7 Super Integral (Non-integer order Integral)

7.1 Super Primitive Function and Super Integral

In $\sum_{j=1}^{m} a_j$, $\prod_{k=1}^{n} b_k$, $\int_{a(n)}^{x} \cdots \int_{a(1)}^{x} f(x)\,dx^n$, $etc.$, $\sum$, $\prod$, $\int$, $etc.$ are operators,

and j , k , $etc.$ are indexes, and $[1,m]$, $[1,n]$, $etc.$ are the domains of the index.
As for these index and its domain, a natural number and the set are usually used.
However, the domain may sometimes be extended. For example,

It is $\sum_{j=0}^{p} a = ap$ which extended the domain of index j of $\sum_{j=1}^{m} a$ to the real number interval $[0,p]$ from the natural number interval $[1,m]$.

It is $\prod_{k=0}^{q} b = b^q$ which extended the domain of index k of $\prod_{k=1}^{n} b$ to the real number interval $[0,q]$ from the natural number interval $[1,n]$.

A fractional and an irrational number are obtained by extending the domain of the index of the operator so that these two examples may show. It is called *analytic continuation* to extend the domain generaly. Although usually analytic continuation is used for extending the domain of a function, it can be used also for extending the domain of the index of a operator as mentioned above.

7.1.1 Super Primitive Function

Definition 7.1.1

$f^{<p>}(x)$ obtained by continuing analytically the index of the integration operator of a higher primitive function $f^{<n>}(x)$ to a complex plane $[0,p]$ from a natural number interval $[1,n]$ is called ***Super Primitive Function of f(x)***. $f^{<p>}(x)$ may mean the Super Indefinite Integral, and may mean a Super Integration Function.

Example

$$(\sin x)^{<p>} = \sin\left(x - \frac{p\pi}{2}\right) + c_p(x)$$

$c_p(x)$ is an arbitrary function.

7.1.2 Super Integral

Definition 7.1.2

We call it ***Super Integral*** to integrate a function f with respect to an independent variable x from $a(0)$ to $a(p)$ continuously. And it is described as follows.

$$\int_{a(p)}^{x} \sim \int_{a(0)}^{x} f(x)\,dx^p \quad \left\{ = \int_{a(p)}^{x} \sim \int_{a(0)}^{x} f(x)\,dx \sim dx \right\}$$

And

when $a(k) = a$ $for\ all\ \ k \in [0,p]$, we call it ***super integral with a fixed lower limit***,
when $a(k) \neq a$ $for\ some\ k \in [0,p]$, we call it ***super integral with variable lower limits***.

Example

$$\int_{0}^{x} \sim \int_{0}^{x} \sin x\,dx^p = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(2r+2+p)} x^{2r+1+p}$$

$$\int_{\frac{p\pi}{2}}^{x}\sim\int_{\frac{0\pi}{2}}^{x}\sin x\,dx^{p}=\sin\left(x-\frac{p\pi}{2}\right)$$

7.1.3 Fundamental Theorem of Super Integral

Continuing analytically the index of the integration operator in Theorem 4.1.3 to a complex plane $[0,p]$ from a natural number interval $[1,n]$, we obtain the following theorem.

Theorem 7.1.3

Let $f^{<r>}$ $r\in[0,p]$ be an continuous function on the closed interval I and be arbitrary the r-th order primitive function of f . And let $a(r)$ be a continuous function on the closed interval $[0,p]$. Then the following expression holds for $a(r),x\in I$.

$$\int_{a(p)}^{x}\sim\int_{a(0)}^{x}f(x)dx^{p}=f^{<p>}(x)-\sum_{r=0}^{p-1}f^{<p-r>}\{a(p-r)\}\int_{a(p)}^{x}\sim\int_{a(p-r+1)}^{x}dx^{r} \tag{1.1}$$

Especially, when $a(r)=a$ $for\ all\ k\in[0,p]$,

$$\int_{a}^{x}\sim\int_{a}^{x}f(x)dx^{p}=f^{<p>}(x)-\sum_{r=0}^{p-1}f^{<p-r>}(a)\frac{(x-a)^{r}}{\Gamma(1+r)} \tag{1.2}$$

Constant-of-integration Function

We call $\sum_{r=0}^{p-1}etc.$ ***Constant-of-integration Function of*** $f(x)$. Since p is a real number, in order to obtain $\sum_{r=0}^{p-1}etc.$ generally, the calculus about the calculus operator $<r>,(s)$ is required. This is very difficult.

However, it becomes easy exceptionally at the time of $f(x)=e^{x}$, and we can obtain the following expression from (1.2) .

$$\sum_{r=0}^{p-1}e^{a}\frac{(x-a)^{r}}{\Gamma(1+r)}=e^{x}-\int_{a}^{x}\sim\int_{a}^{x}e^{x}dx^{p}=e^{a}\sum_{r=0}^{\infty}\left\{\frac{(x-a)^{r}}{\Gamma(1+r)}-\frac{(x-a)^{r+p}}{\Gamma(1+r+p)}\right\}$$

$$\left(\because\quad e^{x}=e^{a}\sum_{r=0}^{\infty}\frac{(x-a)^{r}}{\Gamma(1+r)}\quad,\quad\int_{a}^{x}\sim\int_{a}^{x}e^{x}dx^{p}=e^{a}\sum_{r=0}^{\infty}\frac{(x-a)^{r+p}}{\Gamma(1+r+p)}\right)$$

7.1.4 Lineal and Collateral

Definition 7.1.4

$$\int_{a(p)}^{x}\sim\int_{a(0)}^{x}f(x)dx^{p}=f^{<p>}(x)-\sum_{r=0}^{p-1}f^{<p-r>}\{a(p-r)\}\int_{a(p)}^{x}\sim\int_{a(p-r+1)}^{x}dx^{r} \tag{1.1}$$

In this expression,
when Constant-of-integration Functin is 0,

we call $\int_{a(p)}^{x}\sim\int_{a(0)}^{x}f(x)dx^{p}$ ***Lineal Super Integral*** and

we call the function equal to this ***Lineal Super Primitive Function*** .

when Constant-of-integration Functin is not 0, we call $\int_{a(p)}^{x} \sim \int_{a(0)}^{x} f(x)\,dx^{p}$ ***Collateral Super Integral*** and

we call the function equal to this ***Collateral Super Primitive Function***.

These are the same also in (1.2).

In short , ***Lineal Super Primitive Function*** is what integrated $f(x)$ with respect to x continuously **without considering the constant-of-integration function.**

Example

Left : collateral integral Right : collateral primitive function

$$\int_{p}^{x} \sim \int_{0}^{x} \sin x\,dx^{p} = \sin\left(x - \frac{p\pi}{2}\right) - \sum_{r=0}^{p-1} \sin\left\{(p-r) - \frac{p\pi}{2}\right\} \int_{p}^{x} \sim \int_{p-r+1}^{x} dx^{r}$$

$$\int_{0}^{x} \sim \int_{0}^{x} e^{x}dx^{p} = e^{x} - \sum_{r=0}^{\infty}\left\{\frac{x^{r}}{\Gamma(1+r)} - \frac{x^{r+p}}{\Gamma(1+r+p)}\right\} \quad \left(= \sum_{r=0}^{\infty} \frac{x^{r+p}}{\Gamma(1+r+p)}\right)$$

Left : lineal integral Right : lineal primitive function

$$\int_{\frac{p\pi}{2}}^{x} \sim \int_{\frac{0\pi}{2}}^{x} \sin x\,dx^{p} = \sin\left(x - \frac{p\pi}{2}\right)$$

$$\int_{-\infty}^{x} \sim \int_{-\infty}^{x} e^{x}dx^{p} = e^{x}$$

7.1.5 The Necessary Conditions for the Super Integral being Lineal

In Theorem 7.1.3 (1.1), since the higher integral of 1 can be arbitrary value, in order for Constant-of-integration Function to be 0, we find out that it must be $f^{<r>}\{a(r)\} = 0 \;\; for\ all\ \ r \in [0,p]$.
Since this is important, it is stated here as a theorem.

Theorem 7.1.5

$$\int_{a(p)}^{x} \sim \int_{a(0)}^{x} f(x)dx^{p} = f^{<p>}(x) - \sum_{r=0}^{p-1} f^{<p-r>}\{a(p-r)\} \int_{a(p)}^{x} \sim \int_{a(p-r+1)}^{x} dx^{r} \qquad (1.1)$$

$$\int_{a}^{x} \sim \int_{a}^{x} f(x)dx^{p} = f^{<p>}(x) - \sum_{r=0}^{p-1} f^{<p-r>}(a)\frac{(x-a)^{r}}{\Gamma(1+r)} \qquad (1.2)$$

The necessary condition for the super integral being lineal is as follows respectively.

$$f^{<r>}\{a(r)\} = 0 \qquad for\ all\ \ r \in [0,p]$$
$$f^{<r>}(a) \quad = 0 \qquad for\ all\ \ r \in [0,p]$$

That is, **the necessary condition for the super integral being lineal is** that $\boldsymbol{a(r)}$ *or* $\boldsymbol{a}$ **are zeros of the super primitive function $f^{<r>}$ for all continuous r from 0 to p.**
In addition, this is not sufficient condition as well as the case of Higher Integral.

Note

Although I was taught to Mr. Sugimoto and knew in 2005, seemingly such integration is the field which is called "Fractinal Integral" and studied in Europe in recent years. I thought I follow this term in this text. However, although the number of times of integration is extensible even to a complex plane, wrapping cloth of "Fractional" is too small. So, in this text, I decided to use "Super Integral" that I was using from before.

7.2 Fractional Integral

7.2.1 Fractional Integral & Riemann-Liouville Integral

There seems to be Fractional Integral for about 200 years, and the most general form is as follows.

$$ {}_aD_x^{-p}f(x) = \frac{1}{\Gamma(p)}\int_a^x (x-t)^{p-1}f(t)\,dt \quad (a \text{ is a zero of the left side }) \tag{2.0}$$

The left side ${}_aD_x^{-p}f(x)$ is *Non-integer order Primitive Function*, and this is the same as *Super Primitive Function* $f^{<p>}(x)$ defined in the previous section. Notation ${}_aD_x^{-p}$ is called Riemann-Liouville operator. On the other hand, the integration of the right side is called *Riemann-Liouville Integral*, and this is equivalent to *Super Integral* defined in the previous section.

Although many of super primitive function (non-integer order primitive function) cannot be expressed with an elementary function, the some can be expressed with elementary functions. In traditional *Fractional Integral*, the super primitive function expressed with these elementary functions are drawn from Riemann-Liouville integral. However, the method is very difficult. Two examples are shown below.

(1) super primitive function of $f(x)=x^{\alpha}$

Let

$$f(t) = t^{\alpha} \ , \ g(x-t) = \frac{(x-t)^{p-1}}{\Gamma(p)}$$

then

$$\left(x^{\alpha}\right)^{<p>} = \frac{1}{\Gamma(p)}\int_0^x t^{\alpha}(x-t)^{p-1}dt = \int_0^x f(t)g(x-t)\,dt$$

We find out that this is a convolution $(f*g)(x)$. Then we take the Laplace transform of $(f*g)(x)$;

$$(f*g)(x) \longrightarrow F(s)\cdot G(s)$$

$$f(x) = x^{\alpha} \longrightarrow \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} = F(s)$$

$$g(x) = \frac{x^{p-1}}{\Gamma(p)} \longrightarrow \frac{1}{\Gamma(p)}\frac{\Gamma(p)}{s^p} = \frac{1}{s^p} = G(s)$$

$$\therefore \ F(s)\cdot G(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\frac{1}{s^p} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+p+1)}\frac{\Gamma(\alpha+p+1)}{s^{\alpha+p+1}}$$

Finally, taking the inverse Laplace transform, we obtain $\dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+p+1)}x^{a+p}$.

(2) super primitive function of $f(x)=e^x$

$$\left(e^x\right)^{<p>} = \frac{1}{\Gamma(p)}\int_{-\infty}^x (x-t)^{p-1}e^t dt$$

The indefinite integral of the right-hand side is expressed as follows using $\Gamma(p,x)=\int_x^{\infty}\tau^{p-1}e^{-\tau}d\tau$.

$$\int (x-t)^{p-1}e^t dt = e^x\Gamma(p, x-t)$$

Then,

$$\left(e^x\right)^{<p>} = \frac{1}{\Gamma(p)}\int_{-\infty}^{x}(x-t)^{p-1}e^t dt = \frac{e^x}{\Gamma(p)}\left[\Gamma(p,x-t)\right]_{-\infty}^{x}$$
$$= \frac{e^x}{\Gamma(p)}\left\{\Gamma(p,x-x) - \Gamma(p,x+\infty)\right\} = \frac{e^x}{\Gamma(p)}\Gamma(p)$$
$$= e^x$$

7.2.2 The demerit and the strong point of Fractional Integral

As seen in two upper examples, Fractional Integral is difficult like this. Calculation of the power function of the first example is a masterful performance. I do not know whether it will solve by this method also in the case of $(ax+b)^{<p>}$. Calculationof the exponential function of the 2nd example is by force. In fact, I borrowed the power of mathematical software for this. Although I also challenged Fractional Integral of a logarithmic function, in spite of the help of mathematical software, it was too difficult for me. When p is non-integer, the calculation which obtains the super-primitive function expressed with the elementary function by Riemann-Liouville integration looks just like the trial which makes infinite decimals with unknown rational number or irrational number a fraction.

Next, how to take the lower limit of Riemann-Liouville integral in Fractional Integral cannot understand well to me. Probably, this originates in the fact that (2.0) holds for any lower limit a . However, the greatest cause will be that there is no concept of *Lineal* and *Collateral* in Fractional Integral. Even if it is the usual integral and is Fractional Integral, I think that original asks for a lineal primitive function.

Finally, since Fractional Integral is dependent on Riemann-Liouville integral, it cannot treat super integral with a variable lower limit. Specifically, it cannot perform lineal super integral of trigonometric functions or hyperbolic functions. In order to make these possible, as for us, it is unescapable to stop use of Riemann-Liouville integral.

Although only the demerit of Fractional Integral was mentioned above, there is nothing beyond this for the super integral with a fixed lower limit. The super integral which is the continued type of the higher order integral can be expressed with single integral. Numerical integral is possible. These merits and powers are greatest. If the concept of super integral is taken in and suitable usage is carried out, Riemann-Liouville integral will serve as a very powerful tool.

7.2.3 Fractional Integral and Super Integral

Theorem 7.2.3

When $f(x)$ denotes a continuously differentiable function and $\Gamma(z)$ denotes Gamma Function, the following expression holds for any $p>0$.

$$\int_a^x \sim \int_a^x f(x)\,dx^p = \frac{1}{\Gamma(p)}\int_a^x (x-t)^{p-1} f(t)\,dt \tag{2.1}$$

Proof

Analytically continuing the index of the integration operator in Theorem 4.2.3 to $[0,p]$ from $[1,n]$, we obtain the desired expression.

Remark

This theorem means that Super Integral with a fixed lower limit is equivalent to Fractional Integral. However, in Super Integral, instead of Riemann-Liouville integral, the Higher Integral is used to draw the super primitive function expressed with elementary functions. That is, we ask for the function form of the higher order primitive function by the higher integral first, and extend the order to the real number from an integer. In Super Integral, by this way, we can easily obtain the super primitive function which is expressed with elementary functions, without the difficult calculation such as previous examples. And in order to verify the result numerically, Riemann-Liouville integral is used.

Proof of the pudding is in the eating.

In order to show the justification of Theorem 7.2.3, we describe the formulas in the following 3 sections here in advance.

$$\int_0^x \sim \int_0^x x^\alpha dx^p = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p)} x^{\alpha+p} = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} t^\alpha dt$$

$$\int_{\mp\infty}^x \sim \int_{\mp\infty}^x e^{\pm x} dx^p = (\pm 1)^p e^{\pm x} = \frac{1}{\Gamma(p)} \int_{\mp\infty}^x (x-t)^{p-1} e^{\pm t} dt$$

$$\int_0^x \sim \int_0^x \log x\, dx^p = \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} x^p = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} \log t\, dt$$

Middle is the Super Integral obtained by analytically continuing the index of the operator of the Higher Integral to real number from natural number, and the right side is Riemann-Liouville Integral. These formulas are inputted into mathematical software, arbitrary one point is chosen suitably, and the function values are calculatied. The value of bothe sides are in agreement and it is shown numerically that Super Integral and Riemann-Liouville Integral are equal.

3/2 times integral of x^3

```
a = 3;  p = 3 / 2;
```

$$\text{f1}[x\_] := \frac{\text{Gamma}[1+a]}{\text{Gamma}[1+a+p]} x^{a+p}$$

$$\text{fr}[x\_] := \frac{1}{\text{Gamma}[p]} \int_0^x (x-t)^{p-1} t^a\, dt$$

```
N[f1[3], 10]          N[fr[3], 10]
16.08200268           16.08200268
```

4/3 times integral of e^x

```
p = 4 / 3;
f1[x_] := e^x
```

$$\text{fr}[x\_] := \frac{1}{\text{Gamma}[p]} \int_{-\infty}^x (x-t)^{p-1} e^t\, dt$$

```
N[f1[3], 10]          N[fr[3], 10]
20.08553692           20.08553692
```

5/4 times integral of log x

```
p = 5 / 4;
```

$$\text{f1}[x\_] := \frac{\text{Log}[x] - \text{PolyGamma}[1+p] - \text{EulerGamma}}{\text{Gamma}[1+p]} x^p$$

$$\text{fr}[x\_] := \frac{1}{\text{Gamma}[p]} \int_0^x (x-t)^{p-1} \text{Log}[t]\, dt$$

```
N[f1[7], 10]          N[fr[7], 10]
8.000834211           8.000834211
```

7.3 Super Integral of Power Function

7.3.1 Super Integral of Power Function

Analytically continuing the index of the integration operator in Formula 4.3.1 to $[0,p]$ from $[1,n]$, we obtain the following formula. In addition, Rieamnn-Liouville integrals are also expressed together

Formula 7.3.1

(1) Basic form

$$\int_0^x \sim \int_0^x x^\alpha dx^p = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p)} x^{\alpha+p} = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} t^\alpha dt \qquad (\alpha \geqq 0)$$

$$\int_\infty^x \sim \int_\infty^x x^\alpha dx^p = (-1)^p \frac{\Gamma(-\alpha-p)}{\Gamma(-\alpha)} x^{\alpha+p} = \frac{1}{\Gamma(p)} \int_\infty^x (x-t)^{p-1} t^\alpha dt \qquad (\alpha < -p)$$

(2) Linear form

$$\int_{-\frac{b}{a}}^x \sim \int_{-\frac{b}{a}}^x (ax+b)^\alpha dx^p = \left(\frac{1}{a}\right)^p \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p)} (ax+b)^{\alpha+p} \qquad (\alpha \geqq 0)$$

$$= \frac{1}{\Gamma(p)} \int_{-\frac{b}{a}}^x (x-t)^{p-1} (at+b)^\alpha dt$$

$$\int_\infty^x \sim \int_\infty^x (ax+b)^\alpha dx^p = \left(-\frac{1}{a}\right)^p \frac{\Gamma(-\alpha-p)}{\Gamma(-\alpha)} (ax+b)^{\alpha+p} \qquad (\alpha < -p)$$

$$= \frac{1}{\Gamma(p)} \int_\infty^x (x-t)^{p-1} (at+b)^\alpha dt$$

Note

When $-p \leqq \alpha < 0$, we do not define the super-integral of a power function. It is because a lower limit of the super integral changes irregularly at this time.

Example 1 : When $\alpha \geqq 0$

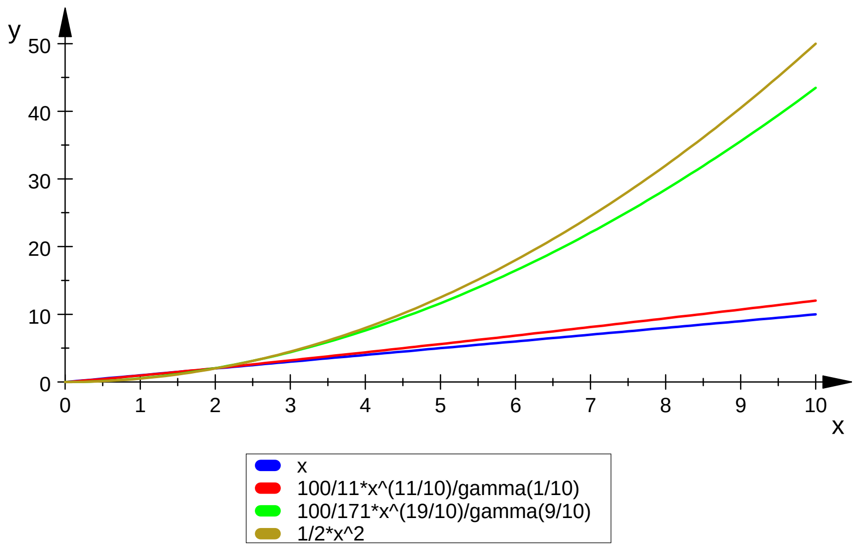
$$\left(x^1\right)^{\left\langle \frac{1}{10} \right\rangle} = \frac{\Gamma(1+1)}{\Gamma\left(1+1+\frac{1}{10}\right)} x^{1+\frac{1}{10}} = \frac{100}{11\Gamma\left(\frac{1}{10}\right)} x^{\frac{11}{10}} = 0.955579\, x^{\frac{11}{10}}$$

$$\left(x^1\right)^{\left\langle \frac{9}{10} \right\rangle} = \frac{\Gamma(1+1)}{\Gamma\left(1+1+\frac{9}{10}\right)} x^{1+\frac{9}{10}} = \frac{100}{171\Gamma\left(\frac{9}{10}\right)} x^{\frac{19}{10}} = 0.547239\, x^{\frac{19}{10}}$$

$$\left(x^2\right)^{\langle e \rangle} = \frac{\Gamma(1+2)}{\Gamma(1+2+e)} x^{2+e} = \frac{2}{\Gamma(3+e)} x^{2+e} = 0.026755\, x^{4.718281828}$$

$$\left(x^1\right)^{\langle i \rangle} = \frac{\Gamma(1+1)}{\Gamma(1+1+i)} x^{1+i} = \frac{x^{1+i}}{(1+i)!} = (1.200176 - 0.630568\, i)\, x^{1+i}$$

When x^1, $\left(x^1\right)^{<1/10>}$, $\left(x^1\right)^{<9/10>}$, $x^2/2$ are drawn on a figure side by side, it is as follows.



Example 2 : When $\alpha < -p$

$$\left\{\frac{1}{(5x+4)^{5/2}}\right\}^{\left\langle\frac{3}{2}\right\rangle} = \left(-\frac{1}{5}\right)^{\frac{3}{2}} \frac{\Gamma(5/2\ -3/2)}{\Gamma(5/2)}(5x+4)^{-\frac{5}{2}+\frac{3}{2}}$$

$$= -\frac{4\sqrt{5}\,i}{75\sqrt{\pi}\,(5x+4)} = -\frac{0.0672835}{5x+4}\,i$$

$$\left(x^{-2}\right)^{<1-i>} = (-1)^{1-i}\frac{\Gamma\{2-(1-i)\}}{\Gamma(2)}\,x^{-2+1-i}$$

$$= -\frac{1}{(-1)^{i}}\,\frac{i\,!}{x^{1+i}} = \frac{-11.524427+3.585646\,i}{x^{1+i}}$$

$$\left(x^{-1}\right)^{\left\langle\frac{1}{2}\right\rangle} = (-1)^{\frac{1}{2}}\frac{\Gamma(1-1/2)}{\Gamma(1)}\,x^{-1+\frac{1}{2}} = i\sqrt{\pi}\,x^{-\frac{1}{2}}$$

Example 3 : Outside of a definition (improper)

$$\left(x^{-\frac{1}{2}}\right)^{\left\langle\frac{1}{2}\right\rangle} = \int_0^x \sim \int_\infty^x x^{-\frac{1}{2}}\,dx^{\frac{1}{2}} \qquad (-p=\alpha<0)$$

$$\left(x^{-1}\right)^{<2>} = \int_0^x\int_1^x x^{-1}\,dx^2 = x(\log x-1) \qquad (-p<\alpha<0)$$

7.3.2 Half Integral of a power function

Especially, Super Integral of order 1/2 is called Half Integral.

Formula 7.3.2

Let n be a non-negative integer, $-1!! \equiv 1,\ (2n-1)!! \equiv 1\times3\times5\times\cdots\times(2n-1)$,

$0!! \equiv 1,\quad 2n\,!! \equiv 2\times4\times6\times\cdots\times2n$,

then following expressions hold.

(1) Basic form

$$\left(x^{n}\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{2(2n)!!}{(2n+1)!!\sqrt{\pi}}\, x^{n+\frac{1}{2}}$$

$$\left(x^{n+\frac{1}{2}}\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{(2n+1)!!\sqrt{\pi}}{2(2n)!!(n+1)}\, x^{n+1}$$

(2) Linear form

$$\left\{(ax+b)^{n}\right\}^{\left\langle\frac{1}{2}\right\rangle} = \left(\frac{1}{a}\right)^{\frac{1}{2}} \frac{2(2n)!!}{(2n+1)!!\sqrt{\pi}}(ax+b)^{n+\frac{1}{2}}$$

$$\left\{(ax+b)^{n+\frac{1}{2}}\right\}^{\left\langle\frac{1}{2}\right\rangle} = \left(\frac{1}{a}\right)^{\frac{1}{2}} \frac{(2n+1)!!\sqrt{\pi}}{2(2n)!!(n+1)}(ax+b)^{n+1}$$

Proof

From Formula 7.3.1 (2) Linear form,

$$\left\{(ax+b)^{n}\right\}^{\left\langle\frac{1}{2}\right\rangle} = \left(\frac{1}{a}\right)^{\frac{1}{2}} \frac{\Gamma(1+n)}{\Gamma\left(1+n+\frac{1}{2}\right)}(ax+b)^{n+\frac{1}{2}}$$

$$\left\{(ax+b)^{n+\frac{1}{2}}\right\}^{\left\langle\frac{1}{2}\right\rangle} = \left(\frac{1}{a}\right)^{\frac{1}{2}} \frac{\Gamma\left\{1+\left(n+\frac{1}{2}\right)\right\}}{\Gamma\left\{1+\left(n+\frac{1}{2}\right)+\frac{1}{2}\right\}}(ax+b)^{n+1}$$

Where, since $\Gamma\left(n+\frac{3}{2}\right) = \frac{(2n+1)!!}{2^{n+1}}\sqrt{\pi}$,

$$\frac{\Gamma(1+n)}{\Gamma\left(1+n+\frac{1}{2}\right)} = \frac{\Gamma(1+n)}{\Gamma\left(n+\frac{3}{2}\right)} = \frac{2^{n+1}\, n!}{(2n+1)!!\sqrt{\pi}} = \frac{2(2n)!!}{(2n+1)!!\sqrt{\pi}}$$

$$\frac{\Gamma\left\{1+\left(n+\frac{1}{2}\right)\right\}}{\Gamma\left\{1+\left(n+\frac{1}{2}\right)+\frac{1}{2}\right\}} = \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+2)}$$

$$= \frac{(2n+1)!!\sqrt{\pi}}{2^{n+1}(n+1)!}x^{n+1} = \frac{(2n+1)!!\sqrt{\pi}}{2(2n)!!(n+1)}x^{n+1}$$

Substituting these for above expressions, we obtain the linear form. And giving $a=1, b=0$, we obtain the basic form.

Example 1

$$\left(x^{0}\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{2\cdot 0!!}{1!!\sqrt{\pi}}\, x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}\, x^{\frac{1}{2}}$$

$$\left(x^1\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{2\cdot 2!!}{3!!\sqrt{\pi}}\,x^{\frac{3}{2}} = \frac{4}{3\sqrt{\pi}}\,x^{\frac{3}{2}}$$

$$\left(x^2\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{2\cdot 4!!}{5!!\sqrt{\pi}}\,x^{\frac{5}{2}} = \frac{16}{15\sqrt{\pi}}\,x^{\frac{5}{2}}$$

$$\left(x^3\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{2\cdot 6!!}{7!!\sqrt{\pi}}\,x^{\frac{7}{2}} = \frac{32}{35\sqrt{\pi}}\,x^{\frac{7}{2}}$$

Example 2

$$\left(x^{\frac{1}{2}}\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{1!!\sqrt{\pi}}{2\cdot 0!!}x^1 = \frac{\sqrt{\pi}}{2}x^1$$

$$\left(x^{\frac{3}{2}}\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{3!!\sqrt{\pi}}{2\cdot 2!!\cdot 2}x^2 = \frac{3\sqrt{\pi}}{8}x^2$$

$$\left(x^{\frac{5}{2}}\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{5!!\sqrt{\pi}}{2\cdot 4!!\cdot 3}x^3 = \frac{15\sqrt{\pi}}{48}x^3$$

$$\left(x^{\frac{7}{2}}\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{7!!\sqrt{\pi}}{2\cdot 6!!\cdot 4}x^4 = \frac{35\sqrt{\pi}}{128}x^4$$

7.3.3 Half Integral of an integer power function

Next, using Riemann-Liouville integral, we obtain the Half Integral of an integer power function.

Formula 7.3.3

When n denots a natural number, the following expression holds.

$$\left(x^n\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}\cdot x^{n+\frac{1}{2}}$$

Proof

Let n be a natural number, $\alpha=n$, $p=1/2$. Then since $\alpha>-p$, Applying the Formula 7.3.1 to these, we obtain the following expression.

$$\left(x^n\right)^{\left\langle\frac{1}{2}\right\rangle} = \frac{1}{\Gamma(1/2)}\int_0^x (x-t)^{-\frac{1}{2}}t^n\,dt = \frac{1}{\sqrt{\pi}}\int_0^x \frac{t^n}{\sqrt{x-t}}dt$$

Where, the following equation is known. (岩波数学公式Ⅰ p96).

$$\int \frac{t^n}{\sqrt{x-t}}dt = \frac{2\sqrt{x-t}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{2n-2r+1}$$

Then

$$\int_0^x \frac{t^n}{\sqrt{x-t}}dt = \left[\frac{2\sqrt{x-t}}{(-1)^{n+1}} \sum_{r=0}^{n} (-x)^r \binom{n}{r} \frac{(x-t)^{n-r}}{2n-2r+1} \right]_0^x$$

$$= \frac{2}{(-1)^n} x^{\frac{1}{2}} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{x^n}{2n-2r+1}$$

$$= 2\sum_{r=0}^{n} \frac{(-1)^{r-n}}{2n-2r+1} \binom{n}{r} \cdot x^{n+\frac{1}{2}}$$

$$\therefore \qquad (x^n)^{\left\langle \frac{1}{2} \right\rangle} = \frac{2}{\sqrt{\pi}} \sum_{r=0}^{n} \frac{(-1)^{r-n}}{2n-2r+1} \binom{n}{r} \cdot x^{n+\frac{1}{2}}$$

Where, we devise further.

$$\sum_{r=0}^{n} \frac{(-1)^{r-n}}{2n-2r+1} \binom{n}{r} = \sum_{r=0}^{n} \frac{(-1)^{n-r}}{2(n-r)+1} \binom{n}{n-r} = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k}$$

Using this, we obtain

$$(x^n)^{\left\langle \frac{1}{2} \right\rangle} = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} \cdot x^{n+\frac{1}{2}}$$

By-product

According to Formula 7.3.2,

$$(x^n)^{\left\langle \frac{1}{2} \right\rangle} = \frac{2(2n)!!}{(2n+1)!!\sqrt{\pi}} x^{n+\frac{1}{2}}$$

Since this must be consistent with Formula 7.3.3, then

$$\frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} = \frac{2(2n)!!}{(2n+1)!!\sqrt{\pi}}$$

From this,

$$\sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} = \frac{(2n)!!}{(2n+1)!!} \tag{3.2}$$

If expanded, it is as follows.

$$\frac{1}{1}\binom{0}{0} = \frac{0!!}{1!!}$$

$$\frac{1}{1}\binom{1}{0} - \frac{1}{3}\binom{1}{1} = \frac{2!!}{3!!}$$

$$\frac{1}{1}\binom{2}{0} - \frac{1}{3}\binom{2}{1} + \frac{1}{5}\binom{2}{2} = \frac{4!!}{5!!}$$

$$\vdots$$

Regrettably this was already known.

Although it is a digression, the following equations hold.

$$\sum_{k=0}^{n} \frac{(-1)^k}{1k+1} \binom{n}{k} = \frac{(1n)!}{(1n+1)!} = \frac{1}{n+1} \tag{3.1}$$

$$\sum_{k=0}^{n}\frac{(-1)^k}{3k+1}\binom{n}{k} = \frac{(3n)!!!}{(3n+1)!!!}$$!!! means triple factorial. (3.3)

$$\sum_{k=0}^{n}\frac{(-1)^k}{mk+1}\binom{n}{k} = \frac{(mn)!_m}{(mn+1)!_m}$$ $!_m$ means multi factorial. (3.n)

Seemingly, it is not known m= 3 or more.

7.3.4 Fractional Integral of an integer power function

Generalizing Formula 7.3.3, we calculate a fractional integral of an integer power function.
First, we prepare the following lemma.

Lemma

When m,n are natural numbers, the following expression holds.

$$\int (x-t)^{-\frac{m-1}{m}} t^n dt = \frac{m(x-t)^{\frac{1}{m}}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+1}$$

Proof

$$F(t) = \frac{m(x-t)^{\frac{1}{m}}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+1}$$

Differentiate this with respect to t. Then

$$\frac{d}{dt}F(t) = \frac{d}{dt}\frac{m(x-t)^{\frac{1}{m}}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+1}$$

$$+\frac{m(x-t)^{\frac{1}{m}}}{(-1)^{n+1}}\frac{d}{dt}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+1}$$

$$= -\frac{(x-t)^{\frac{1}{m}-1}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+1}$$

$$-\frac{m(x-t)^{\frac{1}{m}}}{(-1)^{n+1}}\sum_{r=0}^{n}(n-r)(-x)^r\binom{n}{r}\frac{(x-t)^{n-r-1}}{m(n-r)+1}$$

$$= -\frac{(x-t)^{\frac{1}{m}-1}}{(-1)^{n+1}}\sum_{r=0}^{n}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+1}$$

$$-\frac{(x-t)^{\frac{1}{m}-1}}{(-1)^{n+1}}\sum_{r=0}^{n}m(n-r)(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+1}$$

$$= \frac{(x-t)^{\frac{1}{m}-1}}{(-1)^{n}}\sum_{r=0}^{n}\{1+m(n-r)\}(-x)^r\binom{n}{r}\frac{(x-t)^{n-r}}{m(n-r)+1}$$

$$= \frac{(x-t)^{\frac{1}{m}-1}}{(-1)^n} \sum_{r=0}^{n} \binom{n}{r} (-x)^r (x-t)^{n-r} = \frac{(x-t)^{\frac{1}{m}-1}}{(-1)^n} (-t)^n$$

$$= (x-t)^{-\frac{m-1}{m}} t^n$$

Q.E.D

Using this lemma, we obtain the following formula.

Formula 7.3.4

When m,n are natural numbers, the following expressions hold

$$\left(x^n\right)^{\left\langle \frac{1}{m} \right\rangle} = \frac{m}{\Gamma(1/m)} \sum_{k=0}^{n} \frac{(-1)^k}{mk+1} \binom{n}{k} \cdot x^{n+\frac{1}{m}} \tag{4.1}$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{mk+1} \binom{n}{k} = \frac{B(1+n\,,\,1/m)}{m} \tag{4.2}$$

$B(\ )$ denots beta function.

Proof

$$\left(x^\alpha\right)^{<p>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p)} x^{\alpha+p} = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} t^\alpha dt \qquad (\alpha \geqq 0)$$

Giving $\alpha=n\ ,\ p=1/m$,

$$\left(x^n\right)^{\left\langle \frac{1}{m} \right\rangle} = \frac{1}{\Gamma(1/m)} \int_0^x (x-t)^{-\frac{m-1}{m}} t^n dt$$

Using above Lemma, we calculate as follows.

$$\int_0^x (x-t)^{-\frac{m-1}{m}} t^n dt = \left[\frac{m(x-t)^{\frac{1}{m}}}{(-1)^{n+1}} \sum_{r=0}^{n} (-x)^r \binom{n}{r} \frac{(x-t)^{n-r}}{m(n-r)+1} \right]_0^x$$

$$= \frac{m}{(-1)^n} x^{\frac{1}{m}} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{x^n}{m(n-r)+1}$$

$$= m \sum_{r=0}^{n} \frac{(-1)^{r-n}}{m(n-r)+1} \binom{n}{r} \cdot x^{n+\frac{1}{m}}$$

Furthermore, since r,n are integer,

$$\sum_{r=0}^{n} \frac{(-1)^{r-n}}{m(n-r)+1} \binom{n}{r} = \sum_{r=0}^{n} \frac{(-1)^{n-r}}{m(n-r)+1} \binom{n}{n-r} = \sum_{k=0}^{n} \frac{(-1)^k}{mk+1} \binom{n}{k}$$

Using this,

$$\int_0^x (x-t)^{-\frac{m-1}{m}} t^n dt = m \sum_{k=0}^{n} \frac{(-1)^k}{mk+1} \binom{n}{k} \cdot x^{n+\frac{1}{m}}$$

Thus, we obtain

$$\left(x^n\right)^{\left\langle \frac{1}{m} \right\rangle} = \frac{m}{\Gamma(1/m)} \sum_{k=0}^{n} \frac{(-1)^k}{mk+1} \binom{n}{k} \cdot x^{n+\frac{1}{m}} \tag{4.1}$$

Next,

$$\left(x^n\right)^{\left\langle \frac{1}{m}\right\rangle} = \frac{\Gamma(1+n)}{\Gamma(1+n+1/m)}\,x^{n+\frac{1}{m}} = \frac{\Gamma(1+n)}{\Gamma(1/m)}\,\frac{\Gamma(1/m)}{\Gamma(1+n+1/m)}\,x^{n+\frac{1}{m}}$$

Since this must be equal to (4.1), we obtain

$$\sum_{k=0}^{n}\frac{(-1)^k}{mk+1}\binom{n}{k} = \frac{\Gamma(1+n)\Gamma(1/m)}{m\Gamma(1+n+1/m)} = \frac{B(1+n,\,1/m)}{m} \tag{4.2}$$

Remark

(4.2) shows that (4.1) can be expressed with a beta function and n can be the real number. Actuality,

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p)} = \frac{1}{\Gamma(p)}\,\frac{\Gamma(1+\alpha)\Gamma(p)}{\Gamma(1+\alpha+p)} = \frac{B(1+\alpha\,,\,p)}{\Gamma(p)}$$

Then,

$$\left(x^\alpha\right)^{<p>} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p)}\,x^{\alpha+p} = \frac{B(1+\alpha\,,\,p)}{\Gamma(p)}\,x^{\alpha+p} \qquad (\alpha\geqq 0) \tag{4.3}$$

Therefore, also,

$$\int_0^x (x-t)^{p-1}t^\alpha dt = B(1+\alpha\,,\,p)\,x^{\alpha+p} \qquad (\alpha\geqq 0) \tag{4.4}$$

Example 1

$$\left(x^2\right)^{\left\langle \frac{1}{3}\right\rangle} = \frac{\Gamma(1+2)}{\Gamma(1+2+1/3)}\,x^{2+\frac{1}{3}} = \frac{2}{\Gamma(10/3)}\,x^{\frac{7}{3}} \qquad = 0.7199\,x^{\frac{7}{3}}$$

$$\left(x^2\right)^{\left\langle \frac{1}{3}\right\rangle} = \frac{3}{\Gamma(1/3)}\sum_{k=0}^{2}\frac{(-1)^k}{3k+1}\binom{2}{k}\cdot x^{\frac{7}{3}}$$

$$= \frac{3}{\Gamma(1/3)}\left\{\frac{1}{1}\binom{2}{0}-\frac{1}{4}\binom{2}{1}+\frac{1}{7}\binom{2}{2}\right\}\cdot x^{\frac{7}{3}} \qquad = 0.7199\,x^{\frac{7}{3}}$$

$$\left(x^3\right)^{\left\langle \frac{1}{4}\right\rangle} = \frac{4}{\Gamma(1/4)}\sum_{k=0}^{3}\frac{(-1)^k}{4k+1}\binom{3}{k}\cdot x^{\frac{13}{4}}$$

$$= \frac{4}{\Gamma(1/4)}\left\{\frac{1}{1}\binom{3}{0}-\frac{1}{5}\binom{3}{1}+\frac{1}{9}\binom{3}{2}-\frac{1}{13}\binom{3}{3}\right\}\cdot x^{\frac{13}{4}} = 0.7241\,x^{\frac{13}{4}}$$

Example 2

$$\frac{1}{1}\binom{2}{0}-\frac{1}{4}\binom{2}{1}+\frac{1}{7}\binom{2}{2} = \frac{B(1+2,\,1/3)}{3} \qquad = \frac{9}{14}$$

$$\frac{1}{1}\binom{9}{0}-\frac{1}{4}\binom{9}{1}+\frac{1}{7}\binom{9}{2}-+\cdots-\frac{1}{28}\binom{9}{9} = \frac{B(1+9,\,1/3)}{3} = \frac{1{,}594{,}323}{3{,}803{,}800}$$

$$\binom{3}{0}-\frac{1}{5}\binom{3}{1}+\frac{1}{9}\binom{3}{2}-\frac{1}{13}\binom{3}{3} = \frac{B(1+3,\,1/4)}{4} \qquad = \frac{128}{195}$$

7.3.5 Super Integral of an integer power function

Replacing $1/m$ with p , we obtain the following formula.

Formula 7.3.5

When n is a natural number, the following expressions hold for $p>0$.

$$\left(x^n\right)^{<p>} = \frac{1}{\Gamma(p)}\sum_{k=0}^{n}\frac{(-1)^k}{p+k}\binom{n}{k}\cdot x^{n+p} \tag{5.1}$$

$$B(n,p) = \sum_{k=0}^{n-1}\frac{(-1)^k}{p+k}\binom{n-1}{k} \qquad B(\ )\text{denots beta function.} \tag{5.2}$$

Example 1

$$\left(x^2\right)^{<e>} = \frac{\Gamma(1+2)}{\Gamma(1+2+e)}x^{2+e} = \frac{2}{\Gamma(3+e)}x^{2+e} \qquad = 0.026755\,x^{2+e}$$

$$\left(x^2\right)^{<e>} = \frac{1}{\Gamma(e)}\sum_{k=0}^{2}\frac{(-1)^k}{e+k}\binom{2}{k}\cdot x^{2+e}$$

$$= \frac{1}{\Gamma(e)}\left\{\frac{1}{e}\binom{2}{0} - \frac{1}{e+1}\binom{2}{1} + \frac{1}{e+2}\binom{2}{2}\right\}x^{2+e} = 0.026755\,x^{2+e}$$

Example 2

$$B(2,e) = \frac{1}{e}\binom{1}{0} - \frac{1}{e+1}\binom{1}{1} = \frac{1}{e} - \frac{1}{e+1} = 0.098938\cdots$$

$$B(3,\pi) = \frac{1}{\pi}\binom{2}{0} - \frac{1}{\pi+1}\binom{2}{1} + \frac{1}{\pi+2}\binom{2}{2} = 0.029896\cdots$$

7.3.6 Super Integral of an positive power function

Since Formula 7.3.5 are binomial forms, the further generalization is possible.

Formula 7.3.6

The following expressions hold for positive numbers p, q.

$$\left(x^q\right)^{<p>} = \frac{x^{q+p}}{\Gamma(p)}\sum_{r=0}^{\infty}\frac{(-1)^r}{p+r}\binom{q}{r} \tag{6.1}$$

$$B(q,p) = \sum_{r=0}^{\infty}\frac{(-1)^r}{p+r}\binom{q-1}{r} \qquad B(\ )\text{is Beta Function} \tag{6.2}$$

7.3.7 Super Integral of a polynomial

In the case of a polynomial $f(x) = \sum_{k=0}^{m}c_{m-k}x^{m-k}$, as for the zero of the super primitive function, it is good to perform it as follows. It is based on experience of a writer.

(1) When $f(x)$ is factored by primary formula $ax+b$, let $-b/a$ be the zero.
(2) When $f(x)$ is not factored by primary formula $ax+b$, let 0 be the zero.

For example, in the case of the 1/2 times integral of $f(x) = x^2-2\pi x+\pi^2$, the following calculation is right in many case.

$$\left(x^2-2\pi x+\pi^2\right)^{\left\langle \frac{1}{2}\right\rangle} = \left\{(x-\pi)^2\right\}^{\left\langle \frac{1}{2}\right\rangle} = \frac{16}{15\sqrt{\pi}}(x-\pi)^{\frac{5}{2}}$$

If this is calculated termwise as follows, the result is different from the former.

$$\left(x^2-2\pi x+\pi^2\right)^{\left\langle\frac{1}{2}\right\rangle} = \left(x^2\right)^{\left\langle\frac{1}{2}\right\rangle} - 2\pi\left(x^1\right)^{\left\langle\frac{1}{2}\right\rangle} + \pi^2\left(x^0\right)^{\left\langle\frac{1}{2}\right\rangle}$$

$$= \frac{2}{\sqrt{\pi}}\, x^{\frac{1}{2}}\left(\frac{8}{15}x^2 - \frac{4\pi}{3}x + \pi^2\right)$$

Needless to say, this cause is the difference between

$$\int_{\pi}^{x}\sim\int_{\pi}^{x}\left(x^2-2\pi x+\pi^2\right)dx^{\frac{1}{2}} \text{ and } \int_{0}^{x}\sim\int_{0}^{x}\left(x^2-2\pi x+\pi^2\right)dx^{\frac{1}{2}}.$$

That is, it is because the latter regarded it as 0 although the former regarded the zero of the super primitive function as pi.

The latter is right when there is a special reason why the zero of the super primitive function should be 0. However, such a case is rare, and in almost all cases the former is right.

7.4 Super Integral of Exponential Function

7.4.1 Super Integral of Exponential Function

Analytically continuing the index of the integration operator in Formula 4.3.2 to $[0,p]$ from $[1,n]$, we obtain the following formula. In addition, Rieamnn-Liouville integrals are also expressed together

Formula 7.4.1

(1) Basic form

$$\int_{\mp\infty}^{x}\sim\int_{\mp\infty}^{x} e^{\pm x}dx^p \quad = (\pm 1)^p e^{\pm x} \quad = \frac{1}{\Gamma(p)}\int_{\mp\infty}^{x}(x-t)^{p-1}e^{\pm t}dt$$

(2) Linear form

$$\int_{\mp\infty}^{x}\sim\int_{\mp\infty}^{x} e^{ax+b}dx^p = \left(\frac{1}{a}\right)^p e^{ax+b} = \frac{1}{\Gamma(p)}\int_{\mp\infty}^{x}(x-t)^{p-1}e^{at+b}dt$$

$(a>0:-\ ,\ a<0:+)$

(3) General form

$$\int_{\mp\infty}^{x}\sim\int_{\mp\infty}^{x} \alpha^{ax+b}dx^p = \left(\frac{1}{a\log\alpha}\right)^p \alpha^{ax+b} = \frac{1}{\Gamma(p)}\int_{\mp\infty}^{x}(x-t)^{p-1}\alpha^{at+b}dt$$

$(a>0:-\ ,\ a<0:+)$

Proof of the general form

Let $c=a\log\alpha\ ,\ d=b\log\alpha$, then

$$e^{cx+d} = e^{xa\log\alpha+b\log\alpha} = e^{ax\log\alpha}e^{b\log\alpha} = \left(e^{\log\alpha}\right)^{ax}\left(e^{\log\alpha}\right)^{b} = \alpha^{ax}\alpha^{b} = \alpha^{ax+b}$$

Applying this to (2) Linear form, we obtain (3) General form immediately.

Example

$$\left(e^{-x}\right)^{\left\langle\frac{1}{2}\right\rangle} = (-1)^{\frac{1}{2}}e^{-x} = i\,e^{-x}$$

$$\left(e^{3x-4}\right)^{\langle\sqrt{2}\rangle} = \left(\frac{1}{3}\right)^{\sqrt{2}}e^{3x-4} = 0.211469\,e^{3x-4}$$

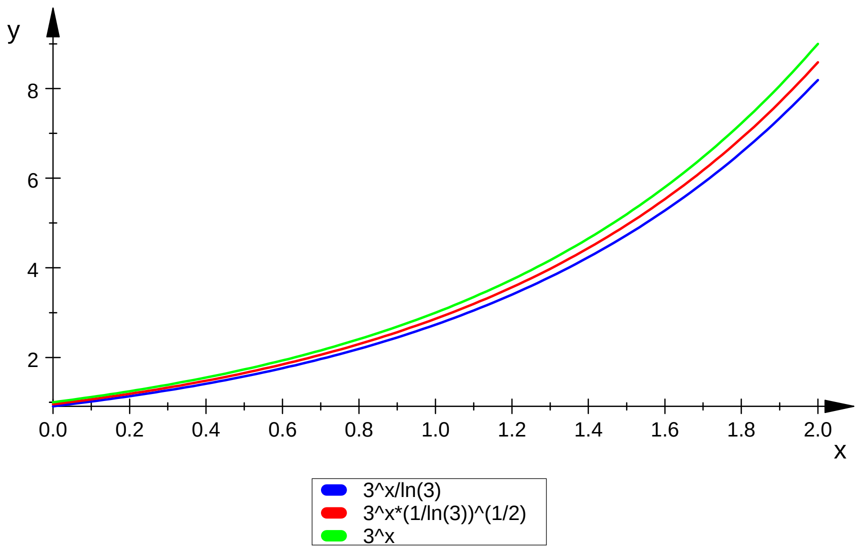
$$\left(3^x\right)^{\left\langle\frac{1}{2}\right\rangle} = \left(\frac{1}{\log 3}\right)^{1/2}3^x = 0.954064\times 3^x$$

$$\left\{(-3)^x\right\}^{\left\langle\frac{1}{2}\right\rangle} = \left(\frac{1}{\log(-3)}\right)^{\frac{1}{2}}(-3)^x$$

$$= (0.447018 - 0.317241\,i)\times(-3)^x$$

$$\left(2^x\right)^{<i>} = \left(\frac{1}{\log 2}\right)^{i}2^x = (0.933582 + 0.358362\,i)\times 2^x$$

When 3^x , $\left(3^x\right)^{<1/2>}$, $\left(3^x\right)^{<1>}$ are drawn on a figure side by side, it is as follows.



7.4.2 Super Integral of Exponential Function (hyperbolic function form)

Also as follows, the super integral of an exponential function is expressed using a hyperbolic function.

Formula 7.4.2

(1) Basic form

$$\int_{\mp\infty}^{x} \sim \int_{\mp\infty}^{x} e^{\pm x} dx^p = i^p \left\{ \cosh\left(x - \frac{p\pi i}{2} \right) \pm \sinh\left(x - \frac{p\pi i}{2} \right) \right\}$$

$$= (\cosh x)^{<p>} \pm (\sinh x)^{<p>}$$

(2) Linear form

$$\int_{\mp\infty}^{x} \sim \int_{\mp\infty}^{x} e^{ax+b} dx^p = \left(\frac{i}{a} \right)^p \cosh\left\{ \left(ax+b - \frac{p\pi i}{2} \right)^p \pm \sinh\left(ax+b - \frac{p\pi i}{2} \right) \right\}$$

$$= \{\cosh(ax+b)\}^{<p>} \pm \{\cosh(ax+b)\}^{<p>}$$

Proof

From the Formula 7.7.1 mentioned later, the following expressions hold.

$$(\cosh x)^{<p>} = i^p \cosh\left(x - \frac{p\pi i}{2} \right) = \frac{1}{2}\{e^x + (-1)^p e^{-x}\}$$

$$(\sinh x)^{<p>} = i^p \sinh\left(x - \frac{p\pi i}{2} \right) = \frac{1}{2}\{e^x - (-1)^p e^{-x}\}$$

Adding and subtracting these two formulas, we obtain the following expressions.

$$i^p \left\{ \cosh\left(x - \frac{p\pi i}{2} \right) + \sinh\left(x - \frac{p\pi i}{2} \right) \right\} = e^x$$

$$i^p \left\{ \cosh\left(x - \frac{p\pi i}{2} \right) - \sinh\left(x - \frac{p\pi i}{2} \right) \right\} = (-1)^p e^{-x}$$

Substituting these for the Formula 7.4.1, we obtain (1) basic form.
In a similar way, (2) linea form is obtained

Note

Therefore, in the concept, the following expression holds.

$$\int_{\mp\infty}^{x}\sim\int_{\mp\infty}^{x}e^{\pm x}dx^{p} = \int_{\frac{(p-1)\pi i}{2a}}^{x}\sim\int_{\frac{-1\pi i}{2a}}^{x}\cosh x\,dx^{p} \pm \int_{\frac{p\pi i}{2}}^{x}\sim\int_{\frac{0\pi i}{2}}^{x}\sinh x\,dx^{p}$$

Actually, when p is a natural number n, both sides recover the function of operation immediately and the result in the higher integral as follows..

$$\int_{\mp\infty}^{x}\cdots\int_{\mp\infty}^{x}e^{\pm x}dx^{n} = \int_{\frac{(n-1)\pi i}{2a}}^{x}\cdots\int_{\frac{0\pi i}{2a}}^{x}\cosh x\,dx^{n} \pm \int_{\frac{n\pi i}{2}}^{x}\cdots\int_{\frac{1\pi i}{2}}^{x}\sinh x\,dx^{n}$$

7.5 Super Integral of Logarithmic Function

Analytically continuing the index of the integration operator in Formula 4.3.3 to $[0,p]$ from $[1,n]$, we obtain the following formula. In addition, Rieamnn-Liouville integrals are also expressed together

Formula 7.5.1

(1) Basic form

$$\int_0^x \sim \int_0^x \log x\, dx^p = \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} x^p = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} \log t\, dt$$

(2) Linear form

$$\int_{-\frac{b}{a}}^x \sim \int_{-\frac{b}{a}}^x \log(ax+b)\, dx^p = \frac{\log(ax+b) - \psi(1+p) - \gamma}{\Gamma(1+p)} \left(x+\frac{b}{a}\right)^p$$

$$= \frac{1}{\Gamma(p)} \int_{-\frac{b}{a}}^x (x-t)^{p-1} \log(at+b)\, dt$$

Proof

Formula 4.3.3 in "**04 Higher Integral**" was as follows.

$$\int_0^x \cdots \int_0^x \log x\, dx^n = \frac{x^n}{n!}\left(\log x - \sum_{k=1}^n \frac{1}{k}\right)$$

$$\int_{-\frac{b}{a}}^x \cdots \int_{-\frac{b}{a}}^x \log(ax+b)\, dx^n = \frac{1}{n!}\left(x+\frac{b}{a}\right)^n \left\{\log(ax+b) - \sum_{k=1}^n \frac{1}{k}\right\}$$

On the other hand, in "**01 Gamma Function & Digamma Function**" there were the following formulas.

$$n! = \Gamma(1+n), \qquad \sum_{k=1}^n \frac{1}{k} = \psi(1+n) + \gamma$$

Then, subsutituting these for the upper formulas and analytically continuing the index of the integration operator to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Example

$$(\log x)^{<1>} = \frac{\log x - \psi(1+1) - \gamma}{\Gamma(1+1)} x^1 = x(\log x - 1)$$

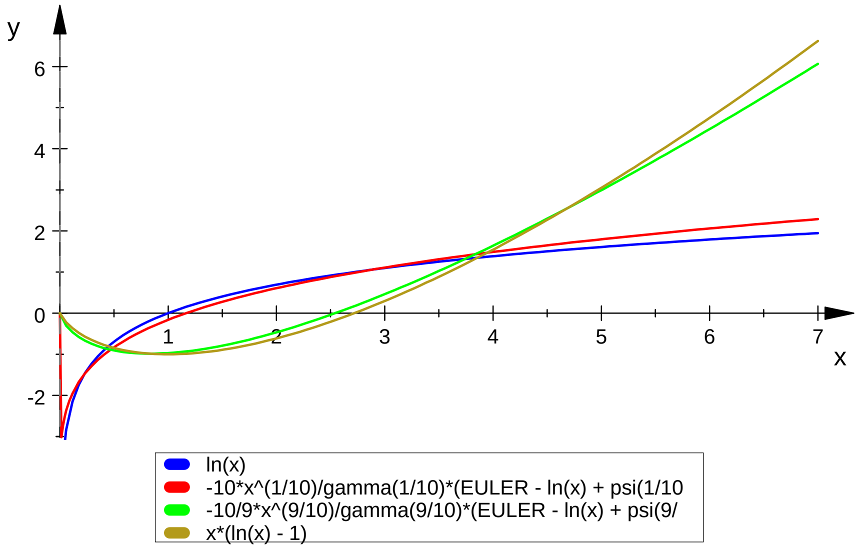
$$\{\log(3x+4)\}^{\left\langle \frac{1}{2} \right\rangle} = \frac{\log(3x+4) - \psi\left(1+\frac{1}{2}\right) - \gamma}{\Gamma\left(1+\frac{1}{2}\right)} \left(x+\frac{4}{3}\right)^{\frac{1}{2}}$$

$$= 1.1283(\log x - 0.6137)\sqrt{x+1.3333}$$

$$(\log x)^{\left\langle \frac{1}{10} \right\rangle} = \frac{\log x - \psi\left(1+\frac{1}{10}\right) - \gamma}{\Gamma\left(1+\frac{1}{10}\right)} x^{\frac{1}{10}} = 1.0511 \sqrt[10]{x}\,(\log x - 0.1534)$$

$$(\log x)^{\langle \frac{9}{10} \rangle} = \frac{\log x - \psi\left(1+\frac{9}{10}\right) - \gamma}{\Gamma\left(1+\frac{9}{10}\right)} x^{\frac{9}{10}} = 1.0397\, x^{\frac{9}{10}} (\log x - 0.9333)$$

When $\log x$, $(\log x)^{<1/10>}$, $(\log x)^{<9/10>}$, $(\log x)^{<1>}$ are drawn on a figure side by side, it is as follows. We can see slightly that the zero of $(\log x)^{<1/10>}$ (red) is also $x=0$.



7.6 Super Integral of Trigonometric Function

7.6.1 Super Integrals of sin x , cos x

Analytically continuing the index of the integration operator in Formula 4.3.4 to $[0,p]$ from $[1,n]$, we obtain the following formula.

Formula 7.6.1

(1) Basic form

$$\int_{\frac{p\pi}{2}}^{x} \sim \int_{\frac{0\pi}{2}}^{x} \sin x\, dx^{p} = \sin\left(x-\frac{p\pi}{2}\right)$$

$$\int_{\frac{(p-1)\pi}{2}}^{x} \sim \int_{\frac{-1\pi}{2}}^{x} \cos x\, dx^{p} = \cos\left(x-\frac{p\pi}{2}\right)$$

(2) Linear form

$$\int_{\frac{p\pi}{2a}-\frac{b}{a}}^{x} \sim \int_{\frac{0\pi}{2a}-\frac{b}{a}}^{x} \sin(ax+b)\, dx^{p} = \left(\frac{1}{a}\right)^{p} \sin\left(ax+b-\frac{p\pi}{2}\right)$$

$$\int_{\frac{(p-1)\pi}{2a}-\frac{b}{a}}^{x} \sim \int_{-\frac{1\pi}{2a}-\frac{b}{a}}^{x} \cos(ax+b)\, dx^{p} = \left(\frac{1}{a}\right)^{p} \cos\left(ax+b-\frac{p\pi}{2}\right)$$

Example

$$(\sin x)^{\langle\frac{1}{2}\rangle} = \sin\left(x-\frac{1}{2}\frac{\pi}{2}\right) = \sin\left(x-\frac{\pi}{4}\right)$$

$$\left((\sin x)^{\langle\frac{1}{2}\rangle}\right)^{\langle\frac{1}{2}\rangle} = \left(\sin\left(x-\frac{\pi}{4}\right)\right)^{\langle\frac{1}{2}\rangle}$$

$$= \left(\frac{1}{1}\right)^{\frac{1}{2}} \sin\left(x-\frac{\pi}{4}-\frac{1}{2}\frac{\pi}{2}\right) = \sin\left(x-\frac{\pi}{2}\right) = -\cos x$$

$$\left\{\sin\left(x+\frac{\pi}{2}\right)\right\}^{<1>} = \left(\frac{1}{1}\right)^{1} \sin\left(x+\frac{\pi}{2}-\frac{1\cdot\pi}{2}\right) = \sin x$$

$$(\cos x)^{\langle\frac{1}{2}\rangle} = \cos\left(x-\frac{1}{2}\frac{\pi}{2}\right) = \cos\left(x-\frac{\pi}{4}\right)$$

$$(\cos x)^{\langle\frac{2}{\pi}\rangle} = \cos\left(x-\frac{2}{\pi}\frac{\pi}{2}\right) = \cos(x-1)$$

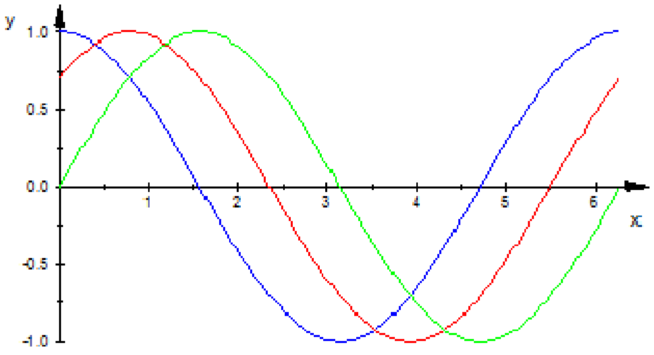
When $\cos x$, $(\cos x)^{1/2}$, $\sin x$ are drawn on a figure side by side, it is as follows. Red shows 1/2 order super integral. It is clear also in the figure that super integral which is the easiest to understand is super integral of trigonometric functions.

Lineal Super Integral of cos x

```
• cos1 := p-> cos(x-p*PI/2)
```

$$p \to \cos\left(x-\frac{\pi \cdot p}{2}\right)$$

```
plotfunc2d(cos(x),cosl(1/2),sin(x), x=0..2*PI)
```



7.6.2 Termwise Super Integrals of sin x , cos x

If common lower limit 0 is employed as the lower limits of the super integral, we obtain the following termwise super integral. These are **collateral super integrals** as understood from the constant-of-integration function in the right side

$$\int_0^x \sim \int_0^x \sin x\, dx^p = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k+2+p)} x^{2k+1+p} = \sin\left(x - \frac{p\pi}{2}\right) + C(p,x)$$

$$C(p,x) = \sum_{k=1}^{p} \frac{x^{p-k}}{\Gamma(1+p-k)} \sin\frac{k\pi}{2}$$

$$\int_0^x \sim \int_0^x \cos x\, dx^p = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k+1+p)} x^{2k+p} = \cos\left(x - \frac{p\pi}{2}\right) + C(p,x)$$

$$C(p,x) = \sum_{k=1}^{p} \frac{x^{p-k}}{\Gamma(1+p-k)} \cos\frac{k\pi}{2}$$

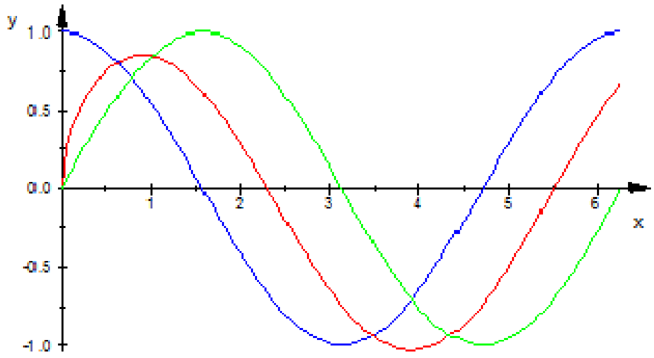
When the 1/2th order collateral super integral of cos x is drawn as cos x and sin x side by side, it is as follows. Red shows the 1/2th order collateral super integral.

Collateral Super Integral of cos x

```
cosc := p-> sum((-1)^k/gamma(2*k+1+p)*x^(2*k+p), k=0..50)
```

$$p \to \sum_{k=0}^{50} \frac{(-1)^k}{\Gamma(2\cdot k+1+p)} \cdot x^{2\cdot k+p}$$

```
plotfunc2d(cos(x),cosc(1/2),sin(x), x=0..2*PI)
```



7.7 Super Integral of Hyperbolic Function

7.7.1 Super Integrals of sinh x , cosh x

Analytically continuing the index of the integration operator in Formula 4.3.5 to $[0,p]$ from $[1,n]$, we obtain the following formula.

Formula 7.7.1

(1) Basic form

$$\int_{\frac{p\pi i}{2}}^{x} \sim \int_{\frac{0\pi i}{2}}^{x} \sinh x\, dx^p = i^p \sinh\left(x - \frac{p\pi i}{2}\right) = \frac{e^x - (-1)^p e^{-x}}{2}$$

$$\int_{\frac{(p-1)\pi i}{2}}^{x} \sim \int_{\frac{-1\pi i}{2}}^{x} \cosh x\, dx^p = i^p \cosh\left(x - \frac{p\pi i}{2}\right) = \frac{e^x + (-1)^p e^{-x}}{2}$$

(2) Linear form

$$\int_{\frac{p\pi i}{2a}-\frac{b}{a}}^{x} \sim \int_{\frac{0\pi i}{2a}-\frac{b}{a}}^{x} \sinh(ax+b)\, dx^p = \left(\frac{i}{a}\right)^p \sinh\left(ax+b - \frac{p\pi i}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a}\right)^p \left\{e^{ax+b} - (-1)^p e^{-(ax+b)}\right\}$$

$$\int_{\frac{(p-1)\pi i}{2a}-\frac{b}{a}}^{x} \sim \int_{\frac{-1\pi i}{2a}-\frac{b}{a}}^{x} \cosh(ax+b)\, dx^p = \left(\frac{i}{a}\right)^p \cosh\left(ax+b - \frac{p\pi i}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a}\right)^p \left\{e^{ax+b} + (-1)^p e^{-(ax+b)}\right\}$$

Example 1

$$(\sinh x)^{\left\langle \frac{1}{2} \right\rangle} = i^{\frac{1}{2}} \sinh\left(x - \frac{1}{2}\frac{\pi i}{2}\right) = i^{\frac{1}{2}} \sinh\left(x - \frac{\pi i}{4}\right)$$

$$\left((\sinh x)^{\left\langle \frac{1}{2} \right\rangle}\right)^{\left\langle \frac{1}{2} \right\rangle} = \left(i^{\frac{1}{2}} \sinh\left(x - \frac{\pi i}{4}\right)\right)^{\left\langle \frac{1}{2} \right\rangle}$$

$$= i^{\frac{1}{2}} \left(\frac{i}{1}\right)^{\frac{1}{2}} \sinh\left(x - \frac{\pi i}{4} - \frac{1}{2}\frac{\pi i}{2}\right)$$

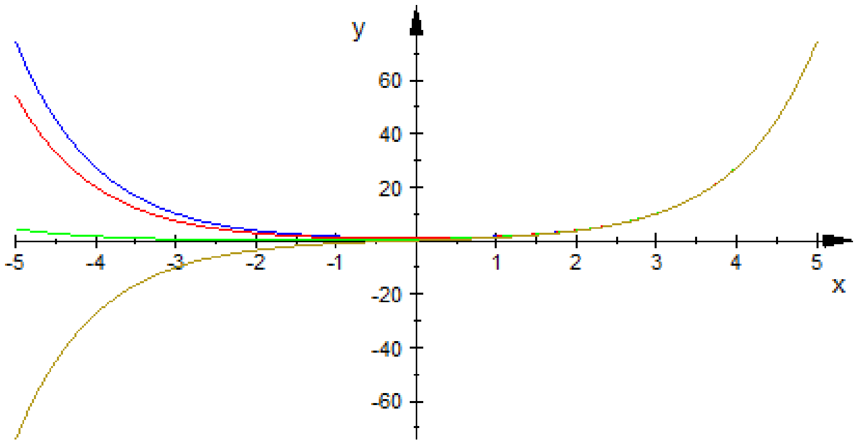
$$= i \sinh\left(x - \frac{\pi i}{2}\right) = \cosh x$$

$$\left\{\cosh\left(x + \frac{\pi i}{2}\right)\right\}^{<1>} = \left(\frac{i}{1}\right)^1 \cosh\left(x + \frac{\pi i}{2} - \frac{1 \cdot \pi i}{2}\right) = i \cosh x$$

$$(\cosh x)^{\left\langle \frac{i}{10} \right\rangle} = i^{\frac{i}{10}} \cosh\left(x + \frac{\pi}{20}\right) = 0.854635 \times \cosh\left(x + \frac{\pi}{20}\right)$$

$$(\cosh x)^{\langle \frac{9i}{10} \rangle} = i^{\frac{9i}{10}} \cosh\left(x+\frac{9\pi}{20}\right) = 0.243237 \times \cosh\left(x+\frac{9\pi}{20}\right)$$

Super integral of hyperbolic function is the most incomprehensible in super integrals. The reason is that the super primitive function turns into a complex function except order p is an integer or a purely imaginary number. Then, when $\cosh x$, $(\cosh x)^{\langle \frac{i}{10} \rangle}$, $(\cosh x)^{\langle \frac{9i}{10} \rangle}$, $\sinh x$ which can be displayed on a real number domain are drawn on a figure side by side, it is as follows.



All of four curves have overlapped in the positive area. It is natural that $(\cosh x)^{\langle \frac{i}{10} \rangle}$ is near $\cosh x$ in a negative area. But $(\cosh x)^{\langle \frac{9i}{10} \rangle}$ is far apart from $\sinh x$ in why.

Example 2

$$(\sinh x)^{\langle \frac{1}{2} \rangle} = \frac{e^x - (-1)^{\frac{1}{2}} e^{-x}}{2} = \frac{e^x - i e^{-x}}{2}$$

$$\left((\sinh x)^{\langle \frac{1}{2} \rangle}\right)^{\langle \frac{1}{2} \rangle} = \left(\frac{e^x - i e^{-x}}{2}\right)^{\langle \frac{1}{2} \rangle} = \frac{e^x}{2} - \frac{i}{2} \times (e^{-x})^{\langle \frac{1}{2} \rangle}$$

$$= \frac{e^x}{2} - \frac{i}{2} \times i e^{-x} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\left\{\cosh\left(x+\frac{\pi i}{2}\right)\right\}^{<1>} = \frac{1^{-1}}{2}\left\{e^{x+\frac{\pi i}{2}} + (-1)^1 e^{-\left(x+\frac{\pi i}{2}\right)}\right\}$$

$$= \frac{1}{2}\left(e^{\frac{\pi i}{2}} e^x - e^{-\frac{\pi i}{2}} e^{-x}\right) = \frac{1}{2}\left(i e^x - \frac{1}{i} e^{-x}\right)$$

$$= \frac{i}{2}(e^x + e^{-x}) = i \cosh x$$

$$(\sinh x)^{\langle \frac{i}{\pi} \rangle} = \frac{e^x - (-1)^{\frac{i}{\pi}} e^{-x}}{2} = \frac{e^x - (e^{\pi i})^{\frac{i}{\pi}} e^{-x}}{2} = \frac{e^x - e^{-1-x}}{2}$$

Note

Super integral of a hyperbolic function is the integral with variable lower limits, and super integral of an exponential function is the integral with a fixed lower limit. So, for example, the p-th order super integral of $\sinh x$ is as follows in the concept.

$$(\sinh x)^{<p>} = \int_{\frac{p\pi i}{2}}^{x} \sim \int_{\frac{0\pi i}{2}}^{x} \sinh x\,dx^p = \int_{\frac{p\pi i}{2}}^{x} \sim \int_{\frac{0\pi i}{2}}^{x} \frac{e^x - e^{-x}}{2} dx^p$$

But the example forgets such a thing and perform super integral of e^x, e^{-x} with lower limits $-\infty$, ∞ . The result is all right. Truly, this is convenience and interesting.

7.7.2 Termwise Super Integrals of sinh x , cosh x

If common lower limit 0 is employed as the lower limits of the super integral, we obtain the following termwise super integral. These are **collateral super integrals** as understood from the constant-of-integration function in the right side

$$\int_0^x \sim \int_0^x \sinh x\,dx^p = \sum_{k=0}^{\infty} \frac{x^{2k+1+p}}{\Gamma(2k+2+p)} = i^p \sinh\left(x - \frac{p\pi i}{2}\right) + C(p,x)$$

$$C(p,x) = \sum_{k=1}^{p} \frac{x^{p-k}}{\Gamma(1+p-k)} \sinh\frac{k\pi i}{2}$$

$$\int_0^x \sim \int_0^x \cosh x\,dx^p = \sum_{k=0}^{\infty} \frac{x^{2k+p}}{\Gamma(2k+1+p)} = i^p \operatorname{cohs}\left(x - \frac{p\pi i}{2}\right) + C(p,x)$$

$$C(p,x) = \sum_{k=1}^{p} \frac{x^{p-k}}{\Gamma(1+p-k)} \cosh\frac{k\pi i}{2}$$

When the values of the lineal and the collateral the 1.2th order integral of cosh x on x=1, x=8 are calculated respectively, it is as follows.

Lineal Super Integral of cosh x

```
• coshl := p-> I^p*cosh(x-p*PI/2*I)
```

$$p \to i^p \cdot \cosh\left(x - i \cdot \frac{\pi \cdot p}{2}\right)$$

Collateral Super Integral of cosh x

```
• coshc := p-> sum(x^(2*k+p)/gamma(2*k+1+p), k=0..50)
```

$$p \to \sum_{k=0}^{50} \frac{x^{2 \cdot k + p}}{\Gamma(2 \cdot k + 1 + p)}$$

x = 1

```
• float(subs(coshl(1.2), x=1)); float(subs(coshc(1.2), x=1))
```

$1.210330554 - 0.1081170551 \cdot i$

1.042561641

x = 8

```
• float(subs(coshl(1.2), x=8)); float(subs(coshc(1.2), x=8))
```

$1490.478858 - 0.00009858999269 \cdot i$

1490.436428

Though the difference of both is large where x is small, both are almost corresponding ing where x is large. Since this is similar also in sinh x, it is thought that the termwise super integrals of sinh x and cosh x are asymptotic expansions of the lineal super integrals.

7.8 Super Integral of Inv-Trigonometric Function etc.

7.8.1 Super Integrals of arctan x , arccot x

Analytically continuing the index of the integration operator in Formula 4.3.6 to $[0,p]$ from $[1,n]$, we obtain the following formula.

Formula 7.8.1

When $\Gamma(x), \psi(x)$ denote gamma function and digamma function respectively, the following expressions hold for $|x| \geqq 1$.

$$\int_0^x \sim \int_0^x \tan^{-1}x\, dx^p = \frac{\tan^{-1}x}{\Gamma(1+p)} \sum_{k=0}^{\infty} (-1)^k \binom{p}{p-2k} x^{p-2k}$$
$$+ \frac{\log(1+x^2)}{2\Gamma(1+p)} \sum_{k=1}^{\infty} (-1)^k \binom{p}{p+1-2k} x^{p+1-2k}$$
$$- \frac{1}{\Gamma(1+p)} \sum_{r=1}^{\infty} (-1)^r \binom{p}{p+1-2r} \{\psi(1+p) - \psi(2r)\} x^{p+1-2r}$$

$$\int_0^x \sim \int_0^x \cot^{-1}x\, dx^p = \frac{x^p}{\Gamma(1+p)} \cot^{-1}x - \frac{\tan^{-1}x}{\Gamma(1+p)} \sum_{k=1}^{\infty} (-1)^k \binom{p}{p-2k} x^{p-2k}$$
$$- \frac{\log(1+x^2)}{2\Gamma(1+p)} \sum_{k=1}^{\infty} (-1)^k \binom{p}{p+1-2k} x^{p+1-2k}$$
$$+ \frac{1}{\Gamma(1+p)} \sum_{r=1}^{\infty} (-1)^r \binom{p}{p+1-2r} \{\psi(1+p) - \psi(2r)\} x^{p+1-2r}$$

Example: The 5/2 th order integral of $\cot^{-1}x$

```
p = 5 / 2 ;  m = 100 ;
```

$$\text{f1}[x\_] := \frac{x^p \,\text{ArcCot}[x]}{\text{Gamma}[1+p]} - \frac{\text{ArcTan}[x]}{\text{Gamma}[1+p]} \sum_{k=1}^{m} (-1)^k \,\text{Binomial}[p, p-2k]\, x^{p-2k}$$
$$- \frac{\text{Log}[1+x^2]}{2\,\text{Gamma}[1+p]} \sum_{k=1}^{m} (-1)^k \,\text{Binomial}[p, p+1-2k]\, x^{p+1-2k}$$
$$+ \frac{1}{\text{Gamma}[1+p]} \sum_{r=1}^{m} (-1)^r \,\text{Binomial}[p, p+1-2r] \times (\text{PolyGamma}[1+p] - \text{PolyGamma}[2r])\, x^{p+1-2r}$$

$$\text{fr}[x\_] := \frac{1}{\text{Gamma}[p]} \int_0^x (x-t)^{p-1} \,\text{ArcCot}[t]\, dt$$

```
N[f1[1.1]]          N[fr[1.1]]
0.489079            0.489079  + 0. i
```

7.8.2 Super Integrals of arctanh x , arccoth x

Analytically continuing the index of the integration operator in Formula 4.3.7 to $[0,p]$ from $[1,n]$, we obtain the following formula.

Formula 7.8.2

When $\Gamma(x), \psi(x)$ denote gamma function and digamma function respectively, the following expression holds for $|x| \geqq 1$.

$$\int_0^x \sim \int_0^x \tanh^{-1}x\, dx^p = \frac{\tanh^{-1}x}{\Gamma(1+p)} \sum_{k=0}^{\infty} \binom{p}{p-2k} x^{p-2k}$$

$$+ \frac{\log(1-x^2)}{2\Gamma(1+p)} \sum_{k=1}^{\infty} \binom{p}{p+1-2k} x^{p+1-2k}$$

$$- \frac{1}{\Gamma(1+p)} \sum_{r=1}^{\infty} \binom{p}{p+1-2r} \{\psi(1+p) - \psi(2r)\} x^{p+1-2r}$$

Example: The 3/2 th order integral of $\tanh^{-1}x$

```
p = 3 / 2;  m = 100;

fl[x_] := ArcTanh[x] / Gamma[1 + p] Sum[Binomial[p, p - 2 k] x^(p-2 k), {k, 0, m}]
   + Log[1 - x^2] / (2 Gamma[1 + p]) Sum[Binomial[p, p + 1 - 2 k] x^(p+1-2 k), {k, 1, m}]
   - 1 / Gamma[1 + p] Sum[Binomial[p, p + 1 - 2 r]
          × (PolyGamma[1 + p] - PolyGamma[2 r]) x^(p+1-2 r), {r, 1, m}]

fr[x_] := 1 / Gamma[p] Integrate[(x - t)^(p-1) ArcTanh[t], {t, 0, x}]

N[fl[1.1]]                        N[fr[1.1]]
0.510394 - 0.0373666 i            0.510394 - 0.0373666 i
```

2010.07.07

K. Kono

Alien's Mathematics