21 Super Calculus of the product of many functions

21.1 Super Integrals of the product of many functions

(1) Generalized binomial theorem and Super integral of the product of 2 functions

According to the generalized binomial theorem in **3.2** , the following expression holds for the real numbers x_1, x_2 such that $|x_1| > |x_2|$ and a positive number p .

$$(x_1+x_2)^{-p} = \sum_{r=0}^{\infty} \binom{-p}{r} x_1^{-p-r} x_2^{r}$$

On the other hand, according to Formula 17.1.2 in **17.1** , the following expression holds for the functions f_1, f_2 of x and a positive number p .

$$\int_a^x \sim \int_a^x (f_1 f_2)^{\langle p \rangle} dx^2 = \sum_{r=0}^{m-1} \binom{-p}{r} f_1^{\langle p+r \rangle} f_2^{(r)} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{p-1}{k} \int_a^x \sim \int_a^x f_1^{\langle m+k \rangle} f_2^{(m+k)} dx^p$$

Reversing the sign of the index of the differentiation operator (p) in the Leibniz Rule about the Super Differentiation in **19.1** ,

$$(f_1 f_2)^{(-p)} = \sum_{r=0}^{m-1} \binom{-p}{r} f_1^{(-p-r)} f_2^{(r)} + R_m^{-p}$$

$$R_m^{-p} = \frac{(-1)^m}{B(p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{p-1}{k} \left\{ f_1^{\langle m+k \rangle} f_2^{(m+k)} \right\}^{(-p)}$$

And, replacing $(-p)$ with the intagration operator $\langle p \rangle$, we obtain the above formula.

(2) Generalized multinomial theorem and Super Integral of the product of many functions

According to the generalized multinomial theorem in **3.4** , the following expression holds for real numbers $x_1, x_2, \cdots, x_\lambda$ such that $|x_1| > |x_2 + x_3 + \cdots + x_\lambda|$ and a positive number p .

$$(x_1+x_2+\cdots+x_\lambda)^{-p} = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} x_1^{-p-r_1} x_2^{r_1-r_2} \cdots x_\lambda^{r_{\lambda-1}}$$

Therefore, the following expression must hold for functions $f_1, f_2, \cdots, f_\lambda$ of x and a positive number p .

$$\int_a^x \sim \int_a^x f_1 f_2 \cdots f_\lambda dx^p = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{\langle p+r_1 \rangle} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^p$$

21.1.1 Super Integral of the product of many functions

Theorem 21.1.1

Let p, r are positive numbers, m be a natural number, $f_k^{(r)}$ be the r th order derivative function of $f_k(x)$ $(k=1,2,\cdots,\lambda)$, $f_k^{\langle r \rangle}$ be arbitrary r th order primitive function of $f_k(x)$ and $B(p,m)$ be the beta function. At this time, if there is a number a such that

$$f_1^{\langle r \rangle}(a) = 0 \quad r \in [0, m+p] \quad or \quad f_k^{(s)}(a) = 0 \quad s \in [0, m+p-1]$$ for at least one $k > 1$,

the following expression holds

$$\int_a^x \sim \int_a^x f_1 f_2 \cdots f_\lambda dx^p = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{\langle p+r_1 \rangle} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{1}{m+k_1} \binom{p-1}{k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$

$$\times \int_a^x \sim \int_a^x f_1^{\langle m+k_1 \rangle} f_2^{(m+k_1-k_2)} f_3^{(k_2-k_3)} \cdots f_\lambda^{(k_{\lambda-1})} dx^p$$

Proof

Analytically continuing the index of the integration operator in Formula 20.2.1 in **20.2** to $[0,p]$ from $[1,n]$ we obtain the desired expression.

Example computation of the product of three functions

Example1 Super Integral of $x^\alpha e^x \sin x$

Since the zero of the higher integral of $x^\alpha e^x \sin x$ is $x=-\infty$, the zero of the super integral is also the same. Then let $f_1 = x^\alpha$, $f_2 = e^x$, $f_3 = \sin x$

$$(x^\alpha)^{<p+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} \quad , \quad (x^\alpha)^{<m+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r}$$

$$(e^x)^{(r-s)} = (e^x)^{(m+r-s)} = e^x \quad , \quad (\sin x)^{(s)} = \sin\left(x+\frac{s\pi}{2}\right)$$

Substituting these for Theorem 21.1.1 , we obtain

$$\int_{-\infty}^x \sim \int_{-\infty}^x x^\alpha e^x \sin x \, dx^p$$

$$= \sum_{r=0}^{m-1} \sum_{s=0}^{r} \binom{-p}{r} \binom{r}{s} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} e^x \sin\left(x+\frac{s\pi}{2}\right) + R_m^p \tag{1.1}$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{r=0}^{\infty} \sum_{s=0}^{m+r} \frac{1}{m+r} \binom{p-1}{r} \binom{m+r}{s}$$

$$\times \int_{-\infty}^x \sim \int_{-\infty}^x \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} e^x \sin\left(x+\frac{s\pi}{2}\right) dx^p \tag{1.1r}$$

As seen in Theorem 20.2.1 in **20.2** , this R_m^p is not converged on 0 at the time of $m \rightarrow \infty$. That is, this polynomial is an asymptotic expansion. When $\alpha=3, p=5/2$, the values of the both sides on arbitrary point $x=20$ are as follows.

```
a = 3;  p = 5 / 2;  m = 100;

fl[x_] := 1/Gamma[p] ∫_{-∞}^{x} (x - t)^(p-1) t^a e^t Sin[t] dt

fr[x_] := Σ_{r=0}^{m-1} Σ_{s=0}^{r} Binomial[-p, r] Binomial[r, s] Gamma[1 + a]/Gamma[1 + a + p + r] x^(a+p+r) e^x Sin[x + s π/2]
```

```
N[fl[20]]
-7.89712 ×10^11 + 0. i
```

```
N[fr[20]]
-7.89712 ×10^11
```

Example2 Super Integral of $x^{\alpha} e^{x} \log x$

Since the zero of the higher integral of $x^{\alpha} e^{x} \log x$ is $x=-\infty$, the zero of the super integral is also the same. Then let $f_1 = x^{\alpha}$, $f_2 = e^{x}$, $f_3 = \log x$

$$\left(x^{\alpha}\right)^{<p+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} \quad , \quad \left(x^{\alpha}\right)^{<m+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r}$$

$$\left(e^{x}\right)^{(r-s)} = e^{x} \quad , \quad \left(e^{x}\right)^{(m+r-s)} = e^{x}$$

$$(\log x)^{(0)} = \log x \quad (s=0) \quad , \quad (\log x)^{(s)} = (-1)^{s-1}(s-1)!\, x^{-s} \quad (s \neq 0)$$

Separating the terms containing $f_3^{(0)}$ from Theorem 21.1.1 and substituting these fot it,

$$\int_{-\infty}^{x} \sim \int_{-\infty}^{x} x^{\alpha} e^{x} \log x \, dx^{p} = e^{x} \log x \sum_{r=0}^{m-1} \binom{-p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r}$$

$$- \sum_{r=1}^{m-1} \sum_{s=1}^{r} (-1)^{s} \binom{-p}{r} \binom{r}{s} \frac{\Gamma(1+\alpha)\Gamma(s)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r-s} e^{x} + R_m^p \qquad \textbf{(1.2)}$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{r=0}^{\infty} \frac{1}{m+r} \binom{p-1}{r} \binom{m+r}{0} \int_{-\infty}^{x} \sim \int_{-\infty}^{x} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} e^{x} \log x \, dx^{p}$$

$$+ \frac{(-1)^m}{B(p,m)} \sum_{r=0}^{\infty} \sum_{s=1}^{m+r} \frac{(-1)^{s-1}}{m+r} \binom{p-1}{r} \binom{m+r}{s} \int_{-\infty}^{x} \sim \int_{-\infty}^{x} \frac{\Gamma(1+\alpha)\Gamma(s)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r-s} e^{x} dx^{p}$$

(1.2r)

When $\alpha=3/2, p=2.7$, the values of the both sides on arbitrary point $x=17$ are as follows. This also seems to be asymptotic expansion.

```
a = 3 / 2; p = 2.7; m = 55;

fl[x_] := 1/Gamma[p] ∫_{-∞}^{x} (x - t)^(p-1) t^a e^t Log[t] dt

fr[x_] := e^x Log[x] Σ_{r=0}^{m-1} Binomial[-p, r] Gamma[1 + a]/Gamma[1 + a + p + r] x^(a+p+r) -
    Σ_{r=1}^{m-1} Σ_{s=1}^{r} (-1)^s Binomial[-p, r] Binomial[r, s] Gamma[1 + a] Gamma[s]/Gamma[1 + a + p + r] x^(a+p+r-s) e^x
```

```
N[fl[17]]
3.50988 ×10^9 - 103.255 i
```

```
N[fr[17]]
3.50988 ×10^9
```

21.1.2 Super Integral of the power of a function

Especially, if $f_1 = f_2 = \cdots = f_{\lambda}$ in Theorem 21.1.1 , the following theorem holds immediately.

Theorem 21.1.2

Let p , r are positive numbers, m be a natural number, $f^{(r)}$ be the r th order derivative function of $f(x)$, $f^{<r>}$ be arbitrary r th order primitive function of $f(x)$ and $B(p,m)$ be the beta function. At this time, if there is a number a such that

$$f^{<r>}(a) = 0 \quad r \in [0, m+p] \quad or \quad f^{(s)}(a) = 0 \quad s \in [0, m+p-1]$$

the following expression holds for $\lambda = 2,3,4,\cdots$.

$$\int_a^x \sim \int_a^x f^{\lambda} dx^p = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f^{\langle p+r_1 \rangle} f^{(r_1-r_2)} \cdots f^{(r_{\lambda-1})} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{1}{m+k_1} \binom{p-1}{k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$
$$\times \int_a^x \sim \int_a^x f^{\langle m+k_1 \rangle} f^{(m+k_1-k_2)} f^{(k_1-k_2)} \cdots f^{(k_{\lambda-1})} dx^p$$

Example Super Integral of $log^3 x$

Since the zero of the higher integral of $log^3 x$ is $x=0$, the zero of the super integral is also the same. Then let $f = log\,x$

$$(log\,x)^{<p+r>} = \frac{log\,x - \psi(1+p+r) - \gamma}{\Gamma(1+p+r)} x^{p+r} \;,\; (log\,x)^{<m+r>} = \frac{log\,x - \psi(1+m+r) - \gamma}{\Gamma(1+m+r)} x^{m+r}$$

$$(log\,x)^{(r-s)} = log\,x \quad (r=s) \;,\quad (log\,x)^{(r-s)} = (-1)^{r-s-1}(r-s-1)!\,x^{-r+s} \quad (r \neq s)$$

$$(log\,x)^{(0)} = log\,x \quad (s=0) \;,\quad (log\,x)^{(s)} = (-1)^{s-1}(s-1)!\,x^{-s} \quad (r \neq 0)$$

Separating the terms containing $f^{(0)}$ from Theorem 21.1.2 and substituting these fot it,

$$\int_0^x \sim \int_0^x (log\,x)^3 dx^p = \frac{log\,x - \psi(1+p) - \gamma}{\Gamma(1+p)} x^p (log\,x)^2$$
$$- 2x^p log\,x \sum_{r=1}^{m-1} (-1)^r \binom{-p}{r} \frac{log\,x - \psi(1+p+r) - \gamma}{\Gamma(1+p+r)} \Gamma(r)$$
$$+ x^p \sum_{r=2}^{m-1} \sum_{s=1}^{r-1} (-1)^r \binom{-p}{r} \binom{r}{s} \frac{log\,x - \psi(1+p+r) - \gamma}{\Gamma(1+p+r)} \Gamma(r-s) \Gamma(s)$$
$$+ R_m^p \qquad (2.1)$$

$$R_m^p = -\frac{1}{B(p,m)} \sum_{r=0}^{\infty} \sum_{s=0}^{m+r} \frac{(-1)^{r-s}}{m+r} \binom{p-1}{r} \binom{m+r}{s}$$
$$\times \int_0^x \sim \int_0^x \frac{log\,x - \psi(1+m+r) - \gamma}{\Gamma(1+m+r)} \Gamma(m+r-s) x^s dx^p \qquad (2.1r)$$

And $\lim_{m \to \infty} R_m^n = 0$ holds although the proof is difficult. Then

$$\int_0^x \sim \int_0^x (log\,x)^3 dx^p = \frac{log\,x - \psi(1+p) - \gamma}{\Gamma(1+p)} x^p (log\,x)^2$$
$$- 2x^p log\,x \sum_{r=1}^{\infty} (-1)^r \binom{-p}{r} \frac{log\,x - \psi(1+p+r) - \gamma}{\Gamma(1+p+r)} \Gamma(r)$$
$$+ x^p \sum_{r=2}^{\infty} \sum_{s=1}^{r-1} (-1)^r \binom{-p}{r} \binom{r}{s} \frac{log\,x - \psi(1+p+r) - \gamma}{\Gamma(1+p+r)} \Gamma(r-s) \Gamma(s)$$
$$(2.1')$$

However, the convergence speed is very slow.

When $p=3/2, m=400$, the values of the both sides on arbitrary point $x=0.2$ are as follows. Even if calculated so far, both sides are corresponding only to 1 digit below the decimal point. Although (2.1') is not suitable for the calculation, it is meaningful that the integral can be expressed by the double series.

```
p = 3 / 2; m = 400;
```

```
fl[x_] := 1/Gamma[p] ∫_0^x (x - t)^(p-1) Log[t]^3 dt

fr[x_] := (Log[x] - PolyGamma[1 + p] - EulerGamma)/Gamma[1 + p] x^p Log[x]^2 -
   2 x^p Log[x] Sum_{r=1}^{m-1} (-1)^r Binomial[-p, r] (Log[x] - PolyGamma[1 + p + r] - EulerGamma)/Gamma[1 + p + r] Gamma[r] +
   x^p Sum_{r=2}^{m-1} Sum_{s=1}^{r-1} (-1)^r Binomial[-p, r] Binomial[r, s] (Log[x] - PolyGamma[1 + p + r] - EulerGamma)/Gamma[1 + p + r]
                                                                          × Gamma[r - s] Gamma[s]
```

```
N[fl[0.2]]              N[fr[0.2]]
-2.44308                -2.40009
```

21.1.3 Super Integrls of $cos^m x$, $sin^m x$

Formula 21.1.3

Let m be a natural number, p be a positive number and $a(s)$ $s \in [0,p]$ be the zero of the lineal super primitive function of $cos^{2m+1}x$ or $sin^{2m+1}x$. Then the following expressions hold.

$$\int_{a(p)}^{x} \sim \int_{a(0)}^{x} cos^{2m+1}x\, dx^p = \frac{1}{2^{2m}} \sum_{r=0}^{m} \frac{{}_{2m+1}C_r}{(2m-2r+1)^p} cos\left\{ (2m-2r+1)x - \frac{p\pi}{2} \right\} \qquad (3.c)$$

$$\int_{a(p)}^{x} \sim \int_{a(0)}^{x} sin^{2m+1}x\, dx^p = \frac{1}{2^{2m}} \sum_{r=0}^{m} \frac{(-1)^{m-r}\, {}_{2m+1}C_r}{(2m-2r+1)^p} sin\left\{ (2m-2r+1)x - \frac{p\pi}{2} \right\} \qquad (3.s)$$

Proof

Analytically continuing the index of the integration operator in Formula 20.2.3c (1) and Formula 20.2.3s (1) in **20.2** to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Note

For example, in the case of $cos^{2m+1}x$, $a(s)$ is a solution of the following transcendental equation.

$$\sum_{r=0}^{m} \frac{{}_{2m+1}C_r}{(2m-2r+1)^s} cos\left\{ (2m-2r+1)x - \frac{s\pi}{2} \right\} = 0 \qquad s \in [0,p]$$

Although it is difficult to calculate this solution, the necessity does not exist in this formula.

Example The 1.5th order intagral of $cos^3 x$

```
• m:=1: a1:=PI*0/2: a2:=PI*1/2:
```

1st order integral

```
• f1 := x-> int(cos(t)^(2*m+1),t=a1..x)
```

$$x \to \int_{a1}^{x} \cos(t)^{2\cdot m+1}\, dt$$

1.5th order Integral

```
• p:=3/2:
• fh := x-> 1/2^(2*m)*sum(binomial(2*m+1,r)/(2*m-2*r+1)^p
                          *cos((2*m-2*r+1)*x-p*PI/2), r=0..m)
```

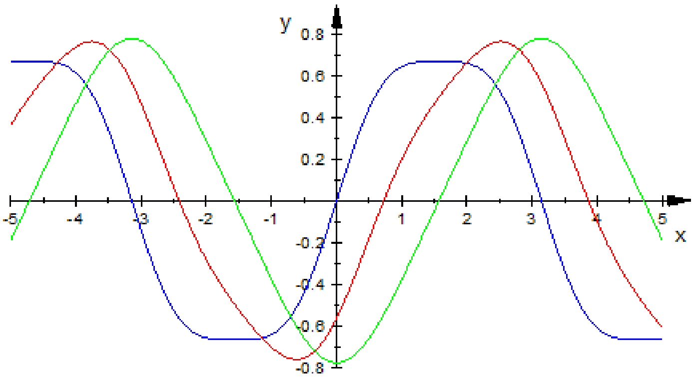
$$x \rightarrow \frac{1}{2^{2\cdot m}} \cdot \left(\sum_{r=0}^{m} \frac{\binom{2\cdot m+1}{r}}{(2\cdot m-2\cdot r+1)^{p}} \cdot \cos\left((2\cdot m-2\cdot r+1)\cdot x - \frac{\pi\cdot p}{2} \right) \right)$$

2nd order Integral

- `f2 := x-> int(int(cos(t)^(2*m+1) ,t=a1..u),u=a2..x)`

$$x \rightarrow \int_{a2}^{x} \int_{a1}^{u} \cos(t)^{2\cdot m+1} \, dt \, du$$

Blue: 1st order, Red: 1.5th order, Green: 2nd order



21.2 Super Derivatives of the product of many functions

(1) Generalized binomial theorem and Leibniz Rule about Super Differentiation

According to the generalized binomial theorem in **3.2** , the following expression holds for the real numbers x_1, x_2 such that $|x_1| > |x_2|$ and a positive number p .

$$(x_1+x_2)^p = \sum_{r=0}^{\infty} \binom{p}{r} x_1^{p-r} x_2^r$$

On the other hand, according to Formula 19.1.1 in **19.1** , the following expression holds for the functions f_1, f_2 of x and a positive number p .

$$(f_1 f_2)^{(p)} = \sum_{r=0}^{m-1} \binom{p}{r} f_1^{(p-r)} f_2^{(r)} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{-p-1}{k} \left(f_1^{<m+k>} f_2^{(m+k)} \right)^{(p)}$$

(2) Generalized multinomial theorem and Super Derivative of the product of many functions

According to the generalized multinomial theorem in **3.4** , the following expression holds for real numbers $x_1, x_2, \cdots, x_\lambda$ such that $|x_1| > |x_2 + x_3 + \cdots + x_\lambda|$ and a positive number p .

$$(x_1+x_2+\cdots+x_\lambda)^p = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} x_1^{p-r_1} x_2^{r_1-r_2} \cdots x_\lambda^{r_{\lambda-1}}$$

Therefore, the following expression must hold for functions $f_1, f_2, \cdots, f_\lambda$ of x and a positive number p .

$$(f_1 f_2 \cdots f_\lambda)^{(p)} = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{(p-r_1)} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^p$$

21.2.1 Super Derivative of the product of many functions

Theorem 21.2.1

Let p, r are positive numbers, m be a natural number, $f_k^{(r)}$ be the r th order derivative function of $f_k(x)$ $(k=1,2,\cdots,\lambda)$, $f_k^{<r>}$ be arbitrary r th order primitive function of $f_k(x)$ and $B(p,m)$ be the beta function.
At this time, if there is a number a such that

$$f_1^{<r>}(a) = 0 \quad r \in [0, m-p] \quad or \quad f_k^{(s)}(a) = 0 \quad s \in [0, m-p-1] \text{ for at least one } k > 1,$$

the following expression holds

$$(f_1 f_2 \cdots f_\lambda)^{(p)} = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{(p-r_1)} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{1}{m+k_1} \binom{-p-1}{k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$
$$\times \left\{ f_1^{\langle m+k_1 \rangle} f_2^{(m+k_1-k_2)} f_3^{(k_2-k_3)} \cdots f_\lambda^{(k_{\lambda-1})} \right\}^{(p)}$$

Proof

Reversing the sign of the index of the integration operator $<p>$ in Formula 21.1.1 ,

$$\int_a^x \sim \int_a^x f_1 f_2 \cdots f_\lambda dx^{-p} = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{\langle -p+r_1 \rangle} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^{-p}$$

$$R_m^{-p} = \frac{(-1)^m}{B(-p,m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{1}{m+k_1} \binom{-p-1}{k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$

$$\times \int_a^x \sim \int_a^x f_1^{\langle m+k_1 \rangle} f_2^{(m+k_1-k_2)} f_3^{(k_2-k_3)} \cdots f_\lambda^{(k_{\lambda-1})} dx^{-p}$$

Then, replacing the integration operator $<-p>$ with the differentiation operator (p), we obtain the desired expression.

Example computation of the product of three functions

Example1 Super Derivative of $x^\alpha e^x \sin x$

Let $f_1 = x^\alpha$, $f_2 = e^x$, $f_3 = \sin x$, then

$$\left(x^\alpha\right)^{(p-r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r} \quad , \quad \left(x^\alpha\right)^{<m+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r}$$

$$\left(e^x\right)^{(r-s)} = \left(e^x\right)^{(m+r-s)} = e^x \quad , \quad (\sin x)^{(s)} = \sin\left(x + \frac{s\pi}{2}\right)$$

Substituting these for Theorem 21.2.1 , we obtain

$$\left(x^\alpha e^x \sin x\right)^{(p)} = \sum_{r=0}^{m-1} \sum_{s=0}^{r} \binom{p}{r} \binom{r}{s} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r} e^x \sin\left(x + \frac{s\pi}{2}\right) + R_m^p \tag{1.1}$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{r=0}^{\infty} \sum_{s=0}^{m+r} \frac{1}{m+r} \binom{-p-1}{r} \binom{m+r}{s} \times \left\{ \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} e^x \sin\left(x + \frac{s\pi}{2}\right) \right\}^{(p)} \tag{1.1r}$$

When $\alpha=3/2, p=1/2$, the values of the both sides on arbitrary point $x=1.4$ are asfollows.

```
a = 3 / 2; p = 1 / 2; m = 10;
```

$$\texttt{f1[x\_] := } \frac{1}{\texttt{Gamma[1 - p]}} \int_{-\infty}^{x} (x - t)^{1-p-1} \partial_t \left(t^a e^t \texttt{Sin[t]}\right) dt$$

$$\texttt{fr[x\_] := } \sum_{r=0}^{m-1} \sum_{s=0}^{r} \texttt{Binomial[p, r] Binomial[r, s]} \frac{\texttt{Gamma[1 + a]}}{\texttt{Gamma[1 + a - p + r]}} x^{a-p+r} e^x \texttt{Sin}\left[x + \frac{s\pi}{2}\right]$$

```
N[f1[1.4]]                 N[fr[1.4]]
10.327 - 0.0369064 i       10.327
```

Example2 Super Derivative of $x^\alpha e^x \log x$

Let $f_1 = x^\alpha$, $f_2 = e^x$, $f_3 = \log x$, then

$$\left(x^\alpha\right)^{(p-r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r} \quad , \quad \left(x^\alpha\right)^{<m+r>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r}$$

$$\left(e^x\right)^{(r-s)} = e^x \quad , \quad \left(e^x\right)^{(m+r-s)} = e^x$$

$$(\log x)^{(0)} = \log x \quad (s=0) \qquad , \qquad (\log x)^{(s)} = (-1)^{s-1}(s-1)!\,x^{-s} \quad (s \neq 0)$$

Separating the terms containing $f_3^{(0)}$ from Theorem21.2.1 and substituting these for it,

$$\left(x^{\alpha} e^{x} \log x\right)^{(p)} = e^{x} \log x \sum_{r=0}^{m-1} \binom{p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r}$$
$$- e^{x} \sum_{r=1}^{m-1} \sum_{s=1}^{r} (-1)^{s} \binom{p}{r} \binom{r}{s} \frac{\Gamma(1+\alpha)\Gamma(s)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r-s} + R_m^p \qquad \textbf{(1.2)}$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{r=0}^{\infty} \frac{1}{m+r} \binom{-p-1}{r} \left\{ \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} e^{x} \log x \right\}^{(p)}$$
$$- \frac{(-1)^m}{B(-p,m)} \sum_{r=0}^{\infty} \sum_{s=1}^{m+r} \frac{(-1)^s}{m+r} \binom{-p-1}{r} \binom{m+r}{s} \left\{ \frac{\Gamma(1+\alpha)\Gamma(s)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r-s} e^{x} \right\}^{(p)}$$

(1.2r)

Polynomials in (1.2) seem to be asymptotic expansion. when $\alpha=3/2, p=1/2$, the values of the both sides on arbitrary point $x=16$ are as followas.

```
a = 3 / 2; p = 1 / 2; m = 50;
fl[x_] := 1/Gamma[1 - p] ∫_{-∞}^{x} (x - t)^(1-p-1) ∂_t (t^a e^t Log[t]) dt
fr[x_] := e^x Log[x] Sum_{r=0}^{m-1} Binomial[p, r] Gamma[1 + a]/Gamma[1 + a - p + r] x^(a-p+r) -
    e^x Sum_{r=1}^{m-1} Sum_{s=1}^{r} (-1)^s Binomial[p, r] Binomial[r, s] Gamma[1 + a] Gamma[s]/Gamma[1 + a - p + r] x^(a-p+r-s)
N[fl[16]]                          N[fr[16]]
1.66734 × 10^9 - 0.0658101 i       1.66734 × 10^9
```

21.2.2 Super Derivative of the power of a function

Especially, if $f_1 = f_2 = \cdots = f_\lambda$ in Theorem 21.2.1, the following theorem holds immediately.

Theorem 21.2.2

Let p, r are positive numbers, m be a natural number, $f^{(r)}$ be the r th order derivative function of $f(x)$, $f^{<r>}$ be arbitrary r th order primitive function of $f(x)$ and $B(p,m)$ be the beta function. At this time, if there is a number a such that

$$f^{<r>}(a) = 0 \quad r \in [0, m-p] \quad or \quad f^{(s)}(a) = 0 \quad s \in [0, m-p-1]$$

the following expression holds for $\lambda = 2, 3, 4, \cdots$.

$$\left(f^{\lambda}\right)^{(p)} = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f^{\langle p+r_1 \rangle} f^{(r_1-r_2)} \cdots f^{(r_{\lambda-1})} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{1}{m+k_1} \binom{-p-1}{k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$
$$\times \left\{ f^{\langle m+k_1 \rangle} f^{(m+k_1-k_2)} f^{(k_2-k_3)} \cdots f^{(k_{\lambda-1})} \right\}^{(p)}$$

Example Super Derivative of $\log^3 x$

Let $f = \log x$, then

$$(\log x)^{(p-r)} = \frac{\log x - \psi(1-p+r) - \gamma}{\Gamma(1-p+r)} x^{-p+r} \ , \quad (\log x)^{<m+r>} = \frac{\log x - \psi(1+m+r) - \gamma}{\Gamma(1+m+r)} x^{m+r}$$

$$(\log x)^{(r-s)} = \log x \quad (r=s) \ , \qquad (\log x)^{(r-s)} = (-1)^{r-s-1}(r-s-1)!\, x^{-r+s} \quad (r \neq s)$$

$$(\log x)^{(0)} = \log x \quad (s=0) \ , \qquad (\log x)^{(s)} = (-1)^{s-1}(s-1)!\, x^{-s} \quad (r \neq 0)$$

Separating the terms containing $f_3^{(0)}$ from Theorem21.2.2 and substituting these for it , we obtain

$$\begin{aligned}\{(\log x)^3\}^{(p)} = &\frac{\log x - \psi(1-p) - \gamma}{\Gamma(1-p)} x^{-p} (\log x)^2 \\ &- 2x^{-p} \log x \sum_{r=1}^{m-1} (-1)^r \binom{p}{r} \frac{\log x - \psi(1-p+r) - \gamma}{\Gamma(1-p+r)} \Gamma(r) \\ &+ x^{-p} \sum_{r=2}^{m-1} \sum_{s=1}^{r-1} (-1)^r \binom{p}{r} \binom{r}{s} \frac{\log x - \psi(1-p+r) - \gamma}{\Gamma(1-p+r)} \Gamma(r-s)\Gamma(s) \\ &+ R_m^p \end{aligned} \qquad \textbf{(2.1)}$$

$$\begin{aligned} R_m^p = &-\frac{1}{B(-p,m)} \sum_{r=0}^{\infty} \frac{(-1)^r 2}{(m+r)^2} \binom{-p-1}{r} \{\{\log x - \psi(1+m+r) - \gamma\} \log x\}^{(p)} \\ &+ \frac{1}{B(-p,m)} \sum_{r=0}^{\infty} \sum_{s=1}^{m+r-1} \frac{(-1)^r}{m+r} \binom{-p-1}{r} \binom{m+r}{s} \\ &\times \left\{ \frac{\log x - \psi(1+m+r) - \gamma}{\Gamma(1+m+r)} \Gamma(m+r-s)\Gamma(s) \right\}^{(p)} \end{aligned} \qquad \textbf{(2.1r)}$$

And $\lim_{m \to \infty} R_m^n = 0$ holds although the proof is difficult. Then

$$\begin{aligned}\{(\log x)^3\}^{(p)} = &\frac{\log x - \psi(1-p) - \gamma}{\Gamma(1-p)} x^{-p} (\log x)^2 \\ &- 2x^{-p} \log x \sum_{r=1}^{\infty} (-1)^r \binom{p}{r} \frac{\log x - \psi(1-p+r) - \gamma}{\Gamma(1-p+r)} \Gamma(r) \\ &+ x^{-p} \sum_{r=2}^{\infty} \sum_{s=1}^{r-1} (-1)^r \binom{p}{r} \binom{r}{s} \frac{\log x - \psi(1-p+r) - \gamma}{\Gamma(1-p+r)} \Gamma(r-s)\Gamma(s) \end{aligned}$$

(2.1')

However, the convergence speed is very slow.

When $p=1/2, m=120$, the values of the both sides on arbitrary point $x=4$ are as follows. Even if calculated so far, both sides are corresponding only to 1 digit below the decimal point.

```
p = 1 / 2;
```

Riemann-Liouville differintegral

$$\texttt{f[x\_] :=} \frac{1}{\texttt{Gamma[1 - p]}} \int_0^x (x-t)^{1-p-1} \texttt{Log[t]}^3 \, dt \qquad \texttt{f1 = Limit}\left[\frac{\texttt{f[4 + h] - f[4]}}{\texttt{h}}, \texttt{h} \to 0\right];$$

Series

```
m = 120;
```

$$\texttt{fr[x\_] :=} \frac{\texttt{Log[x] - PolyGamma[1 - p] - EulerGamma}}{\texttt{Gamma[1 - p]}} x^{-p}\,\texttt{Log[x]}^2 -$$

$$2\, x^{-p}\,\texttt{Log[x]} \sum_{r=1}^{m-1} (-1)^r\,\texttt{Binomial[p, r]} \frac{\texttt{Log[x] - PolyGamma[1 - p + r] - EulerGamma}}{\texttt{Gamma[1 - p + r]}}\,\texttt{Gamma[r]} +$$

$$x^{-p} \sum_{r=2}^{m-1} \sum_{s=1}^{r-1} (-1)^r\,\texttt{Binomial[p, r] Binomial[r, s]} \frac{\texttt{Log[x] - PolyGamma[1 - p + r] - EulerGamma}}{\texttt{Gamma[1 - p + r]}}$$

$$\times\,\texttt{Gamma[r - s] Gamma[s]}$$

```
N[f1]          N[fr[4]]
2.36224        2.30139
```

21.2.3 Super Derivatives of $\cos^m x$, $\sin^m x$

Formula 21.2.3

When m is a natural number, p is a positive number and $\downarrow$ is the floor function,

$$\left(\cos^m x\right)^{(p)} = \frac{1}{2^{m-1}} \sum_{r=0}^{m/2\downarrow} {}_mC_r\,(m-2r)^p \cos\left\{ (m-2r)x + \frac{p\pi}{2} \right\} \tag{3.c}$$

$$\left(\sin^m x\right)^{(p)} = \frac{1}{2^{m-1}} \sum_{r=0}^{m/2\downarrow} {}_mC_r\,(m-2r)^p \cos\left\{ (m-2r)\left(x - \frac{\pi}{2}\right) + \frac{p\pi}{2} \right\} \tag{3.s}$$

Proof

Analytically continuing the index of the differentiation operator in Formula 20.1.3 in **20.1** to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Example The 2.5th order derivative of $\cos^4 x$

- m:=4:

2nd order derivative

- f2 := $x \to \frac{\partial^2}{\partial x^2}\cos(x)^m$

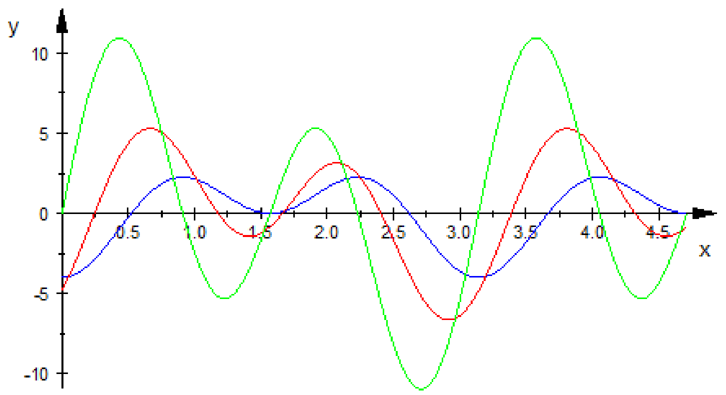
2.5th order derivative

- p:=2.5:
- fh := $x \to \frac{1}{2^{m-1}} \cdot \left(\sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{r} \cdot (m-2\cdot r)^p \cdot \cos\left((m-2\cdot r)\cdot x + \frac{\pi \cdot p}{2}\right)\right)$

3rd order derivative

- f3 := $x \to \frac{\partial^3}{\partial x^3}\cos(x)^m$

Blue: 2nd orderA Red: 2.5th orderA Green: 3rd order



Collateral Super Derivatives of $\cos^m x$, $\sin^m x$

Unlike the higher derivative, in the super derivative, the lineal super derivative and the collateral one exist. Although this was described in 12.1.4 , I describe it once now.

For example, if we differenciate $\cos^3 x$ with respect to x p times according to Theorem 21.2.2 , it is as follows.

$$\left(\cos^3 x\right)^{(p)} = \sum_{r=0}^{m-1}\sum_{s=0}^{r}\binom{p}{r}\binom{r}{s}\cos\left\{x+\frac{(p-r)\pi}{2}\right\}\cos\left\{x+\frac{(r-s)\pi}{2}\right\}\cos\left(x+\frac{s\pi}{2}\right) + R_m^p \qquad \textbf{(4.c')}$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)}\sum_{r=0}^{\infty}\sum_{s=0}^{m+r}\frac{1}{m+r}\binom{-p-1}{r}\binom{m+r}{s} \times\left\{\cos\left\{x-\frac{(m+r)\pi}{2}\right\}\cos\left\{x+\frac{(m+r-s)\pi}{2}\right\}\cos\left(x+\frac{s\pi}{2}\right)\right\}^{(p)} \qquad \text{(4.cr')}$$

Then, this is a collateral super derivative. If it is why, this theorem was drawn out as a reverse-operation of the super integral of the product of many functions as seen during the proof of Theorem 21.2.1 . And the base was the super integral with a fixed lower limit.

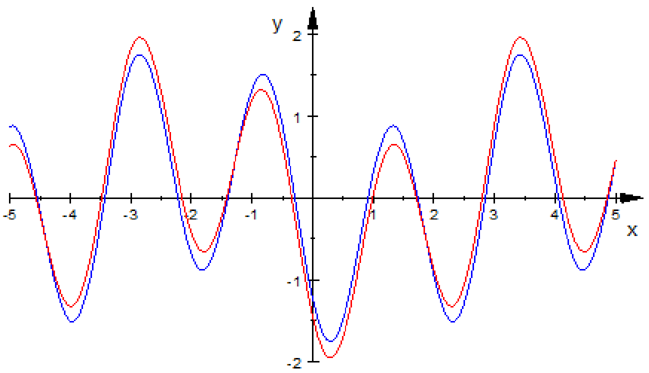
However, the lineal super integral of $\cos^3 x$ is one with a variable lower limit. Therefore, (4.c') and (4.cr') derived based on the fixed lower limit cannot be the lineal super derivative. Althogh this holds as a equation, the polynomial of (4.c') is not well behaved.

By reference, let us compare (4.c') with the following lineal super derivative (4.c) derived from Formula 21.2.3 .

$$\left(\cos^3 x\right)^{(p)} = \frac{3}{4}\cos\left(x+\frac{p\pi}{2}\right) + \frac{3^p}{4}\cos\left(3x+\frac{p\pi}{2}\right) \qquad \textbf{(4.c)}$$

Althogh (4.c') fits (4.c) most at the time of $m=3$, a big difference is still seen by both.

Blue: Collateral , Red: Lin eal



However, since $1/B(-p,m)=0$ at $p=m-1$, $m=1,2,3,\cdots$, $R_m^p = 0$. Therefore

$$\left(\cos^3 x\right)^{(m-1)} = \sum_{r=0}^{m-1}\sum_{s=0}^{r}\binom{m-1}{r}\binom{r}{s}\cos\left\{x+\frac{(m-1-r)\pi}{2}\right\}\cos\left\{x+\frac{(r-s)\pi}{2}\right\}\cos\left(x+\frac{s\pi}{2}\right)$$

Furthermore, replacing $m-1$ with n ,

$$\left(\cos^3 x\right)^{(n)} = \sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}\cos\left\{x+\frac{(n-r)\pi}{2}\right\}\cos\left\{x+\frac{(r-s)\pi}{2}\right\}\cos\left(x+\frac{s\pi}{2}\right)$$

And this results in the following lineal super derivative. (The proof is long, then omitted.)

$$\left(\cos^3 x\right)^{(n)} = \frac{3}{4}\cos\left(x+\frac{n\pi}{2}\right) + \frac{3^n}{4}\cos\left(3x+\frac{n\pi}{2}\right)$$

2010.12.25

K. Kono

Alien's Mathematics