$

Next,

$$\int_a^x f(x)\,dx \quad = f_a^0\cdot\frac{(x-a)^1}{1!}+f_a^1\cdot\frac{(x-a)^2}{2!}+f_a^2\cdot\frac{(x-a)^3}{3!}+f_a^3\cdot\frac{(x-a)^4}{4!}+\cdots$$

$$-\int_a^x\int_a^x\int_a^x f(x)\,dx^3 = -f_a^0\cdot\frac{(x-a)^3}{3!}-f_a^1\cdot\frac{(x-a)^4}{4!}-f_a^2\cdot\frac{(x-a)^5}{5!}-f_a^3\cdot\frac{(x-a)^6}{6!}-\cdots$$

$$\int_a^x\cdots\int_a^x f(x)\,dx^5 = f_a^0\cdot\frac{(x-a)^5}{5!}+f_a^1\cdot\frac{(x-a)^6}{6!}+f_a^2\cdot\frac{(x-a)^7}{7!}+f_a^3\cdot\frac{(x-a)^8}{8!}+\cdots$$

$$-\int_a^x\cdots\int_a^x f(x)\,dx^7 = -f_a^0\cdot\frac{(x-a)^7}{7!}-f_a^1\cdot\frac{(x-a)^8}{8!}-f_a^2\cdot\frac{(x-a)^9}{9!}-f_a^3\cdot\frac{(x-a)^{10}}{10!}-\cdots$$

$$\vdots$$

Adding these perpendicularly.

$$\int_a^x f(x)\,dx - \int_a^x\int_a^x\int_a^x f(x)\,dx^3 + \int_a^x\cdots\int_a^x f(x)\,dx^5 - \int_a^x\cdots\int_a^x f(x)\,dx^7 +- \cdots$$

$$= f_a^0\left\{\frac{(x-a)^1}{1!}-\frac{(x-a)^3}{3!}+\frac{(x-a)^5}{5!}-\frac{(x-a)^7}{7!}+-\cdots\right\}$$

$$+ f_a^1\left\{\frac{(x-a)^2}{2!}-\frac{(x-a)^4}{4!}+\frac{(x-a)^6}{6!}-\frac{(x-a)^8}{8!}+-\cdots\right\}$$

$$+ f_a^2\left\{\frac{(x-a)^3}{3!}-\frac{(x-a)^5}{5!}+\frac{(x-a)^7}{7!}-\frac{(x-a)^9}{9!}+-\cdots\right\}$$

$$+ f_a^3\left\{\frac{(x-a)^4}{4!}-\frac{(x-a)^6}{6!}+\frac{(x-a)^8}{8!}-\frac{(x-a)^{10}}{10!}+-\cdots\right\}$$

$$\vdots$$

$$= \sum_{r=0}^{\infty} f_a^r \sum_{s=0}^{\infty}(-1)^s\frac{(x-a)^{2s+1+r}}{(2s+1+r)!}$$

i.e.

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}(-1)^s\frac{(x-a)^{2s+1+r}}{(2s+1+r)!} \tag{1.2}$$

Furthermore,

$$\sum_{s=0}^{\infty}(-1)^s\frac{(x-a)^{2s+1+r}}{(2s+1+r)!} = \int_a^x\cdots\int_a^x sin(x-a)\,dx^r$$

Then,

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x \sin(x-a)\,dx^r \qquad (1.2')$$

Example1 $\int_0^x \tan x\,dx + \int_0^x\int_0^x\int_0^x \tan x\,dx^3 + \int_0^x\cdots\int_0^x \tan x\,dx^5 + \cdots$

The higher differential quotient of $\tan x$ on $x=0$ is as follows according to Theorem 9.2.6 (**9.2**)

$$(\tan x)^{(2n-1)}\Big|_{x=0} = T_{2n-1} = \frac{2^{2n}\left(2^{2n}-1\right)|B_{2n}|}{2n} \quad , \quad (\tan x)^{(2n)}\Big|_{x=0} = 0$$

Where, T_{2n-1} is the tangent number and B_{2n} is the Bernoulli number. Thus, from (1.1), (1.1') ,

$$\sum_{r=1}^{\infty}\int_0^x\cdots\int_0^x \tan x\,dx^{2r-1} = \sum_{r=1}^{\infty} T_{2r-1}\sum_{s=0}^{\infty}\frac{x^{2s+2r}}{(2s+2r)!} = \sum_{r=1}^{\infty} T_{2r-1}\int_0^x\cdots\int_0^x \sinh x\,dx^{2r-1}$$

$$= 1\int_0^x \sinh x\,dx + 2\int_0^x\int_0^x\int_0^x \sinh x\,dx^3 + 16\int_0^x\cdots\int_0^x \sinh x\,dx^5 + \cdots$$

When higher integrals are replaced with Riemann-Liouville Integral and one arbitrary point $x=0.9$ is given, it is as follows.

```
m = 10;
fl[x_] := Sum[1/Gamma[2 r - 1] Integrate[(x - t)^(2 r - 2) Tan[t], {t, 0, x}], {r, 1, m}]
T[r_] := 2^(2 r) (2^(2 r) - 1) Abs[BernoulliB[2 r]] / (2 r)
fm[x_] := Sum[T[r] Sum[x^(2 s + 2 r)/(2 s + 2 r)!, {s, 0, ∞}], {r, 1, m}]
fr[x_] := Sum[T[r]/Gamma[2 r - 1] Integrate[(x - t)^(2 r - 2) Sinh[t], {t, 0, x}], {r, 1, m}]

N[fl[0.9]]                          N[fm[0.9]]     N[fr[0.9]]
0.505231 + 2.40581 × 10^-18 i       0.505231       0.505068
```

Example2 $\int_0^x \sec x\,dx - \int_0^x\int_0^x\int_0^x \sec x\,dx^3 + \int_0^x\cdots\int_0^x \sec x\,dx^5 - \int_0^x\cdots\int_0^x \sec x\,dx^7 +- \cdots$

The higher differential quotient of $\sec x$ on $x=0$ is as follows according to Theorem 9.2.8 (**9.2**)

$$(\sec x)^{(2n)}\Big|_{x=0} = |E_{2n}| \quad , \quad (\sec x)^{(2n+1)}\Big|_{x=0} = 0$$

Here, E_{2n} is an Euler number. Thus, from (1.2), (1.2') ,

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_0^x\cdots\int_0^x \sec x\,dx^{2r-1} = \sum_{r=0}^{\infty}|E_{2r}|\sum_{s=0}^{\infty}(-1)^s\frac{x^{2s+1+2r}}{(2s+1+2r)!}$$

$$= \sum_{r=0}^{\infty}|E_{2r}|\int_0^x\cdots\int_0^x \sin x\,dx^{2r}$$

$$= \sin x + \int_0^x\int_0^x \sin x\,dx^2 + 5\int_0^x\cdots\int_0^x \sin x\,dx^4 + 61\int_0^x\cdots\int_0^x \sin x\,dx^6 + \cdots$$

24.6 Integrals Series Expansion (cos x)

Theorem 24.6.1

When $f(x)$ is analytic function defined on a closed interval I , the following expressions hold for $a, x \in I$.

$$\sum_{r=0}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}\frac{(x-a)^{2s+r}}{(2s+r)!} \tag{1.1}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x \cosh(x-a)\,dx^{2r} \tag{1.1'}$$

$$\sum_{r=0}^{\infty}(-1)^r\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}(-1)^s\frac{(x-a)^{2s+r}}{(2s+r)!} \tag{1.2}$$

$$= \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x \cos(x-a)\,dx^{r} \tag{1.2'}$$

Proof

Let $f^{(r)}(a) = f_a^r$ and expand $f(x)$ into Taylor series around a . Then

$$f(x) = f_a^0\cdot\frac{(x-a)^0}{0!} + f_a^1\cdot\frac{(x-a)^1}{1!} + f_a^2\cdot\frac{(x-a)^2}{2!} + f_a^3\cdot\frac{(x-a)^3}{3!} + \cdots \tag{0}$$

Integrating both sides of this with respect to x from a to x alternately,

$$\int_a^x\int_a^x f(x)\,dx^2 = f_a^0\cdot\frac{(x-a)^2}{2!} + f_a^1\cdot\frac{(x-a)^3}{3!} + f_a^2\cdot\frac{(x-a)^4}{4!} + f_a^3\cdot\frac{(x-a)^5}{5!} + \cdots$$

$$\int_a^x\cdots\int_a^x f(x)\,dx^4 = f_a^0\cdot\frac{(x-a)^4}{4!} + f_a^1\cdot\frac{(x-a)^5}{5!} + f_a^2\cdot\frac{(x-a)^6}{6!} + f_a^3\cdot\frac{(x-a)^7}{7!} + \cdots$$

$$\int_a^x\cdots\int_a^x f(x)\,dx^6 = f_a^0\cdot\frac{(x-a)^6}{6!} + f_a^1\cdot\frac{(x-a)^7}{7!} + f_a^2\cdot\frac{(x-a)^8}{8!} + f_a^3\cdot\frac{(x-a)^9}{9!} + \cdots$$

$\vdots$

Adding these also including (0) perpendicularly.

$$f(x) + \int_a^x\int_a^x f(x)\,dx^2 + \int_a^x\cdots\int_a^x f(x)\,dx^4 + \int_a^x\cdots\int_a^x f(x)\,dx^6$$

$$= f_a^0\left\{\frac{(x-a)^0}{0!} + \frac{(x-a)^2}{2!} + \frac{(x-a)^4}{4!} + \frac{(x-a)^6}{6!} + \cdots\right\}$$

$$+ f_a^1\left\{\frac{(x-a)^1}{1!} + \frac{(x-a)^3}{3!} + \frac{(x-a)^5}{5!} + \frac{(x-a)^7}{7!} + \cdots\right\}$$

$$+ f_a^2\left\{\frac{(x-a)^2}{2!} + \frac{(x-a)^4}{4!} + \frac{(x-a)^6}{6!} + \frac{(x-a)^8}{8!} + \cdots\right\}$$

$\vdots$

$$= \sum_{r=0}^{\infty} f_a^r\sum_{s=0}^{\infty}\frac{(x-a)^{2s+r}}{(2s+r)!}$$

i.e.

$$\sum_{r=0}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}\frac{(x-a)^{2s+r}}{(2s+r)!} \tag{1.1}$$

Furthermore,

$$\sum_{s=0}^{\infty}\frac{(x-a)^{2s+r}}{(2s+r)!} = \int_a^x\cdots\int_a^x \cosh(x-a)\,dx^r$$

Then,

$$\sum_{r=0}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x \cosh(x-a)\,dx^r \tag{1.1'}$$

Next,

$$f(x) = f_a^0\cdot\frac{(x-a)^0}{0!}+f_a^1\cdot\frac{(x-a)^1}{1!}+f_a^2\cdot\frac{(x-a)^2}{2!}+f_a^3\cdot\frac{(x-a)^3}{3!}+\cdots$$

$$-\int_a^x\int_a^x f(x)\,dx^2 = -f_a^0\cdot\frac{(x-a)^2}{2!}-f_a^1\cdot\frac{(x-a)^3}{3!}-f_a^2\cdot\frac{(x-a)^4}{4!}-f_a^3\cdot\frac{(x-a)^5}{5!}-\cdots$$

$$\int_a^x\cdots\int_a^x f(x)\,dx^4 = f_a^0\cdot\frac{(x-a)^4}{4!}+f_a^1\cdot\frac{(x-a)^5}{5!}+f_a^2\cdot\frac{(x-a)^6}{6!}+f_a^3\cdot\frac{(x-a)^7}{7!}+\cdots$$

$$-\int_a^x\cdots\int_a^x f(x)\,dx^6 = -f_a^0\cdot\frac{(x-a)^6}{6!}-f_a^1\cdot\frac{(x-a)^7}{7!}-f_a^2\cdot\frac{(x-a)^8}{8!}-f_a^3\cdot\frac{(x-a)^9}{9!}-\cdots$$

$$\vdots$$

Adding these perpendicularly and doing calculation similar to Theorem 24.5.1, we obtain (1.2) and (1.2') .

Example1 $\tan^{-1}x + \int_1^x\int_1^x \tan^{-1}x\,dx^2 + \int_1^x\cdots\int_1^x \tan^{-1}x\,dx^4 + \cdots$

According to " 岩波数学公式 I " p39, the following expression holds for a natural number n .

$$\left(\tan^{-1}x\right)^{(n)} = (n-1)!\cos^n\!\left(\tan^{-1}x\right)\sin\left(n\left(\tan^{-1}x+\frac{\pi}{2}\right)\right)$$

From this,

$$\left(\tan^{-1}x\right)^{(n)}\Big|_{x=1} = \frac{(n-1)!}{2^{n/2}}\sin\left(\frac{3n\pi}{4}\right)$$

Substituting this for (1.1), (1.1') ,

$$\sum_{r=0}^{\infty}\int_1^x\cdots\int_1^x \tan^{-1}x\,dx^{2r} = \frac{\pi}{4}\sum_{s=0}^{\infty}\frac{(x-1)^{2s}}{(2s)!}+\sum_{r=1}^{\infty}\frac{(r-1)!}{2^{r/2}}\sin\left(\frac{3r\pi}{4}\right)\sum_{s=0}^{\infty}\frac{(x-1)^{2s+r}}{(2s+r)!}$$

$$= \frac{\pi}{4}\cosh(x-1)+\sum_{r=1}^{\infty}\frac{(r-1)!}{2^{r/2}}\sin\left(\frac{3r\pi}{4}\right)\int_1^x\cdots\int_1^x \cosh(x-1)\,dx^r$$

$$= \frac{\pi}{4}\cosh(x-1)+\frac{1}{2}\int_1^x \cosh(x-1)\,dx-\frac{1}{2}\int_1^x\int_1^x \cosh(x-1)\,dx^2+\cdots$$

When higher integrals are replaced with Riemann-Liouville Integral and one arbitrary point $x=1.8$ is given, it is as follows.

```
m = 15;
f1[x_] := ArcTan[x] + Sum[1/Gamma[2 r] Integrate[(x - t)^(2 r-1) ArcTan[t], {t, 1, x}], {r, 1, m}]

fm[x_] := π/4 Sum[(x - 1)^(2 s)/(2 s)!, {s, 0, ∞}] + Sum[(r - 1)!/2^(r/2) Sin[3 r π/4] Sum[(x - 1)^(2 s+r)/(2 s + r)!, {s, 0, ∞}], {r, 1, m}]

fr[x_] := π/4 Cosh[x - 1] + Sum[(r - 1)!/2^(r/2) Sin[3 r π/4] 1/Gamma[r] Integrate[(x - t)^(r-1) Cosh[t - 1], {t, 1, x}], {r, 1, m}]
```

```
N[fl[1.8]]
1.36536
```

```
N[fm[1.8]]
1.36536
```

```
N[fr[1.8]]
1.36536
```

Example2 $\sin^{-1}x - \int_0^x\int_0^x \sin^{-1}x\,dx^2 + \int_0^x\cdots\int_0^x \sin^{-1}x\,dx^4 - \int_0^x\cdots\int_0^x \sin^{-1}x\,dx^6 +- \cdots$

The higher differential quotient of $\sin^{-1}x$ on $x=0$ is as follows according to Theorem 9.3.2 (**9.3**)

$$\left(\sin^{-1}x\right)^{(2n+1)}\Big|_{x=0} = {}_{2n}C_0(2n-1)!!\,(2n-1)!!\,0^0 = (2n-1)!!^2$$

Then, from (1.2), (1.2'),

$$\sum_{r=0}^{\infty}(-1)^r\int_0^x\cdots\int_0^x \sin^{-1}x\,dx^{2r} = \sum_{r=0}^{\infty}(2r-1)!!^2\sum_{s=0}^{\infty}(-1)^s\frac{x^{2s+2r+1}}{(2s+2r+1)!}$$

$$= \sum_{r=0}^{\infty}(2r-1)!!^2\int_0^x\cdots\int_0^x \cos x\,dx^{2r+1}$$

$$= (-1)!!^2\int_0^x \cos x\,dx + 1!!^2\int_0^x\int_0^x\int_0^x \cos x\,dx^3 + 3!!^2\int_0^x\cdots\int_0^x \cos x\,dx^5 + \cdots$$

During the proof of the previous theorem, if the calculation is done without including (0) , we obtain the following theorem.

Theorem 24.6.1'

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $a, x \in I$.

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty}f^{(r)}(a)\sum_{s=1}^{\infty}\frac{(x-a)^{2s+r}}{(2s+r)!} \tag{1.3}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty}f^{(r)}(a)\sum_{s=1}^{\infty}(-1)^{s-1}\frac{(x-a)^{2s+r}}{(2s+r)!} \tag{1.4}$$

Note

This theorem is not so beautiful. However, this is useful for the calculation.

Example $\int_0^x\int_0^x \sin^{-1}x\,dx^2 - \int_0^x\cdots\int_0^x \sin^{-1}x\,dx^4 + \int_0^x\cdots\int_0^x \sin^{-1}x\,dx^6 - \int_0^x\cdots\int_0^x \sin^{-1}x\,dx^8 +- \cdots$

Since the higher differential quotient of $\sin^{-1}x$ on $x=0$ is same as Example2, from (1.4),

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_0^x\cdots\int_0^x \sin^{-1}x\,dx^{2r} = \sum_{r=0}^{\infty}(2r-1)!!^2\sum_{s=1}^{\infty}(-1)^{s-1}\frac{x^{2s+2r+1}}{(2s+2r+1)!}$$

If the left side is replaced with Riemann-Liouville Integral and one arbitrary point $x=0.6$ is given, then

```
m = 7;
```

$$\text{fl}[x\_] := \sum_{r=1}^{m}\frac{(-1)^{r-1}}{\text{Gamma}[2r]}\int_0^x (x-t)^{2r-1}\,\text{ArcSin}[t]\,dt$$

$$\text{fr}[x\_] := \sum_{r=0}^{m}((2r-1)!!)^2\sum_{s=1}^{\infty}(-1)^{s-1}\frac{x^{2s+2r+1}}{(2s+2r+1)!}$$

```
N[fl[0.6]]
0.0360572
```

```
N[fr[0.6]]
0.0360572
```

24.7 Series of Higher Integrals with coefficients

Theorem 24.1.1 is extensible to the higher integral series with coefficients ${}_{n-1+r}C_{n-1}$. Although these coefficients are numbers on the hypotenuse of Pascal's triangle, coefficients such as $n \geqq 3$ is not so interesting. Then, in this section, we ask for the sum of the higher integral series at n=2 i.e.

$$1\int_a^x f(x)\,dx \pm 2\int_a^x\!\!\int_a^x f(x)\,dx^2 + 3\int_a^x\!\!\int_a^x\!\!\int_a^x f(x)\,dx^3 \pm 4\int_a^x\!\!\cdots\!\int_a^x f(x)\,dx^4 +\pm \cdots$$

Theorem 24.7.1

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $x\in I$ and $a\in I\ \ s.t.\ \ f(a)=0$.

$$\sum_{r=1}^{m} r\int_a^x\!\!\cdots\!\int_a^x f(x)\,dx^r = e^x\int_a^x f(x)e^{-x}dx + e^x\int_a^x\!\!\int_a^x f(x)e^{-x}dx^2$$
$$- e^x(m+1)\int_a^x\!\!\int_a^x\left\{\int_a^x\!\!\cdots\!\int_a^x f(x)\,dx^{m-1}\right\}e^{-x}dx^2$$
$$+ e^x m\int_a^x\!\!\int_a^x\left\{\int_a^x\!\!\cdots\!\int_a^x f(x)\,dx^{m}\right\}e^{-x}dx^2 \qquad (1.1)$$

$$\sum_{r=1}^{\infty} r\int_a^x\!\!\cdots\!\int_a^x f(x)\,dx^r = e^x\int_a^x f(x)e^{-x}dx + e^x\int_a^x\!\!\int_a^x f(x)e^{-x}dx^2 \qquad (1.2)$$

Proof

When $f^{<r>}(a)=0\ \ (r=1,2,\cdots,m+n-1)$, Theorem 16.1.2 (**16.1**) was as follows.

$$\int_a^x\!\!\cdots\!\int_a^x f^{<0>}g^{(0)}dx^n = \sum_{r=0}^{m-1}\binom{-n}{r}f^{<n+r>}g^{(r)}$$
$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\!\!\cdots\!\int_a^x f^{<m+k>}g^{(m+k)}dx^n$$

Here, let $g=e^{-x}$, $n=2$. Then, since $g^{(r)}=(-1)^r e^{-x}$,

$$\int_a^x\!\!\int_a^x f^{<0>}e^{-x}dx^2 = \sum_{r=0}^{m-1}\binom{-2}{r}f^{<2+r>}(-1)^r e^{-x}$$
$$+\frac{(-1)^m}{B(2,m)}\frac{{}_1C_0}{m+0}\int_a^x\!\!\int_a^x f^{<m+0>}(-1)^{m+0}e^{-x}dx^2$$
$$+\frac{(-1)^m}{B(2,m)}\frac{{}_1C_1}{m+1}\int_a^x\!\!\int_a^x f^{<m+1>}(-1)^{m+1}e^{-x}dx^2$$
$$=\sum_{r=0}^{m-1}(1+r)f^{<2+r>}e^{-x} + (m+1)\int_a^x\!\!\int_a^x f^{<m>}e^{-x}dx^2 - m\int_a^x\!\!\int_a^x f^{<m+1>}e^{-x}dx^2$$
$$\left\{\because\ \ (-1)^r\binom{-2}{r} = 1+r\ ,\ \ \frac{1}{B(2,m)} = m(m+1)\right\}$$

Here, subtracting 1 from the index <r> of the integration operator of the function f,

$$\int_a^x\!\!\int_a^x f^{<-1>}e^{-x}dx^2 = \sum_{r=0}^{m-1}(1+r)f^{<1+r>}e^{-x}$$
$$+ (m+1)\int_a^x\!\!\int_a^x f^{<m-1>}e^{-x}dx^2 - m\int_a^x\!\!\int_a^x f^{<m>}e^{-x}dx^2$$

Since $f(a)=0$, Left side becomes as follows.

$$\int_a^x\int_a^x f^{<-1>}e^{-x}dx^2 = \int_a^x\left\{\left[f^{<0>}e^{-x}\right]_a^x + \int_a^x f^{<0>}e^{-x}dx\right\}dx$$

$$= \int_a^x f^{<0>}e^{-x}dx + \int_a^x\int_a^x f^{<0>}e^{-x}dx^2$$

Then,

$$e^{-x}\sum_{r=1}^{m} r f^{<r>}(x)= \int_a^x f(x)e^{-x}dx + \int_a^x\int_a^x f(x)e^{-x}dx^2$$

$$-(m+1)\int_a^x\int_a^x f^{<m-1>}e^{-x}dx^2 + m\int_a^x\int_a^x f^{<m>}e^{-x}dx^2$$

Here, let

$$f^{<r>} = \int_a^x\cdots\int_a^x f(x)dx^r\ ,\ \ f^{<m-1>} = \int_a^x\cdots\int_a^x f(x)dx^{m-1}\ ,\ \ f^{<m>} = \int_a^x\cdots\int_a^x f(x)dx^m$$

Since these satisfy the condition $f^{<r>}(a)=0\ (r=1,2,\cdots,m+1)$, substituting these for the above and multiplying by e^x the both sides,

$$\sum_{r=1}^{m} r\int_a^x\cdots\int_a^x f(x)dx^r= e^x\int_a^x f(x)e^{-x}dx + e^x\int_a^x\int_a^x f(x)e^{-x}dx^2$$

$$-e^x(m+1)\int_a^x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^{m-1}\right\}e^{-x}dx^2$$

$$+e^x m\int_a^x\int_a^x\left\{\int_a^x\cdots\int_a^x f(x)dx^m\right\}e^{-x}dx^2 \qquad (1.1)$$

And since $\lim_{m\to\infty}\int_a^x\cdots\int_a^x f(x)dx^m=0$ holds by assumption (See proof of Theorem 24.1.1), we obtain (1.2).

Example 1 $1\int_a^x dx + 2\int_a^x\int_a^x dx^2 + 3\int_a^x\int_a^x\int_a^x dx^3 + \cdots$

$$f(x)=1\quad ,\quad \int_a^x\cdots\int_a^x dx^{m-1} = \frac{(x-a)^{m-1}}{(m-1)!}\quad ,\quad \int_a^x\cdots\int_a^x dx^m = \frac{(x-a)^m}{m!}$$

Substituting these for (1.1) ,

$$\sum_{r=1}^{m} r\int_a^x\cdots\int_a^x dx^r= e^x\int_a^x e^{-x}dx + e^x\int_a^x\int_a^x e^{-x}dx^2$$

$$-\frac{e^x(m+1)}{(m-1)!}\int_a^x\int_a^x (x-a)^{m-1}e^{-x}dx^2$$

$$+\frac{e^x}{(m-1)!}\int_a^x\int_a^x (x-a)^m e^{-x}dx^2$$

$$= e^x\left[-e^{-x}\right]_a^x + e^x\int_a^x\left[-e^{-x}\right]_a^x dx$$

$$-\frac{e^x(m+1)}{(m-1)!}\int_a^x\left[-e^{-a}\Gamma(m,x-a)\right]_a^x dx$$

$$+\frac{e^x}{(m-1)!}\int_a^x \left[-e^{-a}\Gamma(m+1,x-a)\right]_a^x dx$$

$\vdots$

That is,

$$\sum_{r=1}^{m} r\int_a^x\cdots\int_a^x dx^r = e^{x-a}(x-a)$$
$$+\frac{e^{x-a}(m+1)}{(m-1)!}\{(x-a)\Gamma(m,x-a)-\Gamma(m+1,x-a)-(x-a)\Gamma(m)\}$$
$$-\frac{e^{x-a}}{(m-1)!}\{(x-a)\Gamma(m+1,x-a)-\Gamma(m+2,x-a)-(x-a)\Gamma(m+1)\}$$

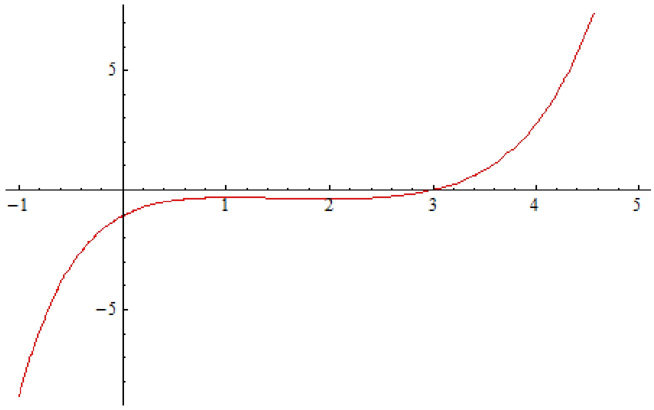
When $m\to\infty$, these remainders converge to 0, it is as follows.

$$\sum_{r=1}^{\infty} r\int_a^x\cdots\int_a^x dx^r = e^{x-a}(x-a)$$

Left side is expressed as $\sum_{r=1}^{m} r\int_a^x\cdots\int_a^x dx^r = \sum_{r=1}^{m}\frac{r(x-a)^r}{r!}$.

When $a=3, m=7$ and both sides are illustrated, it is as follows. Both overlap exactly and blue (left) can not be seen. Of course, both sides are completely equal also at the time of $m\to\infty$.

```
fl[a_, m_] := Sum[r (x - a)^r / r!, {r, 1, m}]
fr[a_, m_] := e^(x-a) (x - a)
   + e^(x-a) (m + 1) / (m - 1)! ((x - a) Gamma[m, x - a] - Gamma[m + 1, x - a] - (x - a) Gamma[m])
   - e^(x-a) / (m - 1)! ((x - a) Gamma[m + 1, x - a] - Gamma[m + 2, x - a] - (x - a) Gamma[m + 1])
```



Example 2 $1\int_a^x \cos x\,dx + 2\int_a^x\int_a^x \cos x\,dx^2 + 3\int_a^x\int_a^x\int_a^x \cos x\,dx^3 + \cdots$

Substituting $f(x)=\cos x$ for (1.2) ,

$$\sum_{r=1}^{\infty} r\int_a^x\cdots\int_a^x \cos x\,dx^r = e^x\int_a^x \cos x\,e^{-x}dx + e^x\int_a^x\int_a^x \cos x\,e^{-x}dx^2$$

Integrating the right side with respect to x from a to x by force,

$$\sum_{r=1}^{\infty} r \int_a^x \cdots \int_a^x \cos x \, dx^r = \frac{e^{x-a}(x-a)(\cos a - \sin a)}{2} + \frac{e^{x-a}\cos a - \cos x}{2}$$

The higher integral of the left side is as follows according to Theorem 4.1.3 (**4.1**)

$$\int_a^x \cdots \int_a^x \cos x \, dx^n = \cos\left(x - \frac{\pi n}{2}\right) - \sum_{s=0}^{n-1} \frac{(x-a)^s}{s!} \cos\left\{a - \frac{\pi(n-s)}{2}\right\}$$

Then, the left side is

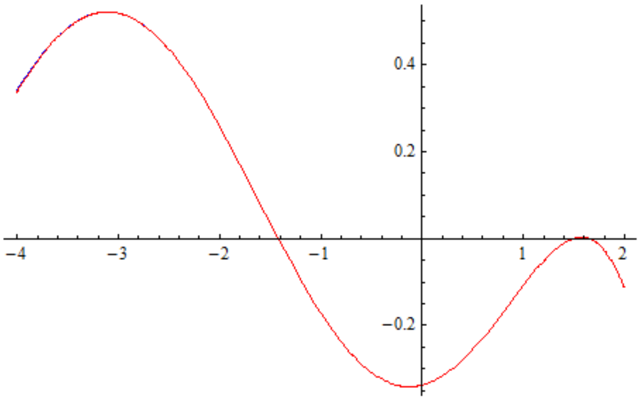
$$\sum_{r=1}^{\infty} r \left\{ \cos\left(x - \frac{\pi r}{2}\right) - \sum_{s=0}^{r-1} \frac{(x-a)^s}{s!} \cos\left\{a - \frac{\pi(r-s)}{2}\right\} \right\}$$

When $a = 3/2$ and the first 18 terms of $\sum$ are calculated and both sides are illustrated, it is as follows. Both sides overlap exactly and blue (left) can not be seen.

```
m = 18;
```

$$\text{fl}[a\_] := \sum_{r=1}^{m} r \left[\text{Cos}\left[x - \frac{\pi r}{2}\right] - \sum_{s=0}^{r-1} \frac{(x-a)^s}{s!} \text{Cos}\left[a - \frac{\pi (r-s)}{2}\right] \right]$$

$$\text{fr}[a\_] := \frac{e^{x-a} (x-a) (\text{Cos}[a] - \text{Sin}[a])}{2} + \frac{e^{x-a} \text{Cos}[a] - \text{Cos}[x]}{2}$$



Theorem 24.7.2

When $f(x)$ is analytic function defined on a closed interval I, the following expressions hold for $x \in I$ and $a \in I$ s.t. $f(a) = 0$.

$$\sum_{r=1}^{m} (-1)^{r-1} r \int_a^x \cdots \int_a^x f(x) dx^r = e^{-x} \int_a^x f(x) e^x dx - e^{-x} \int_a^x \int_a^x f(x) e^x dx^2$$

$$+ (-1)^m (m+1) e^x \int_a^x \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x) dx^{m-1} \right\} e^x dx^2$$

$$+ (-1)^m m e^x \int_a^x \int_a^x \left\{ \int_a^x \cdots \int_a^x f(x) dx^m \right\} e^x dx^2 \qquad (2.1)$$

$$\sum_{r=1}^{\infty} (-1)^{r-1} r \int_a^x \cdots \int_a^x f(x) dx^r = e^{-x} \int_a^x f(x) e^x dx - e^{-x} \int_a^x \int_a^x f(x) e^x dx^2 \qquad (2.2)$$

Proof

In a way similar to Theorem 24.7.1, we obtain the desired expressioons

Example $1\int_0^x \log x\,dx - 2\int_0^x\int_0^x \log x\,dx^2 + 3\int_0^x\int_0^x\int_0^x \log x\,dx^3 - 4\int_0^x \cdots \int_0^x \log x\,dx^4 + - \cdots$

Substituting $f(x) = \log x$, $a=0$ for (2.2) ,

$$\sum_{r=1}^{\infty} (-1)^{r-1} r \int_0^x \cdots \int_0^x \log x\,dx^r = e^{-x}\int_0^x \log x\,e^x dx - e^{-x}\int_a^x\int_a^x \log x\,e^x dx^2$$

The integrals of the rigjht side are as follows.

$$\int_0^x \log x\,e^x dx = \left[e^x \log x - Ei(x)\right]_0^x = e^x \log x - Ei(x) + \gamma$$

$$\int_0^x\int_0^x \log x\,e^x dx = \int_0^x \left\{e^x \log x - Ei(x) + \gamma\right\} dx$$

$$= \left[e^x(\log x + 1) - (x+1)Ei(x) + \gamma x\right]_0^x$$

$$= e^x(\log x + 1) - (x+1)Ei(x) + \gamma x - (1-\gamma)$$

Where, $Ei(x) = \int_{-\infty}^x \frac{e^t}{t} dt$, $\gamma = 0.5772\cdots$ (Euler-Mascheroni Constant).

From these,

$$e^{-x}\int_0^x \log x\,e^x dx - e^{-x}\int_a^x\int_a^x \log x\,e^x dx^2 = e^{-x} x\{Ei(x) - \gamma\} + e^{-x} - 1$$

That is,

$$\sum_{r=1}^{\infty} (-1)^{r-1} r \int_0^x \cdots \int_0^x \log x\,dx^r = e^{-x} x\{Ei(x) - \gamma\} + e^{-x} - 1$$

The higher integral of the left side is the following lineal integral.

$$\int_0^x \cdots \int_0^x \log x\,dx^n = \frac{x^n}{n!}\left(\log|x| - \sum_{s=1}^{n} \frac{1}{s}\right)$$

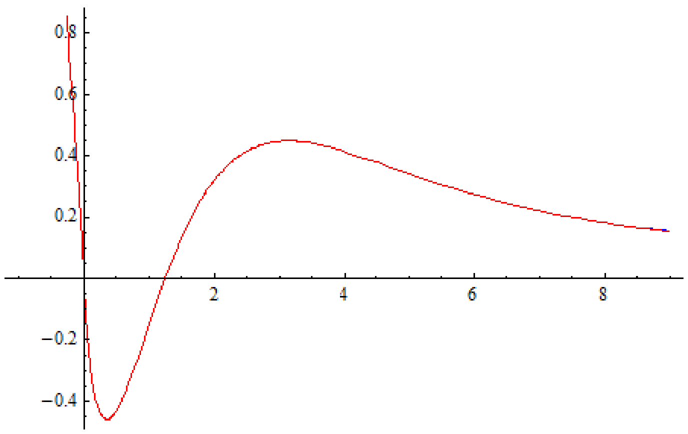
Then, the left side is

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^r}{(r-1)!}\left(\log|x| - \sum_{s=1}^{r} \frac{1}{s}\right)$$

When the first 30 terms of $\sum$ are calculated and both sides are illustrated, it is as follows.
Both sides overlap exactly and blue (left) can not be seen.

$$\texttt{fl[x\_] := } \sum_{r=1}^{30} (-1)^{r-1} \frac{x^r}{(r-1)!}\left(\texttt{Log[Abs[x]]} - \sum_{s=1}^{r} \frac{1}{s}\right)$$

$$\texttt{fr[x\_] := } e^{-x}\, x\, (\texttt{ExpIntegralEi[x]} - \texttt{EulerGamma}) + e^{-x} - 1$$



c.f.

Mr. Sugioka showed the following theorem (**Sugioka's Theorem 3-1**) just like Theorem24.7.1 .

$$1\int_a^x\int_a^x f(x)dx^2 + 2\int_a^x\int_a^x\int_a^x f(x)dx^3 + 3\int_a^x\cdots\int_a^x f(x)dx^4 + \cdots = e^x\int_a^x\int_a^x f(x)e^{-x}dx^2$$

Adding this to Theorem 24.1.1 (1.2) , we obtain Theorem 24.7.1 (1.2) .

2012.09.01

K. Kono

Alien's Mathematics