Summary of Riemann Zeta Function

1 Zeta Generating Functions

Both of hyperbolic functions and trigonometric functions can be expanded to Fourier series and Taylor series. And if the termwise higher order integration of these is carried out, Riemann Zeta Functions are obtained.

Where, these are automorphisms which are expressed by lower zetas. However, in this chapter, we stop those so far. The work that obtain the non-automorphism formulas by removing lower zetas from these are performed subsequent to Chapter2 .

In this chapter, we obtain the following polynomials from the zeta generating functions

Where, Riemann Zeta, Dirichlet Eta and Dirichlet Lambda are as follows.

$$\zeta(x)=\sum_{r=1}^{\infty}\frac{1}{r^x}\ ,\quad \eta(x)=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^x}\ ,\quad \lambda(x)=\sum_{r=1}^{\infty}\frac{1}{(2r-1)^x}$$

Bernoulli numbers and Euler numbers are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$$

$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$$

Harmonic number is $H_s=\sum_{t=1}^{s}1/t=\psi(1+s)+\gamma$

$$\zeta(n)=\frac{(-x)^{n-1}}{(n-1)!}\left(\log x-H_{n-1}\right)+\frac{(-1)^n}{2}\frac{x^n}{n!}+\sum_{r=1}^{\infty}\frac{e^{-rx}}{r^n}$$
$$-(-1)^n\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+n-1}}{2r(2r+n-1)!}-\sum_{s=1}^{n-2}\frac{(-1)^s x^s}{s!}\zeta(n-s)$$

$$\eta(n)=-\frac{(-1)^n}{2}\frac{x^n}{n!}-\sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^n}$$
$$+(-1)^n\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)B_{2r}x^{2r+n-1}}{2r(2r+n-1)!}-\sum_{s=1}^{n-1}\frac{(-1)^s x^s}{s!}\eta(n-s)$$

$$\lambda(n)=\frac{(-1)^{n-1}}{2}\frac{x^{n-1}}{(n-1)!}\left(\log\frac{x}{2}-H_{n-1}\right)+\sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^n}$$
$$+\frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+n-1}}{2r(2r+n-1)!}-\sum_{s=1}^{n-2}\frac{(-1)^s x^s}{s!}\lambda(n-s)$$

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^{2n-1}}-\frac{(-1)^n x^{2n-2}}{(2n-2)!}\left(\log x-H_{2n-2}\right)+(-1)^n\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+2n-2}}{2r(2r+2n-2)!}$$
$$=\sum_{s=0}^{n-2}\frac{(-1)^s x^{2s}}{(2s)!}\zeta(2n-1-2s)$$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^{2n}}-\frac{(-1)^n x^{2n-1}}{(2n-1)!}\left(\log x-H_{2n-1}\right)+(-1)^n\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+2n-1}}{2r(2r+2n-1)!}$$
$$=-\sum_{s=1}^{n-1}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\zeta(2n+1-2s)$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^{2n-1}}-(-1)^n\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r+2n-2}}{2r(2r+2n-2)!}=\sum_{s=0}^{n-1}\frac{(-1)^s x^{2s}}{(2s)!}\eta(2n-1-2s)$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^{2n}}-(-1)^n\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r+2n-1}}{2r(2r+2n-1)!}=-\sum_{s=1}^{n}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\eta(2n+1-2s)$$

$$\sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n-1}}-\frac{(-1)^n x^{2n-2}}{2(2n-2)!}\left(\log\frac{x}{2}-H_{2n-2}\right)-\frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)|B_{2r}|x^{2r+2n-2}}{2r(2r+2n-2)!}$$
$$=\sum_{s=0}^{n-2}\frac{(-1)^s x^{2s}}{(2s)!}\lambda(2n-1-2s)$$

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n}}-\frac{(-1)^n x^{2n-1}}{2(2n-1)!}\left(\log\frac{x}{2}-H_{2n-1}\right)-\frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)|B_{2r}|x^{2r+2n-1}}{2r(2r+2n-1)!}$$
$$=-\sum_{s=1}^{n-1}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\lambda(2n+1-2s)$$

Furthermore, if the termwise higher order differentiation of the Fourier series of each family of tanh, cot and tan are carried out, the following expressions are obtained.

$$\zeta(-n)=\frac{1}{2^{n+1}(1-2^{n+1})}\sum_{r=1}^{n}(-1)^{r-1}{}_nD_r \qquad n=1,2,3,\cdots$$

$$\zeta(1-2n)=\frac{(-1)^{n-1}}{2^{2n}(1-2^{2n})}T_{2n-1} \qquad n=1,2,3,\cdots$$

$$=(-1)^{2n-1}\frac{B_{2n}}{2n} \qquad n=1,2,3,\cdots$$

Where, ${}_nD_r$ are the Eulderian Numbers and T_{n-1} are the tangent numbers. These are defined as follows respectively.

$${}_nD_r=\sum_{k=0}^{r-1}(-1)^k\binom{n+1}{k}(r-k)^n \quad , \quad T_{n-1}=2^n(2^n-1)\frac{|B_n|}{n}$$

By-products

$$-\log 0=\zeta(1)=\zeta(1)-\frac{\pi i}{2}$$

2 Formulas for Riemann Zeta at natural number

In this chapter, removing the lower zetas from automorphism formulas in the previous chapter, we obtain non-automorphism formulas for Riemann Zeta at natural number.
Where, Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$
$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$

And Harmonic number is $H_s=\sum_{t=1}^{s}1/t=\psi(1+s)+\gamma$

For $0<x\leq 2\pi$,

$$\zeta(n)=\frac{x^{n-1}}{(n-1)!}\left(\frac{1}{n-1}-\frac{x}{2n}\right)+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}}-\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

$$\zeta(n)=\frac{x^{n-1}}{(n-1)!}\left\{\frac{1}{n-1}+\frac{(n-1)x}{2n}-\log x\right\}$$
$$+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}}-\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

For $0<x\leq\pi$,

$$\zeta(n)=\frac{2^{n-1}}{2^{n-1}-1}\left\{\frac{x^n}{2n!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}\frac{(-1)^r}{r^n e^{xr}}+\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{(2^{2r}-1)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{x^{n-1}}{2(n-1)!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{\{x(2r-1)\}^s}{s\,!}\frac{e^{-x(2r-1)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{x^{n-1}}{2(n-1)!}\left(\frac{1}{n-1}-\log\frac{x}{2}\right) + \sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{\{x(2r-1)\}^s}{s\,!}\frac{e^{-x(2r-1)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{\left(2^{2r}-2\right)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Especially,

$$\zeta(n) = \frac{n+1}{2n\,!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{r^s}{s\,!}\frac{1}{r^n e^r} - \sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}}{2r(n-1+2r)!}$$

$$\zeta(n) = \frac{2^{n-1}}{n!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2r)^s}{s!}\frac{1}{r^n e^{2r}} - \sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}2^{n-1+2r}}{2r(n-1+2r)!}$$

$$\zeta(n) = \frac{n^2+1}{2n!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{r^s}{s\,!}\frac{1}{r^n e^r} - \sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{B_{2r}}{2r(n-1+2r)!}$$

$$\zeta(n) = \frac{2^{n-1}}{2^{n-1}-1}\left\{\frac{1}{2n\,!} - \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{r^s}{s!}\frac{(-1)^r}{r^n e^r} + \sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{1}{2(n-1)!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2r-1)^s}{s\,!}\frac{e^{-(2r-1)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{1}{2(n-1)!}\left(\frac{1}{n-1}+\log 2\right) + \sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(2r-1)^s}{s\,!}\frac{e^{-(2r-1)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{2^{n-2}}{(n-1)!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(4r-2)^s}{s\,!}\frac{e^{-(4r-2)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{\left(2^{2r}-2\right)B_{2r}2^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Example

$$\zeta(5) = \frac{6}{2\cdot 5!\,4} + \sum_{r=1}^{\infty}\left(1+\frac{r^1}{1!}+\frac{r^2}{2!}+\frac{r^3}{3!}+\frac{r^4}{4!}\right)\frac{1}{r^5 e^r} - \sum_{r=1}^{\infty}\binom{-5}{2r-1}\frac{B_{2r}}{2r(4+2r)!}$$

$$\zeta(5) = \frac{2^4}{5!\,4} + \sum_{r=1}^{\infty}\left\{1+\frac{2r}{1!}+\frac{(2r)^2}{2!}+\frac{(2r)^3}{3!}+\frac{(2r)^4}{4!}\right\}\frac{1}{r^5 e^{2r}} - \sum_{r=1}^{\infty}\binom{-5}{2r-1}\frac{B_{2r}2^{4+2r}}{2r(4+2r)!}$$

$$\zeta(5) = \frac{5^2+1}{2\cdot 5!\,4} + \sum_{r=1}^{\infty}\left(1+\frac{r^1}{1!}+\frac{r^2}{2!}+\frac{r^3}{3!}\right)\frac{1}{r^5 e^r} - \sum_{r=1}^{\infty}\binom{-4}{2r}\frac{B_{2r}}{2r(4+2r)!}$$

$$\zeta(5) = \frac{2^4}{2^4-1}\left\{\frac{1}{2\cdot 5!} - \sum_{r=1}^{\infty}\left(1+\frac{r^1}{1!}+\frac{r^2}{2!}+\frac{r^3}{3!}+\frac{r^4}{4!}\right)\frac{(-1)^r}{r^5 e^r} + \sum_{r=1}^{\infty}\binom{-5}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}}{2r(4+2r)!}\right\}$$

$$\zeta(3) = \frac{8}{7}\left\{\frac{1}{8}+\sum_{r=1}^{\infty}\left(1+\frac{2r-1}{1!}+\frac{(2r-1)^2}{2!}\right)\frac{e^{-(2r-1)}}{(2r-1)^3}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-3}{2r-1}\frac{(2^{2r}-2)B_{2r}}{2r(2+2r)!}\right\}$$

$$\zeta(3) = \frac{8}{7}\left\{\frac{1}{4}\left(\frac{1}{2}+\log 2\right)+\sum_{r=1}^{\infty}\left(1+\frac{2r-1}{1!}\right)\frac{e^{-(2r-1)}}{(2r-1)^3}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-2}{2r}\frac{(2^{2r}-2)B_{2r}}{2r(2+2r)!}\right\}$$

$$\zeta(3) = \frac{8}{7}\left\{\frac{1}{2}+\sum_{r=1}^{\infty}\left(1+\frac{4r-2}{1!}\right)\frac{e^{-(4r-2)}}{(2r-1)^3}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-2}{2r}\frac{(2^{2r}-2)B_{2r}2^{2+2r}}{2r(2+2r)!}\right\}$$

3 Formulas for Riemann Zeta at odd number

In this chapter, we obtain non-automorphism formulas for Riemann Zeta at odd number. Where, Bernoulli numbers, Euler numbers and tangent numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\ \cdots$

$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$

$T_1=1,\ T_3=2,\ T_5=16,\ T_7=272,\ T_9=7936,\ \cdots$

And Harmonic number is $H_s=\sum_{t=1}^{s}1/t=\psi(1+s)+\gamma$

For $0<x<2\pi$,

$$\zeta(2n+1) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^s\frac{(2^{2s}-2)B_{2s}(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+2}}$$
$$+(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

$$\zeta(2n+1) = \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n+1}}$$
$$-(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

For $0<x\le\pi$,

$$\zeta(2n+1) = \frac{2^{2n}}{2^{2n}-1}\left\{\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s B_{2s}(2^{2s}-2)(rx)^{2s}}{(2s)!}\frac{(-1)^r\sin rx}{r^{2n+2}}\right.$$
$$\left.-(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{(2^{2r}-1)|B_{2r}|x^{2r}}{2r}\right\}$$

$$\zeta(2n+1) = -\frac{2^{2n}}{2^{2n}-1}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{(-1)^r\cos rx}{r^{2n+1}}\right.$$
$$\left.-(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{(2^{2r}-1)|B_{2r}|x^{2r}}{2r}\right\}$$

$$\zeta(2n+1) = \frac{2^{2n+1}}{2^{2n+1}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|\{(2r-1)x\}^{2s}}{(2s)!}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n+1}}$$
$$-\frac{(-1)^n(2x)^{2n}}{2^{2n+1}-1}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}(2^{2r}-2)|B_{2r}|}{(2s)!(2n+2r-2s)!}\frac{x^{2r}}{2r}\right\}$$

Especially,

$$\zeta(2n+1) = (-1)^n\pi^{2n}\left\{\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|\pi^{2r}}{2r}\right\}$$

$$\zeta(2n+1) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2^{2n}-1}\sum_{r=1}^{\infty}\left\{\sum_{s=0}^{n}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n+1+2r-2s)!}\right\}T_{2r-1}\left(\frac{\pi}{2}\right)^{2r}$$

$$\zeta(2n+1) = \frac{(-1)^{n-1}\pi^{2n}}{2^{2n+1}-1}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!} - \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}(2^{2r}-2)|B_{2r}|}{(2s)!(2n+2r-2s)!}\frac{1}{2r}\left(\frac{\pi}{2}\right)^{2r}\right\}$$

Example

$$\zeta(5) = \pi^4\left\{\frac{269}{21600} + \sum_{r=1}^{\infty}\left(-\frac{1}{(2r+5)!} + \frac{1}{6(2r+3)!} - \frac{7}{360(2r+1)!}\right)\frac{|B_{2r}|\pi^{2r}}{2r}\right\}$$

$$\zeta(5) = -\frac{16\pi^4}{15}\sum_{r=1}^{\infty}\left\{-\frac{1}{(2r+5)!} + \frac{1}{6(2r+3)!} - \frac{7}{360(2r+1)!}\right\}\frac{(2^{2r}-1)|B_{2r}|\pi^{2r}}{2r}$$

$$\zeta(5) = \frac{\pi^4}{31}\left\{\frac{83}{288} + \sum_{r=1}^{\infty}\left(\frac{1}{(4+2r)!} - \frac{1}{2(2+2r)!} + \frac{5}{24(2r)!}\right)\frac{(2^{2r}-2)|B_{2r}|}{2r}\left(\frac{\pi}{2}\right)^{2r}\right\}$$

4 Formulas for Riemann Zeta at even number

In this chapter, we obtain non-automorphism formulas for Riemann Zeta at even number.
Where, Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30, \cdots$

$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385, \cdots$

For $0 < x < 2\pi$ **,**

$$\zeta(2n) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}(2^{2s}-2)(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+1}} - \frac{|B_{2n}|x^{2n}}{2(2n)!}\left\{(2^{2n}-2) - \frac{\pi(2^{2n+1}-2)}{x}\right\}$$

$$\zeta(2n) = \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n}} - \frac{|E_{2n}|x^{2n}}{2(2n)!} + \frac{\pi}{2}\frac{2^{2n}(2^{2n}-1)|B_{2n}|x^{2n-1}}{(2n)!}$$

For $0 < x \le \pi$ **,**

$$\zeta(2n) = \frac{2^{2n}}{2^{2n}-2}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}(2^{2s}-2)(rx)^{2s-1}}{(2s)!}\frac{(-1)^r\sin rx}{r^{2n}} + \frac{|B_{2n}|(2x)^{2n}}{2(2n)!}$$

$$\zeta(2n) = -\frac{2^{2n}}{2^{2n}-2}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{(-1)^r\cos rx}{r^{2n}} - \frac{|E_{2n}|x^{2n}}{2(2n)!}\right\}$$

$$\zeta(2n) = -\frac{2^{2n}}{2^{2n}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}(2^{2s}-2)\{(2r-1)x\}^{2s-1}}{(2s)!}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n}} + \frac{\pi}{2}\frac{2^{2n}|B_{2n}|x^{2n-1}}{(2n)!}$$

$$\zeta(2n) = \frac{2^{2n}}{2^{2n}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\{(2r-1)x\}^{2s}}{(2s)!}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n}} + \frac{\pi}{4}\frac{2^{4n}|B_{2n}|x^{2n-1}}{(2n)!}$$

Especially,

$$\zeta(2n) = \frac{|B_{2n}|(2\pi)^{2n}}{2(2n)!}$$

$$\zeta(2n) = -\frac{1}{2^{2n}-2}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(\pi r)^{2s}}{(2s)!}\frac{(-1)^r}{r^{2n}} - \frac{|E_{2n}|\pi^{2n}}{2(2n)!}\right\}$$

By-products

$$\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n-2s)!} = -\frac{(2^{2n+1}-2)B_{2n}}{(2n)!0!}$$

$$\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!\,(2n-1-2s)!} = \frac{2^{2n}\left(2^{2n}-1\right)B_{2n}}{(2n)!}$$

5 Formulas for Riemann Zeta at complex number

In this chapter, we obtain the formulas for Riemann Zeta at a complex number by processing " 2 Formulas for Riemann Zeta at natural number "

Where, Bernoulli numbers are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$$

And gamma function and incomplete gamma function are

$$\Gamma(p)=\int_0^\infty t^{p-1}e^{-t}dt \quad , \quad \Gamma(p,x)=\int_x^\infty t^{p-1}e^{-t}dt$$

And p is a complex number such that $p\neq 1,0,-1,-2,\cdots$.

For $x=u+vi$ s.t. $0<|x|\leqq 2\pi$, $u\geqq 0$,

$$\zeta(p) = \frac{x^{p-1}}{\Gamma(p)}\left(\frac{1}{p-1}-\frac{x}{2p}\right)+\sum_{r=1}^{\infty}\frac{\Gamma(p,xr)}{\Gamma(p)r^p}-\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}$$

$$\zeta(p) = \frac{x^{p-1}}{\Gamma(p)}\left\{\frac{1}{p-1}+\frac{(p-1)x}{2p}-\log x\right\}+\sum_{r=1}^{\infty}\frac{\Gamma(p-1,xr)}{\Gamma(p-1)r^p}-\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}$$

For $x=u+vi$ s.t. $0<|x|\leqq \pi$, $u\geqq 0$,

$$\zeta(p)=\frac{2^{p-1}}{2^{p-1}-1}\left\{\frac{x^p}{2\Gamma(p+1)}-\sum_{r=1}^{\infty}\frac{\Gamma(p,xr)}{\Gamma(p)}\frac{(-1)^r}{r^p}+\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}\right\}$$

$$\zeta(p) = \frac{2^p}{2^p-1}\left\{\frac{x^{p-1}}{2(p-1)\Gamma(p)}+\sum_{r=1}^{\infty}\frac{\Gamma\{p,x(2r-1)\}}{\Gamma(p)(2r-1)^p}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}\right\}$$

$$\zeta(p) = \frac{2^p}{2^p-1}\left\{\frac{x^{p-1}}{2\Gamma(p)}\left(\frac{1}{p-1}-\log\frac{x}{2}\right)+\sum_{r=1}^{\infty}\frac{\Gamma\{p-1,x(2r-1)\}}{\Gamma(p-1)(2r-1)^p}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{\left(2^{2r}-2\right)B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}\right\}$$

Especially,

$$\zeta(p) = \frac{p+1}{2(p-1)\Gamma(p+1)}+\sum_{r=1}^{\infty}\frac{\Gamma(p,r)}{\Gamma(p)r^p}-\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{B_{2r}}{2r\Gamma(p+2r)}$$

$$\zeta(p) = \frac{p^2+1}{2(p-1)\Gamma(p+1)}+\sum_{r=1}^{\infty}\frac{\Gamma(p-1,r)}{\Gamma(p-1)r^p}-\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{B_{2r}}{2r\Gamma(p+2r)}$$

$$\zeta(p) = \frac{2^{p-1}}{2^{p-1}-1}\left\{\frac{1}{2\Gamma(p+1)}-\sum_{r=1}^{\infty}\frac{\Gamma(p,r)}{\Gamma(p)}\frac{(-1)^r}{r^p}+\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}}{2r\Gamma(p+2r)}\right\}$$

$$\zeta(p) = \frac{2^p}{2^p-1}\left\{\frac{1}{2(p-1)\Gamma(p)}+\sum_{r=1}^{\infty}\frac{\Gamma(p,2r-1)}{\Gamma(p)(2r-1)^p}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}}{2r\Gamma(p+2r)}\right\}$$

$$\zeta(p) = \frac{2^p}{2^p-1}\left\{\frac{1}{2\Gamma(p)}\left(\frac{1}{p-1}+\log 2\right)+\sum_{r=1}^{\infty}\frac{\Gamma(p-1,2r-1)}{\Gamma(p-1)(2r-1)^p}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{\left(2^{2r}-2\right)B_{2r}}{2r\Gamma(p+2r)}\right\}$$

$$\zeta(p) = \frac{2^p}{2^p-1}\left\{\frac{2^{p-2}}{(p-1)\Gamma(p)} + \sum_{r=1}^{\infty}\frac{\Gamma(p-1,4r-2)}{\Gamma(p-1)(2r-1)^p} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{(2^{2r}-2)B_{2r}2^{p-1+2r}}{2r\Gamma(p+2r)}\right\}$$

6 Global definition of Riemann Zeta, and generalization of related coefficients

From Euler to Riemann, the zeta function was defined with patches as the domain was expanded.

$$\zeta(p) = \begin{cases} \displaystyle\sum_{r=1}^{\infty}\frac{1}{r^p} & Re(p)>1 \\ \displaystyle\frac{1}{1-2^{1-p}}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^p} & 0 \leqq Re(p) \leqq 1 \\ & p \neq 1 \\ \displaystyle\frac{2\Gamma(1-p)}{(2\pi)^{1-p}}\sin\left(\frac{p\pi}{2}\right)\frac{1}{1-2^p}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^{1-p}} & Re(p) \leqq 0 \\ & p \neq 0 \end{cases}$$

This is inconvenient. so, we focus on the following sequence.

$${}_nB_r = \sum_{s=1}^{r}(-1)^{r-s}\binom{r}{s}s^n \qquad r=0,1,2,\cdots,n$$

Using this sequence, we can define the Zeta function on the whole complex plane as follows.

Definition 6.2.1

We difine the Riemann Zeta Function on the complex plane as follows.

$$\zeta(p) = \frac{1}{1-2^{1-p}}\sum_{r=1}^{\infty}\frac{1}{2^{r+1}}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}s^{-p} \qquad p \neq 1$$

Furthermore, by using this sequence, the following various coefficients can be generalized.

Generalized Stirling Number of the 2nd kind

$$S_2(p,r) = \frac{1}{r!}\sum_{s=1}^{\infty}(-1)^{s-1}\binom{r}{s}s^p \qquad r=1,2,3,\cdots$$

Generalized Tangent Number

$$T_p = \begin{cases} 0 & p=0 \\ \displaystyle\sum_{r=1}^{\infty}2^{p-r}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}s^p & p \neq 0 \end{cases}$$

Generalized Bernoulli Number

$$B_p = \begin{cases} -\dfrac{1}{2} & p = 1 \\ \displaystyle\frac{p}{2^p-1}\sum_{r=1}^{\infty}\frac{1}{2^{r+1}}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}s^{p-1} & p \neq 1 \end{cases}$$

7 Completed Riemann Zeta

In **7.1**, we consider even function and odd function for complex function. Generally, we obtain the same results as for real-valued functions, but the results unique to complex-valued functions are as follows.

Theorem 7.1.3

Let $f(z)$ be a complex function in the domain D.

(1) If $f(z)$ is an even function , both the real part and the imaginary part are even functions.

(2) If $f(z)$ is an odd function , both the real part and the imaginary part are odd functions.

Theorem 7.1.4

Let $f(z)$ be a complex function in the domain D. Then,

if $f(z)$ is an even function or an odd function , $|f(z)|^2$ is an even function.

In **7.2**, we study complex conjugate properties. Especially when the function $f(z)$ is an even function or an odd function with complex conjugate properties, two important theorems are obtained.

Theorem 7.2.3

When $f(x,y)=u(x,y)+iv(x,y)$ is a function with the complex conjugate property in the domain D ,

(1) if $f(x,y)$ is an even function,

$$u(x,y) = u(x,-y) = u(-x,y) = u(-x,-y)$$
$$v(x,y) = -v(x,-y) = -v(-x,y) = v(-x,-y)$$

(2) if $f(x,y)$ is an odd function,

$$u(x,y) = u(x,-y) = -u(-x,y) = -u(-x,-y)$$
$$v(x,y) = -v(x,-y) = v(-x,y) = -v(-x,-y)$$

Corollary 7.2.3

Let $f(x,y)=u(x,y)+iv(x,y)$ be a function with the complex conjugate property in the domain D .
Then, the followings hold for any real number $x,y \in D$.

(1) When $f(x,y)$ is an even function , $v(x,0)=0$, $v(0,y)=0$.

(2) When $f(x,y)$ is an odd function , $u(0,y)=0$, $v(x,0)=0$.

Theorem 7.2.4

When $f(z)$ is a function with the complex conjugate property in the domain D and has a zero $z_1=x_1+iy_1$ $(x_1\neq 0)$,

(1) if $f(z)$ is an even function, $-x_1-iy_1$, x_1-iy_1 , $-x_1+iy_1$ are also zeros of $f(z)$.

(2) if $f(z)$ is an odd function, $-x_1-iy_1$, x_1-iy_1 , $-x_1+iy_1$ are also zeros of $f(z)$.

In **7.3**, symmetric functional equations are derived from functional equations.

Formula 7.3.1 (Riemann)

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \qquad z\neq 0,1$$

$$\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) = \pi^{-\frac{1}{2}\left(\frac{1}{2}-z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-z\right)\right\}\zeta\left(\frac{1}{2}-z\right)$$

Where, $z\neq \pm 1/2$

In **7.4**, we define the completed Riemann zeta functions $\xi(z), \Xi(z)$ as follows, respectively. These are a little different from Landau's definition.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$

Then, the following equations hold from Formula 7.3.1 .

$$\xi(z) = \xi(1-z)$$
$$\Xi(z) = \Xi(-z)$$

From the latter, we can see that $\Xi(z)$ is an even function. Therefore, Thoerem 7.2.3 (1) and Corollary 7.2.3 hold for the real part $u(x,y)$ and the imaginary part $v(x,y)$ of $\Xi(z)$ as they are. And from Theorem 7.2.4 , the following very important theorem is obtained.

Theorem 7.4.1

If Riemann zeta function $\zeta(z)$ has a non-trivial zero whose real part is not $1/2$, the one set consists of the following four.

$$1/2+\alpha_1 \pm i\beta_1 \quad , \quad 1/2-\alpha_1 \pm i\beta_1 \qquad (0 < \alpha_1 < 1/2)$$

08 Factorization of Completed Riemann Zeta

In **8.1**, the following Hadamard product is shown.

Formula 8.1.1 (Hadamard product of $\xi(z)$)

Let completed zeta function be as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

When non-trivial zeros of $\zeta(z)$ are $z_k = x_k \pm iy_k \quad k=1,2,3,\cdots$ and γ is Euler-Mascheroni constant, $\xi(z)$ is expressed by the Hadamard product as follows.

$$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

$$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}}$$

And, the following special values are obtained.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}} = 1.02336448\cdots$$

$$\prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\} = \frac{\pi}{3}$$

In **8.2**, we consider how the formulas in the previous section are expressed when non-trivial zeros whose real part is 1/2 and non-trivial zeros whose real part is not 1/2 are mixed. Then, we obtain the following theorems.

Theorem 8.2.2

Let γ be Euler-Mascheroni constant, non-trivial zeros of Riemann zeta function are $x_n+iy_n \quad n=1,2,3,\cdots$. Among them, zeros whose real part is 1/2 are $1/2 \pm iy_r \quad r=1,2,3,\cdots$ and zeros whose real parts is not 1/2 are $1/2 \pm \alpha_s \pm i\beta_s \ (0<\alpha_s<1/2) \quad s=1,2,3,\cdots$. Then the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right) = 1$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = \sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} + \sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = 1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2} = 0.0230957\cdots$$

Formula 8.2.3 (Special values)

When non-trivial zeros of Riemann zeta function are $x_k \pm iy_k \quad k=1,2,3,\cdots$, the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right) = 1$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right) = \frac{\pi}{3}$$

Theorem 8.2.4

Let non-trivial zeros of Riemann zeta function are $x_n+iy_n \quad n=1,2,3,\cdots$ and γ be Euler-Mascheroni constant. If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = 1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2} = 0.0230957\cdots$$

Incidentally, when this was calculated using 200000 y_r, both sides coincided with the decimal point 4 digits.

In **8.3**, we show that $\xi(z)$ is factored completely.

Theorem 8.3.1 (Factorization of $\xi(z)$)

Let Riemann zeta function be $\zeta(z)$, the non-trivial zeros are $z_n = x_n \pm iy_n \quad n=1,2,3,\cdots$ and completed zeta function be as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Then, $\xi(z)$ is factorized as follows.

$$\xi(z) = \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)$$

In **8.4**, we first derive the factorization of $\Xi(z)$.

Theorem 8.4.1 (Factorization of $\Xi(z)$)

Let Riemann zeta function be $\zeta(z)$, the non-trivial zeros are $z_n = x_n \pm iy_n \quad n=1,2,3,\cdots$ and completed zeta function be as follows.

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$

Then, $\Xi(z)$ is factorized as follows.

$$\Xi(z) = \Xi(0)\prod_{n=1}^{\infty}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\}$$

Where, $\Xi(0) = \prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2} = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.99424155\cdots$

And, using this theorem and Theorem 7.4.1 in the previous section, we obtaine the following theorem.

Theorem 8.4.4

When Riemann zeta function is $\zeta(z)$ and the non-trivial zeros sre $z_n = x_n \pm iy_n \quad n=1,2,3,\cdots$,
If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\prod_{r=1}^{\infty}\frac{y_r^2}{1/4+y_r^2} = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.99424155\cdots$$

Incidentally, when this was calculated using 100000 y_r, both sides coincided with the decimal point 5 digits.

09 Maclaurin Series of Completed Riemann Zeta

In **9.1**, completed Riemann zeta $\xi(z)$ is expanded in Maclaurin series.

Theorem 9.1.3 (Maclaurin series of $\xi(z)$)

Let completed Riemann zeta be

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Then, the following expression holds on the whole complex plane.

$$\xi(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{(-1)^{s-t}g_{s-t}(3/2)}{2^{s-t}(s-t)!}c_t z^r$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ -\dfrac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

The first few are as folows.

$$\begin{aligned}
\xi(z) = 1 &+ \left(\frac{\log^1\pi}{2^1 1!} - \frac{g_1(3/2)}{2^1 1!} - \frac{\gamma_0}{0!}\right)z^1 \\
&+ \left(\frac{\log^2\pi}{2^2 2!} + \frac{g_2(3/2)}{2^2 2!} - \frac{\gamma_1}{1!}\right. \\
&\qquad \left. - \frac{\log^1\pi}{2^1 1!}\frac{g_1(3/2)}{2^1 1!} + \frac{g_1(3/2)}{2^1 1!}\frac{\gamma_0}{0!} - \frac{\log^1\pi}{2^1 1!}\frac{\gamma_0}{0!}\right)z^2 \\
&+ \left(\frac{\log^3\pi}{2^3 3!} - \frac{g_3(3/2)}{2^3 3!} - \frac{\gamma_2}{2!}\right. \\
&\qquad - \frac{\log^2\pi}{2^2 2!}\frac{g_1(3/2)}{2^1 1!} - \frac{\log^2\pi}{2^2 2!}\frac{\gamma_0}{0!} - \frac{g_2(3/2)}{2^2 2!}\frac{\gamma_0}{0!} \\
&\qquad + \frac{\log^1\pi}{2^1 1!}\frac{g_2(3/2)}{2^2 2!} - \frac{\log^1\pi}{2^1 1!}\frac{\gamma_1}{1!} + \frac{g_1(3/2)}{2^1 1!}\frac{\gamma_1}{1!} \\
&\qquad \left. + \frac{\log^1\pi}{2^1 1!}\frac{g_1(3/2)}{2^1 1!}\frac{\gamma_0}{0!}\right)z^3 \\
&+ \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
= {} & 1. - 0.0230957\,z + 0.0233439\,z^2 - 0.000497984\,z^3 + 0.000253182\,z^4 \\
& - 5.05025\times 10^{-6}\,z^5 + 1.72099\times 10^{-6}\,z^6 - 3.23784\times 10^{-8}\,z^7 + 8.31597\times 10^{-9}\,z^8 \\
& \vdots
\end{aligned}$$

In **9.2**, completed Riemann zeta $\Xi(z)$ is expanded in Maclaurin series.

Theorem 9.2.3 (Maclaurin Series of $\Xi(z)$)

Let completed Riemann zeta be

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$

Then, the following expression holds on the whole complex plane.

$$\Xi(z) = \Xi(0)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{g_{s-t}(5/4)}{2^{s-t}(s-t)!}c_t z^r$$

$$\Xi(0) = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.9942415563\cdots$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \ldots, \psi_{r-1}\left(\frac{5}{4}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ \frac{2}{\zeta(1/2)} \sum_{s=r}^{\infty} (-1)^r \frac{\gamma_{s-1}}{(s-1)!} \binom{s}{r} \left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

The first few are as folows.

$$\begin{aligned} \Xi(z) = \Xi(0)\Bigg\{ 1 &+ \left(-\frac{\log^1 \pi}{2^1 1!} + \frac{g_1(5/4)}{2^1 1!} + c_1 \right) z^1 \\ &+ \left(\frac{\log^2 \pi}{2^2 2!} + \frac{g_2(5/4)}{2^2 2!} + c_2 - \frac{\log^1 \pi}{2^1 1!}\frac{g_1(5/4)}{2^1 1!} + \frac{g_1(5/4)}{2^1 1!} c_1 - \frac{\log^1 \pi}{2^1 1!} c_1 \right) z^2 \\ &+ \left(-\frac{\log^3 \pi}{2^3 3!} + \frac{g_3(5/4)}{2^3 3!} + c_3 + \frac{\log^2 \pi}{2^2 2!}\frac{g_1(5/4)}{2^1 1!} + \frac{\log^2 \pi}{2^2 2!} c_1 + \frac{g_2(5/4)}{2^2 2!} c_1 \right. \\ &\qquad \left. - \frac{\log^1 \pi}{2^1 1!}\frac{g_2(5/4)}{2^2 2!} - \frac{\log^1 \pi}{2^1 1!} c_2 + \frac{g_1(5/4)}{2^1 1!} c_2 - \frac{\log^1 \pi}{2^1 1!}\frac{g_1(5/4)}{2^1 1!} c_1 \right) z^3 \\ &+ \cdots \Bigg\} \end{aligned}$$

$$\begin{aligned} = 0.994242\{1. &+ 4.44089\times 10^{-16} z + 0.023105 z^2 + 1.38778\times 10^{-16} z^3 \\ &+ 0.000248334 z^4 + 2.08167\times 10^{-17} z^5 + 1.67435\times 10^{-6} z^6 \\ &+ 7.37257\times 10^{-18} z^7 + 8.0307\times 10^{-9} z^8 + 1.0842\times 10^{-18} z^9 \\ &+ 2.94014\times 10^{-11} z^{10}\} \end{aligned}$$

We can see that the coefficients of the odd degree are almost zero.

10 Vieta's Formulas on Completed Riemann Zeta

In **10.1**, the relations between the zeros of completed Riemann zeta $\xi(z)$ and the coefficients of the Maclaurin series are shown by the two theorems.

Theorem 10.1.1

Let completed Riemann zeta $\xi(z)$ and the Maclaurin series are as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \sum_{r=0}^{\infty} A_r z^r$$

Then, these coefficients A_r $r=0,1,2,3,\cdots$ are given by

$$A_r = \sum_{s=0}^{r}\sum_{t=0}^{s} \frac{\log^{r-s}\pi}{2^{r-s}(r-s)!} \frac{(-1)^{s-t} g_{s-t}(3/2)}{2^{s-t}(s-t)!} c_t$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ -\frac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

Theorem 10.1.2

Let completed Riemann zeta $\xi(z)$ and the Maclaurin series are as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \sum_{r=0}^{\infty} B_r z^r$$

Then,

(1) The following expressions hold for non-trivial zeros $z_k = x_k \pm i y_k$, $y_k \neq 0$ $k=1,2,3,\cdots$ of $\zeta(z)$.

$$B_1 = -\sum_{r_1=1}^{\infty} \frac{2x_{r_1}}{x_{r_1}^2 + y_{r_1}^2}$$

$$B_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^2 x_{r_1}x_{r_2}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}$$

$$B_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{2^3 x_{r_1}x_{r_2}x_{r_3}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^1\left(x_{r_1}+x_{r_2}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$$B_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} \frac{2^4 x_{r_1}x_{r_2}x_{r_3}x_{r_4}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{2^2\left(x_{r_1}x_{r_2}+x_{r_1}x_{r_3}+x_{r_2}x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} + \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$\vdots$

$$B_{2n-1} = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty} \frac{2^{2n-1} x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-2}=r_{2n-3}+1}^{\infty} \frac{2^{2n-3}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-2}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-2}}^2+y_{r_{2n-2}}^2\right)}$$

$\vdots$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} \frac{2^1\left(x_{r_1}+x_{r_2}+\cdots+x_{r_n}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

$$B_{2n} = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty} \frac{2^{2n} x_{r_1}x_{r_2}\cdots x_{r_{2n}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty} \frac{2^{2n-2}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$\vdots$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} \frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

(2) When A_n is a coefficient in **Theorem 10.1.1** , $B_n = A_n$ $n=1,2,3,\cdots$.

And, if Riemann Hypothesis is true, the following proposition equivalent to this must hold.

Proposition 10.1.3

When $z_k = 1/2 \pm i y_k$, $y_k \neq 0$ ($k=1,2,3,\cdots$) are the non-trivial zeros of Riemann zeta $\zeta(z)$ and

$A_r \;\; r=1,2,3,\cdots$ are constants given by Theorem 10.1.1 , the following expressions hold.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = -A_1 = 0.0230957089\cdots$$

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)} = A_2+A_1 = 0.0002481555\cdots$$

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)} = -A_3 - 2(A_2+A_1) = 0.0000016727\cdots$$

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)\left(1/4+y_u^2\right)}$$
$$= A_4+3A_3+5(A_2+A_1) = 8.021073428\times 10^{-9}$$

Proposition 10.1.3'

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2 = A_1^2 - 2(A_1+A_2) = 0.00003710063\cdots$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^3 = -A_1^3+3(A_1-2)(A_1+A_2) - 3A_3 = 0.00000014367786\cdots$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^4 = A_1^4 - 4A_1^3+4A_1^2\left(\frac{5}{2}-A_2\right)+4A_1(3A_2+A_3-5)$$
$$+\,2A_1^2 - 20A_2 - 12A_3 - 4A_4 = 6.59827915\times 10^{-10}$$

In **10.2**, the relations between the zeros of completed Riemann zeta $\Xi(z)$ and the coefficients of the Maclaurin series are shown by the two theorems.

Theorem 10.2.1

Let completed Riemann zeta $\Xi(z)$ and the Maclaurin series are as follows.

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$
$$= \Xi(0)\left(1+A_1z^1+A_2z^2+A_3z^3+A_4z^4+\cdots\right)$$

Then, these coefficients $A_r \;\; r=0,1,2,3,\cdots$ are given by

$$A_r = \sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{g_{s-t}(5/4)}{2^{s-t}(s-t)!}c_t$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \displaystyle\sum_{k=1}^{r}B_{r,k}\left(\psi_0\left(\frac{5}{4}\right),\psi_1\left(\frac{5}{4}\right),\ldots,\psi_{r-1}\left(\frac{5}{4}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ \displaystyle\frac{2}{\zeta(1/2)}\sum_{s=r}^{\infty}(-1)^r\frac{\gamma_{s-1}}{(s-1)!}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

Theorem 10.2.2

Let completed Riemann zeta $\Xi(z)$ and the Maclaurin series are as follows.

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$
$$= \Xi(0)\left(1+B_1z^1+B_2z^2+B_3z^3+B_4z^4+\cdots\right)$$

Then,

(1) The following expressions hold for non-trivial zeros $z_k = x_k \pm i y_k$, $y_k \neq 0$ $k=1,2,3,\cdots$ of $\zeta(z)$.

$$\Xi(0) = \prod_{n=1}^{\infty} \frac{(x_n - 1/2)^2 + y_n^2}{x_n^2 + y_n^2} = -\frac{1}{4\pi^{1/4}} \Gamma\left(\frac{1}{4}\right) \zeta\left(\frac{1}{2}\right) = 0.9942415563\cdots$$

$$B_1 = -\sum_{r_1=1}^{\infty} \frac{2(x_{r_1} - 1/2)}{(x_{r_1} - 1/2)^2 + y_{r_1}^2}$$

$$B_2 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^2 (x_{r_1} - 1/2)(x_{r_2} - 1/2)}{\{(x_{r_1} - 1/2)^2 + y_{r_1}^2\}\{(x_{r_2} - 1/2)^2 + y_{r_2}^2\}} + \sum_{r_1=1}^{\infty} \frac{2^0}{\{(x_{r_1} - 1/2)^2 + y_{r_1}^2\}}$$

$$B_3 = -\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{2^3 (x_{r_1} - 1/2)(x_{r_2} - 1/2)(x_{r_3} - 1/2)}{\{(x_{r_1} - 1/2)^2 + y_{r_1}^2\}\{(x_{r_2} - 1/2)^2 + y_{r_2}^2\}\{(x_{r_3} - 1/2)^2 + y_{r_3}^2\}}$$

$$- \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^1 \{(x_{r_1} - 1/2) + (x_{r_2} - 1/2)\}}{\{(x_{r_1} - 1/2)^2 + y_{r_1}^2\}\{(x_{r_2} - 1/2)^2 + y_{r_2}^2\}}$$

$$B_4 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \cdots \sum_{r_4=r_3+1}^{\infty} \frac{2^4 (x_{r_1} - 1/2)(x_{r_2} - 1/2)\cdots(x_{r_4} - 1/2)}{\{(x_{r_1} - 1/2)^2 + y_{r_1}^2\}\{(x_{r_2} - 1/2)^2 + y_{r_2}^2\}\cdots\{(x_{r_4} - 1/2)^2 + y_{r_4}^2\}}$$

$$+ \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{2^2 \{(x_{r_1} - 1/2)(x_{r_2} - 1/2) + \cdots + (x_{r_2} - 1/2)(x_{r_3} - 1/2)\}}{\{(x_{r_1} - 1/2)^2 + y_{r_1}^2\}\{(x_{r_2} - 1/2)^2 + y_{r_2}^2\}\{(x_{r_3} - 1/2)^2 + y_{r_3}^2\}}$$

$$+ \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^0}{\{(x_{r_1} - 1/2)^2 + y_{r_1}^2\}\{(x_{r_2} - 1/2)^2 + y_{r_2}^2\}}$$

$\vdots$

(2) When A_n is a coefficient in **Theorem 10.2.1** , $B_n = A_n$ $n=1,2,3,\cdots$.

And, if Riemann Hypothesis is true, the following proposition equivalent to this must hold.

Proposition 10.2.3

When $z_k = 1/2 \pm i y_k$, $y_k \neq 0$ ($k=1,2,3,\cdots$) are the non-trivial zeros of Riemann zeta $\zeta(z)$ and A_r $r=1,2,3,\cdots$ are constants given by Theorem 10.1.1 , the following expressions hold.

$$\sum_{r_1=1}^{\infty} \frac{1}{y_{r_1}^2} = A_2 = 0.0231049931\cdots$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2} = A_4 = 0.0002483340\cdots$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2 y_{r_3}^2} = A_6 = 0.00000167435\cdots$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2 y_{r_3}^2 y_{r_4}^2} = A_8 = 8.030697 \times 10^{-9}$$

$\vdots$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \cdots \sum_{r_{2n}=r_{2n-1}+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2 \cdots y_{r_{2n}}^2} = A_{2n}$$

Proposition 10.2.3'

$$\sum_{r=1}^{\infty}\frac{1}{y_r^{\,4}} = A_2^2 - 2A_4 = 0.00003717259\cdots$$

$$\sum_{r=1}^{\infty}\frac{1}{y_r^{\,6}} = A_2^3 - 3A_2A_4 + 3A_6 = 0.00000014417393\cdots$$

$$\sum_{r=1}^{\infty}\frac{1}{y_r^{\,8}} = A_2^4 + 2A_4^2 - 4A_2^2A_4 + 4A_2A_6 - 4A_8 = 6.6303\times 10^{-10}$$

11 Zeros on the Critical Line of Riemann Zeta

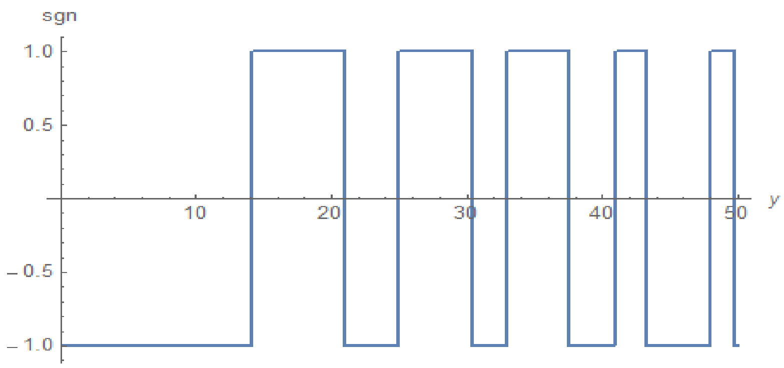
In **11.1**, substituting $z=0+iy$ for the completed Riemann zeta $\Xi(z)$,

$$\Xi_h(y) = -\left(\frac{1}{2}+iy\right)\left(\frac{1}{2}-iy\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+iy\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right)$$

We use this to calculate the zeros on the critical line. However, this function is too small in absolute value and can only find the zeros up to $y=917$.

So we normalize $\Xi_h(y)$ and define the following sign function.

$$sgn(y) = -\frac{\Xi_h(y)}{|\Xi_h(y)|} = \pi^{-\frac{iy}{2}}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right)}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right)\right|}$$

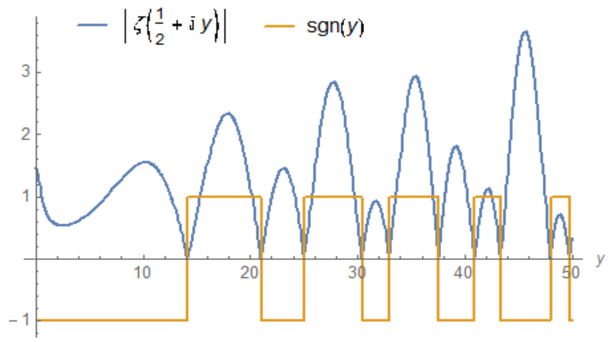


Using this sign function $sgn(y)$, we can find the zeros at large y.

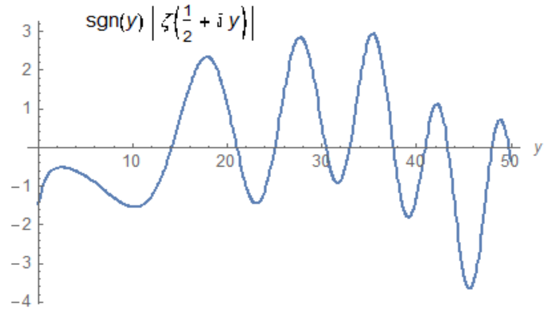
However, this sign function $sgn(y)$ has the disadvantage that it is easy to miss Lehmer's phenomenon.

In **11.2**, multiplying this sign function $sgn(y)$ by the absolute value of the Riemann zeta $\zeta(1/2+iy)$, we obtain a smooth function $Z(y)$.

$$Z(y) = sgn(y)\left|\zeta\left(\frac{1}{2}+iy\right)\right| = \pi^{-\frac{iy}{2}}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\right|}\zeta\left(\frac{1}{2}+iy\right)$$



$\Rightarrow$



Using this $Z(y)$ function, we can find the zeros on the critical line of $\zeta(z)$ by the intersection of the curve and the y-axis Therefore, the risk of missing the Lehmer's phenomenon is reduced.

In **11.3**, first, a lemma is prepared.

Lemma

When $f(z)$ is a complex function defined on the domain D, the following expression holds.

$$e^{i\,Im\,log\,f(z)} = \frac{f(z)}{|f(z)|}$$

Applying this lemma to the gamma function in the **11.2** ,

$$Z(y) = \pi^{-\frac{iy}{2}} \frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\right|} \zeta\left(\frac{1}{2}+iy\right) = \pi^{-\frac{iy}{2}} e^{i\,Im\,log\,\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}} \zeta\left(\frac{1}{2}+iy\right)$$

From this, we obtain

$$Z(y) = e^{i\theta(y)}\zeta\left(\frac{1}{2}+iy\right) \qquad \text{where,} \quad \theta(y) = Im\,log\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\} - \frac{y}{2}log\pi$$

This is a definitional equation of Riemann-Siegel Z function .

12 Zeros of Riemann Zeta and System of Infinite Degree Equations

In **12.1**, the Riemann zeta $\zeta(1/2\pm z)$ is Laurent expanded into real and imaginary parts.

Formula 12.1.3

When the Riemann zeta function is $\zeta(1/2+z)$ ($z=x+iy$) and Stieltjes constansts are γ_s $s=0,1,2,\cdots$, the following expressions hold on the whole complex plane except $z=1/2$.

$$\zeta\left(\frac{1}{2}+z\right) = -\frac{1}{1/2-z} + \sum_{s=0}^{\infty}\gamma_s\frac{(1/2-z)^s}{s\,!} = u_+(z)+i\,v_+(z)$$

$$u_+(x,y) = -\frac{1/2-x}{(1/2-x)^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\gamma_{2r+s}\frac{(1/2-x)^s}{s\,!}\frac{(-1)^r y^{2r}}{(2r)\,!}$$

$$v_+(x,y) = -\frac{y}{(1/2-x)^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\gamma_{2r+s+1}\frac{(1/2-x)^s}{s\,!}\frac{(-1)^r y^{2r+1}}{(2r+1)\,!}$$

Where, $0^0=1$)

Formula 12.1.4

When the Riemann zeta function is $\zeta(1/2-z)$ ($z=x+iy$) and Stieltjes constansts are γ_s $s=0,1,2,\cdots$, the following expressions hold on the whole complex plane except $z=-1/2$.

$$\zeta\left(\frac{1}{2}-z\right) = -\frac{1}{1/2+z} + \sum_{s=0}^{\infty}\gamma_s\frac{(1/2+z)^s}{s\,!} = u_-(z)+i\,v_-(z)$$

$$u_-(x,y) = -\frac{1/2+x}{(1/2+x)^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\gamma_{2r+s}\frac{(1/2+x)^s}{s\,!}\frac{(-1)^r y^{2r}}{(2r)\,!}$$

$$v_-(x,y) = \frac{y}{(1/2+x)^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\gamma_{2r+s+1}\frac{(1/2+x)^s}{s\,!}\frac{(-1)^r y^{2r+1}}{(2r+1)\,!}$$

Where, $0^0=1$)

In **12.2**, these are added and subtracted and rearranged into even and odd functions.

Formula 12.2.1 (Even function)

$$\zeta_e(z) = -\frac{1/2}{1/4-z^2} + \sum_{s=0}^{\infty} f^{(2s)}(0)\frac{z^{2s}}{(2s)!}$$

$$u_e(x,y) = -\frac{1/4-x^2+y^2}{2\left\{(1/4-x^2+y^2)^2+4x^2y^2\right\}} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v_e(x,y) = -\frac{xy}{(1/4-x^2+y^2)^2+4x^2y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Formula 12.2.2 (Odd function)

$$\zeta_o(z) = -\frac{z}{1/4-z^2} - \sum_{s=0}^{\infty} f^{(2s+1)}(0)\frac{z^{2s+1}}{(2s+1)!}$$

$$u_o(x,y) = -\frac{x(1/4-x^2-y^2)}{(1/4-x^2+y^2)^2+4x^2y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v_o(x,y) = -\frac{y(1/4+x^2+y^2)}{(1/4-x^2+y^2)^2+4x^2y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, in the both formulas,

$$f^{(s)}(0) = \sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t \quad \left\{ = \sum_{t=s}^{\infty}\frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s}\right\}, \quad 0^0 = 1$$

In **12.3**, necessary and sufficient conditions for Riemann zeta to have zeros are shown.

Theorem 12.3.1

When the Riemann zeta functions are $\zeta(1/2\pm z)$ and Stieltjes constansts are γ_s $s=0,1,2,\cdots$, $\zeta(1/2\pm z) = 0$ if and only if the following system of equations has a solution.

$$\begin{cases} u_e = -\dfrac{1/4-x^2+y^2}{2\left\{(1/4-x^2+y^2)^2+4x^2y^2\right\}} + \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_e = -\dfrac{xy}{(1/4-x^2+y^2)^2+4x^2y^2} + \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \\ u_o = -\dfrac{x(1/4-x^2-y^2)}{(1/4-x^2+y^2)^2+4x^2y^2} - \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_o = -\dfrac{y(1/4+x^2+y^2)}{(1/4-x^2+y^2)^2+4x^2y^2} - \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \end{cases}$$

Where,

$$f^{(s)}(0) = \sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t \quad \left\{ = \sum_{t=s}^{\infty}\frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s}\right\}, \quad 0^0 = 1$$

In **12.4**, two hypotheses are presented that are equivalent to the Riemann hypothesis.

Hypothesis 12.4.1

When γ_s $s=0,1,2,\cdots$ are Stieltjes constansts and x , y are real numbers, the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} v_o = -\dfrac{y\left(1/4+x^2+y^2\right)}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\dfrac{x^{2s}}{(2s)!}\dfrac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \\ u_o = -\dfrac{x\left(1/4-x^2-y^2\right)}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\dfrac{x^{2s+1}}{(2s+1)!}\dfrac{(-1)^r y^{2r}}{(2r)!} = 0 \end{cases}$$

Hypothesis 12.4.2

When γ_s $s=0,1,2,\cdots$ are Stieltjes constansts and x, y are real numbers,
the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} u_e = -\dfrac{1/4-x^2+y^2}{2\left\{\left(1/4-x^2+y^2\right)^2+4x^2y^2\right\}} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\dfrac{x^{2s}}{(2s)!}\dfrac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_e = -\dfrac{xy}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\dfrac{x^{2s+1}}{(2s+1)!}\dfrac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \end{cases}$$

Where, in the both hypotheses, $f^{(s)}(0) = \sum_{t=0}^{\infty}\dfrac{\gamma_{s+t}}{t!}\left(\dfrac{1}{2}\right)^t \left\{ = \sum_{t=s}^{\infty}\dfrac{\gamma_t}{(t-s)!}\left(\dfrac{1}{2}\right)^{t-s} \right\}$, $0^0=1$

Fractional functions can be ignored when y is large. Then, considering these partial derivatives with respect to y, it seems likely that the above two hypotheses will hold true.

Graphical Proof of the Riemann Hypothesis

This proof is difficult to contain in one chapter, so I write it as a separate paper.

In **Chapter 1**, the definitions of the Riemann zeta function $\zeta(z)$ and the Dirichlet etar function $\eta(z)$ and the relational expression between the two are shown. That is,

$$\zeta(z) = \sum_{r=1}^{\infty} e^{-z\log r} = \frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\cdots \qquad Re(z)>1$$

$$\eta(z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log r} = \frac{1}{1^z}-\frac{1}{2^z}+\frac{1}{3^z}-\frac{1}{4^z}+-\cdots \qquad Re(z)>0$$

$$\zeta(z) = \frac{1}{1-2^{1-z}}\eta(z) \qquad z \neq 1$$

In **Chapter 2**, three equivalent lemmas are presented and proven.

Lemma 2.1

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy,\ x,y\in R)$,
$\eta(z)=0$ in $0<x<1$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \eta(z) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-z\log r} = 0 & (2.1_+) \\ \eta(1-z) = \sum_{r=1}^{\infty}(-1)^{r-1}e^{-(1-z)\log r} = 0 & (2.1_-) \end{cases}$$

Lemma 2.1'

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy,\ x,y\in R)$,
$\eta(1/2\pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \eta\left(\dfrac{1}{2}+z\right) = \sum_{r=1}^{\infty}\dfrac{(-1)^{r-1}}{\sqrt{r}}e^{-z\log r} = 0 & (2.1'_+) \\ \eta\left(\dfrac{1}{2}-z\right) = \sum_{r=1}^{\infty}\dfrac{(-1)^{r-1}}{\sqrt{r}}e^{z\log r} = 0 & (2.1'_-) \end{cases}$$

Lemma 2.2

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy\ ,\ x,y\in R)$,
$\eta(1/2\pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \eta_c(z) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} cosh(z\,log\,r) = 0 & (2.2c) \\ \eta_s(z) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} sinh(z\,log\,r) = 0 & (2.2s) \end{cases}$$

Then, expressing Lemma 2.2 in terms of real and imaginary parts, we obtain the following theorem.

Theorem 2.3

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z=x+iy\ ,\ x,y\in R)$,
$\eta(1/2\pm z)=0$ in $-1/2<x<1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} u_c(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} cosh(x\,log\,r)\ cos(y\,log\,r) = 0 \\ v_c(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} sinh(x\,log\,r)\ sin(y\,log\,r) = 0 \\ u_s(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} sinh(x\,log\,r)\ cos(y\,log\,r) = 0 \\ v_s(x,y) = \sum_{r=1}^{\infty} \dfrac{(-1)^{r-1}}{\sqrt{r}} cosh(x\,log\,r)\ sin(y\,log\,r) = 0 \end{cases}$$

In **Chapter 3**, the amplitude of $v_c(x,y)$ with respect to y is studied and the following law is obtained

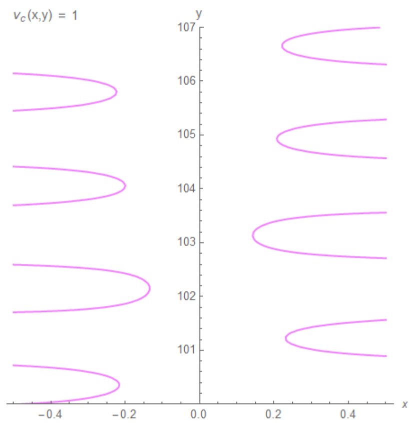
Law 3.4.5

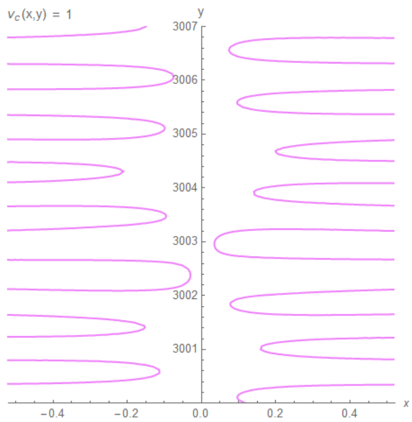
Let x,y are real numbers and function $v_c(x,y)$ be as follows.

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sinh(x\,log\,r)\ sin(y\,log\,r) \qquad (2.4c)$$

Then, given x , the amplitude of $v_c(x,y)$ is generally proportional to the absolute value of y .

This law can explain that when x is given, the tips of the contour line of $v_c(x,y)$ generally approach the y -axis as y becomes larger. For example, if a contour line of height 1 of $v_c(x,y)$ is drawn for $y=100\sim107$ and $y=3000\sim3007$ it is as follows respectively. The left figure is $y=100\sim107$ and the right figure is $y=3000\sim3007$.





In **Chapter 4**, the amplitude of $u_s(x,y)$ with respect to y is studied and the following law is obtained

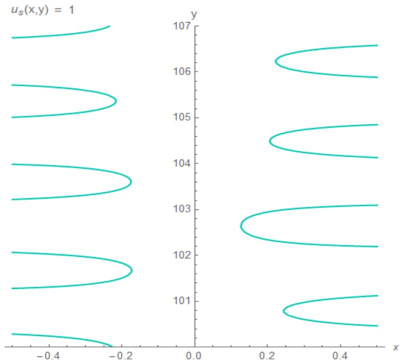
Law 4.4.5

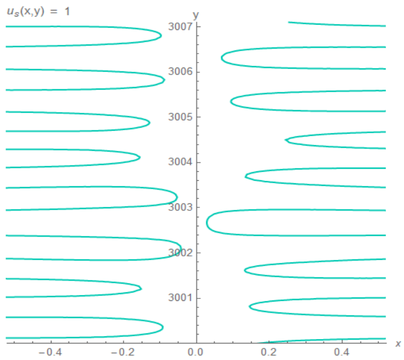
Let x, y are real numbers and function $u_s(x,y)$ be as follows.

$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} \sinh(x \log r) \cos(y \log r) \tag{2.4s}$$

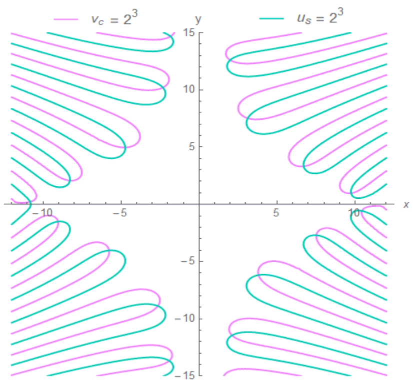
Then, given x, the amplitude of $u_s(x,y)$ is generally proportional to the absolute value of y.

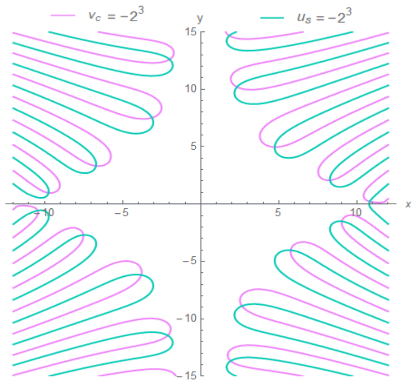
This law can explain that when x is given, the tips of the contour line of $u_s(x,y)$ generally approach the y-axis as y becomes larger. For example, if a contour line of height 1 of $v_c(x,y)$ is drawn for $y=100 \sim 107$ and $y=3000 \sim 3007$ it is as follows respectively. The left figure is $y=100 \sim 107$ and the right figure is $y=3000 \sim 3007$.





In **Chapter 5**, the contour lines of $v_c(x,y) = u_s(x,y) = \pm h \ (h \geq 0)$ are noticed. For example, contour lines of height ± 8 of $v_c(x,y), u_s(x,y)$ are drawn as follows. The left figure is $+8$ and the right figure is -8.



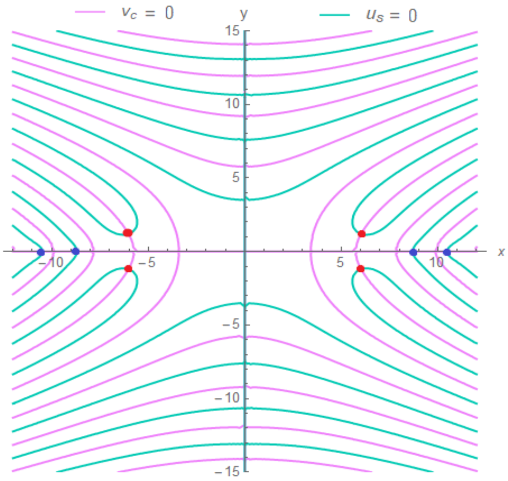


Since $v_c(x,y), u_s(x,y)$ are odd functions with respect to x, the left and right figures are mirror images with respect to the y-axis. Furthermore, since $v_c(x,y)$ is odd function with respect to y, the left and right figures are mirror images with respect to the x-axis. Both figures can never overlap by translation or rotation in the plane.

Nevertheless, at height ± 0, the left and right figures have to overlap without translation or rotation. To do this, the contour lines in both figures must deform as the height approaches ± 0 from above and below. And, at height ± 0, both figures must be symmetrical about both the y-axis and the x-axis.

This forces contour lines that were alternate at height $\neq 0$ to be opposite at height ± 0. This also applies to the x-axis. Thus, at height ± 0, the right and left edges of $\supset$ $\subset$ must be absorbed into the y-axis, and the lower and upper edges of $\cup$ $\cap$ must be absorbed into the x-axis. In fact, if we approach the heights of $v_c(x,y), u_s(x,y)$ from above and below to ± 0, the contour lines become eventually as follows.

↘ ↙



For the animation from $v_c = u_s = \pm 1$ to $v_c = u_s = \pm 0$, click here. AnimZ5219.gif

Consistent with the theory, the contour parts asymmetric with respect to the y and x -axis were absorbed in both axes. As the result, an infinite number of trivial solutions (blue dots) and four non-trivial solutions (red dots) remained. The trivial solutions are on the x -axis, and the non-trivial solutions are outside the critical strip $-1/2 < x < 1/2$.

Though these figures are drawn with $|y| \le 15$, according to **Law 3.4.5 and Law 4.4.5**, the right and left tips of $\supset$ $\subset$ are absorbed more quickly into the y -axis where y is large.

Thus, the system of equations $v_c(x,y) = u_s(x,y) = 0$ has no solution in the critical strip $-1/2 < x < 1/2$ except on the critical line $x = 0$.

In **Chapter 6**, by organizing and summarizing the above, the Riemann hypothesis is proven.

Proposition 6.1 (Riemann Hypothesis)

Let $\zeta(z)$ be the function defined by the following Dirichlet series.

$$\zeta(z) = \sum_{r=1}^{\infty} e^{-z\log r} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots \qquad Re(z) > 1 \qquad (1.\zeta)$$

This function has no non-trivial zeros except on the critical line $Re(z) = 1/2$.

Proof

The problem of finding the zeros of the Riemann zeta function $\zeta(z)$ ultimately reduce to Theorem 2.3 . According to the theorem, the fact that the Dirichlet eta function $\eta(z)$ has zeros in the critical strip is equivalent to the fact that the system of equations $u_c = v_c = u_s = v_s = 0$ has solutions in the critical strip. However, as seen in Chapter 5, $v_c = u_s = 0$ has no solution in the critical strip except on the critical line.

So, according to Theorem 2.3, the Dirichlet eta function $\eta(z)$ has no zeros in the critical strip except on the critical line, therefore, the Riemann zeta function $\zeta(z)$ has no zeros in the critical strip except on the critical line. Q.E.D.

Analytical Proof of the Riemann Hypothesis

In **Chapter 1**, the definitions of the Riemann zeta function $\zeta(z)$ and the Dirichlet etar function $\eta(z)$ and the relational expression between the two are shown. That is,

$$\zeta(z) = \sum_{r=1}^{\infty} e^{-z\log r} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots \qquad Re(z) > 1$$

$$\eta(z) = \sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log r} = \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + - \cdots \qquad Re(z) > 0$$

$$\zeta(z) = \frac{1}{1-2^{1-z}}\eta(z) \qquad z \ne 1$$

In **Chapter 2**, After going through three equivalent lemmas, we finally obtain the following theorem.

Theorem 2.3

When the set of real numbers is R and Dirichlet eta function is $\eta(z)$ $(z = x+iy\ ,\ x,y \in R)$,
$\eta(1/2 \pm z) = 0$ in $-1/2 < x < 1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} u_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} cosh(x\log r)\ cos(y\log r) = 0 \\ v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sinh(x\log r)\ sin(y\log r) = 0 \\ u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sinh(x\log r)\ cos(y\log r) = 0 \\ v_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} cosh(x\log r)\ sin(y\log r) = 0 \end{cases}$$

In **Chapter 3**, it is noted that the first terms $(r=1)$ of the $v_c(x,y)$ and $u_s(x,y)$ series are both 0. So, these first terms are changed from $r=1$ to $r=2$, and the following lemma is proven.

Lemma 3.1

When y is a real number, x is a real number s.t. $-1/2 < x < 1/2$, the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} v_c(x,y) = \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sinh(x\log r)\ sin(y\log r) = 0 & (3.1c) \\ u_s(x,y) = \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}} sinh(x\log r)\ cos(y\log r) = 0 & (3.1s) \end{cases}$$

Proof (overview)

1. Integrating the series (3.1s) term by term from 0 to y with respect to y ,

$$\int u_s(x,y)dy = \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}\ \log r} sinh(x\log r)\ sin(y\log r) \qquad (3.1sy)$$

The y coordinates of the peaks and valleys of $v_c(x,y)$ and $\int u_s(x,y)\,dy$ almost match. ($\because$ $sin(y\log r)$ is shared)
The peaks and valleys of $\int u_s(x,y)\,dy$ and the zeros of $u_s(x,y)$ exactly match. ($\because$ Function and its derivative)
So, the y coordinates of the peaks and valleys of $v_c(x,y)$ and the zeros of $u_s(x,y)$ almost match.

2. Integrating the series (3.1c) term by term from 0 to y with respect to y ,

$$\int v_c(x,y)dy = -\sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{r}\ \log r} sinh(x\log r)\ cos(y\log r) \qquad (3.1cy)$$

The y coordinates of the peaks and valleys of $u_s(x,y)$ and $\int v_c(x,y)\,dy$ almost match. ($\because$ $cos(y\log r)$ is shared)
The peaks and valleys of $\int v_c(x,y)\,dy$ and the zeros of $v_c(x,y)$ exactly match. ($\because$ Function and its derivative)
So, the y coordinates of the peaks and valleys of $u_s(x,y)$ and the zeros of $v_c(x,y)$ almost match.

3. As the result of 1 and 2, $v_c(x,y)$ and $u_s(x,y)$ do not have common zeros in $-1/2 < x < 1/2\ ,\ x \neq 0$.

In **Chapter 4**, by organizing and summarizing the above, the Riemann hypothesis is proven.

Theorem 4.1 (Riemann Hypothesis)

Let $\zeta(z)$ be the function defined by the following Dirichlet series.

$$\zeta(z) = \sum_{r=1}^{\infty} e^{-z\log r} = \frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\cdots \qquad Re(z)>1 \qquad (1.\zeta)$$

This function has no non-trivial zeros except on the critical line $Re(z)=1/2$.

Proof (overview)

According to Lemma 3.1 and Theorem 2.3 , $\eta(1/2+z)$ has no zeros other than $x=0$ in $-1/2<x<1/2$.
That is, Dirichlet eta function $\eta(z)$ has no zeros other than $x=1/2$ in $0<x<1$.
Therefore, Riemann zeta function $\zeta(z)$ also has no zeros other than $x=1/2$ in $0<x<1$.

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