21 Super Calculus of the product of many functions

21.1 Super Integrals of the product of many functions

(1) Generalized binomial theorem and Super integral of the product of 2 functions

According to the generalized binomial theorem in [3,4], the following expression holds for the real numbers $x_1, x_2$ such that $|x_1| > |x_2|$ and a positive number $p$.

\[
(x_1 + x_2)^{-p} = \sum_{r=0}^{\infty} \binom{-p}{r} x_1^{p-r} x_2^r.
\]

On the other hand, according to Formula 17.1.2 in [18], the following expression holds for the functions $f_1, f_2$ of $x$ and a positive number $p$.

\[
\int_a^x \int_a^x (f_1 f_2)^{<p>} \, dx^2 = \sum_{r=0}^{m-1} \binom{-p}{r} f_1^{<p+r>} f_2^{(r)} + R_m^p
\]

\[
R_m^p = \frac{(-1)^m}{B(p, m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \left( \frac{p-1}{k} \right) \int_a^x \int_a^x f_1^{<p+m+k>} f_2^{(m+k)} \, dx^p
\]

Reversing the sign of the index of the differentiation operator $(p)$ in the Leibniz Rule about the Super Differentiation in [19.1],

\[
(f_1 f_2)^{(-p)} = \sum_{r=0}^{m-1} \binom{-p}{r} f_1^{(-p-r)} f_2^{(r)} + R_m^{-p}
\]

\[
R_m^{-p} = \frac{(-1)^m}{B(p, m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \left( \frac{p-1}{k} \right) \left( f_1^{<p+m+k>} f_2^{(m+k)} \right)^{(-p)}
\]

And, replacing $(-p)$ with the integration operator $<p>$, we obtain the above formula.

(2) Generalized multinomial theorem and Super integral of the product of many functions

According to the generalized multinomial theorem in [3,4], the following expression holds for real numbers $x_1, x_2, \ldots, x_\lambda$ such that $|x_1| > |x_2 + x_3 + \ldots + x_\lambda|$ and a positive number $p$.

\[
(x_1 + x_2 + \ldots + x_\lambda)^{-p} = \sum_{r_1=0}^{r} \sum_{r_2=0}^{r} \ldots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-p}{r_1} \binom{r_1}{r_2} \ldots \binom{r_{\lambda-2}}{r_{\lambda-1}} x_1^{p-r_1} x_2^{r_1-r_2} \ldots x_\lambda^{r_{\lambda-1}}
\]

Therefore, the following expression must hold for functions $f_1, f_2, \ldots, f_\lambda$ of $x$ and a positive number $p$.

\[
\int_a^x \int_a^x f_1 f_2 \ldots f_\lambda \, dx^p = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \ldots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-p}{r_1} \binom{r_1}{r_2} \ldots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{(p+r_1)} f_2^{(r_1-r_2)} \ldots f_\lambda^{(r_{\lambda-1})} + R_m^p
\]

21.1.1 Super Integral of the product of many functions

**Theorem 21.1.1**

Let $p, r$ be positive numbers, $m$ be a natural number, $f_k^{(r)}$ be the $r$th order derivative function of $f_k(x)$ $(k=1, 2, \ldots, \lambda)$, $f_k^{(r)}$ be arbitrary $r$th order primitive function of $f_k(x)$ and $B(p, m)$ be the beta function. At this time, if there is a number $a$ such that

\[
f_k^{(r)}(a) = 0 \quad r \in [0, m+p] \quad \text{or} \quad f_k^{(s)}(a) = 0 \quad s \in [0, m+p-1]
\]

for at least one $k > 1$, the following expression holds
\[
\int_a^x \int_a^x f_1 f_2 \cdots f_n \,dx^n = \sum_{r_1=0}^{m-1} \cdots \sum_{r_n=0}^{m-1} \left( \frac{n-p}{r_1} \right) \left( \frac{r_1}{r_2} \right) \cdots \left( \frac{r_{n-2}}{r_{n-1}} \right) f_1^{p+r_1-1} f_2^{r_1-r_2} \cdots f_n^{r_{n-1}}
\]

\[
R_m^p = \frac{(-1)^m}{B(p, m)} \sum_{k_1=0}^{m+k_1} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \frac{1}{m+k_1} \left( \frac{p-1}{k_1} \right) \left( \frac{m+k_1}{k_2} \right) \left( \frac{k_2}{k_3} \right) \left( \frac{k_3}{k_{n-1}} \right)
\]

\[
\times \int_{a}^{x} \int_{a}^{x} f_1^{(m+k_1)} f_2^{(m+k_1-k_2)} f_3^{(m+k_1-k_2-k_3)} \cdots f_n^{(k_{n-1})} \,dx^n
\]

Proof

Analytically continuing the index of the integration operator in Formula 20.2.1 in \[20.2\] to \([0, p]\) from \([1, n]\) we obtain the desired expression.

Example computation of the product of three functions

Example 1: Super Integral of \(x^\alpha e^x \sin x\)

Since the zero of the higher integral of \(x^\alpha e^x \sin x\) is \(x=-\infty\), the zero of the super integral is also the same. Then let \(f_1 = x^\alpha\), \(f_2 = e^x\), \(f_3 = \sin x\)

\[
\begin{align*}
(x^\alpha)^{<p+r>} &= \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r}, \\
(x^\alpha)^{<m+r>} &= \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} \\
(e^x)^{<r-s>} &= (e^x)^{(m+r-s)} = e^x, \\
(sin x)^{(s)} &= \sin \left( x + \frac{s\pi}{2} \right)
\end{align*}
\]

Substituting these for Theorem 21.1.1, we obtain

\[
\int_{-\infty}^{x} \int_{-\infty}^{x} x^\alpha e^x \sin x \,dx^n
\]

\[
= \sum_{r=0}^{m-1} \sum_{s=0}^{r} \left( \frac{n-p}{r} \right) \left( \frac{r}{s} \right) \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} e^x \sin \left( x + \frac{s\pi}{2} \right) + R_m^p \quad (1.1)
\]

\[
R_m^p = \frac{(-1)^m}{B(p, m)} \sum_{r=0}^{m-1} \sum_{s=0}^{r+s} \frac{1}{m+r} \left( \frac{p-1}{r} \right) \left( \frac{m+r}{s} \right)
\]

\[
\times \int_{-\infty}^{x} \int_{-\infty}^{x} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} e^x \sin \left( x + \frac{s\pi}{2} \right) \,dx^n
\]

As seen in Theorem 20.2.1 in \[20.2\], this \(R_m^p\) is not convergent on \(0\) at the time of \(m \to \infty\). That is, this polynomial is an asymptotic expansion. When \(\alpha=3\), \(p=5/2\), the values of the both sides on arbitrary point \(x=20\) are as follows.

\[
a = 3, \quad p = 5/2, \quad m = 100;
\]

\[
\mathfrak{f}l[x_] := \frac{1}{\Gamma[\alpha]} \int_{-\infty}^{\infty} (x - t)^{\alpha-1} e^t \sin t \,dt
\]

\[
\mathfrak{f}r[x_] := \sum_{r=0}^{m-1} \sum_{s=0}^{r} \text{Binomial}[\alpha-p, x] \text{Binomial}[x, s] \frac{\Gamma[\alpha+1]}{\Gamma[1+\alpha+p+r]} x^{\alpha+p+r} e^x \sin \left( x + \frac{s\pi}{2} \right)
\]
Example 2 Super Integral of $x^\alpha e^x \log x$

Since the zero of the higher integral of $x^\alpha e^x \log x$ is $x=-\infty$, the zero of the super integral is also the same. Then let $f_1=x^\alpha$, $f_2=e^x$, $f_3=\log x$

\[
\left(x^\alpha\right)^{(p+r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r}, \quad \left(x^\alpha\right)^{(m+r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r}
\]

\[
\left(e^x\right)^{(r-s)} = e^x, \quad \left(e^x\right)^{(m+r-s)} = e^x
\]

\[
\left(\log x\right)^{(0)} = \log x \quad (s=0), \quad \left(\log x\right)^{(s)} = (-1)^{s-1} (s-1)! x^{-s} \quad (s \neq 0)
\]

Separating the terms containing $f_3^{(0)}$ from Theorem 21.1.1 and substituting these for it,

\[
\int_{-\infty}^{x} \int_{-\infty}^{x} x^\alpha e^x \log x \, dx \, dp = e^x \log x \sum_{r=0}^{m-1} \left( \begin{array}{c} p+r \\ r \end{array} \right) \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} \\
- \sum_{r=1}^{m} \sum_{s=1}^{r} (-1)^s \left( \begin{array}{c} p+r \\ r \end{array} \right) \left( \begin{array}{c} r \\ s \end{array} \right) \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r-s} e^x + R_m^p
\]

\[
R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{r=0}^{\infty} \frac{1}{m+r} \left( \begin{array}{c} p-1 \\ r \end{array} \right) \left( \begin{array}{c} m+r \\ r \end{array} \right) \int_{-\infty}^{x} \int_{-\infty}^{x} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r-s} e^x \, dx \, dp
\]

\[
+ \frac{(-1)^m}{B(p,m)} \sum_{r=0}^{\infty} \frac{1}{m+r} \sum_{s=1}^{r} (-1)^s \left( \begin{array}{c} p-1 \\ r \end{array} \right) \left( \begin{array}{c} m+r \\ s \end{array} \right) \int_{-\infty}^{x} \int_{-\infty}^{x} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r-s} e^x \, dx \, dp
\]

When $\alpha=3/2$, $p=2.7$, the values of the both sides on arbitrary point $x=17$ are as follows. This also seems to be asymptotic expansion.

\[
a = 3/2; \quad p = 2.7; \quad m = 55;
\]

\[
\text{fl}[x_] := \text{Gamma}[p] \int_{-\infty}^{x} (x-t)^{p-1} e^t \log[t] \, dt
\]

\[
\text{fr}[x_] := \text{Gamma}[1+a] \text{Gamma}[1+a+p+x] \left( \begin{array}{c} m-1 \\ r \end{array} \right) \frac{\text{Binomial}[-p, r]}{\text{Gamma}[1+a+p+x]} \frac{\text{Binomial}[a \text{ Gamma}[1+a \text{ Gamma}[s]]]}{\text{Gamma}[1+a+p+x]} x^{a+p+r-s} e^x
\]

\[
N[\text{fl}[17]] \quad N[\text{fr}[17]]
\]

\[
3.50968 \times 10^9 - 103.255 \quad 3.50968 \times 10^9
\]

21.1.2 Super Integral of the power of a function

Especially, if $f_1=f_2=\cdots=f_\lambda$ in Theorem 21.1.1, the following theorem holds immediately.

Theorem 21.1.2

Let $p, r$ be positive numbers, $m$ be a natural number, $f^{(r)}$ be the $r$ th order derivative function of $f(x)$, $f^{(0)}$ be arbitrary $r$ th order primitive function of $f(x)$ and $B(p,m)$ be the beta function. At this time, if there is a number $a$ such that

\[
f^{(r)}(a) = 0 \quad r \in [0,m+p] \quad \text{or} \quad f^{(s)}(a) = 0 \quad s \in [0,m+p-1]
\]
the following expression holds for \( \lambda = 2, 3, 4, \ldots \).

\[
\int_a^x \int_a^x f^\lambda dx^p = \sum_{r_1=0}^{r_2} r_1 \left( \frac{-p}{r_1} \right) f^{(p+r_1)}(r_1) \cdots f^{(r_{\lambda-2})}(r_{\lambda-2}) \right] \right) + R_m^p
\]

\[
R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{1}{m+k_1} \left( \frac{p-1}{k_1} \right) \left( \frac{m+k_1}{k_2} \right) \left( \frac{k_2}{k_3} \right) \cdots \left( \frac{k_{\lambda-2}}{k_{\lambda-1}} \right) \times \int_a^x \int_a^x f^{(m+k_1)}(r_1) \cdots f^{(k_{\lambda-1})}(r_{\lambda-1}) \right] \right) dx^p
\]

**Example** Super Integral of \( \log^3 x \)

Since the zero of the higher integral of \( \log^3 x \) is \( x = 0 \), the zero of the super integral is also the same. Then let \( f = \log x \)

\[
(\log x)^{<p+r>} = \frac{\log x - \psi(1+p+r) \cdot \gamma}{\Gamma(1+p+r)} x^{p+r}, \quad (\log x)^{<m+r>} = \frac{\log x - \psi(1+m+r) \cdot \gamma}{\Gamma(1+m+r)} x^{m+r}
\]

\[
(\log x)^{(r-s)} = \log x \quad (r=s), \quad (\log x)^{(r-s)} = (-1)^{r-s-1} (r-s-1)! x^{-r+s} \quad (r \neq s)
\]

\[
(\log x)^{(0)} = \log x \quad (s=0), \quad (\log x)^{(s)} = (-1)^{s-1} (s-1)! x^{-s} \quad (r \neq 0)
\]

Separating the terms containing \( f^{(0)} \) from Theorem 21.1.2 and substituting these for it,

\[
\int_0^x \int_0^x (\log x)^3 dx^p = \frac{\log x - \psi(1+p+r) \cdot \gamma}{\Gamma(1+p)} x^p (\log x)^2
\]

\[
- 2x^p \log x \sum_{r=1}^{m-1} (-1)^r \left( \frac{-p}{r} \right) \frac{\log x - \psi(1+p+r) \cdot \gamma}{\Gamma(1+p+r)} \Gamma(r)
\]

\[
+ x^p \sum_{r=2}^{m-1} \sum_{s=1}^{r-1} (-1)^r \left( \frac{-p}{r} \right) \left( \frac{s}{r} \right) \frac{\log x - \psi(1+p+r) \cdot \gamma}{\Gamma(1+p+r)} \Gamma(r-s) \Gamma(s)
\]

\[
+ R_m^p
\]

\[
R_m^p = \frac{1}{B(p,m)} \sum_{r=0}^{\infty} \sum_{s=0}^{m+r} \left( (-1)^{r-s} \right) \left( \frac{p-1}{r} \right) \left( \frac{m+r}{s} \right) \times \int_0^x \int_0^x \frac{\log x - \psi(1+m+r) \cdot \gamma}{\Gamma(1+m+r)} \Gamma(m+r-s) x^s dx^p
\]

And \( \lim_{m \to \infty} R_m^p = 0 \) holds although the proof is difficult. Then

\[
\int_0^x \int_0^x (\log x)^3 dx^p = \frac{\log x - \psi(1+p+r) \cdot \gamma}{\Gamma(1+p)} x^p (\log x)^2
\]

\[
- 2x^p \log x \sum_{r=1}^{m-1} (-1)^r \left( \frac{-p}{r} \right) \frac{\log x - \psi(1+p+r) \cdot \gamma}{\Gamma(1+p+r)} \Gamma(r)
\]

\[
+ x^p \sum_{r=2}^{m-1} \sum_{s=1}^{r-1} (-1)^r \left( \frac{-p}{r} \right) \left( \frac{s}{r} \right) \frac{\log x - \psi(1+p+r) \cdot \gamma}{\Gamma(1+p+r)} \Gamma(r-s) \Gamma(s)
\]

\[
(2.1')
\]

However, the convergence speed is very slow.

When \( p = 3/2, m = 400 \), the values of the both sides on arbitrary point \( x = 0.2 \) are as follows. Even if calculated so far, both sides are corresponding only to 1 digit below the decimal point. Although (2.1') is not suitable for the calculation, it is meaningful that the integral can be expressed by the double series.

\[
p = 3/2; \quad m = 400;
\]
21.1.3 Super Integrals of $\cos^m x$, $\sin^m x$

**Formula 21.1.3**

Let $m$ be a natural number, $p$ be a positive number and $a(s) \in [0, p]$ be the zero of the lineal super primitive function of $\cos^{2m+1}x$ or $\sin^{2m+1}x$. Then the following expressions hold.

\[
\int_a^{a(s)} \cos^{2m+1}x \, dx = \frac{1}{2^{2m}} \sum_{r=0}^{m} \frac{2m+1}{(2m-2r+1)^p} \cos \left( (2m-2r+1)x - \frac{p\pi}{2} \right) \quad (3.c)
\]

\[
\int_a^{a(s)} \sin^{2m+1}x \, dx = \frac{1}{2^{2m}} \sum_{r=0}^{m} \frac{(-1)^m}{(2m-2r+1)^p} \sin \left( (2m-2r+1)x - \frac{p\pi}{2} \right) \quad (3.s)
\]

**Proof**

Analytically continuing the index of the integration operator in Formula 20.2.3c (1) and Formula 20.2.3s (1) in $[0, p]$ to $[1, n]$, we obtain the desired expressions.

**Note**

For example, in the case of $\cos^{2m+1}x$, $a(s)$ is a solution of the following transcendental equation.

\[
\sum_{r=0}^{m} \frac{2m+1}{(2m-2r+1)^s} \cos \left( (2m-2r+1)x - \frac{s\pi}{2} \right) = 0 \quad s \in [0, p]
\]

Although it is difficult to calculate this solution, the necessity does not exist in this formula.

**Example**  The 1.5th order integral of $\cos^3 x$

- $m := 1$; $a1 := \pi/2$; $a2 := 3\pi/2$;
- 1st order integral
- $f1 := x \to \int (\cos(t)^{2m+1}, t=a1..x)$
  \[
x \to \int_{a1}^{x} \cos(t)^{2m+1} \, dt
\]

1.5th order Integral
- $p := 3/2$;
- $f1 := x \to 1/2^{(2m)} \sum (\text{binomial}(2m+1, r)/(2m-2r+1)^p \cos((2m-2r+1)x - p\times\pi/2), r=0..m)$
\[ x \rightarrow \frac{1}{2^m} \cdot \left( \sum_{r=0}^{m} \frac{(2 \cdot m+1)}{r} \cos \left( (2 \cdot m - 2 \cdot r + 1) \cdot x - \frac{\pi \cdot p}{2} \right) \right) \]

2nd order Integral

\[ f2 := x \rightarrow \int \int (\cos(t)^{2 \cdot m+1} \cdot (t=a1..u), u=a2..x) \]

\[ x \rightarrow \int_{a2}^{x} \int_{a1}^{u} \cos(t)^{2 \cdot m+1} \, dt \, du \]

Blue: 1st order, Red: 1.5th order, Green: 2nd order
21.2 Super Derivatives of the product of many functions

(1) Generalized binomial theorem and Leibniz Rule about Super Differentiation

According to the generalized binomial theorem in [3.2], the following expression holds for the real numbers \(x_1, x_2\) such that \(|x_1| > |x_2|\) and a positive number \(p\).

\[
(x_1 + x_2)^p = \sum_{r=0}^{\infty} \binom{p}{r} x_1^{p-r} x_2^r
\]

On the other hand, according to Formula 19.1.1 in [19.1], the following expression holds for the functions \(f_1, f_2\) of \(x\) and a positive number \(p\).

\[
(f_1 f_2)^{(p)} = \sum_{r=0}^{p-1} \binom{p}{r} f_1^{(p-r)} f_2^{(r)} + R_m^p
\]

\[
R_m^p = \frac{(-1)^m}{B(-p, m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{-p-1}{k} f_1^{(m+k)} f_2^{(m+k)}
\]

(2) Generalized multinomial theorem and Super Derivative of the product of many functions

According to the generalized multinomial theorem in [3.4], the following expression holds for real numbers \(x_1, x_2, \ldots, x_\lambda\) such that \(|x_1| > |x_2 + x_3 + \cdots + x_\lambda|\) and a positive number \(p\).

\[
(x_1 + x_2 + \cdots + x_\lambda)^p = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} x_1^{p-r_1} x_2^{r_1-r_2} \cdots x_\lambda^{r_{\lambda-1}}
\]

Therefore, the following expression must hold for functions \(f_1, f_2, \ldots, f_\lambda\) of \(x\) and a positive number \(p\).

\[
(f_1 f_2 \cdots f_\lambda)^{(p)} = \sum_{r_1=0}^{p-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{(p-r_1)} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^p
\]

21.2.1 Super Derivative of the product of many functions

**Theorem 21.2.1**

Let \(p, r\) be positive numbers, \(m\) be a natural number, \(f_k^{(r)}\) be the \(r\)th order derivative function of \(f_k(x)\) \((k=1, 2, \ldots, \lambda)\), \(f_k^{<r>}\) be arbitrary \(r\)th order primitive function of \(f_k(x)\) and \(B(p, m)\) be the beta function. At this time, if there is a number \(a\) such that \(f_k^{<r>} (a) = 0 \quad r \in [0, m-p] \quad or \quad f_k^{(s)} (a) = 0 \quad s \in [0, m-p-1] \quad for \ at \ least \ one \ k > 1\), the following expression holds

\[
(f_1 f_2 \cdots f_\lambda)^{(p)} = \sum_{r_1=0}^{p-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{(p-r_1)} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^p
\]

\[
R_m^p = \frac{(-1)^m}{B(-p, m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_{\lambda-1}=0}^{\infty} \frac{1}{m+k_1} \binom{-p-1}{k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}} f_1^{(m+k_1)} f_2^{(m+k_2)} f_3^{(m+k_3)} \cdots f_\lambda^{(m+k_{\lambda-1})}
\]

**Proof**

Reversing the sign of the index of the integration operator \(<p>\) in Formula 21.1.1.
\[
\int_a^x \int_a^x f_1 f_2 \ldots f_n \, dx = \sum_{r_1=0}^{m-1} \ldots \sum_{r_{k-2}=0}^{m-1} \left( p \right) \left( r_1 \right) \ldots \left( r_{k-2} \right) f_1^{(p+r_1)} f_2^{(r_1-r_2)} \ldots f_n^{(r_{k-1})} + R_m^{(p)}
\]

\[
R_m^{(p)} = \frac{(-1)^m}{B(-p, m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{1}{m+k_1} \left( \begin{array}{c}
-p-1 \\
m+k_1
\end{array} \right) f_1^{(m+k_1)} f_2^{(k_2)} \ldots f_n^{(k_{k-1})} \int_a^x \int_a^x f_1 f_2 \ldots f_n \, dx
\]

Then, replacing the integration operator \(<-p>\) with the differentiation operator \((p)\), we obtain the desired expression.

**Example computation of the product of three functions**

**Example 1** Super Derivative of \(x^\alpha e^x \sin x\)

Let \(f_1 = x^\alpha\), \(f_2 = e^x\), \(f_3 = \sin x\), then

\[
\left( x^\alpha \right)^{(p-r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r}, \quad \left( x^\alpha \right)^{(m+r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r}
\]

\[
\left( e^x \right)^{(r-s)} = \left( e^x \right)^{(m+r-s)} = e^x, \quad \left( \sin x \right)^{(s)} = \sin \left( x + \frac{s\pi}{2} \right)
\]

Substituting these for \(\text{Theorem } 21.2.1\), we obtain

\[
\left( x^\alpha e^x \sin x \right)^{(p)} = \sum_{s=0}^{m+r} \sum_{r=0}^{m-r} \left( p \right) \left( r \right) \left( s \right) \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r} e^x \sin \left( x + \frac{s\pi}{2} \right) + R_m^{(p)}
\]

\[
R_m^{(p)} = \frac{(-1)^m}{B(-p, m)} \sum_{r=0}^{m+r} \sum_{s=0}^{m+r} \frac{1}{m+r} \left( \begin{array}{c}
p-1 \\
m+r
\end{array} \right) \left( m+r \right) \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} e^x \sin \left( x + \frac{s\pi}{2} \right)
\]

When \(\alpha = \frac{3}{2}, p = \frac{1}{2}, m = 10\);

\[
\begin{align*}
\text{N}[\text{fl}[x_\cdot]] & := \frac{1}{\Gamma(1-p)} \int_0^\infty (x-t)^{1-p-1} e^t \sum_{s=0}^{m} \sin[t] \, dt \\
\text{N}[\text{fr}[x_\cdot]] & := \sum_{r=0}^{m-1} \sum_{s=0}^{r} \frac{\Gamma[1+a]}{\Gamma[1+a-p+r]} x^{a-p+r} e^x \sin \left( x + \frac{s\pi}{2} \right)
\end{align*}
\]

10.327 - 0.0369064 i 10.327

**Example 2** Super Derivative of \(x^\alpha e^x \log x\)

Let \(f_1 = x^\alpha\), \(f_2 = e^x\), \(f_3 = \log x\), then

\[
\left( x^\alpha \right)^{(p-r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r}, \quad \left( x^\alpha \right)^{(m+r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r}
\]

\[
\left( e^x \right)^{(r-s)} = e^x, \quad \left( e^x \right)^{(m+r-s)} = e^x
\]
\((\log x)^{(0)} = \log x \quad (s=0), \quad (\log x)^{(s)} = (-1)^{s+1}(s-1)!x^{-s} \quad (s \neq 0)\)

Separating the terms containing \(f^{(3)}_3\) from Theorem 21.2.1 and substituting these for it,

\[
\left(x^r e^s \log x\right)^{(p)} = e^x \log x \sum_{r=0}^{m-1} \left(p \frac{\Gamma(1+\alpha)}{r!}\frac{\Gamma(1+\alpha-p+r)}{\Gamma(1+\alpha)}\right)^{x-1} + R_m^p
\]

Polynomials in (1.2) seem to be asymptotic expansion when \(\alpha=3/2, p=1/2\), the values of the both sides on arbitrary point \(x=16\) are as follows.

\[
a = \frac{3}{2}; \quad p = \frac{1}{2}; \quad m = 50;
\]

\[
\begin{align*}
\text{fl} [x_1] & := \frac{1}{\Gamma(1-p)} \int_{-\infty}^{x} (x-t)^{1-p-1} \text{e}^t \log(t) \, dt \\
\text{fr} [x] & := e^x \log[x] \sum_{r=0}^{m-1} \text{Binomial}[p, r] \Gamma[1+a] \frac{\Gamma[1+a-p+r]}{\Gamma[1+a]} x^{a-p+r} \\
\text{fl} [x] & := e^x \log[x] \sum_{r=0}^{m-1} \text{Binomial}[p, r] \Gamma[1+a] \frac{\Gamma[1+a-p+r]}{\Gamma[1+a]} x^{a-p+r} \\
\text{fl} [x] & := e^x \log[x] \sum_{r=0}^{m-1} \text{Binomial}[p, r] \Gamma[1+a] \frac{\Gamma[1+a-p+r]}{\Gamma[1+a]} x^{a-p+r} \\
\end{align*}
\]

Example Super Derivative of \(\log^3 x\)

Let \(f = \log x\), then
\[(\log x)^{(p-r)} = \frac{\log x - \psi(1-p) + r - \gamma}{\Gamma(1-p)} x^{-p+r}, \quad (\log x)^{<m+r>} = \frac{\log x - \psi(1+m) + r - \gamma}{\Gamma(1+m)} x^{m+r}\]

\[(\log x)^{(r-s)} = \log x \quad (r=s), \quad (\log x)^{(r-s)} = (-1)^{r-s-1} (r-s-1)! x^{r-s} \quad (r \neq s)
\]

\[(\log x)^{(s)} = \log x \quad (s=0), \quad (\log x)^{(s)} = (-1)^{s-1} (s-1)! x^{-s} \quad (r \neq 0)\]

Separating the terms containing \(f^{(0)}_3\) from Theorem 2.1.2, and substituting these for it, we obtain

\[
\begin{align*}
\{ (\log x)^3 \}^{(p)} &= \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1-p)} x^{-p}(\log x)^2 \\
& - 2x^{-p}\log x \sum_{r=1}^{\infty} (-1)^r \left( \frac{p}{r} \right) \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} \Gamma(r) \\
& + x^{-p}\sum_{r=2}^{\infty} \sum_{s=1}^{r-1} (-1)^r \left( \frac{p}{r} \right) \left( \frac{r}{s} \right) \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} \Gamma(r-s) \Gamma(s) + R_m^p
\end{align*}
\]

\(R_m^p = -\frac{1}{B(-p,m)} \sum_{r=0}^{\infty} \left( \frac{(-1)^r}{(m+r)^2} \right) \{ \{ \log x - \psi(1+m) - \gamma \} \log x \}^{(p)} \]

\[
\begin{align*}
&+ \frac{1}{B(-p,m)} \sum_{r=1}^{\infty} \sum_{s=1}^{r-1} \left( \frac{(-1)^r}{(m+r)} \right) \left( \frac{p}{r} \right) \left( \frac{r}{s} \right) \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} \Gamma(r-s) \Gamma(s) \}
\end{align*}
\]

(2.1)

And \(\lim_{m \to \infty} R_m^p = 0\) holds although the proof is difficult. Then

\[
\begin{align*}
\{ (\log x)^3 \}^{(p)} &= \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1-p)} x^{-p}(\log x)^2 \\
& - 2x^{-p}\log x \sum_{r=1}^{\infty} (-1)^r \left( \frac{p}{r} \right) \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} \Gamma(r) \\
& + x^{-p}\sum_{r=2}^{\infty} \sum_{s=1}^{r-1} (-1)^r \left( \frac{p}{r} \right) \left( \frac{r}{s} \right) \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} \Gamma(r-s) \Gamma(s) \}
\end{align*}
\]

(2.1')

However, the convergence speed is very slow.

When \(p = 1/2, m = 120\), the values of the both sides on arbitrary point \(x = 4\) are as follows. Even if calculated so far, both sides are corresponding only to 1 digit below the decimal point.

\(p = 1/2\);

Riemann–Liouville differintegral

\[
f[x_] := \frac{1}{\Gamma[1 - p]} \int_0^x (x-t)^{1-p-1} \log[t]^3 \, dt \quad fl_1 = \text{Limit}[\frac{f[4+h] - f[4]}{h}, h \to 0];
\]

Series

\[m = 120:\]

\[
\begin{align*}
fr[x_] &= \frac{\log[x] - \text{PolyGamma}[1-p] - \text{EulerGamma}}{\Gamma[1-p]} x^p \log[x]^2 - \\
& 2 x^{-p} \log[x] \sum_{r=1}^{\infty} (-1)^r \text{Binomial}[p, r] \frac{\log[x] - \text{PolyGamma}[1-p] + \text{EulerGamma}}{\Gamma[1-p+r]} \Gamma[r] + \\
& x^{-p} \sum_{r=2}^{\infty} \sum_{s=1}^{r-1} (-1)^r \text{Binomial}[p, r] \text{Binomial}[r, s] \frac{\log[x] - \text{PolyGamma}[1-p+r] - \text{EulerGamma}}{\Gamma[1-p+r]} x \Gamma[r-s] \Gamma[s]
\end{align*}
\]
21.2.3 Super Derivatives of $\cos^m x$, $\sin^m x$

Formula 21.2.3

When $m$ is a natural number, $p$ is a positive number and $\lfloor \cdot \rfloor$ is the floor function,

$$
(\cos^m x)^{(p)} = \frac{1}{2^{m-1}} \sum_{r=0}^{m/2} m \binom{m}{r} (m-2r)^p \cos \left( (m-2r) x + \frac{p\pi}{2} \right) 
$$

(3.c)

$$
(\sin^m x)^{(p)} = \frac{1}{2^{m-1}} \sum_{r=0}^{m/2} m \binom{m}{r} (m-2r)^p \cos \left( (m-2r) \left( x - \frac{\pi}{2} \right) + \frac{p\pi}{2} \right) 
$$

(3.s)

Proof

Analytically continuing the index of the differentiation operator in Formula 20.1.3 in $[0, p]$ from $[1, n]$, we obtain the desired expressions.

Example 2.5th order derivative of $\cos^4x$

- $m:=4$:

2nd order derivative

- $f_2 := x \rightarrow \frac{\partial^2}{\partial x^2} \cos(x)^m$

2.5th order derivative

- $p:=2.5$:

- $f_3 := x \rightarrow \frac{1}{2^{m-1}} \left( \sum_{r=0}^{\lfloor m/2 \rfloor} \binom{m}{r} (m-2r)^p \cos \left( (m-2r) \left( x - \frac{\pi}{2} \right) + \frac{p\pi}{2} \right) \right)$

3rd order derivative

- $f_3 := x \rightarrow \frac{\partial^3}{\partial x^3} \cos(x)^m$

Blue: 2nd order A Red: 2.5th order A Green: 3rd order

Collateral Super Derivatives of $\cos^m x$, $\sin^m x$

Unlike the higher derivative, in the super derivative, the lineal super derivative and the collateral one exist. Although this was described in [12.1.4], I describe it once now.
For example, if we differentiate \( \cos^3 x \) with respect to \( x \) \( p \) times according to \textbf{Theorem 21.2.2}, it is as follows.

\[
(\cos^3 x)^{(p)} = \sum_{r=0}^{m-1} \sum_{s=0}^{r} \binom{p}{r} \binom{r}{s} \cos \left( x + \frac{(p-r) \pi}{2} \right) \cos \left( x + \frac{(r-s) \pi}{2} \right) \cos \left( x + \frac{s \pi}{2} \right)
\]

\[
R_m^p = \frac{(-1)^m}{B(-p, m)} \sum_{r=0}^{m+r} \binom{-p-1}{r} \binom{m+r}{s} \times \left\{ \cos \left( x - \frac{(m+r) \pi}{2} \right) \cos \left( x + \frac{(m+r-s) \pi}{2} \right) \cos \left( x + \frac{s \pi}{2} \right) \right\}^{(p)}
\]

Then, this is a collateral super derivative. If it is why, this theorem was drawn out as a reverse-operation of the super integral of the product of many functions as seen during the proof of \textbf{Theorem 21.2.1}. And the base was the super integral with a fixed lower limit.

However, the lineal super integral of \( \cos^3 x \) is one with a variable lower limit. Therefore, (4.c') and (4.cr') derived based on the fixed lower limit cannot be the lineal super derivative. Although this holds as an equation, the polynomial of (4.c') is not well behaved.

By reference, let us compare (4.c') with the following lineal super derivative (4.c) derived from \textbf{Formula 21.2.3}.

\[
(\cos^3 x)^{(p)} = \frac{3}{4} \cos \left( x + \frac{p \pi}{2} \right) + \frac{3p}{4} \cos \left( 3x + \frac{p \pi}{2} \right)
\]

Although (4.c') fits (4.c) most at the time of \( m=3 \), a big difference is still seen by both.

\textbf{Blue: Collateral} , \textbf{Red: Lineal}

![Graph](image)

However, since \( \frac{1}{B(-p, m)} = 0 \) at \( p = m-1, m=1, 2, 3, \ldots \), \( R_m^p = 0 \). Therefore

\[
(\cos^3 x)^{(m-1)} = \sum_{r=0}^{m-1} \sum_{s=0}^{r} \binom{m-1}{r} \binom{r}{s} \cos \left( x + \frac{(m-1-r) \pi}{2} \right) \cos \left( x + \frac{(r-s) \pi}{2} \right) \cos \left( x + \frac{s \pi}{2} \right)
\]

Furthermore, replacing \( m-1 \) with \( n \),

\[
(\cos^3 x)^{(n)} = \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n}{r} \binom{r}{s} \cos \left( x + \frac{(n-r) \pi}{2} \right) \cos \left( x + \frac{(r-s) \pi}{2} \right) \cos \left( x + \frac{s \pi}{2} \right)
\]

And this results in the following lineal super derivative. (The proof is long, then omitted.)

\[
(\cos^3 x)^{(n)} = \frac{3}{4} \cos \left( x + \frac{n \pi}{2} \right) + \frac{3n}{4} \cos \left( 3x + \frac{n \pi}{2} \right)
\]

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