17 Super Integral of the Product of Two Functions

17.1 Super Integral of f (x) g (x)

17.1.1 Super Intagration by parts

Formula 17.1.1

Let r,p are positive numbers, $f^{<r>}$ be an arbitrary r th order primitive function of $f(x)$ and $g^{(r)}$ be the r th order derivative function of $g(x)$. At this time, if there is a certain constant a and $f^{<r>}(a)=0 \;\; or \;\; g^{(r)}(a)=0 \;\; for \;\; r\in[0,p]$, the following expression holds.

$$\int_a^x \sim \int_a^x f^{<0>} g^{(0)} dx^p = f^{<p>} g^{(0)} - \sum_{r=1}^{\infty} \binom{p}{r} \int_a^x \sim \int_a^x f^{<r>} g^{(r)} dx^p \tag{1.1}$$

Proof

In Formula 16.1.1 (1.3) in **16.1.1** , let us analytically continue the index of the integration operator to $[0,p]$ from $[1,n]$. Since $\sum$ have to become an infinite series according to this, the upper limit n is expanded to ∞ .

Example Super Intagration by parts of $\int_0^x \sim \int_0^x \sqrt{x}\,\sqrt[3]{x}\; dx^{\frac{1}{6}}$

Let $f^{(0)} = x^{\alpha}$, $g^{(0)} = x^{\beta}$ in (1.1) . Then

Left: $\displaystyle \int_0^x \sim \int_0^x x^{\alpha+\beta} dx^p = \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha+\beta+p)} x^{\alpha+\beta+p}$

Higher Integral of right side is

$$\int_0^x \sim \int_0^x (x^{\alpha})^{<r>} (x^{\beta})^{(r)} dx^p = \int_0^x \sim \int_0^x \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+r)} x^{\alpha+r} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-r)} x^{\beta-r} dx^p$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+r)} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-r)} \int \sim \int x^{\alpha+\beta} dx^p$$

$$= \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha+\beta+p)} x^{\alpha+\beta+p} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+r)} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-r)}$$

Then, from (1.1) ,

Right: $\displaystyle \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p)} x^{\alpha+p} x^{\beta} - \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha+\beta+p)} x^{\alpha+\beta+p} \sum_{r=1}^{\infty} \binom{p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+r)} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-r)}$

Substitute $\alpha=1/2, \beta=1/3, p=1/6$ for both sides. Then

$$\int_0^x \sim \int_0^x x^{\frac{1}{2}} x^{\frac{1}{3}} dx^{\frac{1}{6}} = \frac{\Gamma(11/6)}{\Gamma(2)} x$$

$$= \frac{\Gamma(3/2)}{\Gamma(5/3)} x - \frac{\Gamma(11/6)}{\Gamma(2)} x \sum_{r=1}^{\infty} \binom{1/6}{r} \frac{\Gamma(3/2)}{\Gamma(3/2+r)} \frac{\Gamma(4/3)}{\Gamma(4/3-r)}$$

When rhese are calculated, it is as follows. Both sides are corresponding completely

```
a = 1 / 2; b = 1 / 3;

fl[p_] := Gamma[1 + a + b] / Gamma[1 + a + b + p] x^(a+b+p)
```

```
fr[p_] := Gamma[1 + a]/Gamma[1 + a + p] x^(a+p) x^b - Gamma[1 + a + b]/Gamma[1 + a + b + p] x^(a+b+p)
    × Sum[Binomial[p, r] Gamma[1 + a]/Gamma[1 + a + r] Gamma[1 + b]/Gamma[1 + b - r], {r, 1, ∞}]
N[fl[1/6]]          N[fr[1/6]]
0.940656 x          0.940656 x
```

Note1

The above example is the one for the numerical proof of this formula. Actually, this formula is rarely used directly.

Note2

The more general formulas of Theorem 16.1.1 in **16.1.1** are as follows.

$$\int_{a(p)}^{x}\sim\int_{a(0)}^{x} f^{<0>}g^{(0)}dx^p = f^{<p>}g^{(0)} - \sum_{r=0}^{p-1}\sum_{s=0}^{r}\binom{r}{s} f_{a_{p-r}}^{<p-r+s>} g_{a_{p-r}}^{(s)} \int_{a_p}^{x}\sim\int_{a_{p-r}}^{x} dx^r$$
$$- \sum_{r=1}^{\infty}\binom{p}{r}\int_{a(p)}^{x}\sim\int_{a(0)}^{x} f^{<r>}g^{(r)}dx^p$$

$$\int_{a}^{x}\sim\int_{a}^{x} f^{<0>}g^{(0)}dx^p = f^{<p>}g^{(0)} - \sum_{r=0}^{p-1}\sum_{s=0}^{r}\binom{r}{s} f_{a}^{<p-r+s>} g_{a}^{(s)} \frac{(x-a)^r}{r!}$$
$$- \sum_{r=1}^{\infty}\binom{p}{r}\int_{a}^{x}\sim\int_{a}^{x} f^{<r>}g^{(r)}dx^p$$

These are notionally right. However, it is almost impossible to obtain the constant-of-integration function $\sum_{r=0}^{p-1}\sum_{s=0}^{r}\binom{r}{s}\cdots$ contained in these. So, regrettably below, we build a theory based on the easiest (1.1) .

17.1.2 Super Integral of the Product of Two Functions

Theorem 17.1.2

Let r, p are positive numbers, $f^{<r>}$ be an arbitrary r th order primitive function of $f(x)$, $g^{(r)}$ be the r th order derivative function of $g(x)$, $B(n,m),\Gamma(p)$ are the beta function and the gamma function respectively At this time, if there is a number a such that

$$f^{<r>}(a)=0 \quad r\in[0,m+p] \qquad or \qquad g^{(s)}(a)=0 \quad s\in[0,m+p-1]$$

then the following expression holds.

$$\int_{a}^{x}\sim\int_{a}^{x} f\,g\,dx^p = \sum_{r=0}^{m-1}\binom{-p}{r} f^{<p+r>}g^{(r)} + R_m^p \tag{2.1}$$

$$R_m^p = \frac{(-1)^m}{B(p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_{a}^{x}\sim\int_{a}^{x} f^{<m+k>}g^{(m+k)}dx^p \tag{2.1r}$$

Proof

In Formula 16.1.2 (2.3) in **16.1.2** , let us analytically continue the index of the integration operator to $[0,p]$ from $[1,n]$. Since $\sum$ have to become an infinite series according to this, the upper limit n is expanded to ∞ .

Riemann-Liouville Integral Expression

According to Formula 7.2.3 in **7.2.3** , Super Integral is expressed by Riemann-Liouville Integral as follows.

$$\int_a^x \sim \int_a^x f(x)\,dx^p = \frac{1}{\Gamma(p)}\int_a^x (x-t)^{p-1} f(t)\,dt$$

If Theorem 17.1.2 is rewritten using this, it is as follows. Since the left side does not have operation functions, this right side is indispensable for the calculation.

Theorem 17.1.2'

Let r, p are positive numbers, $f^{<r>}$ be an arbitrary r th order primitive function of $f(x)$, $g^{(r)}$ be the r th order derivative function of $g(x)$, $B(n,m), \Gamma(p)$ are the beta function and the gamma function respectively At this time, if there is a number a such that

$$f^{<r>}(a) = 0 \quad r \in [0, m+p] \qquad or \qquad g^{(s)}(a) = 0 \quad s \in [0, m+p-1]$$

then the following expression holds.

$$\frac{1}{\Gamma(p)}\int_a^x (x-t)^{p-1} f g\,dt = \sum_{r=0}^{m-1} \binom{-p}{r} f^{<p+r>} g^{(r)} + R_m^p \tag{2.1'}$$

$$R_m^p = \frac{(-1)^m}{\Gamma(p)B(p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{p-1}{k} \int_a^x (x-t)^{p-1} f^{<m+k>} g^{(m+k)}\,dt \tag{2.1'r}$$

17.2 Super Integral of x^a f (x) (general)

Formula 17.2.0

Let $\Gamma(z)$ be the gamma function, $B(n,m)$ be the beta function, $f^{<r>}$ be an arbitrary r th primitive function of $f(x)$ and $f_a^{<r>}$ be the function values of $f^{<r>}$ on a. Then the following expressions hold for a positive number p.

(1) When $f^{<r>}(a)=0 \quad r\in[0,m+p] \quad or \quad a=0$

$$\int_a^x\sim\int_a^x f^{<0>}x^{\alpha}dx^p=\sum_{r=0}^{m-1}\binom{-p}{r}f^{<p+r>}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}$$
$$+\frac{(-1)^m}{B(p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_a^x\sim\int_a^x f^{<m+k>}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-k)}x^{\alpha-m-k}dx^p \tag{0.1}$$

Especially, when $\alpha=m=0,1,2,\cdots$

$$\int_a^x\sim\int_a^x f^{<0>}x^{m}dx^p=\sum_{r=0}^{m}\binom{-p}{r}f^{<p+r>}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r} \tag{0.1'}$$

where, if $\alpha=-1,-2,-3,\cdots$, it shall read as follows.

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} \longrightarrow (-1)^{-r}\frac{\Gamma(-\alpha+r)}{\Gamma(-\alpha)} \qquad r=r,\ s,\ m+k$$

(2) When $a=0 \quad or \quad f^{(s)}(a)=0 \quad s\in[0,m+p-1]$

$$\int_a^x\sim\int_a^x x^{\alpha}f^{(0)}dx^p=\sum_{r=0}^{m-1}\binom{-p}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)}x^{\alpha+p+r}f^{(r)}$$
$$+\frac{(-1)^m}{B(p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_a^x\sim\int_a^x\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)}x^{\alpha+m+k}f^{(m+k)}dx^p \tag{0.2}$$

where $\alpha\ne-1,-2,-3,\cdots$ & $\alpha+p\ne-1,-2,-3,\cdots$:

Proof

Analytically continuing the index of the integration operator in (0.3) and (0.5) in Formula 16.2.0 (**16.2**) to $[0,p]$ from $[1,n]$, we obtain (0.1) and (0.2) in this theorem.

Since $\Gamma(1+\alpha-m-k)=\pm\infty \quad k=0,1,2,3,\cdots$ when $\alpha=m-1 \quad m=0,1,2,\cdots$, the remainder term in (0.1) disappears, and is as follows.

$$\int_a^x\sim\int_a^x f^{<0>}x^{m-1}dx^p=\sum_{r=0}^{m-1}\binom{-p}{r}f^{<p+r>}\frac{\Gamma(1+m-1)}{\Gamma(1+m-1-r)}x^{m-1-r}$$

Then, replacing m-1 with m, we obtain (0.1').

17.3 Super Integral of x^a f (x) (particulars)

In this section, substituting various functions f for Formula 17.2.0 in previous section, we obtain various formulas. There are (1) and (2) in Formula 17.2.0, and we may also choose whichever. However, what we wont is the expression or approximation of super integral of $x^{\alpha}f(x)$ by the series. So, in the selection of (1) or (2), we choose the way where such a well-behaved series is obtained.

Moreover, also in which formula, if $\alpha = -1,-2,-3,\cdots$, it shall read as follows.

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} \quad\longrightarrow\quad (-1)^{-r}\frac{\Gamma(-\alpha+r)}{\Gamma(-\alpha)} \qquad r = r,\ s,\ m+k$$

17.3.1 Super Integral of $(ax+b)^p(cx+d)^q$

Formula 17.3.1

The following expressions hold for $p>0$ and $s>0$.

$$\int_{-\frac{b}{a}}^{x}\sim\int_{-\frac{b}{a}}^{x}(ax+b)^p(cx+d)^q dx^s$$

$$= \sum_{r=0}^{m-1}\binom{-s}{r}\frac{(1/a)^{s+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p+s+r)\Gamma(1+q-r)}\frac{(ax+b)^{p+s+r}}{(cx+d)^{r-q}} + R_m^s \tag{1.1}$$

$$R_m^s = \frac{(-1)^m}{B(s,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{s-1}{k}\left(\frac{c}{a}\right)^{m+k}\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p+m+k)\Gamma(1+q-m-k)}$$

$$\times\int_{-\frac{b}{a}}^{x}\sim\int_{-\frac{b}{a}}^{x}(ax+b)^{p+m+k}(cx+d)^{q-m-k}dx^s \tag{1.1r}$$

$$\lim_{m\to\infty}R_m^s = 0$$

Especially, when $m=0,1,2,\cdots$

$$\int_{-\frac{b}{a}}^{x}\sim\int_{-\frac{b}{a}}^{x}(ax+b)^p\,(cx+d)^m dx^s$$

$$= \sum_{r=0}^{m}\binom{-s}{r}\frac{(1/a)^{s+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+m)}{\Gamma(1+p+s+r)\Gamma(1+m-r)}\frac{(ax+b)^{p+s+r}}{(cx+d)^{r-m}} \tag{1.1'}$$

Example1 The 1/2th order integral of $\sqrt{x-2}\,\sqrt[3]{3x+4}$

The zeros of this primitive function are $x=-4/3$, 2 . If -4 / 3 is adopted, since $\sqrt{x-2}$ is a complex number, this higher integral becomes a complex function. It is inconvenient. Then if we assume $x=2$ as the lower limit of the integral, since $a=1, b=-2, p=1/2$, $c=3, d=4, q=1/3$, $n=1$, substituting these for (1.1), (1.1r) , we obtain

$$\int_2^x\sim\int_2^x\sqrt{x-2}\,\sqrt[3]{3x+4}\,dx^{\frac{1}{2}}$$

$$= \sum_{r=0}^{m-1}\binom{-1/2}{r}3^r\frac{\Gamma(3/2)\Gamma(4/3)}{\Gamma(2+r)\Gamma(4/3-r)}(x-2)^{1+r}(3x+4)^{\frac{1}{3}-r} + R_m^{\frac{1}{2}}$$

$$R_m^s = \frac{(-1)^m}{B(1/2,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{-1/2}{k} 3^{m+k} \frac{\Gamma(3/2)\Gamma(4/3)}{\Gamma(3/2+m+k)\Gamma(4/3-m-k)}$$
$$\times \int_2^x \sim \int_2^x (x-2)^{\frac{1}{2}+m+k} (3x+4)^{\frac{4}{3}-m-k} dx^{\frac{1}{2}}$$

When the left side is replaced with Riemann-Liouville Integral and given one arbitrary point $x=3.3$, it is as follows.

- `a:=1: b:=-2: p:=1/2: c:=3: d:=4: q:=1/3: s:=1/2: m:=22:`

Riemann-Liouville Integral

- `g := x->1/gamma(s)*int((x-t)^(s-1)*(a*t+b)^p*(c*t+d)^q,t=-b/a..x)`

$$x \to \frac{1}{\Gamma(s)} \cdot \int_{-\frac{b}{a}}^{x} (x-t)^{s-1} \cdot (a \cdot t+b)^p \cdot (c \cdot t+d)^q \, \mathrm{d}t$$

Series

- `f := x-> sum(binomial(-s,r)*((1/a)^(s+r)/(1/c)^r)`
  `*(gamma(1+p)*gamma(1+q)/(gamma(1+p+s+r)*gamma(1+q-r))`
  `*((a*x+b)^(p+s+r)/(c*x+d)^(r-q)), r=0..m-1)`

$$x \to \sum_{r=0}^{m-1} \binom{-s}{r} \cdot \frac{\left(\frac{1}{a}\right)^{s+r}}{\left(\frac{1}{c}\right)^{r}} \cdot \frac{\Gamma(1+p) \cdot \Gamma(1+q)}{\Gamma(1+p+s+r) \cdot \Gamma(1+q-r)} \cdot \frac{(a \cdot x+b)^{p+s+r}}{(c \cdot x+d)^{r-q}}$$

- `float(g(3.3)),float(f(3.3))`

  2.701947993, 2.701947994

Although this integral cannot be expressed with an elementary function, it can be expressed with a series. Here is the meaning of this formula.

Example1' The 5/2th order integral of $\sqrt[3]{3x+4}\,(x-2)^2$

Since this is the example of application of (1.1'), we assume the noninteger root as the lower limit of the integral. Since $a=3, b=4, p=1/3, c=1, d=-2, s=5/2$, substituting these for (1.1'), we obtain

$$\int_{-\frac{4}{3}}^{x} \sim \int_{-\frac{4}{3}}^{x} \sqrt[3]{3x+4}\,(x-2)^2 dx^{\frac{5}{2}} = \sum_{r=0}^{2} \binom{-5/2}{r} \left(\frac{1}{3}\right)^{\frac{5}{2}+r} \frac{\Gamma(4/3)\Gamma(3)}{\Gamma(23/6+r)\Gamma(3-r)} \frac{(3x+4)^{\frac{17}{6}+r}}{(x-2)^{r-2}}$$
$$= \frac{\Gamma(4/3)\Gamma(3)}{3^{\frac{5}{2}}(3x+4)^{\frac{1}{6}}} \left\{ \frac{(3x+4)^3(x-2)^2}{\Gamma(23/6)\,2!} - \frac{5}{2\cdot 3} \frac{(3x+4)^4(x-2)}{\Gamma(29/6)\,1!} + \frac{35}{8\cdot 3^2} \frac{(3x+4)^5}{\Gamma(35/6)\,0!} \right\}$$

When the left side is replaced with Riemann-Liouville Integral and given one arbitrary point $x=2.1$, it is as follows.

- `a:=3: b:=4: p:=1/3: c:=1: d:=-2: m:=2: s:=5/2:`

Riemann-Liouville Integral

- `g := x->1/gamma(s)*int((x-t)^(s-1)*(a*t+b)^p*(c*t+d)^m,t=-b/a..x)`

$$x \to \frac{1}{\Gamma(s)} \cdot \int_{-\frac{b}{a}}^{x} (x-t)^{s-1} \cdot (a \cdot t+b)^p \cdot (c \cdot t+d)^m \, \mathrm{d}t$$

Series

- `f := x-> sum(binomial(-s,r)*((1/a)^(s+r)/(1/c)^r)*(gamma(1+p)*gamma(1+m)/(gamma(1+p+s+r)*gamma(1+m-r)))*((a*x+b)^(p+s+r)/(c*x+d)^(r-m)), r=0..m)`

$$x \to \sum_{r=0}^{m} \binom{-s}{r} \cdot \frac{\left(\frac{1}{a}\right)^{s+r}}{\left(\frac{1}{c}\right)^{r}} \cdot \frac{\Gamma(1+p)\cdot\Gamma(1+m)}{\Gamma(1+p+s+r)\cdot\Gamma(1+m-r)} \cdot \frac{(a\cdot x+b)^{p+s+r}}{(c\cdot x+d)^{r-m}}$$

- `float(g(2.1)),float(f(2.1))`

  44.54679172, 44.54679172

17.3.2 Super Integral of $x^{\alpha}\log x$

Since the zero of the lineal higher primitive function of $x^{\alpha}\log x$ for α, n such that $\alpha+n>0$ was 0, naturally, the zero of the lineal super primitive function for α, p such that $\alpha+p>0$ is also 0.

Formula 17.3.2

When $\Gamma(z)$, $\psi(z)$ are the gamma function and the psi function respectively, the following expressions hold for α, p such that $\alpha+p>0$.

$$\int_0^x \tilde{} \int_0^x x^{\alpha}\log x\, dx^p = \sum_{r=0}^{m-1}\binom{-p}{r}\frac{\log x-\psi(1+p+r)-\gamma}{\Gamma(1+p+r)}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha+p}+R_m^p \tag{2.1}$$

$$R_m^p=\frac{(-1)^m}{B(p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_0^x \tilde{} \int_0^x \frac{\log x-\psi(1+m+k)-\gamma}{\Gamma(1+m+k)}\frac{\Gamma(1+\alpha)x^{\alpha}}{\Gamma(1+\alpha-m-k)}dx^p$$

$$\lim_{m\to\infty}R_m^p=0$$

Especially, when $m=0,1,2,\cdots$

$$\int_0^x \tilde{} \int_0^x x^{m}\log x\, dx^p = \sum_{r=0}^{m}\binom{-p}{r}\frac{\log x-\psi(1+p+r)-\gamma}{\Gamma(1+p+r)}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m+p} \tag{2.1'}$$

Example1 The 1.9th order integral of $\sqrt{x}\log x$

Substituting $\alpha=1/2$, $p=1.9$ for (2.1), we obtain

$$\int_0^x \tilde{} \int_0^x \sqrt{x}\log x\, dx^{1.9} = \sum_{r=0}^{m-1}\binom{-1.9}{r}\frac{\log x-\psi(2.9+r)-\gamma}{\Gamma(2.9+r)}\frac{\Gamma(3/2)}{\Gamma(3/2-r)}x^{2.4}+R_m^{1.9}$$

When m=20000, if the values of the both sides on arbitrary point x=2 are calculated, it is as follows.
Since the convergence is very slow, even if calculated so far, both sides are corresponding only to 5 digits below a decimal point.

```
a = 1 / 2; p = 1.9; m = 20000;
```

$$\text{fl}[x\_] := \frac{1}{\text{Gamma}[p]}\int_0^x (x-t)^{p-1}\, t^a\, \text{Log}[t]\, dt$$

$$\text{fr}[x\_] := \sum_{r=0}^{m-1}\text{Binomial}[-p, r]\,\frac{\text{Log}[x]-\text{PolyGamma}[1+p+r]-\text{EulerGamma}}{\text{Gamma}[1+p+r]}\,\frac{\text{Gamma}[1+a]}{\text{Gamma}[1+a-r]}\,x^{a+p}$$

```
N[fl[2]]        N[fr[2]]
-0.533354       -0.533357
```

Example1' The 2.1th order integral of $x^3 \log x$

Substituting $m=3$, $p=2.1$ for (2.1'), we obtain

$$\int_0^x \sim \int_0^x x^3 \log x\, dx^{2.1} = \sum_{r=0}^{3} \binom{-2.1}{r} \frac{\log x - \psi(3.1+r) - \gamma}{\Gamma(3.1+r)} \frac{\Gamma(4)}{\Gamma(4-r)} x^{5.1}$$

If the values of the both sides on arbitrary point x=3 are calculated, it is as follows.
All digits in both sides are corresponding this time.

```
m = 3; p = 2.1;
fl[x_] := 1/Gamma[p] ∫_0^x (x - t)^(p-1) t^m Log[t] dt
fr[x_] := Σ_{r=0}^{m} Binomial[-p, r] (Log[x] - PolyGamma[1 + p + r] - EulerGamma)/Gamma[1 + p + r] Gamma[1 + m]/Gamma[1 + m - r] x^(m+p)
N[fl[3]]          N[fr[3]]
7.20417           7.20417
```

Complete Automorphism

Observing well Formula 17.3.2, we notice that the integral of the completely same type as the left side is included in the remainder. In such a case, we can take out the integral of the purpose by transposition.

Formula 17.3.2'

The following expression holds for α, p such that $\alpha+p>0$.

$$\int_0^x \sim \int_0^x x^\alpha \log x\, dx^p$$
$$= \frac{\dfrac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} + p\dfrac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p)} \displaystyle\sum_{k=0}^{\infty} \binom{p-1}{k} \frac{\psi(2+k)+\gamma}{(1+k)^2 B(1+k, \alpha-k)}}{1 + p\displaystyle\sum_{k=0}^{\infty} \binom{p-1}{k} \frac{1}{(1+k)^2 B(1+k, \alpha-k)}} x^{\alpha+p} \tag{2.2}$$

Proof

Analytically continuing the index of the integration operator in Formula 16.3.2' (**16.3**) to $[0,p]$ from $[1,n]$, we obtain the desired expression.

17.3.3 Super Integral of $x^\alpha \sin x$, $x^\alpha \cos x$

When $\alpha \ne 0, -1, -2, \cdots$, the lineal higher integral of $x^\alpha \sin x$ had variable lower limits. Therefore, the lineal super integral of this has also variable lower limits. Since the value of the lower limit to given α becomes the function of p, we denote this a_k, $k \in [0,p]$.

Formula 17.3.3

When $a_0 \sim a_p$ are the zeros of the lineal super primitive of $x^m \sin x$, the follwing expressions hold.

$$\int_{a_p}^x \sim \int_{a_0}^x x^m \sin x\, dx^p = \sum_{r=0}^{m} \binom{-p}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \sin\left\{x - \frac{(p+r)\pi}{2}\right\} \tag{3.1s}$$

$$\int_{a_p}^x \sim \int_{a_0}^x x^m \cos x\, dx^p = \sum_{r=0}^{m} \binom{-p}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \cos\left\{x - \frac{(p+r)\pi}{2}\right\} \tag{3.1c}$$

Proof

Analytically continuing the index of the integration operator in Formula 16.3.3 (3.1') in **16.3.3** to $[0,p]$ from $[1,n]$, we obtain (3.1s). Also, (3.1c) is obtained in a similar way.

Example The 7/2th order integral of $x^3 \sin x$

Substituting $m=3$, $p=7/2$ for (3.1s), we obtain

$$\int_{a_p}^{x}\!\!\sim\!\!\int_{a_0}^{x} x^3 \sin x\, dx^{\frac{7}{2}} = \sum_{r=0}^{3} \binom{-7/2}{r} \frac{\Gamma(4)}{\Gamma(4-r)} x^{3-r} \sin\left\{x - \frac{(7/2+r)\pi}{2}\right\}$$

This super integral can not be examined by Riemann-Liouville Integral. Then we draw this with the 3rd and the 4th order integral on a figure side by side. It is as follows.

Formula 17.3.3' (Collateral Super Integral)

The following expressions hold for α, p such that $\alpha \neq -1,-2,-3,\cdots$ & $\alpha+p>0$.

$$\int_0^{x}\!\!\sim\!\!\int_0^{x} x^\alpha \sin x\, dx^p = \sum_{r=0}^{m-1} \binom{-p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} \sin\left(x+\frac{r\pi}{2}\right) + R_m^p \qquad (3.2s)$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{p-1}{k} \int_0^{x}\!\!\sim\!\!\int_0^{x} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} x^{\alpha+m+k} \sin\left\{x+\frac{(m+k)\pi}{2}\right\} dx^p$$

$$\int_0^{x}\!\!\sim\!\!\int_0^{x} x^\alpha \cos x\, dx^p = \sum_{r=0}^{m-1} \binom{-p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} \cos\left(x+\frac{r\pi}{2}\right) + R_m^p \qquad (3.2c)$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{p-1}{k} \int_0^{x}\!\!\sim\!\!\int_0^{x} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} x^{\alpha+m+k} \cos\left\{x+\frac{(m+k)\pi}{2}\right\} dx^p$$

$$\lim_{m\to\infty} R_m^p = 0$$

Proof

Substituting $f(x)=\sin x$, $a=0$, *etc.* for Formula 17.2.0 (2), we obtain the desired expression.

Example Collateral the 1/2th order integral of $x^{3/2} \sin x$

Substituting $\alpha=3/2$, $p=1/2$ for (3.2s), we obtain

$$\int_0^x \sim \int_0^x x^{\frac{3}{2}} \sin x \, dx^{\frac{1}{2}} = \sum_{r=0}^{m-1} \binom{-1/2}{r} \frac{\Gamma(5/2)}{\Gamma(3+r)} x^{2+r} \sin\left(x + \frac{r\pi}{2}\right) + R_m^{1/2}$$

When the left side is replaced with Riemann-Liouville Integral and given one arbitrary point $x=3$, it is as follows.

```
a = 3 / 2;  p = 1 / 2;  m = 20;
```

$$\texttt{f1[x\_] := } \frac{1}{\texttt{Gamma[p]}} \int_0^x (x - t)^{p-1} \, t^a \, \texttt{Sin[t]} \, dt$$

$$\texttt{fr[x\_] := } \sum_{r=0}^{m-1} \texttt{Binomial[-p, r]} \frac{\texttt{Gamma[1 + a]}}{\texttt{Gamma[1 + a + p + r]}} \, x^{a+p+r} \, \texttt{Sin}\left[x + \frac{r\,\pi}{2}\right]$$

```
N[f1[3]]          N[fr[3]]
2.90726           2.90726
```

17.3.4 Super Integrals of $x^\alpha \sinh x$, $x^\alpha \cosh x$

Formula 17.3.4

When $a_0 \sim a_p$ are the zeros of the lineal super primitive of $x^m \sinh x$, the follwing expression holds.

$$\int_{a_p}^x \sim \int_{a_0}^x x^m \sinh x \, dx^p = \sum_{r=0}^{m} \binom{-p}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \frac{e^x - (-1)^{p+r} e^{-x}}{2} \tag{4.1s}$$

$$\int_{a_p}^x \sim \int_{a_0}^x x^m \cosh x \, dx^p = \sum_{r=0}^{m} \binom{-p}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \frac{e^x + (-1)^{p+r} e^{-x}}{2} \tag{4.1c}$$

Proof

Analytically continuing the index of the integration operator in Formula 16.3.4 (4.1') in **16.3.4** to $[0,p]$ from $[1,n]$, we obtain (4.1s). Also, (4.1c) is obtained in a similar way.

Example The 0.999th order integral of $x^3 \sinh x$

Substituting $m=3$, $p=0.999$ for (4.1s), we obtain

$$\int_{a_p}^x \sim \int_{a_1}^x x^3 \sinh x \, dx^p = \sum_{r=0}^{m} \binom{-0.999}{r} \frac{\Gamma(1+3)}{\Gamma(1+3-r)} x^{3-r} \frac{e^x - (-1)^{0.999+r} e^{-x}}{2}$$

This super integral can not be examined by Riemann-Liouville Integral. Then we calculate on the arbitrary point $x=1$ the value of the 0.999th order integral and the value of the 1st order integral. They are as follows. Although the former becomes a complex number, both real number parts are very near.

```
m = 3;  p = 0.999;
```

0.999th order Integral $\quad \texttt{fp[x\_]:=} \sum_{r=0}^{m} \texttt{Binomial[-p, r]} \frac{\texttt{Gamma[1 + m]}}{\texttt{Gamma[1 + m - r]}} x^{m-r} \frac{e^x - (-1)^{p+r} e^{-x}}{2}$

1st order Integral $\quad \texttt{f1[x\_] = } \int x^m \, \texttt{Sinh[x]} \, dx;$

```
N[fp[1]]                          N[f1[1]]
0.227302  - 0.00923252  i         0.224754
```

Formula 17.3.4' (Collateral Super Integral)

The following expressions hold for α , p such that $\alpha \neq -1,-2,-3,\cdots$ & $\alpha+p>0$.

$$\int_0^x \sim \int_0^x x^\alpha \sinh x \, dx^p = \sum_{r=0}^{m-1} \binom{-p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} \frac{e^x-(-1)^{-r}e^{-x}}{2} \tag{4.2s}$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{p-1}{k} \int_0^x \sim \int_0^x \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} x^{\alpha+m+k} \frac{e^x-(-1)^{-m-k}e^{-x}}{2} dx^p$$

$$\int_0^x \sim \int_0^x x^\alpha \cosh x \, dx^p = \sum_{r=0}^{m-1} \binom{-p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} \frac{e^x+(-1)^{-r}e^{-x}}{2} + R_m^p \tag{4.2c}$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{p-1}{k} \int_0^x \sim \int_0^x \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} x^{\alpha+m+k} \frac{e^x+(-1)^{-m-k}e^{-x}}{2} dx^p$$

$$\lim_{m\to\infty} R_m^p = 0$$

Proof

Substituting $f(x)=\sinh x$, $a=0$, $etc.$ for Formula 17.2.0 (2) , we obtain the desired expression.

Example Collateral the 3/2th order integral of $\sqrt{x}\,\sinh x$

Substituting $\alpha=1/2$, $p=3/2$ for (4.2s), we obtain

$$\int_0^x \sim \int_0^x \sqrt{x} \sinh x \, dx^{\frac{3}{2}} = \sum_{r=0}^{m-1} \binom{-3/2}{r} \frac{\Gamma(3/2)}{\Gamma(3+r)} x^{2+r} \frac{e^x-(-1)^{-r}e^{-x}}{2} + R_m^{3/2}$$

When the left side is replaced with Riemann-Liouville Integral and given one arbitrary point $x=2$, it is as follows.

```
a = 1/2; p = 3/2; m = 20;
fl[x_] := 1/Gamma[p] ∫_0^x (x - t)^(p-1) t^a Cosh[t] dt
fr[x_] := Sum_{r=0}^{m-1} Binomial[-p, r] Gamma[1 + a]/Gamma[1 + a + p + r] x^(a+p+r) (e^x + (-1)^-r e^-x)/2

N[fl[2.1]]        N[fr[2.1]]
3.58623           3.58623
```

17.4 Super Integral of $\log x\, f(x)$

17.4.1 Super Integral of $(\log x)^2$

Since the zero of the lineal higher primitive of $(\log x)^2$ was $x=0$, naturally, the zero of the lineal super primitive is also $x=0$.

Formula 17.4.1

$$\int_0^x \sim \int_0^x \log^2 x\, dx^p = \frac{x^p \log x \{\log x - \psi(1+p) - \gamma\}}{\Gamma(1+p)}$$

$$- x^p \sum_{r=1}^{m-1} (-1)^r \binom{-p}{r} \frac{\{\log x - \psi(1+p+r) - \gamma\}\Gamma(r)}{\Gamma(1+p+r)} + R_m^p \tag{1.1}$$

$$R_m^p = \frac{x^p}{B(p,m)\Gamma(1+p)} \times$$

$$\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(m+k)^2} \binom{p-1}{k} \{\log x - \psi(1+p) - \psi(1+m+k) - 2\gamma\} \tag{1.1r}$$

$$\lim_{m\to\infty} R_m^p = 0$$

Proof

Analytically continuing the index of the integration operator in Formula 16.4.1 in **16.4.1** to $[0,p]$ from $[1,n]$, we obtain the desired expression.

The remainder in the super integral of product of two functions becomes a series of super integrals. So it is very difficult to calculate this. However, the calculation of the remainder in the super integral of $(\log x)^2$ is exceptionally easy. So we show it as follows.

Example The 3/2th order integral of $(\log x)^2$

We substitute $p=3/2$, $m=3$ for (1.1), (1.1r) and calculate the remainder 300 terms. The function values on the arbitrary point $x=0.3$ are as follows. Although the difference of the left side and the series is large, if the remainder is added, all digits in both sides are corresponding.

```
p = 3 / 2;  m = 3;  n = 300;
```

Riemann-Liouville Integral

$$\text{f1}[x\_] := \frac{1}{\text{Gamma}[p]} \int_0^x (x - t)^{p-1} \text{Log}[t]^2 \, dt$$

Series

$$\text{Sr}[x\_] := \frac{x^p \,\text{Log}[x]\, (\text{Log}[x] - \text{PolyGamma}[1+p] - \text{EulerGamma})}{\text{Gamma}[1+p]} -$$

$$x^p \sum_{r=1}^{m-1} (-1)^r \,\text{Binomial}[-p, r] \frac{(\text{Log}[x] - \text{PolyGamma}[1+p+r] - \text{EulerGamma})\, \text{Gamma}[r]}{\text{Gamma}[1+p+r]}$$

Remainder

$$\text{Rm}[x\_] := \frac{x^p}{\text{Beta}[p, m]\, \text{Gamma}[1+p]} \sum_{k=0}^{n} \frac{(-1)^{k-1}}{(m+k)^2} \text{Binomial}[p-1, k] \times$$

$$(\text{Log}[x] - \text{PolyGamma}[1+p] - \text{PolyGamma}[1+m+k] - 2\, \text{EulerGamma})$$

Riemann-Liouville Integral	Series	Remainder	Series + Remainder
N[f1[0.3]]	N[Sr[0.3]]	N[Rm[0.3]]	N[Sr[0.3] + Rm[0.3]]
0.905618	0.667604	0.238014	0.905618

Incidentally, even if this series is calculated to $m=300{,}000$, both are corresponding only to 4 digits below the decimal point. From this, we know that the convergence of the series is extraordinarily slower than the remainder.

17.5 Super Integral of e^x f (x)

17.5.1 Super Integral of $e^x x^\alpha$

Since the zero of the lineal higher primitive of $e^x x^\alpha$ was $-\infty$, naturally, the zero of the lineal super primitive is also $-\infty$.

Formula 17.5.1

When $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$ is the incomplete gamma function, the following expression holds for $x \leqq 0$

$$\int_{-\infty}^{x}\sim\int_{-\infty}^{x} x^\alpha e^x dx^p = \frac{(-1)^\alpha}{\Gamma(p)}\sum_{r=0}^{\infty}\binom{p-1}{r}\Gamma(p-r+\alpha,\ -x)x^r \qquad (1.n)$$

Proof

Analytically continuing the index of the integration operator in Formula 16.5.1 in **16.5.1** to $[0,p]$ from $[1,n]$, we obtain the desired expression.

Example1 The 2.1th order integral of $e^x\sqrt[3]{x}$

When $\alpha=1/3$, $p=2.1$, if the left side is replaced by Riemann-Liouville Integral and an arbitrary point $x=-1.7$ is given to both sides, it is as follows.

```
a = 1 / 3; p = 2.1; m = 250;
fl[x_] := 1/Gamma[p] ∫_{-∞}^{x} (x - t)^(p-1) t^a e^t dt
fr[x_] := (-1)^a/Gamma[p] Σ_{r=0}^{m} Binomial[p - 1, r] Gamma[p - r + a, -x] x^r

N[fl[-1.7]]                    N[fr[-1.7]]
0.140503 + 0.243358 i          0.140503 + 0.243358 i
```

When α is a natural number, x^α can be differentiated completely and the following formula holds.

Formula 17.5.1'

When $m=0,1,2,\cdots$,

$$\int_{-\infty}^{x}\sim\int_{-\infty}^{x} e^x x^m dx^p = e^x\sum_{r=0}^{m}\binom{-p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r} \qquad (1.n')$$

Example1' The 5.1th order integral of $e^x x^4$

Substituting $m=4$, $n=5.1$ for (1.n'), we obtain

$$\int_{-\infty}^{x}\sim\int_{-\infty}^{x} e^x x^4 dx^{5.1} = e^x\sum_{r=0}^{4}\binom{-5.1}{r}\frac{\Gamma(1+4)}{\Gamma(1+4-r)}x^{4-r}$$

If the values of the both sides on arbitrary point $x=2$ are calculated, it is as follows.

```
m = 4; p = 5.1;
fl[x_] := 1/Gamma[p] ∫_{-∞}^{x} (x - t)^(p-1) t^m e^t dt
```

```
fr[x_] := e^x Sum[Binomial[-p, r] Gamma[1 + m]/Gamma[1 + m - r] x^(m-r), {r, 0, m}]
```

```
N[fl[2]]        N[fr[2]]
4592.51         4592.51
```

Collateral Super Integral of $e^x x^\alpha$

For the collateral super integral with the zero x=0, the following formula holds.

Formula 17.5.1"

The following expressions hold for α, p such that $\alpha \ne -1,-2,-3,\cdots$ & $\alpha+p>0$.

$$\int_0^x \sim \int_0^x e^x x^\alpha dx^p = e^x \sum_{r=0}^{m-1} \binom{-p}{r} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} + R_m^p \tag{1.n"}$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k=0}^{\infty} \frac{1}{m+k} \binom{p-1}{k} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)} \int_0^x \sim \int_0^x e^x x^{\alpha+m+k} dx^p$$

$$\lim_{m\to\infty} R_m^p = 0$$

Example1" Collateral the 2.1th order integral of $e^x \sqrt[3]{x}$

Assume $\alpha=1/3$, $p=2.1$ and an arbitrary point $x=-1.7$ are the same as Example1 (Lineal 2.1 th order integral) . The values of the both sides are completely corresponding as follows. However, these are considerably different from the value of Example1.

```
a = 1/3; p = 2.1; m = 30;
fl[x_] := 1/Gamma[p] Integrate[(x - t)^(p-1) t^a e^t, {t, 0, x}]
fr[x_] := e^x Sum[Binomial[-p, r] Gamma[1 + a]/Gamma[1 + a + p + r] x^(a+p+r), {r, 0, m-1}]
```

```
N[fl[-1.7]]                 N[fr[-1.7]]
0.121502 + 0.57162 i        0.121502 + 0.57162 i
```

17.5.2 Super integral of $e^x \log x$

All the polynomials obtained by applying Theorem 17.1.2 to $e^x \log x$ become asymptotic expansions, and they are hardly helpful.

17.5.3 Super integral of $e^x \sin x$, $e^x \cos x$

Formula 17.5.3

$$\int_{-\infty}^x \sim \int_{-\infty}^x e^x \sin x\, dx^p = \left(\sin\frac{\pi}{4}\right)^p e^x \sin\left(x - \frac{p\pi}{4}\right) \tag{3.0s}$$

$$\int_{-\infty}^x \sim \int_{-\infty}^x e^x \cos x\, dx^p = \left(\sin\frac{\pi}{4}\right)^p e^x \cos\left(x - \frac{p\pi}{4}\right) \tag{3.0c}$$

Proof

Analytically continuing the index of the integration operator in Formula 16.5.3 in **16.5.3** to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Example The 1/2th order integral of $e^x \sin x$

If the values of the both sides on arbitrary point $x=2.3$ are calculated, it is as follows.

```
p:=1/2:
```

Left; Riemann-Liouville Integral

```
g := x-> 1/gamma(p)*int((x-t)^(p-1)*E^t*sin(t), t=-infinity..x)
```

$$x \to \frac{1}{\Gamma(p)} \cdot \int_{-\infty}^{x} (x-t)^{p-1} \cdot E^t \cdot \sin(t)\, \mathrm{d}t$$

Right: Formula

```
s := x-> sin(PI/4)^p*E^x*sin(x-p*PI/4)
```

$$x \to \sin\left(\frac{\pi}{4}\right)^p \cdot E^x \cdot \sin\left(x - \frac{\pi \cdot p}{4}\right)$$

Left , Right

```
float(g(2.3)), float(s(2.3))
```

$$7.916851572,\ 7.916851572$$

Trigonometric Series

If Formula 16.5.4 in **16.5.4** is extended to the real number, the following trigonometric series can be obtained

Formula 17.5.4

$$\sum_{k=1}^{\infty} \frac{1}{k} \binom{p-1}{k-1} \sin\frac{k\pi}{2} = \frac{1}{p}\left(\sin\frac{\pi}{4}\right)^{-p} \sin\frac{p\pi}{4} \tag{4.1}$$

$$\sum_{k=1}^{\infty} \frac{1}{k} \binom{p-1}{k-1} \cos\frac{k\pi}{2} = \frac{1}{p}\left(\sin\frac{\pi}{4}\right)^{-p} \cos\frac{p\pi}{4} - \frac{1}{p} \tag{4.2}$$

Alternating Binomial Series

If Formula 16.5.4' in **16.5.4** is extended to the real number, the following alternating binomial series can be obtained.

Formula 17.5.4'

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \binom{p}{2k} = \frac{1}{p+1}\left(\sin\frac{\pi}{4}\right)^{-p-1} \sin\frac{(p+1)\pi}{4} \tag{4.3}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{2k} \binom{p}{2k-1} = \frac{1}{p+1}\left\{\left(\sin\frac{\pi}{4}\right)^{-p-1} \cos\frac{(p+1)\pi}{4} - 1\right\} \tag{4.4}$$

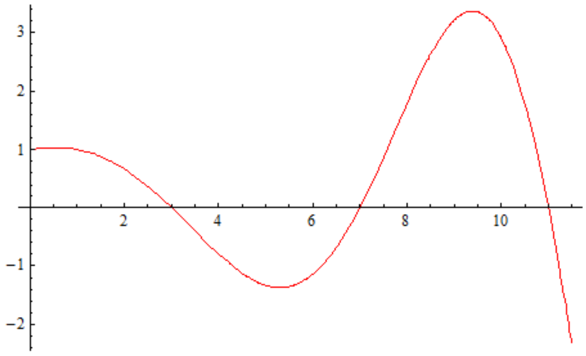
If a horizontal axis is set as p and these are illustrated , it is as follows. Although the left side is blue and right side is red, since both sides overlap exactly, the left side (blue) is not visible.

```
m = 20;
```

sin x series

$$\mathrm{sl}[p\_] := \sum_{k=0}^{m} \frac{(-1)^k}{2k+1}\,\mathrm{Binomial}[p, 2k]$$

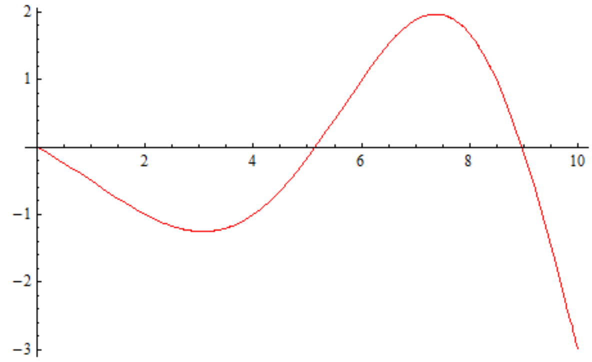
$$\mathrm{sr}[p\_] := \frac{1}{p+1}\left(\mathrm{Sin}\left[\frac{\pi}{4}\right]\right)^{-p-1}\mathrm{Sin}\left[\frac{(p+1)\,\pi}{4}\right]$$



cos x series

$$\mathrm{cl}[p\_] := \sum_{k=1}^{m} \frac{(-1)^k}{2k}\,\mathrm{Binomial}[p, 2k-1]$$

$$\mathrm{cr}[p\_] := \frac{1}{p+1}\left(\left(\mathrm{Sin}\left[\frac{\pi}{4}\right]\right)^{-p-1}\mathrm{Cos}\left[\frac{(p+1)\,\pi}{4}\right]-1\right)$$



17.6 Super Integral of f (x) / e^x

17.6.1 Super Integral of $e^{-x}x^{\alpha}$

Since the zero of the lineal higher primitive of $e^{-x}x^{\alpha}$ was ∞, naturally, the zero of the lineal super primitive is also ∞.

Formula 17.6.1

When $\Gamma(a,x)=\int_x^{\infty} t^{a-1}e^{-t}dt$ is the incomplete gamma function, the following expression holds.

$$\int_{\infty}^{x}\sim\int_{\infty}^{x} e^{-x}x^{\alpha}dx^{p}=\frac{(-1)^{p}}{\Gamma(p)}\sum_{r=0}^{\infty}(-1)^{r}\binom{p-1}{r}\Gamma(p-r+\alpha,x)x^{r} \tag{1.1}$$

Proof

Analytically continuing the index of the integration operator in Formula 16.6.1 in **16.6.1** to $[0,p]$ from $[1,n]$, we obtain the desired expression.

Example1 The 2.1th order integral of $e^{-x}\sqrt[3]{x}$

When $\alpha=1/3$, $p=2.1$, if the left side is replaced by Riemann-Liouville Integral and an arbitrary point $x=1.7$ is given to both sides, it is as follows.

```
a = 1 / 3;  p = 2.1;  m = 250;
fl[x_] := 1/Gamma[p] ∫_∞^x (x - t)^(p-1) t^a e^(-t) dt
fr[x_] := (-1)^p/Gamma[p] Σ_{r=0}^{m} (-1)^r Binomial[p - 1, r] Gamma[p - r + a, x] x^r
N[fl[1.7]]                      N[fr[1.7]]
0.267252 + 0.0868354 i          0.267252 + 0.0868354 i
```

When α is a natural number, x^{α} can be differentiated completely. And the following formula holds.

Formula 17.6.1'

When $m=0,1,2,\cdots$,

$$\int_{\infty}^{x}\sim\int_{\infty}^{x} e^{-x}x^{m}dx^{p}=\frac{(-1)^{p}}{e^{x}}\sum_{r=0}^{m}(-1)^{r}\binom{-p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r} \tag{1.1'}$$

Example1' The 5.1 th order integral of $e^{-x}x^{4}$

Substituting $m=4$, $p=5.1$ for (1.1'), we obtain

$$\int_{\infty}^{x}\sim\int_{\infty}^{x} e^{-x}x^{4}dx^{5.1}=\frac{(-1)^{5.1}}{e^{x}}\sum_{r=0}^{4}(-1)^{r}\binom{-5.1}{r}\frac{\Gamma(1+4)}{\Gamma(1+4-r)}x^{4-r}$$

If the values of the both sides on arbitrary point $x=1.3$ are calculated, it is as follows.

```
m = 4;  p = 5.1;
fl[x_] := 1/Gamma[p] ∫_∞^x (x - t)^(p-1) t^m e^(-t) dt
```

```
fr[x_] := ((-1)^p / e^x) Sum[(-1)^r Binomial[-p, r] Gamma[1 + m] / Gamma[1 + m - r] x^(m-r), {r, 0, m}]
```

N[fl[1.3]]	N[fr[1.3]]
-855.557 - 277.987 i	-855.557 - 277.987 i

Collateral Super Integral of $e^{-x}x^{\alpha}$

For the collateral super integral with the zero x=0, the following formula holds.

Formula 17.6.1"

The following expressions hold for α, p such that $\alpha \neq -1, -2, -3, \cdots$ & $\alpha+p>0$.

$$\int_0^x \!\!\sim\! \int_0^x e^{-x}x^{\alpha}dx^p = \frac{1}{e^x}\sum_{r=0}^{m-1}(-1)^{-r}\binom{-p}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)}x^{\alpha+p+r} + R_m^p \tag{1.1"}$$

$$R_m^p = \frac{1}{B(p,m)}\sum_{k=0}^{n-1}\frac{(-1)^{-k}}{m+k}\binom{p-1}{k}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+k)}\int_0^x \!\!\sim\! \int_0^x \frac{x^{\alpha+m+k}}{e^x}dx^p$$

$$\lim_{m\to\infty} R_m^n = 0$$

Example1" Collateral the 2.1th order integral of $e^{-x}\sqrt[3]{x}$

Assume $\alpha=1/3$, $p=2.1$ and an arbitrary point $x=1.7$ are the same as Example1 (Lineal 2.1 th order integral). The values of the both sides are completely corresponding as follows. However, these are different from the value of Example1 considerably.

```
a = 1/3; p = 2.1; m = 15;
fl[x_] := 1/Gamma[p] Integrate[(x - t)^(p-1) t^a e^-t, {t, 0, x}]
fr[x_] := 1/e^x Sum[(-1)^-r Binomial[-p, r] Gamma[1 + a] / Gamma[1 + a + p + r] x^(a+p+r), {r, 0, m-1}]
```

N[fl[1.7]]	N[fr[1.7]]
0.58439	0.58439

17.6.2 Super integral of $e^{-x}\log x$

All the polynomials obtained by applying Theorem 17.1.2 to $e^{-x}\log x$ become asymptotic expansions, and they are hardly helpful.

17.6.3 Super integral of $e^{-x}\sin x$, $e^{-x}\cos x$

Formula 17.6.3

$$\int_{\infty}^x \!\!\sim\! \int_{\infty}^x e^{-x}\sin x\,dx^p = (-1)^p\left(\sin\frac{\pi}{4}\right)^p e^{-x}\sin\left(x+\frac{p\pi}{4}\right) \tag{3.0s}$$

$$\int_{\infty}^x \!\!\sim\! \int_{\infty}^x e^{-x}\cos x\,dx^p = (-1)^p\left(\sin\frac{\pi}{4}\right)^p e^{-x}\cos\left(x+\frac{p\pi}{4}\right) \tag{3.0c}$$

Proof

Analytically continuing the index of the integration operator in Formula 16.6.3 in **16.6.3** to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Example The 4/3th order integral of $e^{-x}\cos x$

If the values of the both sides on arbitrary point $x=1.2$ are calculated, it is as follows.

p = 4 / 3;

$$\text{fl}[x\_] := \frac{1}{\text{Gamma}[p]} \int_{\infty}^{x} (x - t)^{p-1}\, e^{-t}\, \text{Cos}[t]\, dt$$

$$\text{fr}[x\_] := (-1)^{p} \left(\text{Sin}\left[\frac{\pi}{4}\right]\right)^{p} e^{-x}\, \text{Cos}\left[x + \frac{p\,\pi}{4}\right]$$

N[fl[1.2]]	N[fr[1.2]]
0.0593879 + 0.102863 i	0.0593879 + 0.102863 i

17.7 Series Expansion of Super Integral

17.7.1 Series Expansion of Super Integral with a fixed lower limit

Theorem 17.7.1

Let m be a natural number, $f^{(r)}$ $r=0,1,2,\cdots$ be the r th derivative of f, a be an arbitrary constant on the domain of f, $B(x,y)$ be the beta function. Then the following expressions hold for $p>0$.

$$\int_a^x\sim\int_a^x f(x)\,dx^p=\sum_{r=0}^{m-1}\binom{-p}{r}\frac{(x-a)^{p+r}}{\Gamma(1+p+r)}f^{(r)}(x)+R_m^p \tag{1.1}$$

$$R_m^p=\frac{(-1)^m}{B(p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_a^x\sim\int_a^t\frac{(t-a)^{m+k}}{(m+k)!}f^{(m+k)}(t)\,dt^p \tag{1.1r}$$

$$\int_a^x\sim\int_a^x f(x)\,dx^p=\sum_{r=0}^{m-1}\frac{(x-a)^{p+r}}{\Gamma(1+p+r)}f^{(r)}(a)+R_m^p \tag{1.2}$$

$$R_m^p=\sum_{r=1}^{\infty}\sum_{s=0}^{r-1}(-1)^{p-r+s}\binom{m+p+s}{r}\sum_{t=s}^{r-1}\binom{m+p-1-r+t}{m-1}{}_tC_s\frac{(x-a)^{m+p+s}}{\Gamma(1+m+p+s)}f^{(m+s)}(a)$$

$$+\frac{(-1)^m}{B(p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_a^x\sim\int_a^t\frac{(t-x)^{m+k}}{(m+k)!}f^{(m+k)}(t)\,dt^p \tag{1.2r}$$

Where, there shall be no term of $\Sigma\Sigma$ at the time of $p<2$.

Proof

Analytically continuing the index of the integration operator in Formula 16.7.1 in **16.7.1** to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Series Expansion of Riemann-Liouville Integral

Super Integral with a fixed lower limit is Riemann-Liouville Integral itself. Therefore, the above thorem is also a theorem concerning the series expansion of Riemann-Liouville Integral. If the above thorem is rewritten by Riemann-Liouville Integral, it is as follows.

Thorem 17.7.1'

The following formulas hold for a positive number p and an arbitrary number a on the domain of f.

$$\frac{1}{\Gamma(p)}\int_a^x(x-t)^{p-1}f(t)\,dt=\sum_{r=0}^{m-1}\binom{-p}{r}\frac{(x-a)^{p+r}}{\Gamma(1+p+r)}f^{(r)}(x)+R_m^p \tag{1.1'}$$

$$R_m^p=\frac{(-1)^m}{B(p,m)\Gamma(p)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_a^x\frac{(x-t)^{p-1}(t-a)^{m+k}}{\Gamma(1+m+k)}f^{(m+k)}(t)\,dt^p$$

$$\frac{1}{\Gamma(p)}\int_a^x(x-t)^{p-1}f(t)\,dt=\sum_{r=0}^{m-1}\frac{(x-a)^{p+r}}{\Gamma(1+p+r)}f^{(r)}(a)+R_m^p \tag{1.2'}$$

$$R_m^p=\sum_{r=1}^{\infty}\sum_{s=0}^{r-1}(-1)^{p-r+s}\sum_{t=s}^{r-1}\binom{t}{s}\binom{m+p-1-r+t}{m-1}\binom{m+p+s}{r}\frac{(x-a)^{m+p+s}}{\Gamma(1+m+p+s)}f^{(m+s)}(a)$$

$$-\frac{(-1)^{m+p}}{B(p,m)\Gamma(p)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_a^x\frac{(t-x)^{m+p+k}}{\Gamma(1+m+k)}f^{(m+k)}(t)\,dt^p \tag{1.2'r}$$

Where, there shall be no term of $\Sigma\Sigma$ at the time of $p<2$.

Example1 Series expansion of collateral the 2.1th order integral of e^x

Let $f(x)=e^x$, then

$$f(x)=e^x \quad , \quad f^{(r)}(x)=e^x \,, \quad f^{(r)}(0)=1 \qquad r=1,2,3,\cdots$$

Substituting these for Theorem 17.7.1 ,

$$\int_a^x\!\sim\!\int_a^x e^x\,dx^p = e^x\sum_{r=0}^{m-1}\binom{-p}{r}\frac{(x-a)^{p+r}}{\Gamma(1+p+r)} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_a^x\!\sim\!\int_a^t\frac{(t-a)^{m+k}}{(m+k)!}e^t dt^p$$

$$\int_a^x\!\sim\!\int_a^x e^x\,dx^p = e^a\sum_{r=0}^{m-1}\frac{(x-a)^{p+r}}{\Gamma(1+p+r)} + R_m^p$$

$$R_m^p = \sum_{r=1}^{\infty}\sum_{s=0}^{r-1}(-1)^{p-r+s}\sum_{t=s}^{r-1}\binom{t}{s}\binom{m+p-1-r+t}{m-1}\binom{m+p+s}{r}\frac{(x-a)^{m+p+s}}{\Gamma(1+m+p+s)}e^a$$

$$+\frac{(-1)^m}{B(p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_a^x\!\sim\!\int_a^t\frac{(t-x)^{m+k}}{(m+k)!}e^t dt^p$$

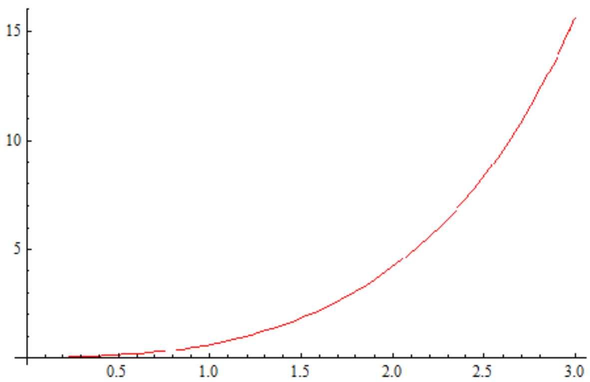
Although the zero of the lineal super integral of e^x is $a=-\infty$, these expressions can not have this zero. That is, by the above formulas, the lineal super integral of e^x can not be expanded to the series. Then, we put $a=0$ and expand the collateral super integral of e^x to the series. When $p=2.1$, if these are illustrated, it is as follows. The left side is calculated by Riemann-Liouville Integral. Both sides overlap exactly and blue (left) can not be seen.

```
a = 0; p = 2.1; m = 20;
```

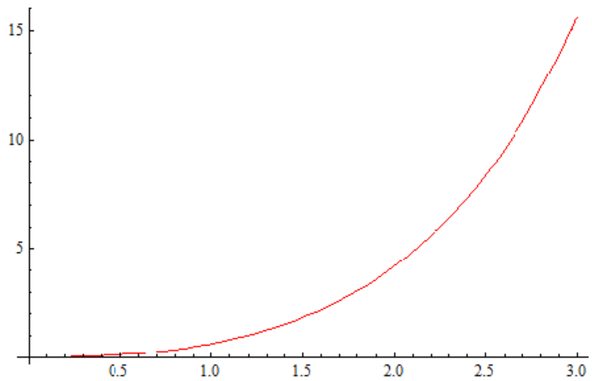
$$\text{f1[x\_] := }\frac{1}{\text{Gamma[p]}}\int_0^x (x-t)^{p-1}\,e^t\,dt$$

$$\text{fr1[x\_] := }e^x\sum_{r=0}^{m-1}\text{Binomial[-p, r]}\frac{(x-a)^{p+r}}{\text{Gamma[1 + p + r]}} \qquad \text{fr2[x\_] := }e^a\sum_{r=0}^{m-1}\frac{(x-a)^{p+r}}{\text{Gamma[1 + p + r]}}$$

```
Plot[{f1[x], fr1[x]}, {x, 0, 3},
 PlotStyle → {RGBColor[0, 0, 1], RGBColor[1, 0, 0]}]
```




```
Plot[{f1[x], fr2[x]}, {x, 0, 3},
 PlotStyle → {RGBColor[0, 0, 1], RGBColor[1, 0, 0]}]
```



Example2 Series expansion of the 2.9th order integral of $\log x$

Let $f(x)=\log x$, then

$$f^{(0)}=\log x \quad , \quad f^{(r)}=(-1)^{r-1}\frac{(r-1)!}{x^r} \qquad r=1,2,3,\cdots$$

Substituting these for Theorem 17.7.1 ,

$$\int_a^x \!\! \sim \!\! \int_a^x \log x \, dx^p = \frac{(x-a)^p}{\Gamma(1+p)} \log x - \sum_{r=1}^{m-1} (-1)^r \binom{-p}{r} \frac{(x-a)^{p+r}}{\Gamma(1+p+r)} \frac{(r-1)!}{x^r} + R_m^p$$

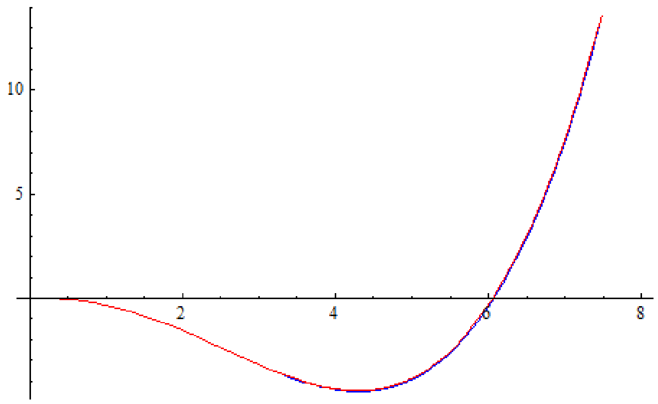
$$\int_a^x \!\! \sim \!\! \int_a^x \log x \, dx^p = \frac{(x-a)^p}{\Gamma(1+p)} \log a - \sum_{r=1}^{m-1} (-1)^r \frac{(x-a)^{p+r}}{\Gamma(1+p+r)} \frac{(r-1)!}{a^r} + R_m^p$$

If $a=0$ then the first formula is lineal super integral, else it is collateral super integral. On the other hand, the second formula can not be lineal super integral.

Let us calculate the left side by the direct integration and compare it with the series by the first formula. Then it is as follows. Although the convergence of this series is slow, as a result of taking very large m, both sids overlapped somehow.

```
a = 0; m = 1000;
fl[p_] := (Log[x] - PolyGamma[1 + p] - EulerGamma) / Gamma[1 + p] x^p
fr[p_] := (x - a)^p / Gamma[1 + p] Log[x] - Sum[(-1)^r Binomial[-p, r] (x - a)^(p+r) / Gamma[1 + p + r] (r - 1)! / x^r, {r, 1, m-1}]
```

```
Plot[{fl[2.9], fr[2.9]}, {x, 0, 8},
 PlotStyle → {RGBColor[0, 0, 1], RGBColor[1, 0, 0]}]
```



2010.07.14

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Alien's Mathematics