19 Super Derivative of the Product of Two Functions

19.1 Super Leibniz Rule

Theorem 19.1.1

Let $B(x,y)$ be the beta function and p be a positive number. And for $r=0,1,2,\cdots$, let $f^{<-p+r>}$ be arbitrary primitive function of $f(x)$ and $g^{(r)}$ be the r th order derivative function of $g(x)$.
Then the following expressions hold.

$$\{f(x)g(x)\}^{(p)} = \sum_{r=0}^{m-1}\binom{p}{r} f^{(p-r)}(x)g^{(r)}(x) + R_m^p \tag{1.1}$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-p-1}{k}\left\{f^{<m+k>}(x)g^{(m+k)}(x)\right\}^{(p)} \tag{1.1r}$$

Especially, when $n=0,1,2,\cdots$

$$\{f(x)g(x)\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r} f^{(n-r)}(x)g^{(r)}(x) \qquad \text{(Leibniz)} \tag{1.1'}$$

Proof

Theorem 16.1.2 (2.1) in **16.1.2** was as follows.

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<0>}g^{(0)}dx^n = \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>}g^{(r)}$$

$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s} f_{a_{n-r}}^{<n-r+s>} g_{a_{n-r}}^{(s)}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$+(-1)^m\sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s\cdot{}_{m+n-1-r+t}C_{m-1}\, f_{a_{n-r}}^{\langle m+n-r+s\rangle} g_{a_{n-r}}^{(m+s)}\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$+\frac{(-1)^m}{B(n,m)}\sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<m+k>}g^{(m+k)}dx^n$$

When $n=1$ $\sum\sum\sum$ of the 3rd line does not exist, when $n=0$ $\sum\sum$ of the 2nd line does not exist also. And the upper limit $n-1$ of $\sum$ of the 4th line can be replaced by ∞. Therefore, if the index n of the integration operator is substituted for $-n$ in consideration of these, it is as follows.

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<0>}g^{(0)}dx^{-n} = \sum_{r=0}^{m-1}\binom{n}{r} f^{<-n+r>}g^{(r)}$$

$$+\frac{(-1)^m}{B(-n,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-n-1}{k}\int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<m+k>}g^{(m+k)}dx^{-n}$$

Analytically continuing the index of the integration operator to $[0,p]$ from $[1,n]$,

$$\int_{a(p)}^{x}\sim\int_{a(0)}^{x} f^{<0>}g^{(0)}dx^{-p} = \sum_{r=0}^{m-1}\binom{p}{r} f^{<-p+r>}g^{(r)}$$

$$+\frac{(-1)^m}{B(-p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-p-1}{k}\int_{a(p)}^{x}\sim\int_{a(0)}^{x} f^{<m+k>}g^{(m+k)}dx^{-p}$$

Then, replacing the integration operators $dx^{-p}, <-p+r>$ with the differentiation operators $(p), (p-r)$ respectivly, we obtain (1.1) and (1.1r) .

Especially, when $p=m-1$, $m=1,2,3,\cdots$, since $B(-p,m)=B(1-m,\, m) = \pm\infty$, $R_m^p=0$.
Then

$$\{f(x)g(x)\}^{(m-1)} = \sum_{r=0}^{m-1} \binom{m-1}{r} f^{(m-1-r)} g^{(r)}$$

Furthermore, replacing $m-1$ with n , we obtain (1.1') .

Q.E.D

19.2 Super Derivative of x^a f (x)

Formula 19.2.0

Let $\Gamma(z)$ be the gamma function, $B(n,m)$ be the beta function, $f^{<r>}$ be arbitrary the r th primitive function of $f(x)$ and $f_a^{<r>}$ be the function values of $f^{<r>}$ on a . Then the following expressions hold for a positive number p .

(1)

$$\left\{x^{\alpha} f(x)\right\}^{(p)} = \sum_{r=0}^{m-1}\binom{p}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}f^{(p-r)}$$

$$+\frac{(-1)^m}{B(-p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-p-1}{k}\left\{\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-k)}x^{\alpha-m-k}f^{<m+k>}\right\}^{(p)} \tag{0.1}$$

Especially, when $m=0,1,2,\cdots$

$$\left\{x^{m} f(x)\right\}^{(p)} = \sum_{r=0}^{m}\binom{p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}f^{(p-r)} \tag{0.1'}$$

Where, if $\alpha = -1,-2,-3,\cdots$, it shall read as follows.

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} \longrightarrow (-1)^{-r}\frac{\Gamma(-\alpha+r)}{\Gamma(-\alpha)}$$

(2) When $\alpha \ne -1,-2,-3,\cdots$ & $\alpha-p \ne -1,-2,-3,\cdots$

$$\left\{x^{m} f(x)\right\}^{(p)} = \sum_{r=0}^{m-1}\binom{p}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)}x^{\alpha-p+r}f^{(r)}$$

$$+\frac{(-1)^m}{B(-p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-p-1}{k}\left\{\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)}x^{\alpha+m+k}f^{(m+k)}\right\}^{(p)} \tag{0.2}$$

Proof

Since differentiation is an inverse operation of integration, replacing the index p of the integration operator with $-p$ in Formula17.2.0 in **17.2** , we obtain the desired expressions.

19.2.1 Super Derivative of $(ax+b)^p(cx+d)^q$

Formula 19.2.1

The following expressions hold for $p>0$, $s>0$ such that $p-s \ne -1,-2,-3,\cdots$

$$\left\{(ax+b)^p(cx+d)^q\right\}^{(s)}$$

$$=\sum_{r=0}^{m-1}\binom{s}{r}\frac{(1/a)^{-s+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p-s+r)\Gamma(1+q-r)}\frac{(ax+b)^{p-s+r}}{(cx+d)^{r-q}}+R_m^s \tag{1.1}$$

$$R_m^s = \frac{(-1)^m}{B(-s,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-s-1}{k}\left(\frac{c}{a}\right)^{m+k}\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p+m+k)\Gamma(1+q-m-k)}$$

$$\times\left\{(ax+b)^{p+m+k}(cx+d)^{q-m-k}\right\}^{(s)} \tag{1.1r}$$

$$\lim_{m\to\infty} R_m^s = 0$$

Especially, when $m=0,1,2,\cdots$

$$\left\{(ax+b)^p (cx+d)^m\right\}^{(s)} = \sum_{r=0}^{m}\binom{s}{r}\frac{(1/a)^{-s+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+m)}{\Gamma(1+p-s+r)\Gamma(1+m-r)}\frac{(ax+b)^{p-s+r}}{(cx+d)^{r-m}} \tag{1.1'}$$

Proof

Although it is an original way to substitute $f(x)=(ax+b)^p$, $g(x)=(cx+d)^q$ for Theorem 19.1.1 , here, we reverse the sign of the index of the integration operator $<s>$ in Formula 17.3.1 (**17.3**) and replace it with the differentiation operator (s) .

Example1 The 1/2th order derivative of $\sqrt{x-2}\,\sqrt[3]{3x+4}$

Substituting $a=1, b=-2, p=1/2, c=3, d=4, q=1/3, s=1/2$ for (1.1) ,

$$\left(\sqrt{x-2}\,\sqrt[3]{3x+4}\right)^{\left(\frac{1}{2}\right)} = \sum_{r=0}^{\infty}\binom{1/2}{r}3^r\frac{\Gamma(3/2)\Gamma(4/3)}{\Gamma(1+r)\Gamma(4/3-r)}(x-2)^r(3x+4)^{\frac{1}{3}-r}$$

The left side is calculated by the expression which replaced the order of integration and differentiation in Riemann-Liouville differintegral. The integration lower limit is taken as $-b/a = 2$ according to Example1 in **17.3** . When m=17, the values of the both sides on arbitrary point $x=5$ are as follows.

```
m = 17;
fl[x_] := 1/Gamma[1 - 1/2] ∫_2^x (x - t)^(1-1/2-1) ∂_t (√(t - 2) ∛(3 t + 4)) dt
fr[x_] := Σ_{r=0}^{m} Binomial[1/2, r] 3^r (Gamma[3/2] Gamma[4/3])/(Gamma[1 + r] Gamma[4/3 - r]) × (x - 2)^r (3 x + 4)^(1/3 - r)
N[fl[5]]      N[fr[5]]
2.5601        2.5601
```

Example1' The 1/2th order derivative of $\sqrt[3]{3x+4}\,(x-2)^2$

Substituting $a=3, b=4, p=1/3, c=1, d=-2, m=2, s=1/2$ for (1.1') ,

$$\left\{\sqrt[3]{3x+4}\,(x-2)^2\right\}^{\left(\frac{1}{2}\right)} = \sum_{r=0}^{2}\binom{1/2}{r}3^{\frac{1}{2}-r}\frac{\Gamma(4/3)\Gamma(3)}{\Gamma(5/6+r)\Gamma(3-r)}\frac{(3x+4)^{-\frac{1}{6}+r}}{(x-2)^{r-2}}$$

The integration lower limit of Riemann-Liouville differintegral in the left side is taken as $-b/a = -4/3$ according to **Example1' in 17.3.1** . The values of the both sides on arbitrary point $x=3$ are as follows.

```
fl[x_] := 1/Gamma[1 - 1/2] ∫_{-4/3}^x (x - t)^(1-1/2-1) ∂_t (∛(3 t + 4) (t - 2)^2) dt
fr[x_] := Σ_{r=0}^{2} Binomial[1/2, r] 3^(1/2 - r) (Gamma[4/3] Gamma[3])/(Gamma[5/6 + r] Gamma[3 - r]) (3 x + 4)^(-1/6 + r)/(x - 2)^(r - 2)
N[fl[3]]      N[fr[3]]
2.79446       2.79446
```

Example2 The 1/2th order derivative of $\sqrt{x-2}\,/\,(3x+4)$

When $q = -1,-2,-3,\cdots$

$$\frac{\Gamma(1+q)}{\Gamma(1+q-r)} = (-1)^{-r}\frac{\Gamma(-q+r)}{\Gamma(-q)}$$

Then (1.1) can be read as follows.

$$\left\{(ax+b)^p(cx+d)^q\right\}^{(s)}$$
$$= \sum_{r=0}^{m-1}\binom{s}{r}\frac{(1/a)^{-s+r}}{(-1/c)^r}\frac{\Gamma(1+p)\Gamma(-q+r)}{\Gamma(1+p-s+r)\Gamma(-q)}\frac{(ax+b)^{p-s+r}}{(cx+d)^{r-q}} + R_m^s$$

Substituting $a=1, b=-2, p=1/2, c=3, d=4, q=-1, s=1/2$ for this ,

$$\left(\frac{\sqrt{x-2}}{3x+4}\right)^{\left(\frac{1}{2}\right)} = \sum_{r=0}^{\infty}\binom{1/2}{r}(-3)^r\frac{\Gamma(3/2)\Gamma(1+r)}{\Gamma(1+r)\Gamma(1)}\frac{(x-2)^r}{(3x+4)^{r+1}}$$

The integration lower limit of Riemann-Liouville differintegral in the left side is $-b/a = 2$ like Example1 .

When m=10, the values of the both sides on arbitrary point $x=4$ are as follows.

```
m = 10;
fl[x_] := 1/Gamma[1 - 1/2] ∫_2^x (x - t)^(1-1/2-1) ∂_t (√(t - 2))/(3 t + 4) dt
fr[x_] := Σ_{r=0}^m Binomial[1/2, r] (-3)^r (Gamma[3/2] Gamma[1 + r])/(Gamma[1 + r] Gamma[1]) (x - 2)^r/(3 x + 4)^(r+1)
N[fl[4]]      N[fr[4]]
0.043789      0.043789
```

19.2.2 Super Derivative of $x^{\alpha}\log x$

Formula 19.2.2

When $\Gamma(z)$, $\psi(z)$ denotes the gamma function and the digamma function respectively, the following expressions hold.

(1)

$$\left(x^{\alpha}\log x\right)^{(p)} = \sum_{r=0}^{m-1}\binom{p}{r}\frac{\log x-\psi(1-p+r)-\gamma}{\Gamma(1-p+r)}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-p} + R_m^p \qquad (2.1)$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-p-1}{k}\left\{\frac{\log x-\psi(1+m+k)-\gamma}{\Gamma(1+m+k)}\frac{\Gamma(1+\alpha)x^{\alpha}}{\Gamma(1+\alpha-m-k)}\right\}^{(p)} \qquad (2.1r)$$

$$\lim_{m\to\infty}R_m^p = 0$$

Especially, when $m=0,1,2,\cdots$

$$\left(x^m\log x\right)^{(p)} = \sum_{r=0}^{m}\binom{p}{r}\frac{\log x-\psi(1-p+r)-\gamma}{\Gamma(1-p+r)}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-p} \qquad (2.1')$$

(2) When $\alpha\neq -1,-2,-3,\cdots$ & $\alpha-p\neq -1,-2,-3,\cdots$

$$\left(x^{\alpha}\log x\right)^{(p)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p)}x^{\alpha-p}\log x$$
$$+\sum_{r=1}^{m-1}(-1)^{r-1}\binom{p}{r}\frac{\Gamma(1+\alpha)\Gamma(r)}{\Gamma(1+\alpha-p+r)}x^{\alpha-p} + R_m^p \qquad (2.2)$$

$$R_m^p = \frac{x^{\alpha-p}}{B(-p,m)}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p)}\sum_{k=0}^{\infty}(-1)^{k-1}\binom{-p-1}{k}\frac{B(1+\alpha,m+k)}{m+k} \tag{2.2r}$$

$$\lim_{m\to\infty} R_m^p = 0$$

Proof

Substituting $f(x)=\log x$ for Formula 19.2.0 , we obtain the desired expressions.

Example1 The 4/5 th order derivative of $x^{3/2}\log x$

Substituting $\alpha=3/2$, $p=4/5$ for (2.1) ,

$$\left(x^{\frac{3}{2}}\log x\right)^{(4/5)} = \sum_{r=0}^{m-1}\binom{4/5}{r}\frac{\log x-\psi(1/5+r)-\gamma}{\Gamma(1/5+r)}\frac{\Gamma(5/2)}{\Gamma(5/2-r)}x^{\frac{7}{10}}+R_m^{\frac{4}{5}}$$

The integration lower limit of Riemann-Liouville differintegral in the left side is $x=0$.

When m=120, the values of both sides on arbitrary point $x=3$ are as follows.

```
a = 3 / 2;  p = 4 / 5;  m = 120;
fl[x_] := 1/Gamma[1 - p] ∫_0^x (x - t)^(1-p-1) ∂t (t^(3/2) Log[t]) dt

fr[x_] := Sum_{r=0}^{m-1} Binomial[p, r] (Log[x] - PolyGamma[1 - p + r] - EulerGamma)/Gamma[1 - p + r]
          × Gamma[1 + a]/Gamma[1 + a - r] x^(a-p)

N[fl[3]]          N[fr[3]]
5.02928           5.02928
```

Example1' The 1.3 th order derivative of $x^3\log x$

Substituting $m=3$, $p=1.3$ for (2.1') ,

$$\left(x^3\log x\right)^{(1.3)} = \sum_{r=0}^{3}\binom{1.3}{r}\frac{\log x-\psi(-0.3+r)-\gamma}{\Gamma(-0.3+r)}\frac{\Gamma(4)}{\Gamma(4-r)}x^{1.7}$$

The values of the both sides on arbitrary point x=2.1 are as follows.

```
m = 3;  p = 1.3;
fl[x_] := 1/Gamma[2 - p] ∫_0^x (x - t)^(2-p-1) ∂t (∂t (t^3 Log[t])) dt

fr[x_] := Sum_{r=0}^{m} Binomial[p, r] (Log[x] - PolyGamma[1 - p + r] - EulerGamma)/Gamma[1 - p + r]
          × Gamma[1 + m]/Gamma[1 + m - r] x^(m-p)

N[fl[2.1]]          N[fr[2.1]]
16.4712             16.4712
```

The remainder in the super derivative of the product of two functions becomes a series of super derivatives. So it is very difficult to calculate this. However, the calculation of the above (2.2r) is exceptionally easy. So we show it as follows.

Example2 The 1/2th order derivative of $\sqrt[3]{x}\ \log x$

Substituting $\alpha=1/3$, $p=1/2$ for the above (2) ,

$$\left(x^{\alpha}\log x\right)^{(p)} = \frac{\Gamma(4/3)}{\Gamma(5/6)}x^{-\frac{1}{6}}\log x$$
$$+\sum_{r=1}^{m-1}(-1)^{r-1}\binom{-1/2}{r}\frac{\Gamma(4/3)\Gamma(r)}{\Gamma(5/6+r)}x^{-\frac{1}{6}} + R_m^{\frac{1}{2}}$$

$$R_m^{\frac{1}{2}} = \frac{x^{-\frac{1}{6}}}{B(-1/2,m)}\frac{\Gamma(4/3)}{\Gamma(5/6)}\sum_{k=0}^{\infty}(-1)^{k-1}\binom{-3/2}{k}\frac{B(4/3,m+k)}{m+k}$$

When m=10, the values of both sides on arbitrary point $x=1$ are as follows.

```
a = 1 / 3;  p = 1 / 2;  m = 10;
```

Riemann-Liouville Differintegrl

$$\text{f1[x\_] := }\frac{1}{\text{Gamma}[1-p]}\int_0^x (x-t)^{1-p-1}\,\partial_t\left(t^{\frac{1}{3}}\,\text{Log}[t]\right)dt$$

Series

$$\text{Sr[x\_] := }\frac{\text{Gamma}[1+a]}{\text{Gamma}[1+a-p]}x^{a-p}\,\text{Log}[x]+\sum^{m-1}(-1)^{r-1}\,\text{Binomial}[p,r]\,\frac{\text{Gamma}[1+a]\,\text{Gamma}[r]}{\text{Gamma}[1+a-p+r]}x^{a-p}$$

Remainder

$$\text{Rm[x\_] := }\frac{x^{a-p}}{\text{Beta}[-p,m]}\frac{\text{Gamma}[1+a]}{\text{Gamma}[1+a-p]}\sum_{k=0}^{\infty}(-1)^{k-1}\,\text{Binomial}[-p-1,k]\,\frac{\text{Beta}[1+a,m+k]}{m+k}$$

Riemann-Liouville Differintegrl	Series	Remainder	Series+Remainder
N[f1[1]]	N[Sr[1]]	N[Rm[1]]	N[Sr[1] + N[Rm[1]]]
0.600201	0.590565	0.00963605	0.600201

Complete Automorphism

Reversing the sign of the index of the integration operator $<p>$ in Formula 17.3.2 in **17.3** , we obtain the following expression without the remainder term. However, it is complicated and the convergence is also very slow.

Formula 19.2.2'

$$\left(x^{\alpha}\log x\right)^{(p)}$$
$$= \frac{\dfrac{\log x-\psi(1-p)-\gamma}{\Gamma(1-p)} - p\dfrac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p)}\displaystyle\sum_{k=0}^{\infty}\binom{-p-1}{k}\frac{\psi(2+k)+\gamma}{(1+k)^2B(1+k\alpha-k)}}{1-p\displaystyle\sum_{k=0}^{\infty}\binom{-p-1}{k}\frac{1}{(1+k)^2B(1+k\alpha-k)}}x^{\alpha-p}$$

19.2.3 Super Derivatives of $x^{\alpha}\sin x$, $x^{\alpha}\cos x$

Formula 19.2.3

When $m=0,1,2,\cdots$, the following expressions hold for $p>0$.

$$\left(x^m \sin x\right)^{(p)} = \sum_{r=0}^{m} \binom{p}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \sin\left\{x+\frac{(p-r)\pi}{2}\right\} \tag{3.1s}$$

$$\left(x^m \cos x\right)^{(p)} = \sum_{r=0}^{m} \binom{p}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \cos\left\{x+\frac{(p-r)\pi}{2}\right\} \tag{3.1c}$$

Proof

Reversing the sign of the index of the integration operator $<p>$ in Formula 17.3.3 in **17.3** and replacing it with the differentiation operator (p), we obtain the desired expressions.

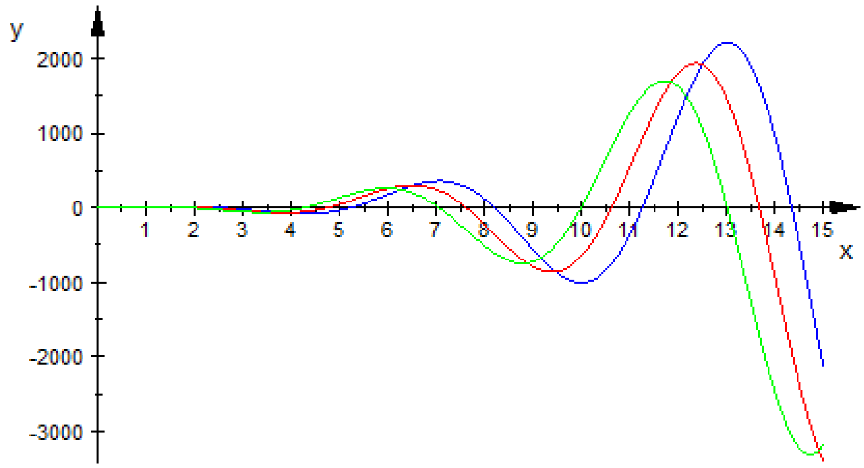
Example The 3/2th order derivative of $x^3 \sin x$

Substituting $m=2$, $p=3/2$ for (3.1s) ,

$$\left(x^3 \sin x\right)^{\left(\frac{3}{2}\right)} = \sum_{r=0}^{3} \binom{3/2}{r} \frac{\Gamma(4)}{\Gamma(4-r)} x^{3-r} \sin\left\{x+\frac{(3/2-r)\pi}{2}\right\}$$

This super derivative can not be examined by Riemann-Liouville differntegral. Then we draw this with the 1st and the 2nd order derivative on a figure side by side. It is as follows.

Blue: 1st order, Red: 3/2th order, Green: 2nd order



Formula 19.2.3' (Collateral Super Derivative)

The following expressions hold for α, p such that $\alpha \neq -1,-2,-3,\cdots$ & $\alpha - p \neq -1,-2,-3,\cdots$.

$$\left(x^\alpha \sin x\right)^{\{p\}} = \sum_{r=0}^{m-1} \binom{p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r} \sin\left(x+\frac{r\pi}{2}\right) + R_m^p \tag{3.2s}$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{-p-1}{k} \left\{ \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} x^{\alpha+m+k} \sin\left\{x+\frac{(m+k)\pi}{2}\right\}\right\}^{(p)}$$

$$\left(x^\alpha \cos x\right)^{\{p\}} = \sum_{r=0}^{m-1} \binom{p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r} \cos\left(x+\frac{r\pi}{2}\right) + R_m^p \tag{3.2c}$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{-p-1}{k} \left\{ \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} x^{\alpha+m+k} \cos\left\{x+\frac{(m+k)\pi}{2}\right\}\right\}^{(p)}$$

$$\lim_{m\to\infty} R_m^p = 0$$

Proof

Reversing the sign of the index of the integration operator $<p>$ in Formula 17.3.3' in **17.3** and replacing it with the differentiation operator (p), we obtain the desired expressions.

Example Collateral the 1/2th order derivative of $x^{3/2}\sin x$

Substituting $\alpha=3/2$, $p=1/2$ for (3.2s), we obtain

$$\left(x^{\frac{3}{2}}\sin x\right)^{\left(\frac{1}{2}\right)}=\sum_{r=0}^{m-1}\binom{1/2}{r}\frac{\Gamma(5/2)}{\Gamma(2-r)}x^{1-r}\sin\left(x+\frac{r\pi}{2}\right)+R_m^{\frac{1}{2}}$$

The integration lower limit of Riemann-Liouville differintegral in the left side is $x=0$.

When m=15, the values of the both sides on arbitrary point $x=5$ are as follows.

```
a = 3 / 2;  p = 1 / 2;  m = 15;
f1[x_] := 1/Gamma[1 - p] ∫_0^x (x - t)^(1-p-1) ∂_t (t^(3/2) Sin[t]) dt
fr[x_] := Sum_{r=0}^{m-1} Binomial[p, r] Gamma[1 + a]/Gamma[1 + a - p + r] x^(a-p+r) Sin[x + r π/2]

N[f1[5]]      N[fr[5]]
-6.80724      -6.80724
```

19.2.4 Super Derivatives of $x^\alpha\sinh x$, $x^\alpha\cosh x$

Formula 19.2.4

When $m=0,1,2,\cdots$, the following expressions hold for $p>0$.

$$\left(x^m\sinh x\right)^{\{p\}}=\sum_{r=0}^{m}\binom{p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\frac{e^x-(-1)^{-p+r}e^{-x}}{2} \qquad (4.1s)$$

$$\left(x^m\cosh x\right)^{\{p\}}=\sum_{r=0}^{m}\binom{p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\frac{e^x+(-1)^{-p+r}e^{-x}}{2} \qquad (4.1c)$$

Example The 0.999th order derivaive of $x^3\cosh x$

Substituting $m=3$, $p=0.999$ for (4.1c), we obtain

$$\left(x^3\cosh x\right)^{\{0.999\}}=\sum_{r=0}^{3}\binom{0.999}{r}\frac{\Gamma(4)}{\Gamma(4-r)}x^{3-r}\frac{e^x+(-1)^{-0.999+r}e^{-x}}{2}$$

This super derivative can not be examined by Riemann-Liouville differntegral. Then we calculate the values on arbitrary point x=2 for the 0.999th order derivative and the 1st order derivative respectively. Although the former is a complex number, the real part is naturally near to the coefficient of the later.

The 0.999th order derivative

```
m = 3;  p = 0.999;
fp[x_] := Sum_{r=0}^{m} Binomial[p, r] Gamma[1 + m]/Gamma[1 + m - r] x^(m-r) (e^x + (-1)^(-p+r) e^-x)/2
```

The 1st order derivative

```
f1[x_] = ∂_x (x^3 Cosh[x]);
```

The 0.999th order derivative

```
N[fp[2]]
74.0981 + 0.000849271 i
```

The 1st order derivative

```
N[f1[2]]
74.1612
```

Formula 19.2.4' (Collateral Super Derivative)

The following expressions hold for α, p such that $\alpha \neq -1, -2, -3, \cdots$ & $\alpha - p \neq -1, -2, -3, \cdots$.

$$\left(x^{\alpha}\sinh x\right)^{\{p\}} = \sum_{r=0}^{m-1}\binom{p}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)}x^{\alpha-p+r}\frac{e^{x}-(-1)^{-r}e^{-x}}{2} + R_m^p \tag{4.2s}$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-p-1}{k}\left\{\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)}x^{\alpha+m+k}\frac{e^{x}-(-1)^{-m-k}e^{-x}}{2}\right\}^{(p)}$$

$$\left(x^{\alpha}\cosh x\right)^{\{p\}} = \sum_{r=0}^{m-1}\binom{p}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)}x^{\alpha-p+r}\frac{e^{x}+(-1)^{-r}e^{-x}}{2} + R_m^p \tag{4.2c}$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-p-1}{k}\left\{\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)}x^{\alpha+m+k}\frac{e^{x}+(-1)^{-m-k}e^{-x}}{2}\right\}^{(p)}$$

$$\lim_{m\to\infty} R_m^n = 0$$

Example Collateral the 3/2th order integral of $\sqrt[3]{x}\,\sinh x$

Substituting $\alpha=1/3$, $p=3/2$ for (4.2s), we obtain

$$\left(\sqrt[3]{x}\,\sinh x\right)^{\left(\frac{3}{2}\right)} = \sum_{r=0}^{m-1}\binom{3/2}{r}\frac{\Gamma(4/3)}{\Gamma(-1/6+r)}x^{-\frac{1}{6}+r}\frac{e^{x}-(-1)^{-r}e^{-x}}{2} + R_m^{3/2}$$

The integration lower limit of Riemann-Liouville differintegral in the left side is $x=0$.

When m=50, the values of the both sides on arbitrary point $x=17$ are as follows.

```
a = 1 / 3;  p = 3 / 2;  m = 50;
fl[x_] := 1/Gamma[2 - p] ∫_0^x (x - t)^(2-p-1) ∂t (∂t (∛t Sinh[t])) dt
fr[x_] := Sum_{r=0}^{m-1} Binomial[p, r] Gamma[1 + a]/Gamma[1 + a - p + r] x^(a-p+r) (e^x - (-1)^-r e^-x)/2
N[fl[17], 8]
3.1958852 ×10^7
N[fr[17], 8]
3.1958852 ×10^7
```

19.3 Super Derivative of $\log x\, f(x)$

19.3.1 Super Derivative of $(\log x)^2$

Formula 19.3.1

$$\left(\log^2 x\right)^{(p)} = \frac{x^{-p}\log x\,\{\log x - \psi(1-p) - \gamma\}}{\Gamma(1-p)}$$

$$- x^{-p}\sum_{r=1}^{m-1}(-1)^r\binom{p}{r}\frac{\{\log x - \psi(1-p+r) - \gamma\}\Gamma(r)}{\Gamma(1-p+r)} + R_m^p \qquad (1.1)$$

$$R_m^p = \frac{x^{-p}}{B(-p,m)\,\Gamma(1-p)} \times$$

$$\sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{(m+k)^2}\binom{-p-1}{k}\{\log x - \psi(1-p) - \psi(1+m+k) - 2\gamma\} \qquad (1.1r)$$

$$\lim_{m\to\infty} R_m^p = 0$$

Proof

Reversing the sign of the index of the integration operator $\langle p\rangle$ in Formula 17.4.1 in **17.4** and replacing it with the differentiation operator (p), we obtain the desired expressions.

Example The 1/2th order derivative of $(\log x)^2$

When m=4000, the values of both sides on arbitrary point $x=2.7$ are as follows. Since the convergence is slow, even if it calculates so far, both sides does not match only up to 3 digits after the decimal point.

```
p = 1 / 2;  h = 10^-6;  m = 4000;
```

Riemann-Liouville Differintegrl

$$\texttt{f[x\_] :=} \frac{1}{\texttt{Gamma[1 - p]}}\int_0^x (x-t)^{1-p-1}\,\texttt{Log[t]}^2\,dt \qquad \texttt{Rld} = \frac{\texttt{f[2.7 + h] - f[2.7]}}{\texttt{h}};$$

Series

$$\texttt{Sr[x\_] :=} \frac{x^{-p}\,\texttt{Log[x] (Log[x] - PolyGamma[1 - p] - EulerGamma)}}{\texttt{Gamma[1 - p]}} -$$

$$x^{-p}\sum_{r=1}^{m-1}(-1)^r\,\texttt{Binomial[p, r]}\frac{\texttt{(Log[x] - PolyGamma[1 - p + r] - EulerGamma) Gamma[r]}}{\texttt{Gamma[1 - p + r]}}$$

```
N[Rld]              N[Sr[2.7]]
0.814566            0.814946
```

19.4 Super Derivative of e^x f (x)

19.4.1 Super Derivative of $e^x x^\alpha$

Formula 19.4.1

$$\left(e^x x^\alpha\right)^{(p)} = e^x \sum_{r=0}^{m-1} \binom{p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} x^{\alpha-r} + R_m^p \tag{1.1}$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{-p-1}{k} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-k)} \left(e^x x^{\alpha-m-k}\right)^{(p)} \tag{1.1r}$$

Especially, when $m=0,1,2,\cdots$

$$\left(e^x x^m\right)^{(p)} = e^x \sum_{r=0}^{m} \binom{p}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \tag{1.1'}$$

Proof

Let $f(x)=e^x$, $g(x)=x^\alpha$. Then

$$\left(x^\alpha\right)^{(r)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} x^{\alpha-r} \qquad (\alpha \ne -1,-2,-3,\cdots)$$

Substituting these for Theorem 19.1.1 , we obtain (1.1) and (1.1r).

Especially, when $\alpha=m-1$ $m=1,2,3,\cdots$, $\Gamma(1+\alpha-m-k)=\pm\infty$ $k=0,1,2,3,\cdots$.
Then, the remainder term disappears and (1.1) is as follows.

$$\left(e^x x^{m-1}\right)^{(p)} = e^x \sum_{r=0}^{m-1} \binom{p}{r} \frac{\Gamma(1+m-1)}{\Gamma(1+m-1-r)} x^{m-1-r}$$

Thus, replacing $m-1$ with m , we obtain (1.1').

Example1 The 1/2.1th order integral of $e^x x^{3/4}$

Substituting $\alpha=3/4$, $p=1/2$ for (1.1) ,

$$\left(e^x x^{3/4}\right)^{\left(\frac{1}{2}\right)} = e^x \sum_{r=0}^{m-1} \binom{1/2}{r} \frac{\Gamma(1+3/4)}{\Gamma(1+3/4-r)} x^{\frac{3}{4}-r} + R_m^{1/2}$$

The left side is calculated by the expression which replaced an order of integration and differentiation in Riemann-Liouville differintegral. When m=10, the values of the both sides on arbitrary point $x=-10$ are as follows.

```
a = 3 / 4;  p = 1 / 2;  m = 10;

fl[x_] := 1/Gamma[1 - p] ∫_{-∞}^{x} (x - t)^(1-p-1) ∂_t (t^a e^t) dt

fr[x_] := e^x Σ_{r=0}^{m-1} Binomial[p, r] Gamma[1 + a]/Gamma[1 + a - r] x^(a-r)

N[fl[-10]]                                   N[fr[-10]]
-0.000173796  + 0.000173796  i               -0.000173796  + 0.000173796  i
```

Example1' The 5/2th order derivative of $e^x x^7$

Substituting $m=7$, $p=5/2$ for (1.1') ,

$$\left(e^x x^7\right)^{\left(\frac{5}{2}\right)} = e^x \sum_{r=0}^{7} \binom{5/2}{r} \frac{\Gamma(8)}{\Gamma(8-r)} x^{7-r}$$

The values of the both sides on arbitrary point $x=0.5$ are as follows.

```
m = 7; p = 5 / 2;
fl[x_] := 1/Gamma[3 - p] ∫_{-∞}^{x} (x - t)^(3-p-1) ∂t (∂t (∂t (t^m e^t))) dt
fr[x_] := e^x Σ_{r=0}^{m} Binomial[p, r] Gamma[1 + m]/Gamma[1 + m - r] x^(m-r)

N[fl[0.5]]                          N[fr[0.5]]
16.6933  - 4.89299 × 10^-16 i       16.6933
```

Formula 19.4.1" (Collateral Super Derivative)

The following expression holds for α, p such that $\alpha \neq -1,-2,-3,\cdots$ & $\alpha - p \neq -1,-2,-3,\cdots$.

$$\left(e^x x^\alpha\right)^{(p)} = e^x \sum_{r=0}^{m-1} \binom{p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)} x^{\alpha-p+r} + R_m^p \qquad (1.1")$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{-p-1}{k} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} \left(e^x x^{\alpha+m+k}\right)^{(p)}$$

$$\lim_{m\to\infty} R_m^p = 0$$

Example1" Collateral the 1/2th order derivative of $e^x x^{3/4}$

Substituting $\alpha=3/4$, $p=1/2$ for (1.1") , we obtain

$$\left(e^x x^{3/4}\right)^{\left(\frac{1}{2}\right)} = e^x \sum_{r=0}^{m-1} \binom{1/2}{r} \frac{\Gamma(3/2)}{\Gamma(3/4+r)} x^{\frac{3}{4}+r} + R_m^{1/2}$$

The values of both sides on the same point $x=-10$ as Example1 are as follows. Though the both sides are corresponding, they are considerably different from the values of Example1 (Lineal the 1/2th order derivative.)

```
a = 3 / 4; p = 1 / 2; m = 30;
fl[x_] := 1/Gamma[1 - p] ∫_0^x (x - t)^(1-p-1) ∂t (t^a e^t) dt
fr[x_] := e^x Σ_{r=0}^{m-1} Binomial[p, r] Gamma[1 + a]/Gamma[1 + a - p + r] x^(a-p+r)

N[fl[-10]]                            N[fr[-10]]
-0.00839073  - 0.00839073 i           -0.00839073  - 0.00839073 i
```

19.4.2 Super derivative of $e^x \log x$

All the polynomials obtained by applying Theorem 19.1.1 to $e^x \log x$ become asymptotic expansions, and they are hardly helpful.

19.4.3 Super derivatives of $e^x \sin x$, $e^x \cos x$

Formula 19.4.3

$$\left(e^{x}\sin x\right)^{(p)} = \left(\sin\frac{\pi}{4}\right)^{-p} e^{x}\sin\left(x+\frac{p\pi}{4}\right) \tag{3.0s}$$

$$\left(e^{x}\cos x\right)^{(p)} = \left(\sin\frac{\pi}{4}\right)^{-p} e^{x}\cos\left(x+\frac{p\pi}{4}\right) \tag{3.0c}$$

Proof

Analytically continuing the index of the differentiation operator in Formula 18.4.3 in **18.4** to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Example The 1/3th order derivative of $e^{x}\sin x$

When $p=1/3$, the values of the both sides of (3.0s) on arbitrary point $x=1.2$ are as follows.

```
p = 1 / 3;
f1[x_] := 1/Gamma[1 - p] ∫_{-∞}^{x} (x - t)^(1-p-1) ∂_t (e^t Sin[t]) dt
fr[x_] := (Sin[π/4])^(-p) e^x Sin[x + p π/4]
N[f1[1.2]]          N[fr[1.2]]
3.70458 + 0. i      3.70459
```

Trigonometric Series

If Formula 18.4.3' in **18.4** is extended to the real number, the following trigonometric series are obtained.

Formula 19.4.3'

$$\sum_{r=0}^{\infty}\binom{p}{r}\sin\left(x+\frac{r\pi}{2}\right) = \left(\sin\frac{\pi}{4}\right)^{-p}\sin\left(x+\frac{p\pi}{4}\right) \tag{3.1s}$$

$$\sum_{r=0}^{\infty}\binom{p}{r}\cos\left(x+\frac{r\pi}{2}\right) = \left(\sin\frac{\pi}{4}\right)^{-p}\cos\left(x+\frac{p\pi}{4}\right) \tag{3.1c}$$

Especially, when $x=0$,

$$\sum_{r=0}^{\infty}\binom{p}{r}\sin\frac{r\pi}{2} = \left(\sin\frac{\pi}{4}\right)^{-p}\sin\frac{p\pi}{4} \tag{3.1's}$$

$$\sum_{r=0}^{n}\binom{p}{r}\cos\frac{r\pi}{2} = \left(\sin\frac{\pi}{4}\right)^{-p}\cos\frac{p\pi}{4} \tag{3.1'c}$$

Alternating Binomial Series

Removing $\sin\frac{r\pi}{2}, \cos\frac{r}{2}$ from (3.1's), (3.1'c), we obtain the following interesting series.

Formula 19.4.3"

$$\sum_{k=0}^{\infty}(-1)^{k}\binom{p}{2k+1} = 2^{\frac{p}{2}}\sin\frac{p\pi}{4} \tag{3.2s}$$

$$\sum_{k=0}^{\infty}(-1)^k\binom{p}{2k} \quad = 2^{\frac{p}{2}}\cos\frac{p\pi}{4} \tag{3.2c}$$

Proof

Since the odd-numbered terms of the left side in (3.1's) are all 0,

$$\sum_{r=0}^{\infty}\binom{p}{r}\sin\frac{r\pi}{2} = \binom{p}{1}-\binom{p}{3}+\binom{p}{5}-\binom{p}{7}+-\cdots = \sum_{k=0}^{\infty}(-1)^k\binom{p}{2k+1}$$

$$\therefore \sum_{k=0}^{\infty}(-1)^k\binom{p}{2k+1} = \left(\sin\frac{\pi}{4}\right)^{-p}\sin\frac{p\pi}{4} = 2^{\frac{p}{2}}\sin\frac{p\pi}{4}$$

Next, since the even-numbered terms of the left side in (3.1'c) are all 0,

$$\sum_{r=0}^{n}\binom{p}{r}\cos\frac{r\pi}{2} = \binom{p}{0}-\binom{p}{2}+\binom{p}{4}-\binom{p}{6}+-\cdots = \sum_{k=0}^{\infty}(-1)^k\binom{p}{2k}$$

$$\therefore \sum_{k=0}^{\infty}(-1)^k\binom{p}{2k} = \left(\sin\frac{\pi}{4}\right)^{-p}\cos\frac{p\pi}{4} = 2^{\frac{p}{2}}\cos\frac{p\pi}{4}$$

If a horizontal axis is set as p and these are illustrated , it is as follows. Although the left side is blue and right side is red, since both sides overlap exactly, the left side (blue) is not visible.

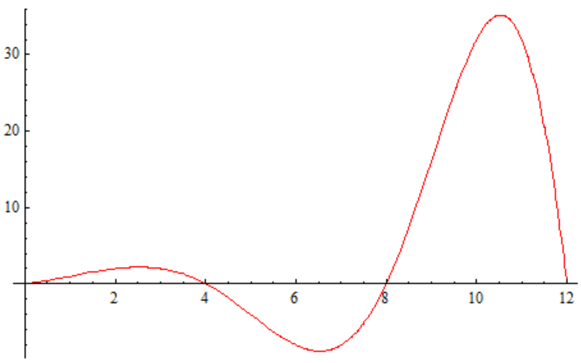
```
m = 20;
```

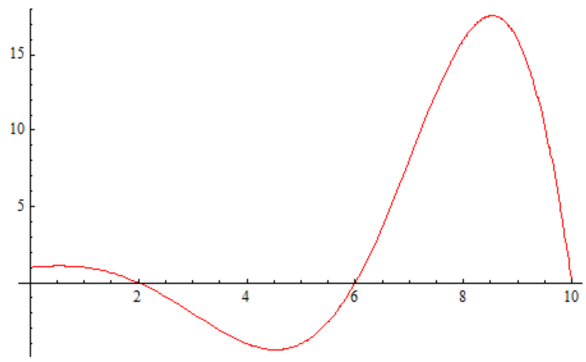
sin x series

```
sl[p_] := Sum[(-1)^k Binomial[p, 2 k + 1], {k, 0, m}]
sr[p_] := 2^(p/2) Sin[p π / 4]
```

cos x series

```
cl[p_] := Sum[(-1)^k Binomial[p, 2 k], {k, 0, m}]
cr[p_] := 2^(p/2) Cos[p π / 4]
```





19.4.4 Super Derivatives of $e^x \sinh x$, $e^x \cosh x$

Formula 19.4.4

$$\left(e^x \sinh x\right)^{(p)} = e^x \sum_{r=0}^{\infty}\binom{p}{r}\frac{e^x-(-1)^{-r}e^{-x}}{2} \tag{4.0s}$$

$$\left(e^x \cosh x\right)^{(p)} = e^x \sum_{r=0}^{\infty}\binom{p}{r}\frac{e^x+(-1)^{-r}e^{-x}}{2} \tag{4.0c}$$

Example The 3/2th order derivative of $e^x \cosh x$

When $p = 3/2$, the values of the both sides of (4.0c) on arbitrary point $x = 1.3$ are as follows.

```
p = 3 / 2; m = 200;
```

$$fl[x\_] := \frac{1}{\text{Gamma}[2-p]} \int_{-\infty}^{x} (x-t)^{2-p-1} \partial_t \left(\partial_t \left(e^t \text{Cosh}[t]\right)\right) dt$$

$$fr[x\_] := e^x \sum_{r=0}^{m} \text{Binomial}[p, r] \frac{e^x + (-1)^{-r} e^{-x}}{2}$$

```
N[fl[1.3]]      N[fr[1.3]]
19.0406         19.0406
```

19.5 Super Derivative of f (x) / e^x

19.5.1 Super Derivative of $e^{-x}x^{\alpha}$

Formula 19.5.1

$$\left(e^{-x}x^{\alpha}\right)^{(p)} = \frac{(-1)^{-p}}{e^x}\sum_{r=0}^{m-1}(-1)^r\binom{p}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r} + R_m^p \tag{1.1}$$

$$R_m^p = \frac{1}{B(-p,m)}\sum_{k=0}^{\infty}\frac{(-1)^k}{m+k}\binom{-p-1}{k}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-k)}\left(\frac{x^{\alpha-m-k}}{e^x}\right)^{(p)} \tag{1.1r}$$

Especially, when $m=0,1,2,\cdots$

$$\left(e^{-x}x^{m}\right)^{(p)} = \frac{(-1)^{-p}}{e^x}\sum_{r=0}^{m}(-1)^r\binom{p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r} \tag{1.1'}$$

Example1 The 1/2th order derivative of $e^{-x}\sqrt{x}$

If we substitute $\alpha=1/2$, $p=1/2$ for (1.1) and calculate the values of the both sides on arbitrary point $x=10$, it is as follows. In addition, this is an asymptotic expansion.

```
a = 1 / 2;  p = 1 / 2;  m = 10;
```

$$\texttt{fl[x\_] := } \frac{1}{\texttt{Gamma[1 - p]}}\int_{\infty}^{x}(x-t)^{1-p-1}\,\partial_t\left(e^{-t}t^{a}\right)dt$$

$$\texttt{fr[x\_] := } \frac{(-1)^{-p}}{e^x}\sum_{r=0}^{m-1}(-1)^r\,\texttt{Binomial[p, r]}\,\frac{\texttt{Gamma[1 + a]}}{\texttt{Gamma[1 + a - r]}}\,x^{a-r}$$

```
N[fl[10]]                 N[fr[10]]
0. - 0.00014002 i         0. - 0.00014002 i
```

Example1' The 5/2th order derivative of $e^{-x}x^7$

If we substitute $m=7$, $p=5/2$ for (1.1') and calculate the values of the both sides on arbitrary point $x=6$, it is as follows.

```
m = 7;  p = 5 / 2;
```

$$\texttt{fl[x\_] := } \frac{1}{\texttt{Gamma[3 - p]}}\int_{\infty}^{x}(x-t)^{3-p-1}\,\partial_t\left(\partial_t\left(\partial_t\left(e^{-t}t^{m}\right)\right)\right)dt$$

$$\texttt{fr[x\_] := } \frac{(-1)^{-p}}{e^x}\sum_{r=0}^{m}(-1)^r\,\texttt{Binomial[p, r]}\,\frac{\texttt{Gamma[1 + m]}}{\texttt{Gamma[1 + m - r]}}\,x^{m-r}$$

```
N[fl[6]]                  N[fr[6]]
0. + 43.4887 i            0. + 43.4887 i
```

Formula 19.5.1" (Collateral Super Derivative)

The following expression holds for α,p such that $\alpha\neq-1,-2,-3,\cdots$ & $\alpha-p\neq-1,-2,-3,\cdots$.

$$\left(e^{-x}x^{\alpha}\right)^{(p)} = \frac{1}{e^x}\sum_{r=0}^{m-1}(-1)^{-r}\binom{p}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p+r)}x^{\alpha-p+r} + R_m^p \tag{1.1"}$$

$$R_m^p = \frac{1}{B(-p,m)} \sum_{k=0}^{\infty} \frac{(-1)^{-k}}{m+k} \binom{-p-1}{k} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} \left(\frac{x^{\alpha-m-k}}{e^x} \right)^{(p)}$$

$$\lim_{m \to \infty} R_m^p = 0$$

Example1" Collateral the 1/2th order derivative of $e^{-x}\sqrt{x}$

If we substitute $\alpha=1/2, p=1/2, m=30$ for (1.1") and calculate the values of the both sides on arbitrary point $x=10$, it is as follows. Though the both sides are corresponding, they are completely different from the values of Example1 (Lineal the 1/2th order derivative.)

```
a = 1 / 2;  p = 1 / 2;  m = 30;
```

$$\texttt{fl[x\_] :=} \frac{1}{\texttt{Gamma[1 - p]}} \int_0^x (x-t)^{1-p-1} \partial_t \left(e^{-t} t^a\right) dt$$

$$\texttt{fr[x\_] :=} \frac{1}{e^x} \sum_{r=0}^{m-1} (-1)^r \texttt{Binomial[p, r]} \frac{\texttt{Gamma[1 + a]}}{\texttt{Gamma[1 + a - p + r]}} x^{a-p+r}$$

```
N[fl[10]]         N[fr[10]]
-0.0107629        -0.0107629
```

19.5.2 Super Derivative of $e^{-x} \log x$

All the polynomials obtained by applying Theorem 19.1.1 to $e^{-x}\log x$ become asymptotic expansions, and they are hardly helpful.

19.5.3 Super derivatives of $e^{-x}\sin x$, $e^{-x}\cos x$

Formula 19.5.3

$$\left(e^{-x}\sin x\right)^{(p)} = (-1)^{-p} \left(\sin\frac{\pi}{4}\right)^{-p} e^{-x} \sin\left(x - \frac{p\pi}{4}\right) \tag{3.0s}$$

$$\left(e^{-x}\cos x\right)^{(p)} = (-1)^{-p} \left(\sin\frac{\pi}{4}\right)^{-p} e^{-x} \cos\left(x - \frac{p\pi}{4}\right) \tag{3.0c}$$

Example The 3/2th order derivative of $e^{-x}\sin x$

The values of the both sides of (3.0s) on arbitrary point $x=1.7$ are as follows.

```
p = 3 / 2;
```

$$\texttt{fl[x\_] :=} \frac{1}{\texttt{Gamma[2 - p]}} \int_\infty^x (x-t)^{2-p-1} \partial_t \left(\partial_t \left(e^{-t} \texttt{Sin[t]}\right)\right) dt$$

$$\texttt{fr[x\_] := } (-1)^{-p} \left(\texttt{Sin}\left[\frac{\pi}{4}\right]\right)^{-p} e^{-x} \texttt{Sin}\left[x - \frac{p\,\pi}{4}\right]$$

```
N[fl[1.7]]           N[fr[1.7]]
0. + 0.153166 i      0. + 0.153166 i
```

19.5.4 Super derivatives of $e^{-x}\sinh x$, $e^{-x}\cosh x$

Formula 19.5.4

$$\left(e^{-x}\sinh x\right)^{(p)}=e^{-x}\sum_{r=0}^{\infty}(-1)^{-p+r}\binom{p}{r}\frac{e^{x}-(-1)^{-r}e^{-x}}{2} \tag{4.0s}$$

$$\left(e^{-x}\cosh x\right)^{(p)}=e^{-x}\sum_{r=0}^{\infty}(-1)^{-p+r}\binom{p}{r}\frac{e^{x}+(-1)^{-r}e^{-x}}{2} \tag{4.0c}$$

Example The 3/2th order derivative of $e^{-x}\cosh x$

The values of the both sides of (4.0c) on arbitrary point $x=1.3$ are as follows.

```
p = 3 / 2; m = 2500;
```

$$\texttt{f1[x\_] :=}\ \frac{1}{\texttt{Gamma}[2-\texttt{p}]}\int_{\infty}^{x}(x-t)^{2-p-1}\,\partial_t\left(\partial_t\left(e^{-t}\,\texttt{Cosh}[t]\right)\right)dt$$

$$\texttt{fr[x\_] :=}\ e^{-x}\sum_{r=0}^{m}(-1)^{-p+r}\,\texttt{Binomial}[p,r]\ \frac{e^{x}+(-1)^{-r}e^{-x}}{2}$$

```
N[f1[1.3]]              N[fr[1.3]]
0. + 0.105039 i         0. + 0.105038 i
```

19.6 Super Derivatives of sin x f (x), cos x f (x)

19.6.1 Super Derivatives of $sin^2 x$, $cos^2 x$

Formula 19.6.1

$$\left(sin^2 x\right)^{(p)} = -2^{p-1} cos\left(2x+\frac{p\pi}{2}\right) \tag{1.0s}$$

$$\left(cos^2 x\right)^{(p)} = 2^{p-1} cos\left(2x+\frac{p\pi}{2}\right) \tag{1.0c}$$

Proof

Analytically continuing the index of the differentiation operator in Formula 18.6.1 in **18.6.1** to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Example

$$\left(sin^2 x\right)^{\left(\frac{1}{2}\right)} = -2^{\frac{1}{2}-1} cos\left(2x+\frac{\pi}{4}\right) = \frac{1}{2}\left(sin\,2x+cos\,2x\right)$$

$$\left(cos^2 x\right)^{\left(\frac{3}{2}\right)} = 2^{\frac{3}{2}-1} cos\left(2x+\frac{3\pi}{4}\right) = -\left(sin\,2x+cos\,2x\right)$$

Analytically continuing the index of the differentiation operator in Formula 18.6.1' and Formula 18.6.1" in **18.6** to $[0,p]$ from $[1,n]$ respectively, we obtain the following two formualas.

Formula 19.6.1'

$$\left(sin^2 x\right)^{(p)} = \sum_{r=0}^{\infty}\binom{p}{r} sin\left\{x+\frac{(p-r)\pi}{2}\right\} sin\left(x+\frac{r\pi}{2}\right) \tag{1.1s}$$

$$\left(cos^2 x\right)^{(p)} = \sum_{r=0}^{\infty}\binom{p}{r} cos\left\{x+\frac{(p-r)\pi}{2}\right\} cos\left(x+\frac{r\pi}{2}\right) \tag{1.1c}$$

Formula 19.6.1"

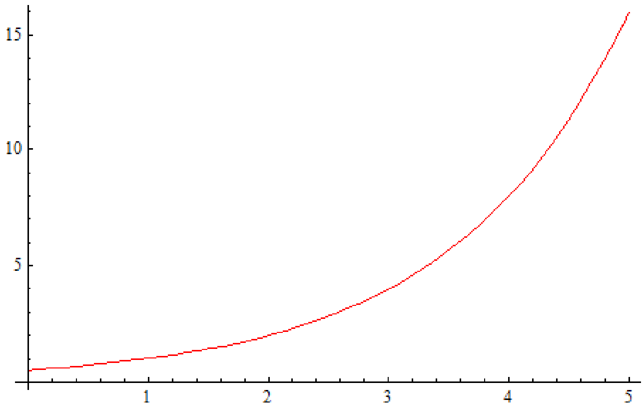
$$\sum_{r=0}^{\infty}\binom{p}{2r} = 2^{p-1} \qquad p>0 \tag{1.2e}$$

$$\sum_{r=0}^{\infty}\binom{p}{2r+1} = 2^{p-1} \qquad p>0 \tag{1.2o}$$

If a horizontal axis is set as p and these are illustrated , it is as follows. (1.2e) , (1.2o) and 2^{p-1} are blue, red and green respectively. Since three curves overlap exactly, only red (2^{p-1}) is visible.

```
fe[p_] := Sum[Binomial[p, 2 r], {r, 0, ∞}]    fo[p_] := Sum[Binomial[p, 2 r], {r, 0, ∞}]    f2[p_] := 2^(p-1)

Plot[{fe[p], fo[p], f2[p]}, {p, 0, 5}, PlotStyle → {Blue, Green, Red}]
```



The following formula follows from Formula 19.6.1 immediately.

Formula 19.6.1'''

$$\sum_{r=0}^{\infty} \binom{p}{r} = 2^p \qquad p>0 \tag{1.3}$$

Note

In fact, Formula 19.6.1'' and Formula 19.6.1''' hold for $p > -1$.

19.6.2 Super Derivatives of $\sin^3 x$, $\cos^3 x$

Formula 19.6.2

$$(\sin^3 x)^{(p)} = \frac{3}{4}\sin\left(x+\frac{p\pi}{2}\right) - \frac{3^p}{4}\sin\left(3x+\frac{p\pi}{2}\right) \tag{2.0s}$$

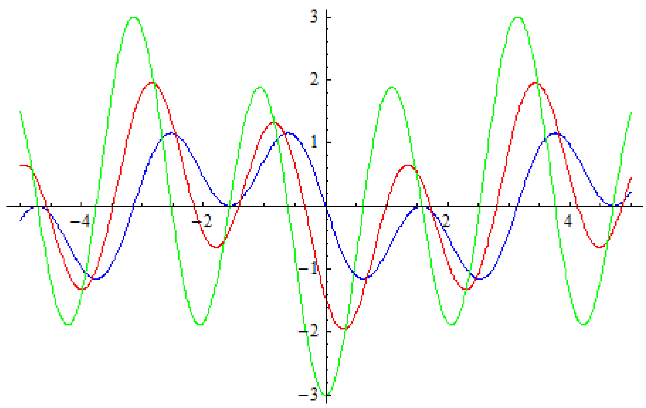
$$(\cos^3 x)^{(p)} = \frac{3}{4}\cos\left(x+\frac{p\pi}{2}\right) + \frac{3^p}{4}\cos\left(3x+\frac{p\pi}{2}\right) \tag{2.0c}$$

Example

$$(\sin^3 x)^{\left(\frac{1}{2}\right)} = \frac{3}{4}\sin\left(x+\frac{\pi}{4}\right) - \frac{\sqrt{3}}{4}\sin\left(3x+\frac{\pi}{4}\right)$$

$$(\cos^3 x)^{\left(\frac{3}{2}\right)} = \frac{3}{4}\cos\left(x+\frac{3\pi}{4}\right) + \frac{3\sqrt{3}}{4}\cos\left(3x+\frac{3\pi}{4}\right)$$

If the later is drawn with the 1st and the 2nd order derivatives on a figure side by side, it is as follows.



19.6.3 Super Derivatives of the product of trigonometric and hyperbolic functions

Formula 19.6.3

$$(\sin x \cdot \sinh x)^{(p)} = \sum_{r=0}^{\infty} \binom{p}{r} \sin\left\{x + \frac{(p-r)\pi}{2}\right\} \frac{e^x - (-1)^{-r} e^{-x}}{2} \tag{3.1}$$

$$(\sin x \cdot \cosh x)^{(p)} = \sum_{r=0}^{\infty} \binom{p}{r} \sin\left\{x + \frac{(p-r)\pi}{2}\right\} \frac{e^x + (-1)^{-r} e^{-x}}{2} \tag{3.2}$$

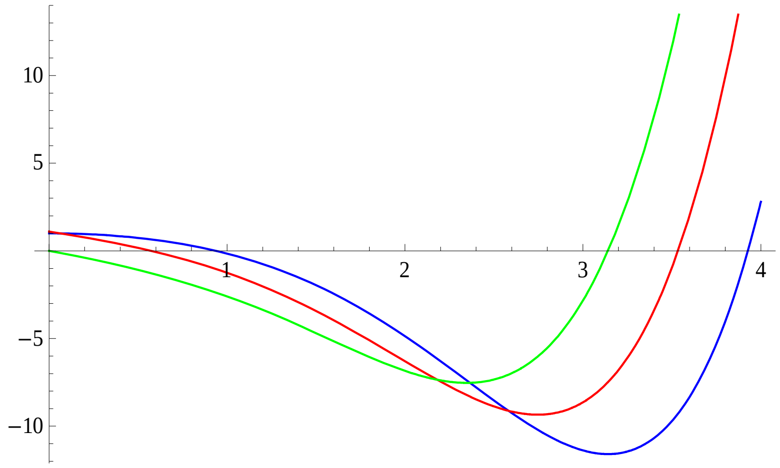
$$(\cos x \cdot \sinh x)^{(p)} = \sum_{r=0}^{\infty} \binom{p}{r} \cos\left\{x + \frac{(p-r)\pi}{2}\right\} \frac{e^x - (-1)^{-r} e^{-x}}{2} \tag{3.3}$$

$$(\cos x \cdot \cosh x)^{(p)} = \sum_{r=0}^{\infty} \binom{p}{r} \cos\left\{x + \frac{(p-r)\pi}{2}\right\} \frac{e^x + (-1)^{-r} e^{-x}}{2} \tag{3.4}$$

Example The 3/2th order derivative of $\cos x \cdot \sinh x$

$$(\cos x \cdot \sinh x)^{(3/2)} = \sum_{r=0}^{\infty} \binom{3/2}{r} \cos\left\{x + \frac{(3/2-r)\pi}{2}\right\} \frac{e^x - (-1)^{-r} e^{-x}}{2}$$

If this is drawn with the 1st and the 2nd order derivatives on a figure side by side, it is as follows.



19.7 Super Derivatives of sinh x f (x), cosh x f (x)

19.7.1 Super Derivatives of $sinh^2 x$, $cosh^2 x$

Formula 19.7.1

$$\left(sinh^2 x\right)^{(p)} = \sum_{r=0}^{\infty} \binom{p}{r} \frac{e^x - (-1)^{-p+r} e^{-x}}{2} \frac{e^x - (-1)^{-r} e^{-x}}{2} \tag{1.1s}$$

$$\left(cosh^2 x\right)^{(p)} = \sum_{r=0}^{\infty} \binom{p}{r} \frac{e^x + (-1)^{-p+r} e^{-x}}{2} \frac{e^x + (-1)^{-r} e^{-x}}{2} \tag{1.1c}$$

Proof

Analytically continuing the index of the differentiation operator in Formula 18.7.1 in **18.7** to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Example: The 0.01th and the 0.99th order derivatives of $sinh^2 x$

According to (1.1s), if each differential coefficient of the 0th order, the 0.01th order, the 0.99th order and the 1st order on arbitrary point $x=2$ are calculated , they are as follows. Naturally, the super differential coefficients turn into complex numbers.

```
f0[x_] = Sinh[x]^2;          f1[x_] = ∂x f0[x];
```

$$fp[p\_, x\_] := \sum_{r=0}^{100} \text{Binomial}[p, r] \frac{e^x - (-1)^{-p+r} e^{-x}}{2} \frac{e^x - (-1)^{-r} e^{-x}}{2}$$

```
N[f0[2]]        N[fp[0.01, 2]]
13.1541         13.2739  + 0.00731029  i

N[f1[2]]        N[fp[0.99, 2]]
27.2899         27.1014  - 0.00028484  i
```

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Alien's Mathematics