6 Superellipse (Lamé curve)

6.1 Equations of a superellipse

A superellipse (horizontally long) is expressed as follows.

Implicit Equation

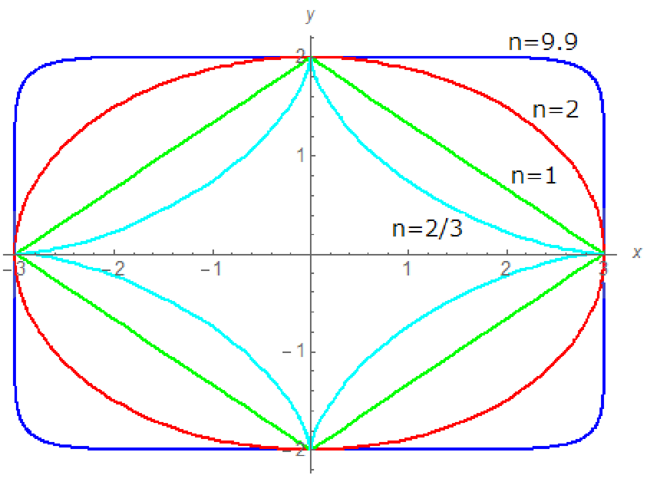
$$\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1 \qquad 0<b\leqq a\ ,\ n>0 \tag{1.1}$$

Explicit Equation

$$y = \pm b\left(1 - \left|\frac{x}{a}\right|^n\right)^{\frac{1}{n}} \qquad 0<b\leqq a\ ,\ n>0 \tag{1.1'}$$

When $a=3,\ b=2$, the superellipses for $n=9.9,\ 2,\ 1\ and\ 2/3$ are drawn as follows.

Fig1



These are called an ellipse when n = 2, are called a diamond when n = 1, and are called an asteroid when n = 2/3. These are known well.

parametric representation of an ellipse

In order to ask for the area and the arc length of a super-ellipse, it is necessary to calculus the equations. However, it is difficult for (1.1) and (1.1'). Then, in order to make these easy, parametric representation of (1.1) is often used.

As far as the 1st quadrant, (1.1) is as follows .

$$\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1 \qquad \begin{matrix} 0\leqq x\leqq a \\ 0\leqq y\leqq b \end{matrix}$$

Let us compare this with the following trigonometric equation.

$$\sin^2\theta + \cos^2\theta = 1$$

Then, we obtain

$$\frac{x^n}{a^n} = \sin^2\theta \quad , \quad \frac{y^n}{b^n} = \cos^2\theta$$

i.e.

$$x = a \sin^{\frac{2}{n}} \theta \quad , \quad y = b \cos^{\frac{2}{n}} \theta$$

The variable and the domain are changed as follows.

$$x: 0 \sim a \quad \rightarrow \quad \theta: 0 \sim \pi/2$$

By this representation, the area and the arc length of a superellipse can be calculated clockwise from the y-axis.

cf.

If we represent this as usual as follows,

$$x = a \cos^{\frac{2}{n}} \theta \quad , \quad y = b \sin^{\frac{2}{n}} \theta$$

the area and the arc length of a superellipse can be calculated counterclockwise from the x-axis. However, from this representation, we cannot obtain the Legendre forms of elliptic integrals

6.2 Area of a superellipse

Formula 6.2.1

When $n,a,b\ (b \leqq a)$ are positive numbers respectively and $\Gamma(z)$ is the gamma function, the area S of the ellipse of degree n is given by the following expression.

$$S = 4ab\frac{\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^2}{\Gamma\left(1+\frac{2}{n}\right)}$$

Proof

In order to obtain the area of a superellipse, we integrate with the following equation in the 1st quadrant, and should just multiply it by 4.

$$y = b\left(1-\frac{x^n}{a^n}\right)^{\frac{1}{n}} \qquad 0<b \leqq a\ ,\ n>0$$

That is,

$$S = 4b\int_0^a \left(1-\frac{x^n}{a^n}\right)^{\frac{1}{n}} dx$$

However, it is known that this integration cannot be expressed with elementary functions. Then, we use the parametric representation mentioned in the previous section. That is,

$$x = a\sin^{\frac{2}{n}}\theta \quad , \quad y = b\cos^{\frac{2}{n}}\theta$$

The variable and the domain are changed as follows.

$$x:\ 0\sim a \quad \rightarrow \quad \theta:\ 0\sim\pi/2$$

And dx is

$$dx = \frac{2a}{n}\sin^{\frac{2}{n}-1}\theta\cdot\cos\theta\, d\theta$$

Then,

$$\frac{S}{4} = \int_0^a y\,dx = \int_0^{\frac{\pi}{2}} b\cos^{\frac{2}{n}}\theta\left(\frac{2a}{n}\sin^{\frac{2}{n}-1}\theta\cdot\cos\theta\right)d\theta$$

$$= ab\cdot\frac{2}{n}\int_0^{\frac{\pi}{2}}\sin^{\frac{2}{n}-1}\theta\cdot\cos^{\frac{2}{n}+1}\theta\, d\theta$$

According to "岩波 数学公式Ｉ" p 243 ,

$$\int_0^{\frac{\pi}{2}}\sin^{\alpha}\theta\cdot\cos^{\beta}\theta\, d\theta = \frac{1}{2}B\left(\frac{\alpha+1}{2},\frac{\beta+1}{2}\right) \qquad \alpha,\beta > -1$$

Using this,

$$\int_0^{\frac{\pi}{2}}\sin^{\frac{2}{n}-1}\theta\cdot\cos^{\frac{2}{n}+1}\theta\, d\theta = \frac{1}{2}B\left(\frac{\frac{2}{n}-1+1}{2},\frac{\frac{2}{n}+1+1}{2}\right) = \frac{1}{2}B\left(\frac{1}{n},\frac{1}{n}+1\right)$$

$$= \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{n}+1\right)}{2\Gamma\left(\frac{1}{n}+\frac{1}{n}+1\right)} = \frac{n}{2}\frac{\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^2}{\Gamma\left(1+\frac{2}{n}\right)}$$

Substituting this for the avove,

$$\frac{S}{4} = ab\cdot\frac{2}{n}\frac{n}{2}\frac{\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^2}{\Gamma\left(1+\frac{2}{n}\right)} = ab\cdot\frac{\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^2}{\Gamma\left(1+\frac{2}{n}\right)}$$

Quadrupling both sides, we obtain the desired expression.

Note

Especially when $a=b=1$,

$$\rho_n = 4\cdot\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^2 / \Gamma\left(1+\frac{2}{n}\right)$$

ρ_n is the area of the unit circle $|x|^n+|y|^n=1 \quad n>0$, and should be called **constant of the circle of degree n**. That is, the area of an ellipse of degree n is given by $ab\rho_n$, where a is the major axis and b is the minor axis.

In fact, when $n=2$,

$$\rho_2 = 4\cdot\frac{\left\{\Gamma\left(1+\frac{1}{2}\right)\right\}^2}{\Gamma\left(1+\frac{2}{2}\right)} = 4\cdot\frac{\left\{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right\}^2}{1!} = 4\left(\frac{1}{2}\sqrt{\pi}\right)^2 = \pi$$

This is the area of the circle $x^2+y^2=1$, and the area of an ellipse of degree2 is given by $ab\pi$.

Example: Area of an ellipse of degree 2.3, $a=3$, $b=2$

As the result of calculating this area in numerical integral and the formula, both were completely consistent.

Numerical Integral

$$\text{sl}[a\_, b\_, n\_] := 4\,b\int_0^a \left(1-\left(\frac{x}{a}\right)^n\right)^{\frac{1}{n}} dx \qquad \text{sl}[3, 2, 2.3]$$

19.7922

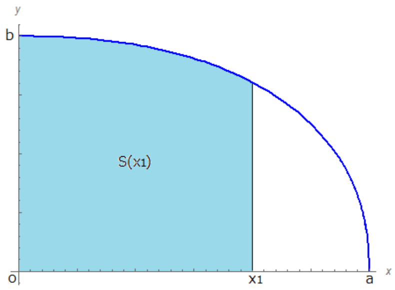
Formula

$$\text{sr}[a\_, b\_, n\_] := 4\,a\,b\,\frac{\left(\text{Gamma}\left[1+\frac{1}{n}\right]\right)^2}{\text{Gamma}\left[1+\frac{2}{n}\right]} \qquad \text{sr}[3, 2, 2.3]$$

19.7922

6.3 A part of area of a superellipse

In this section, we calculate the area of the light-blue portion of the following figure.



Formula 6.3.1

When $n,a,b\ (b \leqq a)$ are positive numbers respectively, the area $s(x)$ from 0 to x of the ellipse of degree n in the 1st quadrant is given by the following expression.

$$s(x) = b\sum_{r=0}^{\infty}\binom{1/n}{r}\frac{(-1)^r}{a^{nr}}\frac{x^{nr+1}}{nr+1} \tag{1.1}$$

Proof

The area $s(x)$ from 0 to x of the superellipse in the 1st quadrant is given by the following integral.

$$s(x) = b\int_0^x\left(1-\frac{x^n}{a^n}\right)^{\frac{1}{n}}dx \qquad 0< x \leqq a$$

This integral is not expressed with elementary functions except for a special case. So, we try the termwise integration.

Applying generalized binomial theorem to this integrand,

$$\left(1-\frac{x^n}{a^n}\right)^{\frac{1}{n}} = \sum_{r=0}^{\infty}(-1)^r\binom{1/n}{r}\left(\frac{x}{a}\right)^{nr}$$

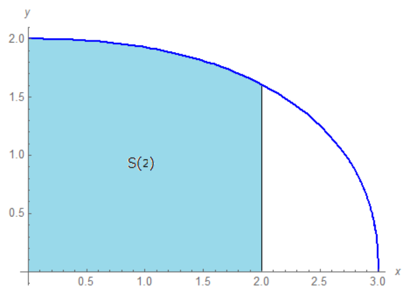
Integrating the right side with respect to x from 0 to x term by term,

$$\int_0^x\left(1-\frac{x^n}{a^n}\right)^{\frac{1}{n}}dx = \sum_{r=0}^{\infty}(-1)^r\binom{1/n}{r}\int_0^x\left(\frac{x}{a}\right)^{nr}dx$$

$$= \sum_{r=0}^{\infty}\binom{1/n}{r}\frac{(-1)^r}{a^{nr}}\frac{x^{nr+1}}{nr+1}$$

Multiplying this by b , we obtain (1.1) .

Example: Area of the ellipse of degree 2.3, $a=3, b=2$ on $x=0\sim 2$



We calculate the area of the light-blue portion of the above figure. As the result of calculating by numerical integral and (1.1) , both were completely consistent.

```
a = 3; b = 2; n = 2.3;
```

Numerical Integral

$$sl[x\_] := b\int_0^x \left(1-\left(\frac{t}{a}\right)^n\right)^{\frac{1}{n}} dt \qquad sl[2] \quad 3.77652$$

Series

$$sr[x\_] := b\sum_{r=0}^{\infty} \mathrm{Binomial}\left[\frac{1}{n}, r\right] \frac{(-1)^r}{a^{nr}} \frac{x^{nr+1}}{nr+1} \qquad sr[2] \quad 3.77652$$

By-products

Especially integrating this to a ,

$$b\int_0^a \left(1-\frac{x^n}{a^n}\right)^{\frac{1}{n}} dx = b\sum_{r=0}^{\infty} \binom{1/n}{r} \frac{(-1)^r}{a^{nr}} \frac{a^{nr+1}}{nr+1} = ab\sum_{r=0}^{\infty} \binom{1/n}{r} \frac{(-1)^r}{nr+1}$$

Then, the area S of the superellipse is

$$S = 4ab\sum_{r=0}^{\infty} \binom{1/n}{r} \frac{(-1)^r}{nr+1} \tag{1.2}$$

Therefore also,

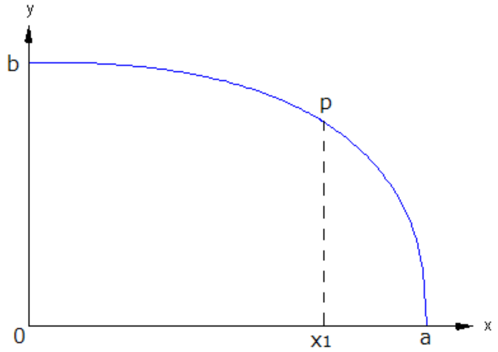
$$\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^2 / \Gamma\left(1+\frac{2}{n}\right) = \sum_{r=0}^{\infty} \binom{1/n}{r} \frac{(-1)^r}{nr+1} \tag{1.3}$$

6.4 Arc length of a superellipse

6.4.1 Arc length of an oblong superellipse

In this sub-section, we calculate the length $\overset{\frown}{bP}$ of the following oblong superellipse.

$$y = b\left(1-\frac{x^n}{a^n}\right)^{\frac{1}{n}} \qquad 0<b\leqq a\ ,\ n>0 \tag{1.0}$$



Formula 6.4.1

Let $n,a,b\ (b\leqq a)$ are positive numbers, x be a number s.t. $0< x \leq a\left\{1+\left(\frac{b}{a}\right)^{\frac{n}{n-1}}\right\}^{-\frac{1}{n}}$ and $(z)_s$ be Pochhammer's symbol. Then, the arc length $l(x)$ from 0 to x of (1.0) is given by the followings.

$$l(x)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{b^{2r}}{a^{2nr+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1} \tag{1.1}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\left(\frac{2(n-1)r}{n}\right)_s\frac{1}{s!}\frac{b^{2r}}{a^{2nr+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1} \tag{1.1'}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{1/2}{r-s}\left(\frac{2(n-1)(r-s)}{n}\right)_s\frac{1}{s!}\frac{b^{2(r-s)}}{a^{2n(r-s)+ns}}\frac{x^{2(n-1)(r-s)+ns+1}}{2(n-1)(r-s)+ns+1} \tag{1.1''}$$

Proof

Length $l(x)$ of the curve $y=f(x)\ \ \alpha\leqq x\leqq\beta$ on a plane is given by the following formula.

$$l(x)=\int_{\alpha}^{x}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \alpha\leqq x\leqq\beta$$

Differentiating both sides of (1.0) with respect to x,

$$\frac{dy}{dx}=\frac{b}{n}\left(1-\frac{x^n}{a^n}\right)^{\frac{1}{n}-1}\left(-\frac{nx^{n-1}}{a^n}\right)=-\frac{b}{a}\frac{x^{n-1}}{a^{n-1}}\left(1-\frac{x^n}{a^n}\right)^{-\frac{n-1}{n}}$$

$$=-\frac{b}{a}\left(\frac{x^n/a^n}{1-x^n/a^n}\right)^{\frac{n-1}{n}}$$

Then,

$$l(x) = \int_0^x \left\{ 1+\frac{b^2}{a^2}\left(\frac{x^n/a^n}{1-x^n/a^n}\right)^{\frac{2(n-1)}{n}} \right\}^{\frac{1}{2}} dx \tag{1}$$

This integral is not expressed with elementary functions except for a special case. So, we expand this integrand to series.

$$0 < x \le a\left\{1+\left(\frac{b}{a}\right)^{\frac{n}{n-1}}\right\}^{-\frac{1}{n}} \implies 0 < \frac{b^2}{a^2}\left(\frac{x^n/a^n}{1-x^n/a^n}\right)^{\frac{2(n-1)}{n}} \le 1$$

According to genralized binomial thorem,

$$\left\{ 1+\frac{b^2}{a^2}\left(\frac{x^n/a^n}{1-x^n/a^n}\right)^{\frac{2(n-1)}{n}} \right\}^{\frac{1}{2}} = \sum_{r=0}^{\infty}\binom{1/2}{r}\left(\frac{b}{a}\right)^{2r}\left(\frac{x^n/a^n}{1-x^n/a^n}\right)^{\frac{2(n-1)r}{n}}$$

Then,

$$l(x) = \sum_{r=0}^{\infty}\binom{1/2}{r}\left(\frac{b}{a}\right)^{2r}\int_0^x \left(\frac{x^n/a^n}{1-x^n/a^n}\right)^{\frac{2(n-1)r}{n}} dx \tag{2}$$

Since this integral is also a non-elementary function, we try the termwise integration.

$$\frac{1}{1-x} = 1+x^1+x^2+\cdots = \sum_{s=0}^{\infty}\binom{s+0}{0}x^s$$

Differentiating both sides of this with respect to x and dividing it by factorial one by one,

$$\frac{1}{(1-x)^2} = \frac{1}{1!}\left(1x^0+2x^1+3x^2+4x^3+\cdots\right) = \sum_{s=0}^{\infty}\binom{s+1}{1}x^s$$

$$\frac{1}{(1-x)^3} = \frac{1}{2!}\left(1x^0+3x^1+6x^2+10x^3+\cdots\right) = \sum_{s=0}^{\infty}\binom{s+2}{2}x^s$$

$$\vdots$$

$$\frac{1}{(1-x)^m} = \sum_{s=0}^{\infty}\binom{s+m-1}{m-1}x^s$$

Multiplying both sides by x^m,

$$\frac{x^m}{(1-x)^m} = \sum_{s=0}^{\infty}\binom{s+m-1}{m-1}x^{m+s}$$

Replacing x with x^n/a^n,

$$\left(\frac{x^n/a^n}{1-x^n/a^n}\right)^m = \sum_{s=0}^{\infty}\binom{s+m-1}{m-1}\left(\frac{x}{a}\right)^{n(m+s)}$$

Integrating both sides with respect to x from 0 to x,

$$\int_0^x \left(\frac{x^n/a^n}{1-x^n/a^n}\right)^m dx = \sum_{s=0}^{\infty}\binom{s+m-1}{m-1}\int_0^x \left(\frac{x}{a}\right)^{n(m+s)} dx$$

i.e.

$$\int_0^x \left(\frac{x^n/a^n}{1-x^n/a^n}\right)^m dx = \sum_{s=0}^{\infty}\binom{m-1+s}{m-1}\frac{1}{a^{nm+ns}}\frac{x^{nm+ns+1}}{nm+ns+1}$$

Since m may be a real number, replacing this with $\dfrac{2(n-1)r}{n}$,

$$\int_0^x\left(\frac{x^n/a^n}{1-x^n/a^n}\right)^{\frac{2(n-1)r}{n}}dx=\sum_{s=0}^{\infty}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{1}{a^{2(n-1)r+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1}$$

Substituting this for (2) , we obtain

$$l(x)=\sum_{r=0}^{\infty}\binom{1/2}{r}\left(\frac{b}{a}\right)^{2r}\sum_{s=0}^{\infty}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{1}{a^{2(n-1)r+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{b^{2r}}{a^{2nr+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1} \tag{1.1}$$

Next,

$$\binom{m-1+s}{m-1}=\binom{m-1+s}{s}=\frac{\Gamma(m+s)}{\Gamma(m)\Gamma(s+1)}=\frac{(m)_s}{s!}$$

Then

$$\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}=\frac{\left(\frac{2(n-1)r}{n}\right)_s}{s!}$$

Substituting thes for (1.1) , we obtain (1.1') . Furthermore, rearranging (1.1') to the diagonal series, we obtain (1.1") .

Example1: Arc length of the ellipse of degree 2.5, $a=3, b=2$ on $x=0\sim2.5449\cdots$

When $a=3, b=2, n=5/2$, the largest x1 calculable in the formula 6.4.1 is as follows.

```
a = 3; b = 2; n = 2.5;
```

$$\mathtt{x1} = \mathtt{a}\left(1+\left(\frac{\mathtt{b}}{\mathtt{a}}\right)^{\frac{\mathtt{n}}{\mathtt{n}-1}}\right)^{-\frac{1}{\mathtt{n}}};$$

```
SetPrecision[x1, 10]
2.544913192
```

If the arc length on $x:0\sim2.5449\cdots$ are calculated by numerical integration and (1.1') , it is as follows. As the result of calculating $\Sigma\ \Sigma$ to 30 respectively, both were consistent until 7 digits below the decimal point.

Numerical Integral

$$\mathtt{l1[x\_]} := \int_0^{\mathtt{x}}\left(1+\frac{\mathtt{b}^2}{\mathtt{a}^2}\left(\frac{\frac{\mathtt{t}^{\mathtt{n}}}{\mathtt{a}^{\mathtt{n}}}}{1-\frac{\mathtt{t}^{\mathtt{n}}}{\mathtt{a}^{\mathtt{n}}}}\right)^{\frac{2(\mathtt{n}-1)}{\mathtt{n}}}\right)^{\frac{1}{2}}d\mathtt{t}$$

```
SetPrecision[l1[x1], 10]
2.7089997275491674525
```

Double Series

```
lp[x_, m_] := Sum[Binomial[1/2, r] Pochhammer[2 (n-1) r/n, s] × 1/s! b^(2 r)/a^(2 n r+n s) x^(2 (n-1) r+n s+1)/(2 (n-1) r+n s+1), {r, 0, m}, {s, 0, m}]
```

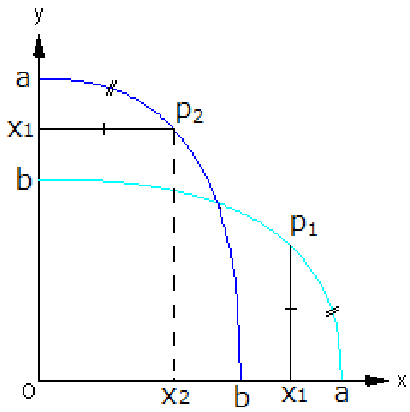
$$lp[x\_, m\_] := \sum_{r=0}^{m}\sum_{s=0}^{m} \text{Binomial}\left[\frac{1}{2}, r\right] \text{Pochhammer}\left[\frac{2(n-1)r}{n}, s\right] \times \frac{1}{s!}\frac{b^{2r}}{a^{2nr+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1}$$

```
SetPrecision[lp[x1, 30], 10]
2.708999702
```

6.4.2 Arc length of a longwise superellipse

In Formula 6.4.1, we cannot calculate the arc length $\overset{\frown}{aP_1}$ of an oblong superellipse. However, this problem is easily solvable. That is, let us replace x and y in (1.0) , as follows.

$$y = a\left(1-\frac{x^n}{b^n}\right)^{\frac{1}{n}} \qquad 0<b\leqq a \ ,\ n>0 \tag{2.0}$$



Then, the length of $\overset{\frown}{aP_1}$ of an oblong superellipse is consistent with the length of $\overset{\frown}{aP_2}$ of a longwise superellipse. And in order to obtain this, we should just replace a and b in Formula 6.4.1.

Formula 6.4.2

Let $n,a,b\ (b\leqq a)$ are positive numbers, x be a number s.t. $0<x\leq b\left\{1+\left(\frac{a}{b}\right)^{\frac{n}{n-1}}\right\}^{-\frac{1}{n}}$ and $(z)_s$ be Pochhammer's symbol. Then, the arc length $l(x)$ from 0 to x of (2.0) is given by the followings.

$$l(x)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{a^{2r}}{b^{2nr+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1} \tag{2.1}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\left(\frac{2(n-1)r}{n}\right)_s\frac{1}{s!}\frac{a^{2r}}{b^{2nr+ns}}\frac{x^{2(n-1)r+ns+1}}{2(n-1)r+ns+1} \tag{2.1'}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{1/2}{r-s}\left(\frac{2(n-1)(r-s)}{n}\right)_s\frac{1}{s!}\frac{a^{2(r-s)}}{b^{2n(r-s)+ns}}\frac{x^{2(n-1)(r-s)+ns+1}}{2(n-1)(r-s)+ns+1} \tag{2.1''}$$

Example2: Arc length of the ellipse of degree 2.5, $a=3, b=2$ on $x=3\sim2.5449\cdots$

When $a=3, b=2, n=5/2$, the largest x2 calculable in the formula 6.4.2 is as follows.

```
a = 3; b = 2; n = 2.5;
```

$$x2 = b\left(1-\left(\frac{a}{b}\right)^{\frac{n}{n-1}}\right)^{-\frac{1}{n}};$$

```
SetPrecision[x2, 10]
1.294754834
```

The arc length of (2.0) on $x: 0\sim1.2947\cdots$ is consistent with the arc length of (1.0) on $x: 3\sim2.5449\cdots$.

If the arc length on $x: 0\sim1.2947\cdots$ are calculated by numerical integral and (2.1"), it is as follows.

As the result of calculating $\sum\sum$ to 15, both were consistent until 9 digits below the decimal point.

Numerical Integral

$$ll[x\_] := \int_0^x \left(1+\frac{a^2}{b^2}\left(\frac{\frac{t^n}{b^n}}{1-\frac{t^n}{b^n}}\right)^{\frac{2(n-1)}{n}}\right)^{\frac{1}{2}} dt$$

```
SetPrecision[ll[x2], 10]
1.4138711054912040613
```

Diagonal Series

$$ld[x\_, m\_] := \sum_{r=0}^{m}\sum_{s=0}^{r} \text{Binomial}\left[\frac{1}{2}, r-s\right] \text{Pochhammer}\left[\frac{2(n-1)(r-s)}{n}, s\right] \times \frac{1}{s!} \frac{a^{2(r-s)}}{b^{2n(r-s)+ns}} \frac{x^{2(n-1)(r-s)+ns+1}}{2(n-1)(r-s)+ns+1}$$

```
SetPrecision[ld[x2, 15], 10]
1.413871105
```

Note

In the longwise superellipse, the convergence speed of the double series (2.1) (or (2.1')) is slow. It is about 1/100 of the convergence speed of the diagonal series (2.1").

6.5 Peripheral length of a superellipse

Formula 6.5.1

When $n,a,b\ (b \leqq a)$ are positive numbers respectively, the peripheral length L of the ellipse of degree n is given by the following expressions.

$$L=4\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{b^{2r}a^{1-2r}A^{-\frac{2(n-1)r+ns+1}{n}}+a^{2r}b^{1-2r}B^{-\frac{2(n-1)r+ns+1}{n}}}{2(n-1)r+ns+1} \tag{1.1}$$

$$=4\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\left(\frac{2(n-1)r}{n}\right)_s\frac{1}{s!}\frac{b^{2r}a^{1-2r}A^{-\frac{2(n-1)r+ns+1}{n}}+a^{2r}b^{1-2r}B^{-\frac{2(n-1)r+ns+1}{n}}}{2(n-1)r+ns+1} \tag{1.1'}$$

$$=4\sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{1/2}{r-s}\left(\frac{2(n-1)(r-s)}{n}\right)_s\frac{1}{s!}\frac{1}{2(n-1)(r-s)+ns+1}\times$$
$$\left\{b^{2(r-s)}a^{1-2(r-s)}A^{-\frac{2(n-1)(r-s)+ns+1}{n}}+a^{2(r-s)}b^{1-2(r-s)}B^{-\frac{2(n-1)(r-s)+ns+1}{n}}\right\} \tag{1.1''}$$

Where, $A=1+\left(\frac{b}{a}\right)^{\frac{n}{n-1}}$, $B=1+\left(\frac{a}{b}\right)^{\frac{n}{n-1}}$

Proof

Substituting $x=a\left\{1+\left(\frac{b}{a}\right)^{\frac{n}{n-1}}\right\}^{-\frac{1}{n}}\equiv aA$ for Formula 6.4.1 , we obtain the arc length from 0 to x as follows.

$$l(aA)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{b^{2r}}{a^{2r}}\frac{aA^{-\frac{2(n-1)r+ns+1}{n}}}{2(n-1)r+ns+1}$$

Next, substituting $x=b\left\{1+\left(\frac{a}{b}\right)^{\frac{n}{n-1}}\right\}^{-\frac{1}{n}}\equiv bB$ for Formula 6.4.2 , we obtain the arc length from a to x as follows.

$$l(bB)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{1/2}{r}\binom{\frac{2(n-1)r}{n}-1+s}{\frac{2(n-1)r}{n}-1}\frac{a^{2r}}{b^{2r}}\frac{bB^{-\frac{2(n-1)r+ns+1}{n}}}{2(n-1)r+ns+1}$$

Adding both, we obtain the arc length from 0 to a . Quadrupling this, we obtain (1.1) . And rewriting this with Pochhammer's symbol, we obtain (1.1') . Furthermore, rearranging this to the diagonal series, we obtain (1.1'') .

Example: Peripheral length of the ellipse of degree 2.5, $a=3$, $b=2$

We calculate this peripheral length by numerical integral and by the above formula respectively. Since the numerical integral can not be accurately calculated, We add two integration values in the previous section, and quadruple it. Formula (1.1'') is the only choice for the convergence speed. As the result of calculating $\sum\sum$ to 45, both were consistent until 9 digits below the decimal point.

Numerical Integral

4 × { 2.708999727549167452512418 + 1.413871105491204061256779 }

16.49148333216148605507679

Diagonal Series

a = 3; b = 2; n = 2.5;

$A = 1 + \left(\frac{b}{a}\right)^{\frac{n}{n-1}};$ SetPrecision[A, 10]

1.508761886

$B = 1 + \left(\frac{a}{b}\right)^{\frac{n}{n-1}};$ SetPrecision[B, 10]

2.965556046

$$Ld[m\_] := 4 \sum_{r=0}^{m} \sum_{s=0}^{r} \mathrm{Binomial}\left[\frac{1}{2}, r-s\right] \mathrm{Pochhammer}\left[\frac{2(n-1)(r-s)}{n}, s\right] \frac{1}{s!}$$
$$\times \frac{b^{2(r-s)} a^{1-2(r-s)} A^{-\frac{2(n-1)(r-s)+ns+1}{n}} + a^{2(r-s)} b^{1-2(r-s)} B^{-\frac{2(n-1)(r-s)+ns+1}{n}}}{2(n-1)(r-s)+ns+1}$$

SetPrecision[Ld[45], 12]

16.4914833320

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