04 Sum of series equivalent to the Riemann hypothesis

As an example of application of the preceding chapter " **03 Vieta's Formulas in Infinite-degree Equation** ", we take up the sum of the series equivalent to the Riemann hypothesis in this chapter. Such a series and its sum are obtained from Vieta's formula in the Maclaurin series of function $-z\zeta(1-z)$ which is a part of completed Riemann zeta function.

4.1 Factorization of $-z\zeta(1-z)$

In this section, we factor $-z\zeta(1-z)$, which is a part of $\xi(z)$, around 0.

Formula 4.4.1

When γ is Euler-Mascheroni constant, $\zeta(z)$ is Riemann zeta function and the non-trivial zeros are $x_n+iy_n \quad n=1,2,3,\cdots$, the following expression holds.

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}} e^{\left(\log 2-1-\frac{\gamma}{2}\right)z} \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right) e^{\frac{2x_n z}{x_n^2+y_n^2}} \tag{1.1}$$

Proof

Let completed zeta function be as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Replacing z with $1-z$,

$$\xi(1-z) = -z(1-z)\pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \tag{w_L}$$

According to Formula 8.1.1 in " **08 Factorization of Completed Riemann Zeta** " (**Riemann Zeta Function**), $\xi(z)$ was represented by the Hadamard product as follows.

$$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

Since functional equation $\xi(z)=\xi(1-z)$ is established,

$$\xi(1-z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{w_R}$$

From (w_L) and (w_R) ,

$$-z(1-z)\pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

From this,

$$-z\zeta(1-z) = \frac{\frac{1}{2}\pi^{\frac{1-z}{2}}}{\frac{1-z}{2}\Gamma\left(\frac{1-z}{2}\right)} e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

Here,

$$\frac{1}{2}\pi^{\frac{1-z}{2}} = \frac{\sqrt{\pi}}{2}\pi^{-\frac{z}{2}} = \frac{\sqrt{\pi}}{2}e^{-z\log\sqrt{\pi}}$$

$$\frac{1-z}{2}\Gamma\left(\frac{1-z}{2}\right) = \Gamma\left(1+\frac{1-z}{2}\right) = \Gamma\{(3-z)/2\}$$

Substituting these for the above,

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}}e^{\left(\log 2-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} \tag{1.0}$$

When the non-trivial zeros are $z_k = x_k \pm i y_k \quad k=1,2,3,\cdots$,

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}}e^{\left(\log 2-1-\frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}} \tag{1.1}$$

If $x_n = 1/2 \quad n=1,2,3,\cdots$

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}}e^{\left(\log 2-1-\frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{z}{1/4+y_n^2}+\frac{z^2}{1/4+y_n^2}\right)e^{\frac{z}{1/4+y_n^2}} \tag{1.1'}$$

Trivial Zeros

Trivial zeros $z=3,5,7,\cdots$ of $\zeta(1-z)$ are contained in $1/\Gamma\{(3-z)/2\}$. The reason is as follows. Formula 11.1.1 (1.3_+) in " **11 Series Expansion of Reciprocal of Gamma Function** " **(A la carte)** was as follows.

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{2n-1}\right)e^{\frac{z}{2n-1}} = \frac{\sqrt{\pi}}{\Gamma\{(1-z)/2\}}e^{\left(\frac{\gamma}{2}+\log 2\right)z} \tag{1.3$_+$}$$

Since the left side is

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{2n-1}\right)e^{\frac{z}{2n-1}} = (1-z)e^{z}\prod_{n=1}^{\infty}\left(1-\frac{z}{2n+1}\right)e^{\frac{z}{2n+1}}$$

dividing both sides of (1.3_+) by $(1-z)e^{z}$,

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{2n+1}\right)e^{\frac{z}{2n+1}} = \frac{\sqrt{\pi}}{(1-z)\Gamma\{(1-z)/2\}}e^{\left(\frac{\gamma}{2}+\log 2-1\right)z}$$

$$= \frac{\sqrt{\pi}/2}{(1-z)/2\,\Gamma\{(1-z)/2\}}e^{\left(-\frac{\gamma}{2}+\log 2-1\right)z}\cdot e^{\gamma z}$$

From this,

$$\frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}}e^{\left(\log 2-1-\frac{\gamma}{2}\right)z} = e^{-\gamma z}\prod_{n=1}^{\infty}\left(1-\frac{z}{2n+1}\right)e^{\frac{z}{2n+1}}$$

Since the exponential function $e^{\pm z}$ has no zero, this formula means that $z=3,5,7,\cdots$ are the zeros of $1/\Gamma\{(3-z)/2\}$

4.2 Maclaurin Expansion by Stieltjes Constants

Formula 4.2.1

When $\zeta(z)$ is Riemann zeta function, the following expression holds on whole complex plane.

$$-z\zeta(1-z) = 1 - \sum_{s=1}^{\infty} \frac{s\gamma_{s-1}}{s!} z^s \tag{2.1}$$

Where, γ_s is Stieltjes constant defined by the following expression.

$$\gamma_s = \lim_{n\to\infty}\left\{ \sum_{k=1}^{n} \frac{(\log k)^s}{k} - \frac{(\log n)^{s+1}}{s+1} \right\}$$

Proof

It is known that the following expression holds on whole complex plane.

$$\zeta(z) = \frac{1}{z-1} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}\gamma_s (z-1)^s \qquad \gamma_s : \textit{Stieltjes constant}$$

Multiplying both sides by $z-1$,

$$(z-1)\zeta(z) = 1 + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}\gamma_s (z-1)^{s+1} = 1 + \sum_{s=1}^{\infty} (-1)^{s-1}\frac{s\gamma_{s-1}}{s!}(z-1)^s$$

Replacing z with $1-z$,

$$(1-z-1)\zeta(1-z) = 1 + \sum_{s=1}^{\infty} (-1)^{s-1}\frac{s\gamma_{s-1}}{s!}(1-z-1)^s$$

$$= 1 + \sum_{s=1}^{\infty} (-1)^{2s-1}\frac{s\gamma_{s-1}}{s!} z^s$$

i.e.

$$-z\zeta(1-z) = 1 - \sum_{s=1}^{\infty} \frac{s\gamma_{s-1}}{s!} z^s$$

Note

The function $-z\zeta(1-z)$ does not have singular point on whole complex plane. So, the convergence radius of (2.1) is infinite.

4.3 Maclaurin Expansion by Hadamard Product

As seen in Section 1, when non-trivial zeros of $\zeta(z)$ are $z_k = x_k \pm i y_k \quad k=1,2,3,\cdots$, the function $-z\zeta(1-z)$ was factored asfollows.

$$-z\zeta(1-z) = \frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}} e^{\left(\log 2 - 1 - \frac{\gamma}{2}\right)z} \prod_{r=1}^{\infty}\left(1 - \frac{2x_r z}{x_r^2+y_r^2} + \frac{z^2}{x_r^2+y_r^2}\right) e^{\frac{2x_r z}{x_r^2+y_r^2}} \tag{1.1}$$

Each function which constitutes this is expanded to Maclaurin series as follows, respectively.

Maclaurin expansion of the reciprocal of gamma function

According to Formula 12.3.3 in " **12 Series Expansion of Gamma Function & the Reciprocal** " **(A la carte)** , when $\psi_n(z)$ is the polygamma function and $B_{n,k}(f_1, f_2, \ldots)$ are Bell polynomials,

$$\frac{\sqrt{\pi}}{2\Gamma\{(3-z)/2\}} = 1 + \sum_{s=1}^{\infty}(-1)^s \frac{\beta_s}{(2s)!!} z^s \tag{3.g}$$

$$= 1 + \frac{z}{2!!}\psi_0\left(\frac{3}{2}\right) + \frac{z^2}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right) - \psi_1\left(\frac{3}{2}\right)\right\}$$

$$+ \frac{z^3}{6!!}\left\{\psi_0^3\left(\frac{3}{2}\right) - 3\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right) + \psi_2\left(\frac{3}{2}\right)\right\} + \cdots$$

where,

$$\beta_s = \sum_{k=1}^{s}(-1)^k B_{s,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{s-1}\left(\frac{3}{2}\right)\right) \qquad s=1,2,3,\cdots$$

Maclaurin expansion of the reciprocal of exponential function

Exponential function of (1.1) is expanded to Maclaurin series as follows.

$$e^{\left(\log 2 - 1 - \frac{\gamma}{2}\right)z} = \sum_{s=1}^{\infty}\frac{z^s}{s!}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^s \tag{3.e}$$

$$= 1 + \frac{z^1}{1!}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^1 + \frac{z^2}{2!}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^2 + \frac{z^3}{3!}\left(\log 2 - 1 - \frac{\gamma}{2}\right)^3 + \cdots$$

Maclaurin expansion of non-trivial zeros

According to Formula 3.6.1 in " **03 Vieta's Formulas in Infinite-degree Equation** " **(Infinite-degree Equation)** , the non-trivial zeros of (1.1) is expanded to Maclaurin series as follows.

$$\prod_{r=1}^{\infty}\left(1 - \frac{2x_r z}{x_r^2+y_r^2} + \frac{z^2}{x_r^2+y_r^2}\right) e^{\frac{2x_r z}{x_r^2+y_r^2}} = 1 - 0z - \left\{\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^2 - \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}\right\} z^2$$

$$- \left\{\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^3 - \sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2} + \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r_1}+x_{r_2})}{(x_{r_1}^2+y_{r_1}^2)(x_{r_2}^2+y_{r_2}^2)}\right\} z^3$$

$$- \cdots \tag{3.z}$$

Maclaurin series of $-z\zeta(1-z)$

Maclaurin series of $-z\zeta(1-z)$ consists of the product of (3.g), (3.e) and (3.z) . That is,

$$-z\zeta(1-z) = \left(1+\sum_{s=1}^{\infty}(-1)^s\frac{\beta_s}{(2s)!!}z^s\right)\times\sum_{s=1}^{\infty}\frac{z^s}{s!}\left(\log 2-1-\frac{\gamma}{2}\right)^s$$
$$\times\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)e^{\frac{2x_r z}{x_r^2+y_r^2}}$$

According to Formula 1.1.2 in " **01 Power of Infinite Series** " (**Infinite-degree Equation**) , The product of the three series of (1.1) is expressed as follows. (Where, $a_0=b_0=c_0=1$)

$$\left(\sum_{r=0}^{\infty}a_r z^r\right)\left(\sum_{r=0}^{\infty}b_r z^r\right)\left(\sum_{r=0}^{\infty}c_r z^r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}a_{r-s}b_{s-t}c_t z^r$$
$$= 1+z^1(a_1+b_1+c_1)$$
$$+z^2(a_2+b_2+c_2+a_1b_1+b_1c_1+c_1a_1)$$
$$+z^3(a_3+b_3+c_3+a_2b_1+a_2c_1+b_2c_1+b_2a_1+c_2a_1+c_2b_1+a_1b_1c_1)$$
$$+$$
$$\vdots$$

So, put

$$a_1=\frac{1}{2!!}\psi_0\left(\frac{3}{2}\right) \quad , a_2=\frac{1}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right\}$$
$$, a_3=\frac{1}{6!!}\left\{\psi_0^3\left(\frac{3}{2}\right)-3\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)+\psi_2\left(\frac{3}{2}\right)\right\}$$

$$b_1=\frac{1}{1!}\left(\log 2-1-\frac{\gamma}{2}\right)^1 , \quad b_2=\frac{1}{2!}\left(\log 2-1-\frac{\gamma}{2}\right)^2 , \quad b_3=\frac{1}{3!}\left(\log 2-1-\frac{\gamma}{2}\right)^3$$

$$c_1=0 \quad , \quad c_2=-\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2+\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2} \quad ,$$

$$c_3=-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r1}+x_{r2})}{(x_{r1}^2+y_{r1}^2)(x_{r2}^2+y_{r2}^2)}$$

Then,

$$-z\zeta(1-z) = 1+z^1(a_1+b_1+c_1)$$
$$+z^2(a_2+b_2+c_2+a_1b_1+b_1c_1+c_1a_1)$$
$$+z^3(a_3+b_3+c_3+a_2b_1+a_2c_1+b_2c_1+b_2a_1+c_2a_1+c_2b_1+a_1b_1c_1)$$
$$+$$
$$\vdots$$

Since $c_1=0$

$$-z\zeta(1-z) = 1 + z^1(a_1+b_1)$$
$$+ z^2(a_2+b_2+c_2+a_1b_1)$$
$$+ z^3(a_3+b_3+c_3+a_2b_1+b_2a_1+c_2a_1+c_2b_1) \tag{3.0}$$
$$+$$
$$\vdots$$

Coefficients of the 1st, 2nd, 3rd degree

Comparing Formula 4.2.1 and (3.0) , we obtain the following formula.

Formula 4.3.1

When γ is Euler-Mascheroni constant, γ_s is Stieltjes constant, $\psi_n(z)$ is the polygamma function and non-trivial zeros of Riemann zeta function are $x_n+iy_n \quad n=1,2,3,\cdots$, the following expressions hold.

$$-\gamma_0 = \frac{1}{2!!}\psi_0\left(\frac{3}{2}\right) + \log 2 - 1 - \frac{\gamma}{2} \tag{3.1$_1$}$$

$$-\gamma_1 = \frac{\gamma_0^2}{2} - \frac{1}{4!!}\psi_1\left(\frac{3}{2}\right) - \frac{1}{2}\sum_{r1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2 + \sum_{r1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2} \tag{3.1$_2$}$$

$$-\frac{\gamma_2}{2} = \frac{\gamma_0^3}{3} + \gamma_0\gamma_1 + \frac{1}{6!!}\psi_2\left(\frac{3}{2}\right)$$
$$-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3 + \sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r1}+x_{r2})}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)} \tag{3.1$_3$}$$

Proof

From Formula 4.2.1 ,

$$-z\zeta(1-z) = 1 - \frac{\gamma_0}{0!}z^1 - \frac{\gamma_1}{1!}z^2 - \frac{\gamma_2}{2!}z^3 - \frac{\gamma_3}{3!}z^4 - \cdots$$

Comparing these coefficients with the coefficients of (3.0) ,

$$-\gamma_0 = a_1+b_1 \tag{1}$$
$$-\gamma_1 = a_2+b_2+a_1b_1 + c_2 \tag{2}$$
$$-\frac{\gamma_2}{2} = a_3+b_3+a_2b_1+b_2a_1+c_2(a_1+b_1) + c_3 \tag{3w}$$

From (2) ,

$$c_2 = -\gamma_1 - (a_2+b_2) - a_1b_1$$

Substituting this and (1) for (3w) ,

$$-\frac{\gamma_2}{2} = a_3+b_3+a_2b_1+b_2a_1+\{-\gamma_1-(a_2+b_2)-a_1b_1\}(-\gamma_0) + c_3$$

i.e.

$$-\frac{\gamma_2}{2} = a_3+b_3+a_2b_1+b_2a_1+\gamma_0\gamma_1+\gamma_0(a_2+b_2)+\gamma_0 a_1b_1+c_3 \tag{3}$$

Substituting the above $a_1 \sim c_3$ for (1), (2) and (3) respectively,

Coefficient of the 1st degree

$$-\gamma_0 = a_1+b_1 = \frac{1}{2^1 1!}\psi_0\left(\frac{3}{2}\right) + \log 2-1-\frac{\gamma}{2}$$

i.e.

$$-\gamma_0 = \frac{1}{2!!}\psi_0\left(\frac{3}{2}\right) + \log 2-1-\frac{\gamma}{2} \tag{3.1$_1$}$$

Coefficient of the 2nd degree

$$\begin{aligned}-\gamma_1 &= a_2+b_2+c_2+a_1b_1\\ &= \frac{1}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right\} + \frac{1}{2!}\left(\log 2-1-\frac{\gamma}{2}\right)^2\\ &\qquad -\frac{1}{2}\sum_{r1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2+\sum_{r1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}\\ &\qquad + \frac{1}{2!!}\psi_0\left(\frac{3}{2}\right)\left(\log 2-1-\frac{\gamma}{2}\right)^1\end{aligned}$$

i.e.

$$\begin{aligned}-\gamma_1 &= \frac{1}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right\}+\frac{1}{2}\left(\log 2-1-\frac{\gamma}{2}\right)^2+\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\left(\log 2-1-\frac{\gamma}{2}\right)\\ &\qquad -\frac{1}{2}\sum_{r1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2+\sum_{r1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}\end{aligned}$$

Here, from (1) ,

$$\log 2-1-\frac{\gamma}{2} = -\gamma_0 - \frac{1}{2}\psi_0\left(\frac{3}{2}\right) \tag{3.1$_1$'}$$

Using this for the 2nd and the 3rd term of the right hand,

$$\begin{aligned}&\frac{1}{2}\left(\log 2-1-\frac{\gamma}{2}\right)^2+\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\left(\log 2-1-\frac{\gamma}{2}\right)\\ &\quad = \frac{1}{2}\left\{\gamma_0+\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\right\}^2-\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\left\{\gamma_0+\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\right\}\\ &\quad = \frac{\gamma_0^2}{2}-\frac{1}{8}\psi_0^2\left(\frac{3}{2}\right)\end{aligned}$$

Substituting this for the right hand of the above,

$$\begin{aligned}-\gamma_1 &= \frac{1}{8}\psi_0^2\left(\frac{3}{2}\right)-\frac{1}{8}\psi_1\left(\frac{3}{2}\right)+\frac{\gamma_0^2}{2}-\frac{1}{8}\psi_0^2\left(\frac{3}{2}\right)\\ &\qquad -\frac{1}{2}\sum_{r1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2+\sum_{r1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}\end{aligned}$$

$$= \frac{\gamma_0^2}{2} - \frac{1}{4!!}\psi_1\left(\frac{3}{2}\right) - \frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^2 + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2} \tag{3.1$_2$}$$

Coefficient of the 3rd degree

$$-\frac{\gamma_2}{2} = a_3+b_3+a_2b_1+b_2a_1+\gamma_0\gamma_1+\gamma_0(a_2+b_2)+\gamma_0 a_1b_1+c_3 \tag{3}$$

Substituting the above $a_1 \sim c_3$ for this respectively,

$$-\frac{\gamma_2}{2} = \frac{1}{3!}\left(\log 2-1-\frac{\gamma}{2}\right)^3 + \frac{1}{2!}\left\{\frac{1}{2}\psi_0\left(\frac{3}{2}\right)+\gamma_0\right\}\left(\log 2-1-\frac{\gamma}{2}\right)^2$$
$$+\left\{\frac{1}{4!!}\left(\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right)+\frac{\gamma_0}{2}\psi_0\left(\frac{3}{2}\right)\right\}\left(\log 2-1-\frac{\gamma}{2}\right)$$
$$+\frac{1}{6!!}\left\{\psi_0^3\left(\frac{3}{2}\right)-3\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)+\psi_2\left(\frac{3}{2}\right)\right\}$$
$$+\frac{\gamma_0}{4!!}\left\{\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right\} + \gamma_0\gamma_1$$
$$-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^3 + \sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r_1}+x_{r_2})}{(x_{r_1}^2+y_{r_1}^2)(x_{r_2}^2+y_{r_2}^2)}$$

Here, using **(3.1$_1$')** for the 1st term ~ the 3rd term,

$$\frac{1}{3!}\left(\log 2-1-\frac{\gamma}{2}\right)^3 + \frac{1}{2!}\left\{\frac{1}{2}\psi_0\left(\frac{3}{2}\right)+\gamma_0\right\}\left(\log 2-1-\frac{\gamma}{2}\right)^2$$
$$+\left\{\frac{1}{4!!}\left(\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right)+\frac{\gamma_0}{2}\psi_0\left(\frac{3}{2}\right)\right\}\left(\log 2-1-\frac{\gamma}{2}\right)$$
$$= -\frac{1}{3!}\left\{\gamma_0+\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\right\}^3 + \frac{1}{2!}\left\{\frac{1}{2}\psi_0\left(\frac{3}{2}\right)+\gamma_0\right\}\left\{\gamma_0+\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\right\}^2$$
$$-\left\{\frac{1}{4!!}\left(\psi_0^2\left(\frac{3}{2}\right)-\psi_1\left(\frac{3}{2}\right)\right)+\frac{\gamma_0}{2}\psi_0\left(\frac{3}{2}\right)\right\}\left\{\gamma_0+\frac{1}{2}\psi_0\left(\frac{3}{2}\right)\right\}$$
$$= \frac{1}{3}\gamma_0^3 + \frac{\gamma_0^2}{2}\psi_0\left(\frac{3}{2}\right) + \frac{\gamma_0}{4}\psi_0^2\left(\frac{3}{2}\right) + \frac{1}{24}\psi_0^3\left(\frac{3}{2}\right)$$
$$-\frac{\gamma_0}{8}\psi_0^2\left(\frac{3}{2}\right) + \frac{\gamma_0}{8}\psi_1\left(\frac{3}{2}\right) - \frac{1}{16}\psi_0^3\left(\frac{3}{2}\right) + \frac{1}{16}\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)$$
$$-\frac{\gamma_0^2}{2}\psi_0\left(\frac{3}{2}\right) - \frac{\gamma_0}{4}\psi_0^2\left(\frac{3}{2}\right)$$
$$= \frac{\gamma_0^3}{3} - \frac{\gamma_0}{8}\psi_0^2\left(\frac{3}{2}\right) - \frac{1}{48}\psi_0^3\left(\frac{3}{2}\right) + \frac{\gamma_0}{8}\psi_1\left(\frac{3}{2}\right) + \frac{1}{16}\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)$$

Substituting this for the right hand of the above,

$$-\frac{\gamma_2}{2} = \frac{\gamma_0^3}{3} - \frac{\gamma_0}{8}\psi_0^2\left(\frac{3}{2}\right) - \frac{1}{48}\psi_0^3\left(\frac{3}{2}\right) + \frac{\gamma_0}{8}\psi_1\left(\frac{3}{2}\right) + \frac{1}{16}\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)$$

$$+\frac{1}{48}\psi_0^3\left(\frac{3}{2}\right)-\frac{1}{16}\psi_0\left(\frac{3}{2}\right)\psi_1\left(\frac{3}{2}\right)+\frac{1}{48}\psi_2\left(\frac{3}{2}\right)$$

$$+\frac{\gamma_0}{8}\psi_0^2\left(\frac{3}{2}\right)-\frac{\gamma_0}{8}\psi_1\left(\frac{3}{2}\right)+\gamma_0\gamma_1$$

$$-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

$$=\frac{\gamma_0^3}{3}+\gamma_0\gamma_1+\frac{1}{6!!}\psi_2\left(\frac{3}{2}\right)$$

$$-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)} \tag{3.1$_3$}$$

4.4 Proposition equivalent to the Riemann Hypothesis

From Formula 4.3.1 in the previous section, a proposition equivalent to the Riemann hypothesis is obtained

Proposition 4.4.1

When γ_s is Stieltjes constant, $\psi_n(z)$ is the polygamma function and non-trivial zeros of Riemann zeta function are $1/2+iy_r \quad r=1,2,3,\cdots$, the following expressions hold.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = \gamma_0 - \frac{1}{2}\log\pi + \frac{1}{2}\psi_0\left(\frac{3}{2}\right) \tag{4.1$_1$}$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2 = \gamma_0^2 + 2\gamma_0 + 2\gamma_1 - \log\pi + \psi_0\left(\frac{3}{2}\right) - \frac{1}{4}\psi_1\left(\frac{3}{2}\right) \tag{4.1$_2$}$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^3 = \gamma_0^3 + 3\gamma_0^2 + 6\gamma_0 + 6\gamma_1 + 3\gamma_0\gamma_1 + \frac{3}{2}\gamma_2 - 3\log\pi$$
$$+3\psi_0\left(\frac{3}{2}\right) - \frac{3}{4}\psi_1\left(\frac{3}{2}\right) + \frac{1}{16}\psi_2\left(\frac{3}{2}\right) \tag{4.1$_3$}$$

Proof of equivalence

According to Theorem 8.2.4 in " **08 Factorization of Completed Riemann Zeta** " (**Riemann Zeta Function**) , If Riemann hypothesis is true, the following expression holds.,

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2} \tag{1.2'}$$

On the other hand, from the previous section ,

$$\log 2 - 1 - \frac{\gamma}{2} = -\gamma_0 - \frac{1}{2}\psi_0\left(\frac{3}{2}\right) \tag{3.1$_1$'}$$

Substituting the latter for the former,

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = \gamma_0 - \frac{\log\pi}{2} + \frac{1}{2}\psi_0\left(\frac{3}{2}\right) \tag{4.1$_1$}$$

Next, substituting $x_{r_1} = 1/2$ for (3.1$_2$) in Formula 4.3.1 ,

$$-\gamma_1 = \frac{\gamma_0^2}{2} - \frac{1}{4!!}\psi_1\left(\frac{3}{2}\right) - \frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^2 + \sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r_1}^2}$$

Substituting (1.2') for the last term and arranging it,

$$\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^2 = \gamma_0^2 + 2\gamma_1 - \frac{1}{4}\psi_1\left(\frac{3}{2}\right) + 2\left(1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2}\right)$$

Further, substituting (3.1$_1$') for a part of the last term,

$$\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^2 = \gamma_0^2 + 2\gamma_0 + 2\gamma_1 - \log\pi + \psi_0\left(\frac{3}{2}\right) - \frac{1}{4}\psi_1\left(\frac{3}{2}\right) \tag{4.1$_2$}$$

According to Formula 3.6.1 and the Corollary 3.6.1 in " **03 Vieta's Formulas in Infinite-degree Equation** ", if $x_{r_1} = 1/2$,

$$-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1(x_{r1}+x_{r2})}{(x_{r1}^2+y_{r1}^2)(x_{r2}^2+y_{r2}^2)}$$

$$=-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2$$

Substituting this for the right side of (3.1_3) in Formula 4.3.1 ,

$$-\frac{\gamma_2}{2}=\frac{\gamma_0^3}{3}+\gamma_0\gamma_1+\frac{1}{6!!}\psi_2\left(\frac{3}{2}\right)-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2$$

Further, substituting (4.1_2) for the last term,

$$-\frac{\gamma_2}{2}=\frac{\gamma_0^3}{3}+\gamma_0\gamma_1+\frac{1}{6!!}\psi_2\left(\frac{3}{2}\right)-\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3$$
$$+\gamma_0^2+2\gamma_0+2\gamma_1-\log\pi+\psi_0\left(\frac{3}{2}\right)-\frac{1}{4}\psi_1\left(\frac{3}{2}\right)$$

From this,

$$\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3=\gamma_0^3+3\gamma_0^2+6\gamma_0+6\gamma_1+3\gamma_0\gamma_1+\frac{3}{2}\gamma_2-3\log\pi$$
$$+3\psi_0\left(\frac{3}{2}\right)-\frac{3}{4}\psi_1\left(\frac{3}{2}\right)+\frac{1}{16}\psi_2\left(\frac{3}{2}\right) \qquad (4.1_3)$$

Numerical Calculation

When we take 20,000 zero points y_r on the critical line and calculate (4.1_1) ~ (4.1_3) using the formula manipulation software *Mathematica*, the results are as follows respectively.

```
γs_ := StieltjesGamma[s]; ψk_[p_] := PolyGamma[k, p];  yn_ := Im[ZetaZero[n]]
```

1st degree

$$f1[m\_] := \sum_{r=1}^{m}\frac{1}{1/4+y_r^2} \qquad g1 := \gamma_0-\frac{1}{2}\mathrm{Log}[\pi]+\frac{1}{2}\psi_0\left[\frac{3}{2}\right]$$

```
N[f1[20000]]
0.0230167
```

```
N[g1]
0.0230957
```

Both sides match up to 3 significant digits.

2nd degree

$$f2[m\_] := \sum_{r=1}^{m}\left(\frac{1}{1/4+y_r^2}\right)^2 \qquad g2 := \gamma_0^2+2\gamma_0+2\gamma_1-\mathrm{Log}[\pi]+\psi_0\left[\frac{3}{2}\right]-\frac{1}{4}\psi_1\left[\frac{3}{2}\right]$$

```
SetPrecision[f2[20000], 10]
0.0000371006364
```

```
SetPrecision[g2, 15]
0.0000371006364
```

Both sides match up to 9 significant digits.

3rd degree

```
f3[m_] := Sum[(1/(1/4 + y_r^2))^3, {r, 1, m}]
```

$$f3[m\_] := \sum_{r=1}^{m} \left(\frac{1}{1/4 + y_r^2} \right)^3$$

$$g3 := \gamma_0^3 + 3\gamma_0^2 + 6\gamma_0 + 6\gamma_1 + 3\gamma_0\gamma_1 + \frac{3}{2}\gamma_2 - 3\,\mathrm{Log}[\pi] + 3\,\psi_0\left[\frac{3}{2}\right] - \frac{3}{4}\psi_1\left[\frac{3}{2}\right] + \frac{1}{16}\psi_2\left[\frac{3}{2}\right]$$

SetPrecision[f3[20000], 16]   SetPrecision[g3, 24]

$1.436778602886916 \times 10^{-7}$ $1.436778602886918 \times 10^{-7}$

Both sides match up to 15 significant digits. The reason that this number of digits does not reach 27 is probably due to the low calculation accuracy on the right side.

(4.1_1) is the same as the following in Theorem 8.2.4 (" **08 Factorization of Completed Riemann Zeta** ").

$$\sum_{r=1}^{\infty} \frac{1}{1/4 + y_r^2} = 1 + \frac{\gamma}{2} - \log 2 - \frac{\log \pi}{2} = 0.0230957\cdots \qquad (1.2')$$

So, (4.1_1) is equivalent to the Riemann hypothesis according Theorem 8.2.4 . Since (4.1_2) & (4.1_3) are derived using (1.2'), these are also respectively equivalent to the Riemann hypothesis.

However, (4.1_2) & (4.1_3) are considerably faster than (4.1_1) at convergence speed. If more non-trivial zeros are computed by a high-speed machine, both sides of (4.1_3) will become infinitely closer.

But even so, it only serves as circumstantial evidence for the Riemann hypothesis. So, Proposition 4.4.1 has to be proved analytically. And for the purpose, imaginary parts y_r of the non-trivialzeros $1/2 \pm i y_r \quad r = 1, 2, 3, \cdots$ have to be obtained as a formula.

2018.06.16
2024.03.24 Updated numerical calculation.

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Alien's Mathematics