12 Super Derivative (Non-integer times Derivative)

12.1 Super Derivative and Super Differentiation

Defintion 12.1.1

$f^{(p)}(x)$ obtained by continuing analytically the index of the differentiation operator of Higher Derivative of a function $f(x)$ to a complex plane $[0,p]$ from a natural number interval $[1,n]$ is called ***Super Derivative of*** $\boldsymbol{f(x)}$.

Example

$$(\sin x)^{(p)} = \sin\left(x+\frac{p\pi}{2}\right) + c_p(x)$$

$c_p(x)$ is an arbitrary function.

12.1.2 Super Differentiation

Definition 12.1.2

We call it ***Super Differentiation*** to differentiate a function f with respect to an independent variable x non-integer times continuously. And it is described as follows.

$$\frac{d^p}{dx^p}f(x) \qquad \left\{ = \frac{d}{dx}\sim\frac{d}{dx}f(x) \qquad \frac{d}{dx}: \text{ p pieces}\right\}$$

Example

$$\frac{d^p}{dx^p}\cos x = \cos\left(x+\frac{p\pi}{2}\right)$$

12.1.3 Fundamental Theorem of Super Differentiation

The following theorem holds from Theorem 7.1.3 in **7.1**.

Theorem 12.1.3

Let $f^{(r)}$ $r\in[0,p]$ be an continuous function on the closed interval I and be arbitrary the r-th order derivative function of f . And let $a(r)$ be a continuous function on the closed interval $[0,p]$.
Then the following expression holds for $a(r), x\in I$.

$$\frac{d^p}{dx^p}f(x) = f^{(p)}(x) + \frac{d^p}{dx^p}\sum_{r=0}^{p-1} f^{(r)}\{a(p-r)\}\int_{a(p)}^{x}\sim\int_{a(p-r)}^{x}dx^r \tag{1.1}$$

Especially, when $a(r)=a \ \ for\ all\ \ k\in[0,p]$,

$$\frac{d^p}{dx^p}f(x) = f^{(p)}(x) + \frac{d^p}{dx^p}\sum_{r=0}^{p-1}f^{(r)}(a)\frac{(x-a)^r}{\Gamma(1+r)} \tag{1.2}$$

Proof

Theorem 7.1.3 in **7.1** can be rewritten as follows.

$$f^{<p>}(x) = \int_{a(p)}^{x}\sim\int_{a(0)}^{x}f(x)dx^p + \sum_{r=0}^{p-1} f^{<p-r>}\{a(p-r)\}\int_{a(p)}^{x}\sim\int_{a(p-r+1)}^{x}dx^r$$

Especially, when $a(r)=a \ \ for\ all\ \ k \in [0,p]$,

$$f^{<p>}(x) = \int_a^x \sim \int_a^x f(x)dx^n + \sum_{r=0}^{p-1} f^{<p-r>}(a)\frac{(x-a)^r}{\Gamma(1+r)}$$

Differentiating these both sides with respect to x p times,

$$\frac{d^p}{dx^p}f^{<p>}(x) = f^{<0>}(x) + \frac{d^p}{dx^p}\sum_{r=0}^{p-1} f^{<p-r>}\{a(p-r)\}\int_{a(p)}^x \sim \int_{a(p-r)}^x dx^r$$

$$\frac{d^p}{dx^p}f^{<p>}(x) = f^{<0>}(x) + \frac{d^p}{dx^p}\sum_{r=0}^{p-1} f^{<p-r>}(a)\frac{(x-a)^r}{\Gamma(1+r)}$$

Shifting by $-p$ the index in the integration operator $<>$ and replacing $<>$ by differentiation operator $(\ )$, we obtain the desired expression.

Constant-of-differentiation Function

We call $\frac{d^p}{dx^p}\sum_{r=0}^{p-1} etc.$ ***Constant-of-differentiation Function of*** $\boldsymbol{f(x)}$. Since p is a real number, generally it is difficult to obtain this. However, it becomes easy exceptionally at the time of $f(x)=e^x$.
That is, **Constant-of-integration Function in 7.1.3** was as follows.

$$\sum_{r=0}^{p-1} e^a\frac{(x-a)^r}{\Gamma(1+r)} = e^a\sum_{r=0}^{\infty}\left\{\frac{(x-a)^r}{\Gamma(1+r)} - \frac{(x-a)^{r+p}}{\Gamma(1+r+p)}\right\}$$

Differentiating both sides with respect to x p times,

$$\frac{d^p}{dx^p}\sum_{r=0}^{p-1} e^a\frac{(x-a)^r}{\Gamma(1+r)} = e^a\sum_{r=0}^{\infty}\frac{d^p}{dx^p}\left\{\frac{(x-a)^r}{\Gamma(1+r)} - \frac{(x-a)^{r+p}}{\Gamma(1+r+p)}\right\}$$

From the **12.3.1** mentioned later, the following expressions hold.

$$\frac{d^p}{dx^p}(x-a)^r = \frac{\Gamma(1+r)}{\Gamma(1+r-p)}(x-a)^{r-p} , \quad \frac{d^p}{dx^p}(x-a)^{r+p} = \frac{\Gamma(1+r+p)}{\Gamma(1+r)}(x-a)^r$$

Substituting this for the above, we obtain the following expression.

$$\frac{d^p}{dx^p}\sum_{r=0}^{p-1} e^a\frac{(x-a)^r}{\Gamma(1+r)} = e^a\sum_{r=0}^{\infty}\left\{\frac{(x-a)^{r-p}}{\Gamma(1+r-p)} - \frac{(x-a)^r}{\Gamma(1+r)}\right\}$$

12.1.4 Lineal and Collateral

In the case of the higher differentiation, since the constant-of-integration polynomial was degree $n-1$, the constant-of-differentiation function which differentiated this n times became 0. However, in the case of the super differentiation, since the constant-of-integration function is expressed by a series in general, the constant-of-differentiation function which differentiated this p times does not become 0. This shows that there are lineal and collateral in the super differentiation.

Definition 12.1.4

$$\frac{d^p}{dx^p}f(x) = f^{(p)}(x) + \frac{d^p}{dx^p}\sum_{r=0}^{p-1} f^{(r)}\{a(p-r)\}\int_{a(p)}^x \sim \int_{a(p-r)}^x dx^r \tag{1.1}$$

In this expression,

when Constant-of-differentiation Functin is 0, we call $\frac{d^p}{dx^p}f(x)$ ***Lineal Super Differentiation*** and

we call the function equal to this ***Lineal Super Derivaive Function***.

when Constant-of-differentiation Functin is not 0,

we call $\frac{d^p}{dx^p}f(x)$ ***Collateral Super Differentiation*** and

we call the function equal to this ***Collateral Super Derivaive Function***.

These are the same also in (1.2) .

In short , ***Lineal Super Derivaive Function*** **is what differentiated $f(x)$ with respect to x continuously without considering the constant-of-differentiation function.**

Example: lineal derivative and collateral derivative of e^x

In the case of easier fixed lower limit, from (1.2) in the theorem

$$\frac{d^p}{dx^p}e^x = e^x + \frac{d^p}{dx^p}\sum_{r=0}^{p-1} e^a \frac{(x-a)^r}{\Gamma(1+r)}$$

Here, using the former expression i.e.

$$\frac{d^p}{dx^p}\sum_{r=0}^{p-1} e^a \frac{(x-a)^r}{\Gamma(1+r)} = e^a \sum_{r=0}^{\infty}\left\{\frac{(x-a)^{r-p}}{\Gamma(1+r-p)} - \frac{(x-a)^r}{\Gamma(1+r)}\right\}$$

we obtain

$$\frac{d^p}{dx^p}e^x = e^x + e^a \sum_{r=0}^{\infty}\left\{\frac{(x-a)^{r-p}}{\Gamma(1+r-p)} - \frac{(x-a)^r}{\Gamma(1+r)}\right\} \qquad \left(= e^a \sum_{r=0}^{\infty}\frac{(x-a)^{r-p}}{\Gamma(1+r-p)}\right)$$

When $a \neq -\infty$, since the constant-of-differentiation functin can not be 0, this is a collateral differentiation.

When $a = -\infty$, since $e^a = 0$, we obtain the following lineal differentiation.

$$\frac{d^p}{dx^p}e^x = e^x$$

When $p=1/2 , a=0$, if the differential quotients on $x=\pm 0.3$ are compared with the calculation result by Riemann-Liouville differintegral (later 12.2.1), it is as follows.

Left: Riemann Liouville differintegral

```
• a:=0: p:=1/2:
• g := x-> 1/gamma(1-p)*int((x-t)^(1-p-1)*E^t , t=a..x):
• h := 10^-9:
•  float((g(-0.3+h)-g(-0.3))/h),   float((g(0.3+h)-g(0.3))/h)
```

$-0.5219916843 \cdot i,\quad 1.787904935$

Right: series

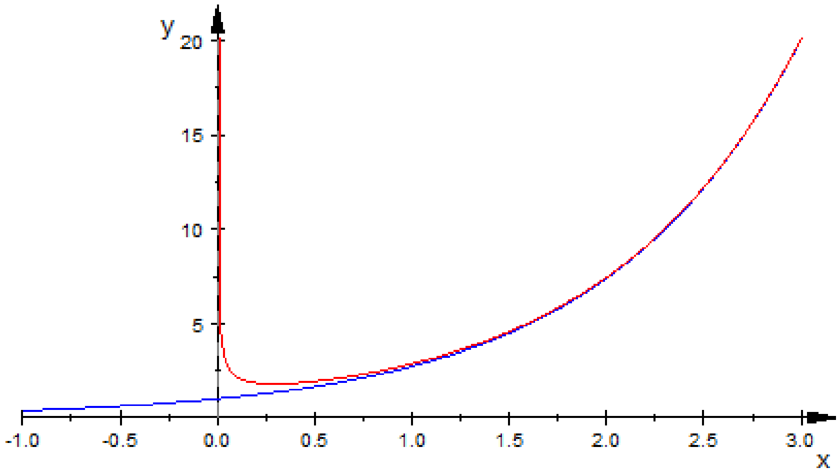
```
• m:=100:
• er := x-> E^x + E^a*sum((x-a)^(r-p)/gamma(1+r-p)
                          -(x-a)^r/gamma(1+r), r=0..m):
• float(er(-0.3)),              float(er(0.3))
```

$1.626303259 \cdot 10^{-19} - 0.5219916832 \cdot i,\quad 1.787904935$

And if the lineal super derivative and the collateral super derivative are illustrated side by side, it is as follows.

Blue : lineal , Red : collateral

```
• plotfunc2d(E^x,er(x), x=-1..3, ViewingBoxYRange=0..20)
```



Remark

It is thought that this collateral super derivative is an asymptotic expansion. And this collateral super derivative is corresponding with the termwise super differentiation. In general, a termwise super differentiation seems to become a collateral super differentiation.

12.1.5 The basic formulas of the Super Differentiation

The following formulas hold like the higher differentiation.

$$\{c\,f(x)\}^{(p)} = c\,f^{(p)}(x) \qquad c \neq 0 \qquad \text{: constant multiple rule}$$

$$\{f(x) + g(x)\}^{(p)} = f^{(p)}(x) + g^{(p)}(x) \qquad \text{: sum rule}$$

12.2 Fractional Derivative

12.2.1 Riemann-Liouville differintegral

Among the super integrals of function f(x), the super integral whose lower limit function $a(k)$ is a constant a was calculable by Riemann-Liouville integral. The super derivativeof such a function f(x) is calculable by Riemann-Liouville integral and integer times differentiation. It is as follows.

Let $n=\lceil p\rceil \ \{=ceil(p)\}$. First, integrate with f (x) $n-p$ times. Next, differentiate it n times. And, since the result is n- (n-p), it means that f(x) was differentiated p times.

$$f^{(p)}(x) = \frac{1}{\Gamma(n-p)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-p-1}f(t)\,dt \qquad n=\lceil p\rceil \tag{2.0}$$

This expression is called Riemann-Liouville differintegral. "differintegral" is a coined word which combined "differential" and "integral".

Although the numerical integration and the numerical differentiation are possible for (2.0) with this, the accuracy of numerical differentiation is bad and the desired result may not be obtained. In this case, the following formula which replaced the calculation order of integration and differentiation is effective.

$$f^{(p)}(x) = \frac{1}{\Gamma(n-p)}\int_a^x (x-t)^{n-p-1}\left\{\frac{d^n}{dt^n}f(t)\right\}dt \qquad n=\lceil p\rceil \tag{2.0'}$$

Although this formula has a possibility of cutting off a constant of integration as a result of differentiating previously, in many cases, it is correctly calculable. This (2.0') is often used in the following chapters.

12.2.2 Riemann-Liouville differintegral expressions of super derivatives of elementary functions

Riemann-Liouville differintegral expressions of super derivatives of some elementary functions are as follows. In the right side, super derivatives obtained by super differentiation are shown in advance. Needless to say, Riemann-Liouville differintegral holds only if the lower limit function $a(k)$ is a constant a . In addition, p is a positive non-integer and $n=\lceil p\rceil \ \{=ceil(p)\}$ in all the expressions.

$$(x^\alpha)^{(p)} = \frac{1}{\Gamma(n-p)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-p-1}t^\alpha dt = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p)}x^{\alpha-p} \qquad (\alpha \geqq 0)$$

$$= \frac{1}{\Gamma(n-p)}\int_\infty^x (x-t)^{n-p-1}\left(\frac{d^n}{dt^n}t^\alpha\right)dt = (-1)^{-p}\frac{\Gamma(-\alpha+p)}{\Gamma(-\alpha)}x^{\alpha-p} \qquad (\alpha<0)$$

$$(e^{\pm x})^{(p)} = \frac{1}{\Gamma(n-p)}\frac{d^n}{dx^n}\int_{\mp\infty}^x (x-t)^{n-p-1}e^{\pm t}dt = (\pm 1)^{-p}e^{\pm x}$$

$$(\log x)^{(p)} = \frac{1}{\Gamma(n-p)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-p-1}\log t\,dt = \frac{\log x-\psi(1-p)-\gamma}{\Gamma(1-p)}x^{-p}$$

Note

When $n=\lceil p\rceil$, $-(n-p)<\alpha<0$, the following expression does not hold.

$$(x^\alpha)^{(p)} = \frac{1}{\Gamma(n-p)}\frac{d^n}{dx^n}\int_\infty^x (x-t)^{n-p-1}t^\alpha dt \tag{2.1}$$

In this case, $n-p$ times integral of x^α $(\alpha<0)$ is carried out, and super primitive function $x^{\alpha+n-p}$ $(\alpha+n-p>0)$ is obtained. According to this, the zero of the super primitive function changes from ∞ to 0. The integral with a fixed lower limit is inapplicable to such an integral.

In this case, the following formula which replaced the calculation order of integration and differentiation is effective.

$$\left(x^{\alpha}\right)^{(p)} = \frac{1}{\Gamma(n-p)}\int_{\infty}^{x}(x-t)^{n-p-1}\left\{\frac{d^n}{dt^n}t^{\alpha}\right\}dt \qquad n=[p] \tag{2.1'}$$

If this formula is used, $n-p$ times integral of $x^{\alpha-n}$ $(\alpha-n<0)$ is carried out, and super primitive function $x^{\alpha-p}(\alpha-p<0)$ is obtained. Since the zero of the super primitive function does not change, the integral with a fixed lower limit is applicable. (See Example 5 in the following chapter).

Super Derivative & Fractional Derivative

In *Super Derivative* which I developed, first, we obtain the higher derivative, next, extending the index of the operator to real number, we obtain the super derivative.

On the contrary, in traditional *Fractional Derivative*, the super derivative is directly drawn from Riemann-Liouville differintegral. However, the calculation is very difficult.

Three examples are shown below. In each example, the 1st is *Super Derivative*, and the 2nd is *Fractional Derivative*.

Example 1

$$\left(x^{1}\right)^{\left(\frac{1}{2}\right)} = \frac{\Gamma(1+1)}{\Gamma(1+1-1/2)}x^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma(3/2)}x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}$$

$$\left(x^{1}\right)^{\left(\frac{1}{2}\right)} = \frac{1}{\Gamma(1-1/2)}\frac{d^1}{dx^1}\int_{0}^{x}(x-t)^{1-\frac{1}{2}-1}t^1\,dt$$

$$= \frac{1}{\Gamma(1/2)}\frac{d}{dx}\int_{0}^{x}(x-t)^{-\frac{1}{2}}t^1\,dt = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\left[-\frac{2}{3}\sqrt{x-t}\,(2x+t)\right]_0^x$$

$$= \frac{1}{\sqrt{\pi}}\frac{d}{dx}\left(\frac{4}{3}x^{\frac{3}{2}}\right) = \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}$$

Example 2

$$\left(e^{-x}\right)^{\left(\frac{1}{2}\right)} = (-1)^{-\frac{1}{2}}e^{-x} = \frac{1}{i}e^{-x}$$

$$\left(e^{-x}\right)^{\left(\frac{1}{2}\right)} = \frac{1}{\Gamma(1-1/2)}\frac{d^1}{dx^1}\int_{+\infty}^{x}(x-t)^{1-\frac{1}{2}-1}e^{-t}dt$$

$$= \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_{+\infty}^{x}(x-t)^{-\frac{1}{2}}e^{-t}dt = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\left[-e^{x}\sqrt{\pi}\,erfi\left(\sqrt{x-t}\right)\right]_{\infty}^{x}$$

$$= \frac{1}{\sqrt{\pi}}\frac{d}{dx}\left\{e^{-x}\sqrt{\pi}\,erfi\left(\sqrt{x-\infty}\right)\right\} = \frac{1}{i}\frac{d}{dx}\left\{e^{-x}erf\left(i\sqrt{x-\infty}\right)\right\}$$

$$= \frac{1}{i}\frac{d}{dx}\left\{e^{-x}erf\left(-\sqrt{\infty}\right)\right\} = \frac{1}{i}\frac{d}{dx}\left(-e^{-x}\right) = \frac{1}{i}e^{-x}$$

Example 3

$$(\log x)^{\left(\frac{1}{2}\right)} = \frac{\log x-\psi(1-1/2)-\gamma}{\Gamma(1-1/2)}x^{-\frac{1}{2}} = \frac{\log x-\psi(1/2)-\gamma}{\Gamma(1/2)}x^{-\frac{1}{2}}$$

$$= \frac{\log x - (-\gamma - 2\log 2) - \gamma}{\sqrt{\pi}} x^{-\frac{1}{2}} = \frac{\log x + 2\log 2}{\sqrt{\pi}} x^{-\frac{1}{2}}$$

$$(\log x)^{\left(\frac{1}{2}\right)} = \frac{1}{\Gamma(1-1/2)} \frac{d^1}{dx^1} \int_0^x (x-t)^{1-\frac{1}{2}-1} \log t\, dt$$

$$= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_0^x (x-t)^{-\frac{1}{2}} \log t\, dt$$

$$= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \left[-4\sqrt{x}\, \tanh^{-1}\left(\frac{\sqrt{x-t}}{\sqrt{x}} \right) - 2\sqrt{x-t}\, (\log t - 2) \right]_0^x$$

Here

$$\tanh^{-1}\frac{x}{n} = \log\sqrt{\frac{n+x}{n-x}} = \frac{1}{2}\{\log(n+x) - \log(n-x)\} \qquad |x| < n$$

Then

$$\tanh^{-1}\left(\frac{\sqrt{x-t}}{\sqrt{x}} \right) = \frac{1}{2}\{\log(\sqrt{x} + \sqrt{x-t}) - \log(\sqrt{x} - \sqrt{x-t})\}$$

From this

$$4\sqrt{x}\tanh^{-1}\left(\frac{\sqrt{x-t}}{\sqrt{x}} \right) = 2\sqrt{x}\{\log(\sqrt{x} + \sqrt{x-t}) - \log(\sqrt{x} - \sqrt{x-t})\}$$

$$= 4\sqrt{x}\log(\sqrt{x}+\sqrt{x-t}) - 2\sqrt{x}\{\log(\sqrt{x}+\sqrt{x-t}) + \log(\sqrt{x} - \sqrt{x-t})\}$$

$$= 4\sqrt{x}\log(\sqrt{x}+\sqrt{x-t}) - 2\sqrt{x}\, \log\{x-(x-t)\}$$

$$= 4\sqrt{x}\log(\sqrt{x}+\sqrt{x-t}) - 2\sqrt{x}\, \log t$$

Therefore

$$4\sqrt{x}\, \tanh^{-1}\left(\frac{\sqrt{x-t}}{\sqrt{x}} \right) + 2\sqrt{x-t}\, (\log t - 2)$$

$$= 4\sqrt{x}\log(\sqrt{x}+\sqrt{x-t}) - 2\sqrt{x}\, \log t + 2\sqrt{x-t}\, \log t - 4\sqrt{x-t}$$

$$= 4\sqrt{x}\log(\sqrt{x}+\sqrt{x-t}) - 2(\sqrt{x} - \sqrt{x-t})\, \log t - 4\sqrt{x-t}$$

Using this

$$(\log x)^{\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \left[-4\sqrt{x}\, \tanh^{-1}\left(\frac{\sqrt{x-t}}{\sqrt{x}} \right) - 2\sqrt{x-t}\, (\log t - 2) \right]_0^x$$

$$= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \left[-4\sqrt{x}\log(\sqrt{x}+\sqrt{x-t}) + 2(\sqrt{x} - \sqrt{x-t})\, \log t + 4\sqrt{x-t} \right]_0^x$$

$$= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \{-4\sqrt{x}\, \log\sqrt{x} + 2\sqrt{x}\, \log x\}$$

$$+ \frac{1}{\sqrt{\pi}} \frac{d}{dx} \{4\sqrt{x}\log(\sqrt{x}+\sqrt{x}) - 2(\sqrt{x} - \sqrt{x})\, \log 0 - 4\sqrt{x}\}$$

$$= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \{4\sqrt{x}\log(2\sqrt{x}) - 4\sqrt{x}\} \qquad (\because\ 2\sqrt{x}\log\sqrt{x} = \sqrt{x}\, \log x)$$

Furthermore

$$4\sqrt{x}\,log\left(2\sqrt{x}\right) - 4\sqrt{x} = 4\sqrt{x}\ log2 + 4\sqrt{x}\ log\sqrt{x} - 4\sqrt{x}$$
$$= 4\sqrt{x}\ log2 + 2\sqrt{x}\ log x - 4\sqrt{x}$$
$$= 4\sqrt{x}\left(log2-1\right) + 2\sqrt{x}\,log x$$

Thus

$$\left(log\,x\right)^{\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\left\{4\sqrt{x}\left(log2-1\right)+2\sqrt{x}\,log x\right\} = \frac{log x+2\,log2}{\sqrt{\pi}}\,x^{-\frac{1}{2}}$$

Reference

$$\int (x-t)^{-\frac{1}{2}}\,t\,dt = -\frac{2}{3}\sqrt{x-t}\,(2x+t)$$

$$\int (x-t)^{-\frac{1}{2}}\,e^{t}\,dt = -e^{x}\sqrt{\pi}\ erf\left(\sqrt{x-t}\right) \qquad erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$$

$$\int (x-t)^{-\frac{1}{2}}\,e^{-t}\,dt = -e^{-x}\sqrt{\pi}\ erf\,i\left(\sqrt{x-t}\right) \qquad erf\,i(z)=erf(iz)/i$$

$$\int (x-t)^{-\frac{1}{2}}\,log\,t\,dt = -4\sqrt{x}\ tanh^{-1}\left(\frac{\sqrt{x-t}}{\sqrt{x}}\right) - 2\sqrt{x-t}\ \left(log\,t-2\right)$$

12.2.3 The demerit and the strong point of Fractional Derivative

As seen in three upper examples, *Fractional Derivative* based on Riemann-Liouville differintegral is difficult like this. Although Example 3 was the 1/2 times derivative of a logarithmic function, for this calculation, the delicate technique was used abundantly, and one day was required. When p is a real number, the p times derivative of this function is hopelessly difficult. Thus, in *Fractional Derivative*, it is very difficult to obtain the derivative from Riemann-Liouville differintegral. Moreover, how to take a lower limit is not clear, and it cannot treat trigonometric functions etc. These are the same as that of *Fractional Integral*.

A peculiar problem to Riemann-Liouville differintegral is not to be able to use it when p is an integer. It is because of becoming $\Gamma(0)=\infty$ of n=p at this time. In this case, it will rely on the Higher Derivative.

However, (2.0) and (2.0') are the powerful tools of numerical computation and can obtain super derivative of the arbitrary points of arbitrary functions easily. And super derivatrive can be verified numerically with these.

12.3 Super Derivative of Power Function

12.3.1 Formula of Super Derivative of Power Function

Analytically continuing the index of the differentiation operator in Formula 9.2.1 (**9.2**) to $[0,p]$ from $[1,n]$ we obtain the following formula. In addition, Rieamnn-Liouville differintegrals are also expressed together

Formula 12.3.1

When $\Gamma(z)$ is gamma function and $n=\lceil p\rceil$ is ceiling function, the following expressions hold

(1) Basic form

$$\left(x^{\alpha}\right)^{(p)}=\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p)}x^{\alpha-p}=\frac{1}{\Gamma(n-p)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-p-1}t^{\alpha}dt \qquad (\alpha\geqq 0)$$

$$=(-1)^{-p}\frac{\Gamma(-\alpha+p)}{\Gamma(-\alpha)}x^{\alpha-p}=\frac{1}{\Gamma(n-p)}\int_{\infty}^x (x-t)^{n-p-1}\left(\frac{d^n}{dt^n}t^{\alpha}\right)dt \qquad (\alpha<0)$$

(2) Linear form

$$\left\{(ax+b)^{\alpha}\right\}^{(p)}=\left(\frac{1}{a}\right)^{-p}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p)}(ax+b)^{\alpha-p} \qquad (\alpha\geqq 0)$$

$$=\frac{1}{\Gamma(n-p)}\frac{d^n}{dx^n}\int_{-\frac{b}{a}}^x (x-t)^{n-p-1}(at+b)^{\alpha}dt$$

$$\left\{(ax+b)^{\alpha}\right\}^{(p)}=\left(-\frac{1}{a}\right)^{-p}\frac{\Gamma(-\alpha+p)}{\Gamma(-\alpha)}(ax+b)^{\alpha-p} \qquad (\alpha<0)$$

$$=\frac{1}{\Gamma(n-p)}\int_{\infty}^x (x-t)^{n-p-1}\left\{\frac{d^n}{dt^n}(at+b)^{\alpha}\right\}dt$$

Caution !

Do not describe $\left(\frac{1}{a}\right)^{-p}$ to be a^p in the upper formula. Because, since the law of exponents $\left(a^p\right)^q=a^{pq}$ does not hold for the arbitrary real numbers p,q at the time $a<0$,

$$\left(\frac{1}{a}\right)^{-p}=\left(a^{-1}\right)^{-p}\neq a^{(-1)(-p)}=a^p \qquad i.e.\quad \left(\frac{1}{a}\right)^{-p}\neq a^p$$

If it is described as $\left(\frac{1}{a}\right)^{-p}=a^p$, the rotation direction on the complex plane becomes reverse and the mistaken result is caused.

For example, when $a=-3$, $p=1/2$,

$$\left(\frac{1}{a}\right)^{-p}=\left(\frac{1}{-3}\right)^{-\frac{1}{2}}=\left\{(-1)\cdot\frac{1}{3}\right\}^{-\frac{1}{2}}=(-1)^{-\frac{1}{2}}3^{\frac{1}{2}}=-i\sqrt{3}$$

$$\neq$$

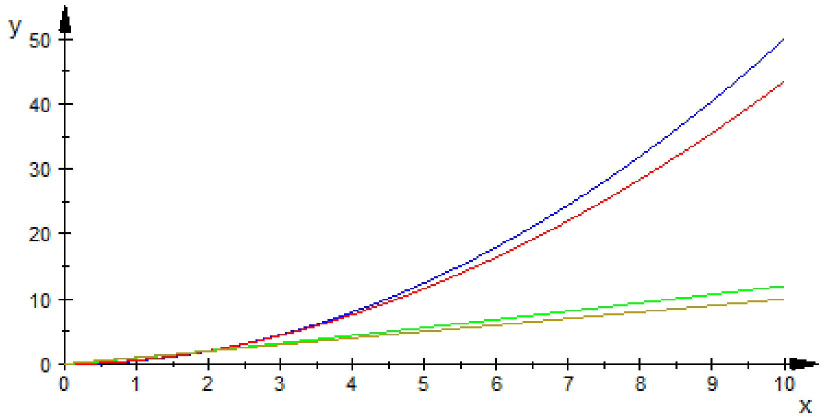
$$a^p=(-3)^{\frac{1}{2}}=\left\{(-1)\cdot 3\right\}^{\frac{1}{2}}=(-1)^{\frac{1}{2}}3^{\frac{1}{2}}=i\sqrt{3}$$

Example 1 $\alpha > p$

$$\left(\frac{x^2}{2}\right)^{\left(\frac{1}{10}\right)} = \frac{\Gamma(1+2)}{2\Gamma(1+2-1/10)}x^{2-\frac{1}{10}} = \frac{100}{171\Gamma(9/10)}x^{\frac{19}{10}} = 0.547239\,x^{\frac{19}{10}}$$

$$\left(\frac{x^2}{2}\right)^{\left(\frac{9}{10}\right)} = \frac{\Gamma(1+2)}{2\Gamma(1+2-9/10)}x^{2-\frac{9}{10}} = \frac{100}{11\Gamma(1/10)}x^{\frac{11}{10}} = 0.955579\,x^{\frac{11}{10}}$$

When $x^2/2$, $(x^2/2)^{<1/10>}$, $(x^2/2)^{<9/10>}$, x^1 are drawn on a figure side by side, it is as follows.



Example 2 $\alpha = p$

$$\left(x^1\right)^{(1)} = \frac{\Gamma(1+1)}{\Gamma(1+1-1)}x^{1-1} = \frac{\Gamma(2)}{\Gamma(1)}x^0 = \frac{1}{1}x^0 = 1$$

$$\left(x^{\frac{1}{2}}\right)^{\left(\frac{1}{2}\right)} = \frac{\Gamma\left(1+\frac{1}{2}\right)}{\Gamma\left(1+\frac{1}{2}-\frac{1}{2}\right)}x^{\frac{1}{2}-\frac{1}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)}x^0 = \frac{\sqrt{\pi}}{2}$$

Example 3 $0 \leqq \alpha < p$

$$\left(x^0\right)^{\left(\frac{1}{2}\right)} = \frac{\Gamma(1+0)}{\Gamma(1+0-1/2)}x^{0-\frac{1}{2}} = \frac{\Gamma(1)}{\Gamma(1/2)}x^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}x^{-\frac{1}{2}}$$

$$\left(x^1\right)^{(2)} = \frac{\Gamma(1+1)}{\Gamma(1+1-2)}x^{1-2} = \frac{\Gamma(2)}{\Gamma(0)}x^{-1} = \frac{1}{\infty}x^{-1} = 0$$

Example 4 $\alpha < 0$

$$\left(x^{-1}\right)^{\left(\frac{1}{2}\right)} = (-1)^{-\frac{1}{2}}\frac{\Gamma\{-(-1)+1/2\}}{\Gamma\{-(-1)\}}x^{-1-\frac{1}{2}} = -i\frac{\sqrt{\pi}}{2}x^{-\frac{3}{2}}$$

$$\left(x^{-1}\right)^{(1)} = (-1)^{-1}\frac{\Gamma\{-(-1)+1\}}{\Gamma\{-(-1)\}}x^{-1-1} = -\frac{\Gamma(2)}{\Gamma(1)}x^{-2} = -x^{-2}$$

$$\left(x^{-\frac{1}{2}}\right)^{(2)} = (-1)^{-2}\frac{\Gamma\{-(-1/2)+2\}}{\Gamma\{-(-1/2)\}}x^{-\frac{1}{2}-2} = \frac{\Gamma(5/2)}{\Gamma(1/2)}x^{-\frac{5}{2}} = \frac{3}{4}x^{-\frac{5}{2}}$$

Example 5 $-(\lceil p\rceil - p) < \alpha < 0$

$$\left(x^{-\frac{1}{3}}\right)^{\left(\frac{1}{2}\right)} = (-1)^{-\frac{1}{2}} \frac{\Gamma\left\{-\left(-\frac{1}{3}\right)+\frac{1}{2}\right\}}{\Gamma\left\{-\left(-\frac{1}{3}\right)\right\}} x^{-\frac{1}{3}-\frac{1}{2}} = -i\cdot\frac{\Gamma(5/6)}{\Gamma(1/3)} x^{-\frac{5}{6}}$$

$$= -\frac{1.1287}{2.6789} i\, x^{-\frac{5}{6}} = -0.42135\, i\, x^{-\frac{5}{6}}$$

$$\left(x^{-\frac{1}{3}}\right)^{\left(\frac{1}{2}\right)} = \frac{1}{\Gamma\left(1-\frac{1}{2}\right)} \int_{\infty}^{x} (x-t)^{-\frac{1}{2}} \left(\frac{d^1}{dt^1} t^{-\frac{1}{3}}\right) dt = -\frac{1}{3\sqrt{\pi}} \int_{\infty}^{x} (x-t)^{-\frac{1}{2}} t^{-\frac{4}{3}} dt$$

This last integral cannot be expressed with an elementary function. Then, if the value on x=1 is numerically integrated by mathematical software, it is as follows. This result is consistent with the previous value $-0.42135i$ exactly.

Riemann-Liouville differintegral

```
• a := -1/3: p := 1/2:
• df := diff(t^a,t)
```

$$-\frac{1}{3\cdot t^{\frac{4}{3}}}$$

```
• fl := x-> 1/gamma(ceil(p)-p)
        *int((x-t)^(ceil(p)-p-1)*df, t=infinity..x)
```

$$x \rightarrow \frac{1}{\Gamma(\lceil p\rceil - p)} \cdot \int_{\infty}^{x} (x-t)^{\lceil p\rceil - p - 1} \cdot \mathrm{df}\ \mathrm{d}t$$

```
• float(fl(1))
```

$$-0.4213560763\cdot \mathrm{i}$$

12.3.2 Half Derivative of a power function

Especially, Super Derivative of order 1/2 is called Half Derivative.

Formula 12.3.2

Let n be a non-negative integer, $-1!! \equiv 1,\ (2n-1)!! \equiv 1\times 3\times 5\times\cdots\times(2n-1)$,

$0!! \equiv 1,\ \ 2n!! \equiv 2\times 4\times 6\times\cdots\times 2n$,

then following expressions hold.

(1) Basic form

$$\left(x^{n}\right)^{\left(\frac{1}{2}\right)} = \frac{(2n)!!}{(2n-1)!!\sqrt{\pi}} x^{n-\frac{1}{2}}$$

$$\left(x^{n+\frac{1}{2}}\right)^{\left(\frac{1}{2}\right)} = \frac{(n+1)(2n+1)!!\sqrt{\pi}}{\{2(n+1)\}!!} x^{n}$$

(2) Linear form

$$\{(ax+b)^n\}^{\left(\frac{1}{2}\right)} = \frac{(2n)!!}{(2n-1)!!}\sqrt{\frac{a}{\pi}}\,(ax+b)^{n-\frac{1}{2}}$$

$$\left\{(ax+b)^{n+\frac{1}{2}}\right\}^{\left(\frac{1}{2}\right)} = \frac{(n+1)(2n+1)!!}{\{2(n+1)\}!!}\sqrt{a\pi}\,(ax+b)^n$$

Proof

Upper rows $(\alpha \geqq 0)$ of the linear form of Formula 12.3.1 were as follows.

$$\{(ax+b)^n\}^{\left(\frac{1}{2}\right)} = \left(\frac{1}{a}\right)^{-\frac{1}{2}}\frac{\Gamma(1+n)}{\Gamma\left(1+n-\frac{1}{2}\right)}(ax+b)^{n-\frac{1}{2}}$$

$$\left\{(ax+b)^{n+\frac{1}{2}}\right\}^{\left(\frac{1}{2}\right)} = \left(\frac{1}{a}\right)^{-\frac{1}{2}}\frac{\Gamma\left(1+n+\frac{1}{2}\right)}{\Gamma(1+n)}(ax+b)^n$$

Here, when n is a non-negative integer,

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \quad , \quad (2n)!! = 2^n\, n!$$

Then

$$\frac{\Gamma(1+n)}{\Gamma\left(1+n-\frac{1}{2}\right)} = \frac{\Gamma(1+n)}{\Gamma\left(n+\frac{1}{2}\right)} = \frac{2^n\, n!}{(2n-1)!!\sqrt{\pi}} = \frac{(2n)!!}{(2n-1)!!\sqrt{\pi}}$$

$$\frac{\Gamma\left(1+n+\frac{1}{2}\right)}{\Gamma(1+n)} = \frac{\Gamma\left\{(1+n)+\frac{1}{2}\right\}}{\Gamma(1+n)} = \frac{1}{n!}\frac{\{2(1+n)-1\}!!\sqrt{\pi}}{2^{1+n}}$$

$$= \frac{(n+1)(2n+1)!!\sqrt{\pi}}{2^{n+1}(n+1)!} = \frac{(n+1)(2n+1)!!\sqrt{\pi}}{\{2(n+1)\}!!}$$

Substituting these for the previous formula, we obtain the linear form, andgiving $a=1, b=0$ to them, we obtain the basic form.

Example 1

$$(x^0)^{\left(\frac{1}{2}\right)} = \frac{0!!}{(-1)!!\sqrt{\pi}}x^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}x^{-\frac{1}{2}} \quad (\equiv a_0 x^{-\frac{1}{2}})$$

$$(x^1)^{\left(\frac{1}{2}\right)} = \frac{2!!}{1!!\sqrt{\pi}}x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}} \quad (\equiv a_1 x^{\frac{1}{2}})$$

$$(x^2)^{\left(\frac{1}{2}\right)} = \frac{4!!}{3!!\sqrt{\pi}}x^{\frac{3}{2}} = \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}} \quad (\equiv a_2 x^{\frac{3}{2}})$$

$$(x^3)^{\left(\frac{1}{2}\right)} = \frac{6!!}{5!!\sqrt{\pi}}x^{\frac{5}{2}} = \frac{16}{5\sqrt{\pi}}x^{\frac{5}{2}} \quad (\equiv a_3 x^{\frac{5}{2}})$$

Example 2

$$\left(x^{\frac{1}{2}}\right)^{\left(\frac{1}{2}\right)} = \frac{1\cdot 1!!\sqrt{\pi}}{2!!}x^0 \quad = \frac{\sqrt{\pi}}{2}x^0 \qquad \left(= \frac{1}{a_1}x^0 \right)$$

$$\left(x^{\frac{3}{2}}\right)^{\left(\frac{1}{2}\right)} = \frac{2\cdot 3!!\sqrt{\pi}}{4!!}x^1 \quad = \frac{3\sqrt{\pi}}{4}x^1 \qquad \left(= \frac{2}{a_2}x^1 \right)$$

$$\left(x^{\frac{5}{2}}\right)^{\left(\frac{1}{2}\right)} = \frac{3\cdot 5!!\sqrt{\pi}}{6!!}x^2 \quad = \frac{15\sqrt{\pi}}{16}x^2 \qquad \left(= \frac{3}{a_3}x^2 \right)$$

$$\left(x^{\frac{7}{2}}\right)^{\left(\frac{1}{2}\right)} = \frac{4\cdot 7!!\sqrt{\pi}}{8!!}x^3 \quad = \frac{35\sqrt{\pi}}{32}x^3 \qquad \left(= \frac{4}{a_4}x^3 \right)$$

12.3.3 Half Derivative of an integer power function (Fractional Derivative)

Next, using Riemann-Liouville differintegral, we obtain the Half Derivative of an integer power function.

Formula 12.3.3

When n denots a natural number, the following expression holds.

$$\left(x^n\right)^{\left(\frac{1}{2}\right)} = \frac{2n+1}{\sqrt{\pi}}\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}\cdot x^{n-\frac{1}{2}}$$

Proof

Let n be a natural number, $\alpha = n$, $p = 1/2$. Then, since $\alpha > p$, from Formula 12.3.1 we obtain the following expression.

$$\left(x^n\right)^{\left(\frac{1}{2}\right)} = \frac{1}{\Gamma(1/2)}\frac{d^1}{dx^1}\int_0^x (x-t)^{-\frac{1}{2}}t^n\,dt = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_0^x \frac{t^n}{\sqrt{x-t}}dt$$

Here, according to「岩波数学公式Ｉ」p96, the following expressions hold

$$\int \frac{t^n}{\sqrt{x-t}}dt = \frac{2\sqrt{x-t}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{2n-2r+1}$$

Then

$$\int_0^x \frac{t^n}{\sqrt{x-t}}dt = \left[\frac{2\sqrt{x-t}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{2n-2r+1}\right]_0^x$$

$$= \frac{2}{(-1)^n}x^{\frac{1}{2}}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{x^n}{2n-2r+1}$$

Differentiate both sides of this with respect to x as follows.

$$\frac{d}{dx}\int_0^x \frac{t^n}{\sqrt{x-t}}dt = \frac{d}{dx}\left\{\frac{2}{(-1)^n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{1}{2n-2r+1}x^{n+\frac{1}{2}}\right\}$$

$$= \frac{2}{(-1)^n}\left(n+\frac{1}{2}\right)\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{1}{2n-2r+1}x^{n-\frac{1}{2}}$$

$$= (2n+1)\sum_{r=0}^{n}\frac{(-1)^{r-n}}{2n-2r+1}\binom{n}{r}\cdot x^{n-\frac{1}{2}}$$

$$\therefore \qquad (x^n)^{\left(\frac{1}{2}\right)} = \frac{2n+1}{\sqrt{\pi}}\sum_{r=0}^{n}\frac{(-1)^{r-n}}{2n-2r+1}\binom{n}{r}\cdot x^{n-\frac{1}{2}}$$

Here, we devise further,

$$\sum_{r=0}^{n}\frac{(-1)^{r-n}}{2n-2r+1}\binom{n}{r} = \sum_{r=0}^{n}\frac{(-1)^{n-r}}{2(n-r)+1}\binom{n}{n-r} = \sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}$$

Using this, we obtain

$$(x^n)^{\left(\frac{1}{2}\right)} = \frac{2n+1}{\sqrt{\pi}}\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}\cdot x^{n-\frac{1}{2}}$$

By-product

Comparing Formula 12.3.2 and Formula 12.3.3 , we obtain the following formula.

$$\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k} = \frac{(2n)!!}{(2n+1)!!}$$

This is the same as the by-product in **7.3.3 .**

12.3.4 Fractional Derivative of an integer power function

Generalizing Formula 12.3.3 , we calculate a fractional derivative of an integer power function. First, we prepare the following lemma.

Lemma

When m,n are natural numbers, the following expression holds.

$$\int (x-t)^{-\frac{1}{m}} t^n\, dt = \frac{m(x-t)^{\frac{m-1}{m}}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+(m-1)} \qquad \textbf{(4.0)}$$

Proof

Let

$$F(t) = \frac{m(x-t)^{\frac{m-1}{m}}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+(m-1)}$$

Differentiate this with respect to t. Then

$$\frac{d}{dt}F(t) = \frac{d}{dt}\frac{m(x-t)^{\frac{m-1}{m}}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+(m-1)}$$

$$+\frac{m(x-t)^{\frac{m-1}{m}}}{(-1)^{n+1}}\frac{d}{dt}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+(m-1)}$$

$$= -\frac{(m-1)(x-t)^{-\frac{1}{m}}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+(m-1)}$$

$$-\frac{m(x-t)^{\frac{m-1}{m}}}{(-1)^{n+1}}\sum_{r=0}^{n}(n-r)(-x)^r\binom{n}{r}\frac{(x-t)^{n-r-1}}{m(n-r)+(m-1)}$$

$$=\frac{(x-t)^{-\frac{1}{m}}}{(-1)^n}\sum_{r=0}^{n}\{(m-1)+m(n-r)\}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+(m-1)}$$

$$=\frac{(x-t)^{-\frac{1}{m}}}{(-1)^n}\sum_{r=0}^{n}\binom{n}{r}(-x)^r(x-t)^{n-r}=\frac{(x-t)^{-\frac{1}{m}}}{(-1)^n}(-t)^n$$

$$=(x-t)^{-\frac{1}{m}}t^n$$

Using this Lemma, we obtain the following formula.

Formula 12.3.4

When n, m are natural numbers such that $n\geqq 1, m\geqq 2$, the following expressions hold.

$$\left(x^n\right)^{\left(\frac{1}{m}\right)}=-\frac{m(mn+m-1)}{\Gamma(-1/m)}\sum_{k=0}^{n}\frac{(-1)^k}{mk+(m-1)}\binom{n}{k}\cdot x^{n-\frac{1}{m}} \tag{4.1}$$

$$=\frac{mn+(m-1)}{\Gamma(1-1/m)}\sum_{k=0}^{n}\frac{(-1)^k}{mk+(m-1)}\binom{n}{k}\cdot x^{n-\frac{1}{m}} \tag{4.1'}$$

$$\sum_{k=0}^{n}\frac{(-1)^k}{mk+(m-1)}\binom{n}{k}=-\frac{B(1+n,\ -1/m)}{m(mn+m-1)}\qquad B(\ ):\ \text{beta function} \tag{4.2}$$

Proof

$$\left(x^\alpha\right)^{(p)}=\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p)}x^{\alpha-p}=\frac{1}{\Gamma(k-p)}\frac{d^k}{dx^k}\int_0^x(x-t)^{k-p-1}t^\alpha dt\qquad(\alpha\geqq p)$$

Give $\alpha=n$, $p=1/m$ to this. Then since $k=\lceil 1/m\rceil=1$,

$$\left(x^n\right)^{\left(\frac{1}{m}\right)}=\frac{1}{\Gamma(1-1/m)}\frac{d^1}{dx^1}\int_0^x(x-t)^{-\frac{1}{m}}t^n dt$$

$$=-\frac{m}{\Gamma(-1/m)}\frac{d}{dx}\int_0^x(x-t)^{-\frac{1}{m}}t^n dt$$

Using the above Lemma, we calculate as follows.

$$\int_0^x(x-t)^{-\frac{1}{m}}t^n dt=\left[\frac{m(x-t)^{\frac{m-1}{m}}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+(m-1)}\right]_0^x$$

$$=\frac{m\,x^{\frac{m-1}{m}}}{(-1)^n}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{x^{n-r}}{m(n-r)+(m-1)}$$

$$= m\sum_{r=0}^{n}\frac{(-1)^{r-n}}{m(n-r)+(m-1)}\binom{n}{r}\cdot x^{n-\frac{1}{m}+1}$$

Furthermore, since r,n are integers,

$$\sum_{r=0}^{n}\frac{(-1)^{r-n}}{m(n-r)+(m-1)}\binom{n}{r} = \sum_{r=0}^{n}\frac{(-1)^{n-r}}{m(n-r)+(m-1)}\binom{n}{n-r}$$

$$= \sum_{k=0}^{n}\frac{(-1)^{k}}{mk+(m-1)}\binom{n}{k}$$

Using this,

$$\int_0^x (x-t)^{-\frac{1}{m}} t^n\,dt = m\sum_{k=0}^{n}\frac{(-1)^{k}}{mk+(m-1)}\binom{n}{k}\cdot x^{n-\frac{1}{m}+1}$$

Differentiating this with respect to x,

$$\frac{d}{dx}\int_0^x (x-t)^{-\frac{1}{m}} t^n\,dt = (mn+m-1)\sum_{k=0}^{n}\frac{(-1)^{k}}{mk+(m-1)}\binom{n}{k}\cdot x^{n-\frac{1}{m}}$$

Thus

$$\left(x^n\right)^{\left(\frac{1}{m}\right)} = -\frac{m(mn+m-1)}{\Gamma(-1/m)}\sum_{k=0}^{n}\frac{(-1)^{k}}{mk+(m-1)}\binom{n}{k}\cdot x^{n-\frac{1}{m}} \tag{4.1}$$

Moreover, (4.1') follows immediately from this.

Next,

$$\left(x^n\right)^{\left(\frac{1}{m}\right)} = \frac{\Gamma(1+n)}{\Gamma(1+n-1/m)}x^{n-\frac{1}{m}} = \frac{\Gamma(1+n)}{\Gamma(-1/m)}\frac{\Gamma(-1/m)}{\Gamma(1+n-1/m)}x^{n-\frac{1}{m}}$$

Since this have to be equal to (4.1) ,

$$-m(mn+m-1)\sum_{k=0}^{n}\frac{(-1)^{k}}{mk+(m-1)}\binom{n}{k} = \frac{\Gamma(1+n)\Gamma(-1/m)}{\Gamma(1+n-1/m)}$$

From this

$$\sum_{k=0}^{n}\frac{(-1)^{k}}{mk+(m-1)}\binom{n}{k} = -\frac{B(1+n,\,-1/m)}{m(mn+m-1)} \tag{4.2}$$

Remark

(4.2) suggests that (4.1) can be expressed with a beta function and n,m can be real numbers. Actuality,

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p)} = \frac{\Gamma(1+\alpha)}{\Gamma(-p)}\frac{\Gamma(-p)}{\Gamma(1+\alpha-p)} = \frac{B(1+\alpha,\,-p)}{\Gamma(-p)}$$

Then,

$$\left(x^\alpha\right)^{(p)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p)}x^{\alpha-p} = \frac{B(1+\alpha,\,-p)}{\Gamma(-p)}x^{\alpha-p} \qquad (\alpha \geqq 0)$$

12.3.5 Super Derivative of an integer power function

Replacing $1/m$ with p in Formula 12.3.4 , we obtain the following formula.

Formula 12.3.5

When n is a natural number, the following expressions hold for $0<p\leqq n$.

$$\left(x^{n}\right)^{(p)} = \frac{n+1-p}{\Gamma(1-p)} \sum_{k=0}^{n} \frac{(-1)^{k}}{k+1-p} \binom{n}{k} \cdot x^{n-p} \tag{5.1}$$

$$B(n,-p) = -\frac{n-p}{p} \sum_{k=0}^{n-1} \frac{(-1)^{k}}{k+1-p} \binom{n-1}{k} \qquad B(\ )\text{ denots beta function.} \tag{5.2}$$

Example 1

$$\left(x^{2}\right)^{(\sqrt{3})} = \frac{\Gamma(1+2)}{\Gamma\left(1+2-\sqrt{3}\right)} x^{2-\sqrt{3}} = \frac{2}{\Gamma\left(3-\sqrt{3}\right)} x^{2-\sqrt{3}} = 2.2151\, x^{2-\sqrt{3}}$$

$$\left(x^{2}\right)^{(\sqrt{3})} = \frac{3-\sqrt{3}}{\Gamma\left(1-\sqrt{3}\right)} \left\{ \frac{1}{1-\sqrt{3}} \binom{2}{0} - \frac{1}{2-\sqrt{3}} \binom{2}{1} + \frac{1}{3-\sqrt{3}} \binom{2}{2} \right\} x^{2-\sqrt{3}}$$

$$= 2.2151\, x^{2-\sqrt{3}}$$

Example 2

$$B\left(2,-\sqrt{3}\right) = -\frac{2-\sqrt{3}}{\sqrt{3}} \left\{ \frac{1}{1-\sqrt{3}} \binom{1}{0} - \frac{1}{2-\sqrt{3}} \binom{1}{1} \right\} = 0.788675\cdots$$

$$B(3,-e) = -\frac{3-e}{e} \left\{ \frac{1}{1-e} \binom{2}{0} - \frac{1}{2-e} \binom{2}{1} + \frac{1}{3-e} \binom{2}{2} \right\} = -0.596137\cdots$$

12.3.6 Super Derivative of a positive power function

Since Formula12.3.5 are binomial forms, the further generalization is possible.

Formula 12.3.6

The following expressions hold for p,q such that $0<p \leqq q$.

$$\left(x^{q}\right)^{(p)} = \frac{q+1-p}{\Gamma(1-p)} \sum_{r=0}^{\infty} \frac{(-1)^{r}}{r+1-p} \binom{q}{r} \cdot x^{q-p} \tag{6.1}$$

$$B(q,-p) = -\frac{q-p}{p} \sum_{r=0}^{\infty} \frac{(-1)^{r}}{r+1-p} \binom{q-1}{r} \qquad B(\ )\text{ denots beta function.} \tag{6.2}$$

12.3.7 Super Derivative of a polynomial

In the case of a polynomial $f(x) = \sum_{k=0}^{m} c_{m-k}\, x^{m-k}$, as for the zero of the super derivative, it is good to perform it as follows. It is based on experience of a writer.

(1) When $f(x)$ is factored by primary formula $ax+b$, let $-b/a$ be the zero.
(2) When $f(x)$ is not factored by primary formula $ax+b$, let 0 be a the zero.

For example, in the case of the 1/2 times derivative of $f(x)=x^{2}-2\pi x+\pi^{2}$, the following calculation is right in many case.

$$\left(x^{2}-2\pi x+\pi^{2}\right)^{\left(\frac{1}{2}\right)} = \left\{ (x-\pi)^{2} \right\}^{\left(\frac{1}{2}\right)} = \frac{8}{3\sqrt{\pi}} (x-\pi)^{\frac{3}{2}}$$

If this is calculated termwise as follows, the result is different from the former.

$$\left(x^2-2\pi x+\pi^2\right)^{\left(\frac{1}{2}\right)} = \left(x^2\right)^{\left(\frac{1}{2}\right)} - 2\pi\left(x^1\right)^{\left(\frac{1}{2}\right)} + \pi^2\left(x^0\right)^{\left(\frac{1}{2}\right)}$$
$$= \frac{1}{\sqrt{\pi}}\left(\frac{8}{3}x^{\frac{3}{2}} - 4\pi x^{\frac{1}{2}} + \pi^2 x^{-\frac{1}{2}}\right)$$

Needless to say, this cause is the difference between

$$\int_{\pi}^{x}\sim\int_{\pi}^{x}\left(x^2-2\pi x+\pi^2\right)dx^{-\frac{1}{2}} \text{ and } \int_{0}^{x}\sim\int_{0}^{x}\left(x^2-2\pi x\right)dx^{-\frac{1}{2}}+\int_{\infty}^{x}\sim\int_{\infty}^{x}\pi^2 dx^{-\frac{1}{2}}$$

That is, it is because the latter regarded it as 0 and ∞ although the former regarded the zero of the super derivative as pi.

The latter is right when there is a special reason why the zero of the super derivative should be 0. However, such a case is rare, and in almost all cases the former is right.

12.4 Super Derivative of Exponential Function

Analytically continuing the index of the differentiation operator in Formula 9.2.2 (**9.2**) to $[0,p]$ from $[1,n]$ we obtain the following formula. In addition, Rieamnn-Liouville diffeintegrals are also expressed together

Formula 12.4.1

When $n=\lceil p\rceil$ is ceiling function, the following expressions hold

(1) Basic form

$$\left(e^{\pm x}\right)^{(p)} = (\pm 1)^{-p}e^{\pm x} = \frac{1}{\Gamma(n-p)}\frac{d^n}{dx^n}\int_{\mp\infty}^{x}(x-t)^{n-p-1}e^{\pm t}dt$$

(2) Linear form

$$\left(e^{ax+b}\right)^{(p)} = \left(\frac{1}{a}\right)^{-p}e^{ax+b} = \frac{1}{\Gamma(n-p)}\frac{d^n}{dx^n}\int_{\mp\infty}^{x}(x-t)^{n-p-1}e^{at+b}dt$$

$$(a>0:-\ ,\ a<0:+)$$

(3) General form

$$\left(\alpha^{ax+b}\right)^{(p)} = \left(\frac{1}{a\log\alpha}\right)^{-p}\alpha^{ax+b} = \frac{1}{\Gamma(n-p)}\frac{d^n}{dx^n}\int_{\mp\infty}^{x}(x-t)^{n-p-1}\alpha^{at+b}dt$$

$$(a>0:-\ ,\ a<0:+)$$

Proof of the general form

Let $c=a\log\alpha\ ,\ d=b\log\alpha$, then

$$e^{cx+d} = e^{xa\log\alpha+b\log\alpha} = e^{ax\log\alpha}e^{b\log\alpha} = \left(e^{\log\alpha}\right)^{ax}\left(e^{\log\alpha}\right)^{b} = \alpha^{ax}\alpha^{b} = \alpha^{ax+b}$$

Applying this to (2) Linear form, we obtain (3) General form immediately.

Example

$$\left(e^{-x}\right)^{\left(\frac{1}{2}\right)} = (-1)^{-\frac{1}{2}}e^{-x} = \frac{1}{i}e^{-x} = -ie^{-x}$$

$$\left(e^{3x-4}\right)^{(\sqrt{2})} = \left(\frac{1}{3}\right)^{-\sqrt{2}}e^{3x-4} = 4.728804\,e^{3x-4}$$

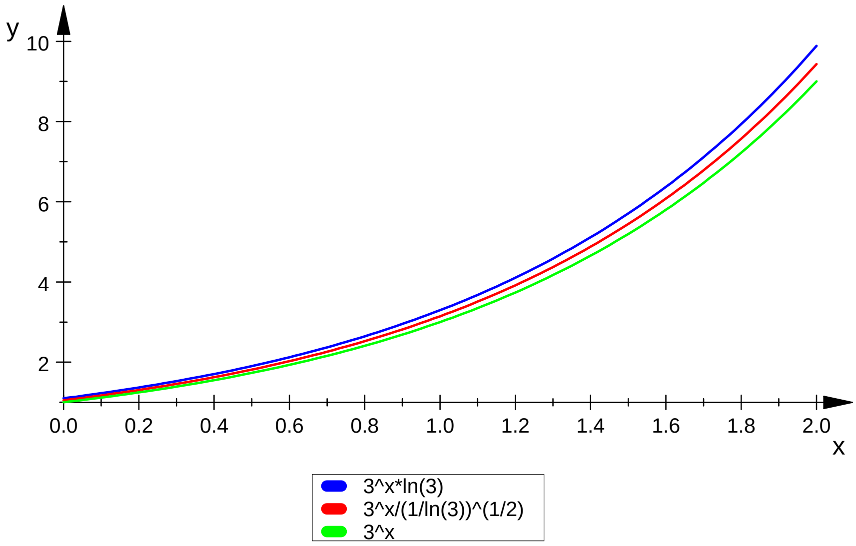
$$\left(2^{x}\right)^{(i)} = \left(\frac{1}{\log 2}\right)^{-i}2^{x} = (0.933582 - 0.358362\,i)\times 2^{x}$$

$$\left\{(-3)^{x}\right\}^{\left(\frac{1}{2}\right)} = \left(\frac{1}{\log(-3)}\right)^{-\frac{1}{2}}(-3)^{x}$$

$$= (1.487742 + 1.05582\,i)\times(-3)^{x}$$

$$\left(3^{x}\right)^{\left(\frac{1}{2}\right)} = \left(\frac{1}{\log 3}\right)^{-1/2}3^{x} = 1.048147\times 3^{x}$$

When $\left(3^{x}\right)^{(1)}$, $\left(3^{x}\right)^{(1/2)}$, 3^{x} are drawn on a figure side by side, it is as follows.


y
10
8
6
4
2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
x
3^x*ln(3)
3^x/(1/ln(3))^(1/2)
3^x

12.5 Super Derivative of Logarithmic Function

Reversing the sign of the index of the integration operator in Formula 7.5.1 (**7.5**), we obtain the following formula. In addition, Rieamnn-Liouville differintegrals are also expressed together

Formula 12.5.1

When $\Gamma(z), \psi(z), n=\lceil p \rceil$ denote zeta function, psi function, ceiling function respectively, the following expressions hold.

(1) Basic form

$$(\log x)^{(p)} = \frac{\log x - \psi(1-p) - \gamma}{\Gamma(1-p)} x^{-p} = \frac{1}{\Gamma(n-p)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-p-1} \log t \, dt$$

(2) Linear form

$$\{\log(ax+b)\}^{(p)} = \frac{\log(ax+b) - \psi(1-p) - \gamma}{\Gamma(1-p)} \left(x + \frac{b}{a} \right)^{-p}$$

$$= \frac{1}{\Gamma(n-p)} \frac{d^n}{dx^n} \int_{-\frac{b}{a}}^x (x-t)^{n-p-1} \log(at+b) \, dt$$

Where, $\dfrac{\psi(1-p)}{\Gamma(1-p)} = (-1)^p (p-1)!$ for $p=1,2,3,\cdots$

Proof

Formula 7.5.1 (**7.5**) was as follows.

$$\int_0^x \sim \int_0^x \log x \, dx^p = \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} x^p$$

$$\int_{-\frac{b}{a}}^x \sim \int_{-\frac{b}{a}}^x \log(ax+b) \, dx^p = \frac{\log(ax+b) - \psi(1+p) - \gamma}{\Gamma(1+p)} \left(x + \frac{b}{a} \right)^p$$

Since differentiation is the reverse operation of integration, replacing the index p of the integration operator with $-p$, we obtain the desired expressions. And Formula 1.3.1 (**1.3**) was

$$\frac{\psi(-n)}{\Gamma(-n)} = (-1)^{n+1} n! \ , \qquad n=0,1,2,3,\cdots$$

Then, replacing n with $p-1$, we obtain

$$\frac{\psi(1-p)}{\Gamma(1-p)} = (-1)^p (p-1)! \ , \qquad p=1,2,3,\cdots$$

Example

$$(\log x)^{(1)} = \frac{\log x - \psi(1-1) - \gamma}{\Gamma(1-1)} x^{-1} = -(-1)^1 0! \, x^{-1} = x^{-1}$$

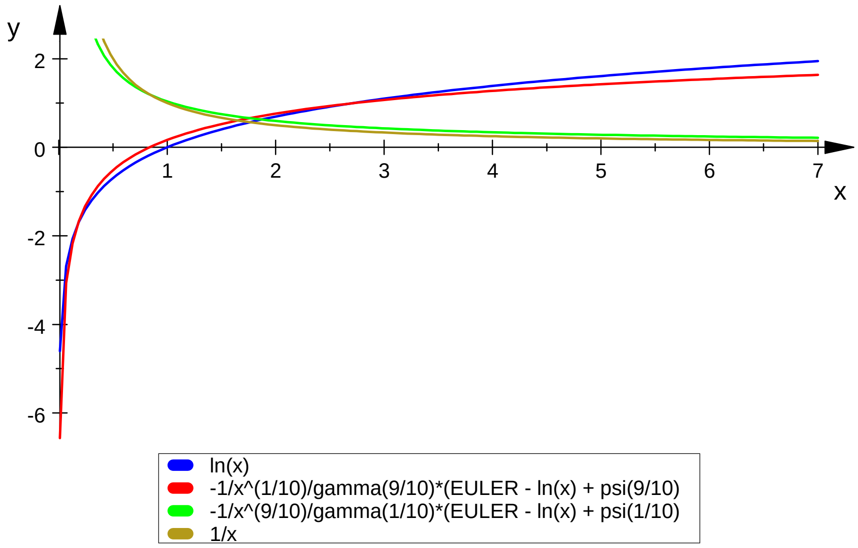
$$\{\log(3x+4)\}^{\left(\frac{1}{2}\right)} = \frac{\log(3x+4) - \psi\left(1-\frac{1}{2}\right) - \gamma}{\Gamma\left(1-\frac{1}{2}\right)} \left(x + \frac{4}{3} \right)^{-\frac{1}{2}}$$

$$= 0.5641(\log x + 1.3862) / \sqrt{x+1.3333}$$

$$(\log x)^{\left(\frac{1}{10}\right)} = \frac{\log x - \psi\left(1-\frac{1}{10}\right) - \gamma}{\Gamma\left(1-\frac{1}{10}\right)} x^{-\frac{1}{10}} = 0.9357 x^{-\frac{1}{10}}(\log x + 0.1777)$$

$$(\log x)^{\left(\frac{9}{10}\right)} = \frac{\log x - \psi\left(1-\frac{9}{10}\right) - \gamma}{\Gamma\left(1-\frac{9}{10}\right)} x^{-\frac{9}{10}} = 0.1051 x^{-\frac{9}{10}}(\log x + 9.8465)$$

When $\log x$, $(\log x)^{(1/10)}$, $(\log x)^{(9/10)}$, $(\log x)^{(1)}$ are drawn on a figure side by side, it is as follows.



12.6 Super Derivative of Trigonometric Function

12.6.1 Super Derivatives of sin x , cos x

Analytically continuing the index of the differentiation operator in Formula 9.2.4 (**9.2**) to $[0,p]$ from $[1,n]$ fwe obtain the following formula.

Formula 12.6.1

(1) Basic form

$$(\sin x)^{(p)} = \sin\left(x+\frac{p\pi}{2}\right)$$

$$(\cos x)^{(p)} = \cos\left(x+\frac{p\pi}{2}\right)$$

(2) Linear form

$$\{\sin(ax+b)\}^{(p)} = \left(\frac{1}{a}\right)^{-p}\sin\left(ax+b+\frac{p\pi}{2}\right)$$

$$\{\cos(ax+b)\}^{(p)} = \left(\frac{1}{a}\right)^{-p}\cos\left(ax+b+\frac{p\pi}{2}\right)$$

Example

$$(\sin x)^{\left(\frac{1}{2}\right)} = \sin\left(x+\frac{1}{2}\frac{\pi}{2}\right) = \sin\left(x+\frac{\pi}{4}\right)$$

$$((\sin x)^{\left(\frac{1}{2}\right)})^{\left(\frac{1}{2}\right)} = (\sin(x+\frac{\pi}{4}))^{\left(\frac{1}{2}\right)}$$

$$= \left(\frac{1}{1}\right)^{-\frac{1}{2}}\sin\left(x+\frac{\pi}{4}+\frac{1}{2}\frac{\pi}{2}\right) = \sin\left(x+\frac{\pi}{2}\right) = \cos x$$

$$\left\{\sin\left(x-\frac{\pi}{2}\right)\right\}^{(1)} = \left(\frac{1}{1}\right)^{-1}\sin\left(x-\frac{\pi}{2}+\frac{1\cdot\pi}{2}\right) = \sin x$$

$$(\cos x)^{\left(\frac{1}{2}\right)} = \cos\left(x+\frac{1}{2}\frac{\pi}{2}\right) = \cos\left(x+\frac{\pi}{4}\right)$$

$$(\cos x)^{\left(\frac{2}{\pi}\right)} = \cos\left(x+\frac{2}{\pi}\frac{\pi}{2}\right) = \cos(x+1)$$

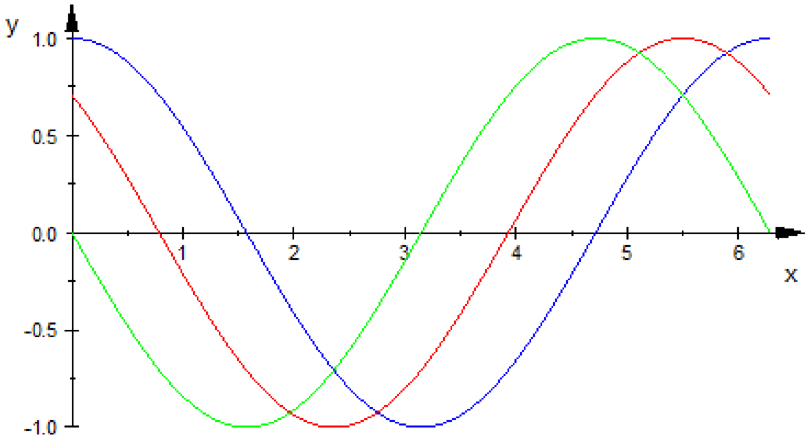
When $\cos x$, $(\cos x)^{(1/2)}$, $-\sin x$ are drawn on a figure side by side, it is as follows. Red shows 1/2 order super derivative. It is clear also in the figure that super derivative which is the easiest to understand is super derivative of trigonometric functions.

Lineal Super Derivative of cos x

```
• cosl := p-> cos(x+p*PI/2)
```

$$p \to \cos\left(x+\frac{\pi\cdot p}{2}\right)$$

- `plotfunc2d(cos(x),cos1(1/2),-sin(x), x=0..2*PI)`



12.6.2 Termwise Super Derivative of sin x , cos x

Reversing the sign of the index of the operator of the collateral super integrals of sin x , cos x in **7.6.2** (**7.6**) , we obtain the following termwise super derivatives. These are ***collateral super derivatives*** as understood from the constant-of-differentiation function in the right side.

$$(\sin x)^{(p)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k+2-p)} x^{2k+1-p} = \sin\left(x+\frac{p\pi}{2}\right) + C(p,x)$$

$$C(p,x) = \sum_{k=1}^{p} \frac{x^{-p-k}}{\Gamma(1-p-k)} \sin\frac{k\pi}{2}$$

$$(\cos x)^{(p)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k+1-p)} x^{2k-p} = \cos\left(x+\frac{p\pi}{2}\right) - C(p,x)$$

$$C(p,x) = \sum_{k=1}^{p} \frac{x^{-p-k}}{\Gamma(1-p-k)} \cos\frac{k\pi}{2}$$

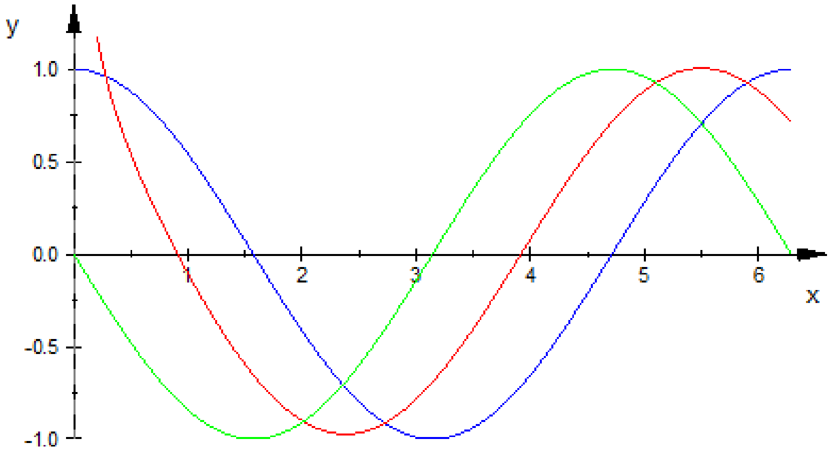
When the 1/2th order collateral super derivative of cos x is drawn as cos x and -sin x side by side, it is as follows. Red shows the 1/2th order collateral super derivative.

Collateral Super derivative of cos x

- `cosc := p-> sum((-1)^k/gamma(2*k+1-p)*x^(2*k-p), k=0..50)`

$$p \to \sum_{k=0}^{50} \frac{(-1)^k}{\Gamma(2\cdot k+1-p)} \cdot x^{2\cdot k-p}$$

- `plotfunc2d(cos(x),cosc(1/2),-sin(x), x=0..2*PI)`



Compared with the upper figure, the collateral super derivative is curving unnaturally near the coordinate origin in this figure. Since this is similar also in $\sin x$, it is thought that the termwie super derivatives of $\sin x$ and $\cos x$ are asymptotic expansions of the lineal super derivatives.

12.7 Super Derivative of Hyperbolic Function

12.7.1 Super Derivatives of sinh x , cosh x

Analytically continuing the index of the differentiation operator in Formula 9.2.5 (**9.2**) to $[0,p]$ from $[1,n]$ we obtain the following formula.

Formula 12.7.1

(1) Basic form

$$(\sinh x)^{(p)} = i^{-p}\sinh\left(x+\frac{p\pi i}{2}\right) = \frac{e^{x}-(-1)^{-p}e^{-x}}{2}$$

$$(\cosh x)^{(p)} = i^{-p}\cosh\left(x+\frac{p\pi i}{2}\right) = \frac{e^{x}+(-1)^{-p}e^{-x}}{2}$$

(2) Linear form

$$\{\sinh(ax+b)\}^{(p)} = \left(\frac{i}{a}\right)^{-p}\sinh\left(ax+b+\frac{p\pi i}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a}\right)^{-n}\left\{e^{ax+b}-(-1)^{-p}e^{-(ax+b)}\right\}$$

$$\{\cosh(ax+b)\}^{(p)} = \left(\frac{i}{a}\right)^{-p}\cosh\left(ax+b+\frac{p\pi i}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a}\right)^{-n}\left\{e^{ax+b}+(-1)^{-p}e^{-(ax+b)}\right\}$$

Example 1

$$(\sinh x)^{\left(\frac{1}{2}\right)} = i^{-\frac{1}{2}}\sinh\left(x+\frac{1}{2}\frac{\pi i}{2}\right) = i^{-\frac{1}{2}}\sinh\left(x+\frac{\pi i}{4}\right)$$

$$\left((\sinh x)^{\left(\frac{1}{2}\right)}\right)^{\left(\frac{1}{2}\right)} = \left(i^{-\frac{1}{2}}\sinh\left(x+\frac{\pi i}{4}\right)\right)^{\left(\frac{1}{2}\right)}$$

$$= i^{-\frac{1}{2}}i^{-\frac{1}{2}}\sinh\left(x+\frac{\pi i}{4}+\frac{1}{2}\frac{\pi i}{2}\right)$$

$$= i^{-1}\sinh\left(x+\frac{\pi i}{2}\right) = \cosh x$$

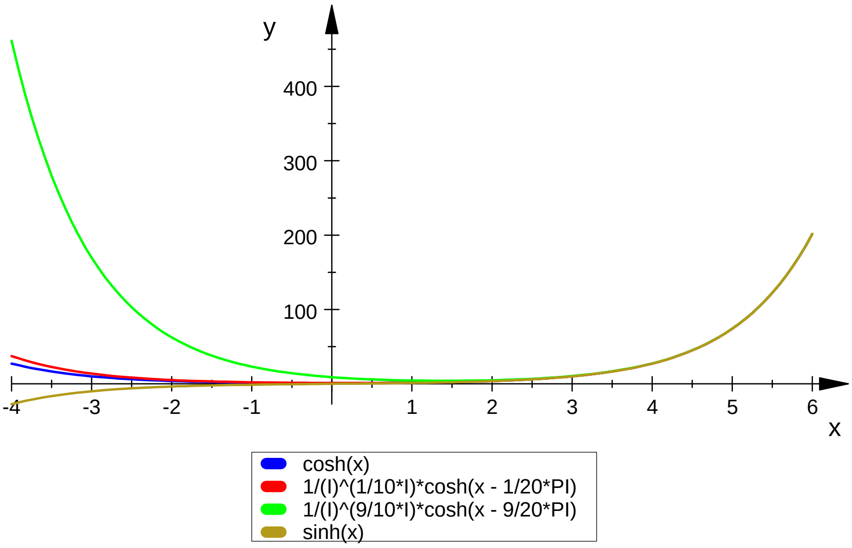
$$\left\{\cosh\left(x+\frac{\pi i}{2}\right)\right\}^{(1)} = \left(\frac{i}{1}\right)^{-1}\cosh\left(x+\frac{\pi i}{2}+\frac{1\cdot\pi i}{2}\right)$$

$$= -i\cosh(x+\pi i) = i\cosh x$$

$$(\cosh x)^{\left(\frac{i}{10}\right)} = i^{-\frac{i}{10}}\cosh\left(x-\frac{\pi}{20}\right) = 1.170088\times\cosh\left(x-\frac{\pi}{20}\right)$$

$$(\cosh x)^{\left(\frac{9i}{10}\right)} = i^{-\frac{9i}{10}}\cosh\left(x-\frac{9\pi}{20}\right) = 4.111207\times\cosh\left(x-\frac{9\pi}{20}\right)$$

Super derivative of hyperbolic function is the most incomprehensible in super derivatives. The reason is that the super derivative turns into a complex function except the order p is an integer or a purely imaginary number. Then, when $\cosh x$, $(\cosh x)^{(i/10)}$, $(\cosh x)^{(9i/10)}$, $\sinh x$ which can be displayed on a real number domain are drawn on a figure side by side, it is as follows.



All of four curves have overlapped in the positive area. It is natural that $(\cosh x)^{(i/10)}$ is near $\cosh x$ in a negative area. But $(\cosh x)^{(9i/10)}$ is far apart from $\sinh x$ in why.

Example 2

$$(\sinh x)^{\left(\frac{1}{2}\right)} = \frac{e^x - (-1)^{-\frac{1}{2}} e^{-x}}{2} = \frac{e^x + i\, e^{-x}}{2}$$

$$\left((\sinh x)^{\left(\frac{1}{2}\right)}\right)^{\left(\frac{1}{2}\right)} = \left(\frac{e^x + i\, e^{-x}}{2}\right)^{\left(\frac{1}{2}\right)} = \frac{e^x}{2} + \frac{i}{2} \times \left(e^{-x}\right)^{\left(\frac{1}{2}\right)}$$

$$= \frac{e^x}{2} + \frac{i}{2}\left(-i\, e^{-x}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\left\{\cosh\left(x + \frac{\pi i}{2}\right)\right\}^{(1)} = \frac{1}{2}\left(\frac{1}{1}\right)^{-1}\left\{e^{x+\frac{\pi i}{2}} + (-1)^{-1} e^{-\left(x+\frac{\pi i}{2}\right)}\right\}$$

$$= \frac{1}{2}\left(e^{\frac{\pi i}{2}} e^x - e^{-\frac{\pi i}{2}} e^{-x}\right) = \frac{1}{2}\left(i\, e^x - \frac{1}{i} e^{-x}\right)$$

$$= \frac{i}{2}\left(e^x + e^{-x}\right) = i \cosh x$$

$$(\sinh x)^{\left(\frac{i}{\pi}\right)} = \frac{e^x - (-1)^{-\frac{i}{\pi}} e^{-x}}{2} = \frac{e^x - \left(e^{\pi i}\right)^{-\frac{i}{\pi}} e^{-x}}{2} = \frac{e^x - e^{1-x}}{2}$$

12.7.2 Termwise Super Derivative of sinh x , cosh x

Reversing the sign of the index of the operator of the collateral super integrals of sinh x , cosh x in **7.7.2** we obtain the following termwise super derivatives. These are ***collateral super derivatives*** as understood from the constant-of-integration function in the right side.

$$(\sinh x)^{(p)} = \sum_{k=0}^{\infty} \frac{x^{2k+1-p}}{\Gamma(2k+2-p)} = i^{-p} \sinh\left(x+\frac{p\pi i}{2}\right) - C(p,x)$$

$$C(p,x) \doteqdot \sum_{k=1}^{p\uparrow} \frac{x^{-p-k}}{\Gamma(1-p-k)} i^{-k} \sinh\frac{k\pi i}{2}$$

$$(\cosh x)^{(p)} = \sum_{k=0}^{\infty} \frac{x^{2k-p}}{\Gamma(2k+1-p)} = i^{-p} cohs\left(x+\frac{p\pi i}{2}\right) - C(p,x)$$

$$C(p,x) \doteqdot \sum_{k=1}^{p\uparrow} \frac{x^{-p-k}}{\Gamma(1-p-k)} i^{-k} \cosh\frac{k\pi i}{2}$$

When the values of the lineal and the collatera l .7th order derivatives of cosh x on x=1, x=6 are calculated respectively, it is as follows.

Lineal Super Derivative of cosh x

```
• coshl := p-> I^-p*cosh(x+p*PI/2*I)
```

$$p \to \frac{1}{i^{p}} \cdot \cosh\left(x + i \cdot \frac{\pi \cdot p}{2}\right)$$

Collateral Super derivative of cosh x

```
• coshc := p-> sum(x^(2*k-p)/gamma(2*k+1-p), k=0..50)
```

$$p \to \sum_{k=0}^{50} \frac{x^{2 \cdot k - p}}{\Gamma(2 \cdot k + 1 - p)}$$

x = 1

```
• float(subs(coshl(1.7), x=1)), float(subs(coshc(1.7), x=1))
```

$1.467257969 + 0.1488103599 \cdot i,$ 1.279969476

x = 6

```
• float(subs(coshl(1.7), x=6)), float(subs(coshc(1.7), x=6))
```

$201.7151252 + 0.001002676318 \cdot i,$ 201.7170572

Though the difference of both is large where x is small, both are almost corresponding ing where x is large. Since this is similar also in sinh x, it is thought that the termwise super derivatives of sinh x and cosh x are asymptotic expansions of the lineal super derivatives.

12.8 Super Derivative of Inverse Trigonometric Function

12.8.1 Super Derivatives of $tan^{-1}x, cot^{-1}x$

Reversing the sign of the index of the integration operator in Formula 7.8.1 (**7.8**), we obtain the following formula.

Formula 12.8.1

When $\Gamma(x), \psi(x)$ denote gamma function and digamma function respectively, the following expressions hold for $|x| \geqq 1$.

$$\left(tan^{-1}x\right)^{(p)} = \frac{tan^{-1}x}{\Gamma(1-p)}\sum_{k=0}^{\infty}(-1)^k\binom{-p}{-p-2k}x^{-p-2k}$$
$$+\frac{log\left(1+x^2\right)}{2\Gamma(1-p)}\sum_{k=1}^{\infty}(-1)^k\binom{-p}{-p+1-2k}x^{-p+1-2k}$$
$$-\frac{1}{\Gamma(1-p)}\sum_{r=1}^{\infty}(-1)^r\binom{-p}{-p+1-2r}\{\psi(1-p)-\psi(2r)\}x^{-p+1-2r}$$

$$\left(cot^{-1}x\right)^{(p)} = \frac{x^{-p}}{\Gamma(1-p)}cot^{-1}x - \frac{tan^{-1}x}{\Gamma(1-p)}\sum_{k=1}^{\infty}(-1)^k\binom{-p}{-p-2k}x^{-p-2k}$$
$$-\frac{log\left(1+x^2\right)}{2\Gamma(1-p)}\sum_{k=1}^{\infty}(-1)^k\binom{-p}{-p+1-2k}x^{-p+1-2k}$$
$$+\frac{1}{\Gamma(1-p)}\sum_{r=1}^{\infty}(-1)^r\binom{-p}{-p+1-2r}\{\psi(1-p)-\psi(2r)\}x^{-p+1-2r}$$

Example: 1/2th order derivative of $cot^{-1}x$

```
• p:=1/2: m:=110:
```

Super derivative of arccot x

```
• f := x-> arccot(x)/(gamma(1-p)*x^p)
            - arctan(x)/gamma(1-p)*sum((-1)^k*binomial(-p,-p-2*k)*x^-
                                                          p-2*k),k=1..m)
            - ln(1+x^2)/(2*gamma(1-p))*sum((-1)^k*binomial(-p,-p+1-
                                              2*k)*x^(-p+1-2*k), k=1..m)
            + 1/gamma(1-p)*sum((-1)^r*binomial(-p,-p+1-2*r)*(psi(1-p
                                         psi(2*r))*x^(-p+1-2*r), r=1..m)
```

$$x \rightarrow \frac{\operatorname{arccot}(x)}{\Gamma(1-p)\cdot x^p} - \frac{\arctan(x)}{\Gamma(1-p)}\cdot\left(\sum_{k=1}^{m}(-1)^k\cdot\binom{-p}{-p-2\cdot k}\cdot x^{-p-2\cdot k}\right)$$
$$-\frac{\ln\left(1+x^2\right)}{2\cdot\Gamma(1-p)}\cdot\left(\sum_{k=1}^{m}(-1)^k\cdot\binom{-p}{1-p-2\cdot k}\cdot x^{1-p-2\cdot k}\right)$$
$$+\frac{1}{\Gamma(1-p)}\cdot\left(\sum_{r=1}^{m}(-1)^r\cdot\binom{-p}{1-p-2\cdot r}\cdot(\operatorname{psi}(1-p)-\operatorname{psi}(2\cdot r))\cdot x^{1-p-2\cdot r}\right)$$

```
• float(f(1.1))
```

0.07073996634

Riemann-Liouville differintegral

```
• g := x-> 1/gamma(1-p)*int((x-t)^(1-p-1)*arccot(t), t=0..x)
```

$$x \to \frac{1}{\Gamma(1-p)} \cdot \int_0^x (x-t)^{1-p-1} \cdot \operatorname{arccot}(t) \, dt$$

```
• h := 10^-11:
• float((g(1.1+h)-g(1.1))/h)
```

0.07073997706

2010.07.07

K. Kono

Alien's Mathematics