21 Taylor Expansion by Real & Imaginary Parts around a Complex Number

In " 14 Taylor Expansion by Real Part & Imaginary Part ", the complex functions were expanded into Taylor series around a real number. In this chapter, we expand complex functions into Taylor series around a complex number. Furthermore, as a special case of this, the complex function is expanded into Taylor series for each real part and imaginary part on the vertical line in the complex plane.

Formulas to use

" **14 Taylor Expansion by Real Part & Imaginary Part** " **Formula 14.1.2** . This reprint is as follows.

Formula 14.1.2 (Reprint)

Suppose that a complex function $f(z)$ $(z=x+iy)$ is expanded around a real number a into a Taylor series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a)\frac{(z-a)^s}{s!}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$.

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$.

21.1 Taylor Expansion by Real & Imaginary Parts around a Complex Number

21.1.1 Taylor Expansion around a Complex Number

Formula 14.1.2 can be generalized as follows.

Theorem 21.1.1

Let $f(z)$ $(z=x+iy)$ be a complex function, $u(x,y)$, $v(x,y)$ are the real & imaginary parts. Further, let Re , Im are symbols representing the real & imaginary parts. Then, if $f(z)$ is holomorphic in the whole domain D , the following expressions hold for arbitrary point $(a,b) \in D$.

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a+ib)\frac{\{z-(a+ib)\}^s}{s!} \tag{1.1}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{Re\left[f^{(2r+s)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!} - Im\left[f^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{Im\left[f^{(2r+s)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!} + Re\left[f^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Where, $0^0 = 1$.

Proof

From (1.1),

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a+ib) \frac{\{x-a+i(y-b)\}^s}{s!}$$

Put $X = x-a$, $Y = y-b$. Then, this is expressed as follows.

$$f(z) = \sum_{r=0}^{\infty} f^{(s)}(a+ib) \frac{(X+iY)^s}{s!}$$

$$= \sum_{s=0}^{\infty} Re\left[f^{(s)}(a+ib)\right] \frac{(X+iY)^s}{s!} + i \sum_{s=0}^{\infty} Im\left[f^{(s)}(a+ib)\right] \frac{(X+iY)^s}{s!}$$

Applying Formula 14.1.2 to the first term on the right side,

$$\sum_{s=0}^{\infty} Re\left[f^{(s)}(a+ib)\right] \frac{(X+iY)^s}{s!} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} Re\left[f^{(2r+s)}(a+ib)\right] \frac{X^s}{s!} \frac{(-1)^r Y^{2r}}{(2r)!}$$

$$+ i \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} Re\left[f^{(2r+s+1)}(a+ib)\right] \frac{X^s}{s!} \frac{(-1)^r Y^{2r+1}}{(2r+1)!}$$

Applying Formula 14.1.2 to the second term on the right side,

$$i \sum_{s=0}^{\infty} Im\left[f^{(s)}(a+ib)\right] \frac{(X+iY)^s}{s!} = i \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} Im\left[f^{(2r+s)}(a+ib)\right] \frac{X^s}{s!} \frac{(-1)^r Y^{2r}}{(2r)!}$$

$$+ i^2 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} Im\left[f^{(2r+s+1)}(a+ib)\right] \frac{X^s}{s!} \frac{(-1)^r Y^{2r+1}}{(2r+1)!}$$

Therefore,

$$f(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} Re\left[f^{(2r+s)}(a+ib)\right] \frac{X^s}{s!} \frac{(-1)^r Y^{2r}}{(2r)!}$$

$$- \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} Im\left[f^{(2r+s+1)}(a+ib)\right] \frac{X^s}{s!} \frac{(-1)^r Y^{2r+1}}{(2r+1)!}$$

$$+ i \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} Re\left[f^{(2r+s+1)}(a+ib)\right] \frac{X^s}{s!} \frac{(-1)^r Y^{2r+1}}{(2r+1)!}$$

$$+ i \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} Im\left[f^{(2r+s)}(a+ib)\right] \frac{X^s}{s!} \frac{(-1)^r Y^{2r}}{(2r)!}$$

i.e.

$$f(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^r \frac{X^s}{s!} \left\{ Re\left[f^{(2r+s)}(a+ib)\right] \frac{Y^{2r}}{(2r)!} - Im\left[f^{(2r+s+1)}(a+ib)\right] \frac{Y^{2r+1}}{(2r+1)!} \right\}$$

$$+ i \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^r \frac{X^s}{s!} \left\{ Im\left[f^{(2r+s)}(a+ib)\right] \frac{Y^{2r}}{(2r)!} + Re\left[f^{(2r+s+1)}(a+ib)\right] \frac{Y^{2r+1}}{(2r+1)!} \right\}$$

Returning X, Y to the original symbols,

$$f(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(x-a)^s}{s!} \left\{ Re\left[f^{(2r+s)}(a+ib)\right] \frac{(-1)^r (y-b)^{2r}}{(2r)!} \right.$$

$$- Im\left[f^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\Bigg\}$$

$$+i\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\Bigg\{Im\left[f^{(2r+s)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!}$$

$$+ Re\left[f^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\Bigg\}$$

Replacing the real and imaginary parts with $u(x,y), v(x,y)$, we obtain the desired expressions. Q.E.D.

Note

When $f^{(s)}(a)$ $(s=0,1,2,\cdots)$ are real numbers, this theorem reduce to Formula 14.1.2 .

21.1.2 Taylor Expansion on the Vertical Line

Putting $x=a$ in Theorem 21.1.1 , we obtain the following theorem.

Theorem 21.1.2

Let $f(z)$ $(z=x+iy)$ be a complex function, $u(x,y)$, $v(x,y)$ are the real & imaginary parts. Further, let Re , Im are symbols representing the real & imaginary parts. Then, if $f(z)$ is holomorphic in the whole domain D , the following expressions hold for arbitrary point $(a,b)\in D$.

$$f(a+iy) = \sum_{s=0}^{\infty}\frac{f^{(s)}(a+ib)}{s!}\{(a-a)+i(y-b)\}^s \tag{1.2}$$

$$u(a,y) = \sum_{r=0}^{\infty}(-1)^r\left\{Re\left[\frac{f^{(2r)}(a+ib)}{(2r)!}\right](y-b)^{2r} - Im\left[\frac{f^{(2r+1)}(a+ib)}{(2r+1)!}\right](y-b)^{2r+1}\right\}$$

$$v(a,y) = \sum_{r=0}^{\infty}(-1)^r\left\{Im\left[\frac{f^{(2r)}(a+ib)}{(2r)!}\right](y-b)^{2r} + Re\left[\frac{f^{(2r+1)}(a+ib)}{(2r+1)!}\right](y-b)^{2r+1}\right\}$$

Where, $0^0=1$.

Proof

Putting $x=a$ in Theorem 21.1.1 ,

$$f(a+iy) = \sum_{s=0}^{\infty}f^{(s)}(a+ib)\frac{\{a+iy-(a+ib)\}^s}{s!}$$

$$u(a,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{0^s}{s!}\Bigg\{Re\left[f^{(2r+s)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!}$$

$$- Im\left[f^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\Bigg\}$$

$$v(a,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{0^s}{s!}\Bigg\{Im\left[f^{(2r+s)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!}$$

$$+ Re\left[f^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\Bigg\}$$

Here, let us abbreviate the symbols in { } of u as follows

$$Re\left[f^{(2r+s)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!}=R_{2r+s}$$

$$Im\left[f^{(2r+s+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}=I_{2r+s+1}$$

Then,

$$u(a,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{0^s}{s!}\left(R_{2r+s}-I_{2r+s+1}\right)$$

$$=\sum_{r=0}^{\infty}\frac{0^0}{0!}\left(R_{2r+0}-I_{2r+0+1}\right)+\sum_{r=0}^{\infty}\sum_{s=1}^{\infty}\frac{0^s}{s!}\left(R_{2r+s}-I_{2r+s+1}\right)$$

Since, $0^0=1,\ 0^n=0\ (n=1,2,3,\cdots)$,

$$u(a,y)=\sum_{r=0}^{\infty}\left(R_{2r+0}-I_{2r+0+1}\right)$$

Returning R_{2r}, I_{2r+1} to the original symbols,

$$u(a,y)=\sum_{r=0}^{\infty}\left\{Re\left[f^{(2r)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!}-Im\left[f^{(2r+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

In a similar way,

$$v(a,y)=\sum_{r=0}^{\infty}\left\{Im\left[f^{(2r)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!}+Re\left[f^{(2r+1)}(a+ib)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

With a slight rewrite of these, we obtain the desired expressions.

Remark

(1.2) is the Taylor series around b of the function $f(a+iy)$ on the vertical line $x=a$ in the domain D. And $u(a,y), v(a,y)$ are sub-series of this, the former is the real part and the latter is the imaginary part. Graphically speaking, $u(a,y), v(a,y)$ are 2D figures in which the 3D figures of $u(x,y), v(x,y)$ in Theorem 21.1.1 are cut at $x=a$. Though these can be expressed only on the vertical line, the entire domain D can be expressed by changing a parametrically. In this way, the domain of $u(a,y), v(a,y)$ is narrowed but at the cost of this, they are represented by a simpler single series.

Method of calculation

In the following sections, some numerical examples will be calculated, but they are difficult to calculate by hand. So,

(1) Formula manipulation software ***Mathematica*** is used for calculation and drawing.

(2) Functions $Re[z]$, $Im[z]$ of ***Mathematica*** are used to calculate $Re\left[f^{(s)}(z)\right]$, $Im\left[f^{(s)}(z)\right]$.

(3) $0^0 = 1$ is assumed. So the following options are specified prior to calculation and drawing in ***Mathematica***.

Unprotect [Power] ; Power [0,0] = 1 ;

21.2 Example 1: sin z

As examples of Taylor expansion by real and imaginary parts around complex numbers, this section takes up $\sin z$. This is so-called entire function that has no singularity.

21.2.1 Expansion around $a+bi$

$$\sin z = \sum_{s=0}^{\infty} \sin\left(a+ib+\frac{s\pi}{2}\right)\frac{\{z-(a+ib)\}^s}{s!}$$

$$u(x,y) = \sum_{s=0}^{\infty} \sin\left(a+\frac{s\pi}{2}\right)\frac{(x-a)^s}{s!}\cdot\sum_{r=0}^{\infty}\left\{\cosh b\frac{(y-b)^{2r}}{(2r)!}+\sinh b\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$v(x,y) = \sum_{s=0}^{\infty} \cos\left(a+\frac{s\pi}{2}\right)\frac{(x-a)^s}{s!}\cdot\sum_{r=0}^{\infty}\left\{\sinh b\frac{(y-b)^{2r}}{(2r)!}+\cosh b\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Where, $0^0 = 1$.

Derivation

Higher order derivative of $f(z) = \sin z$ is

$$f^{(s)}(z) = \sin\left(z+\frac{s\pi}{2}\right)$$

From this,

$$f^{(s)}(a+ib) = \sin\left(a+ib+\frac{s\pi}{2}\right)$$

$$f^{(2r+s)}(a+ib) = \sin\left(a+ib+\frac{(2r+s)\pi}{2}\right)$$

$$f^{(2r+s+1)}(a+ib) = \sin\left(a+ib+\frac{(2r+s+1)\pi}{2}\right)$$

Substituting these for Theorem 21.1.1 ,

$$\sin z = \sum_{s=0}^{\infty} \sin\left(a+ib+\frac{s\pi}{2}\right)\frac{\{z-(a+ib)\}^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{Re\left[\sin\left(a+ib+\frac{(2r+s)\pi}{2}\right)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!}\right.$$
$$\left.-Im\left[\sin\left(a+ib+\frac{(2r+s+1)\pi}{2}\right)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{Im\left[\sin\left(a+ib+\frac{(2r+s)\pi}{2}\right)\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!}\right.$$
$$\left.+Re\left[\sin\left(a+ib+\frac{(2r+s+1)\pi}{2}\right)\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Although these can be left as they are,

$$\sin\left(a+ib+\frac{(2r+s)\pi}{2}\right) = \sin\left(a+\frac{(2r+s)\pi}{2}\right)\cosh b$$
$$+\, i\cos\left(a+\frac{(2r+s)\pi}{2}\right)\sinh b$$

$$\sin\left(a+ib+\frac{(2r+s+1)\pi}{2}\right) = \sin\left(a+\frac{(2r+s+1)\pi}{2}\right)\cosh b + i\cos\left(a+\frac{(2r+s+1)\pi}{2}\right)\sinh b$$

Therefore,

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{\sin\left(a+\frac{(2r+s)\pi}{2}\right)\cosh b\,\frac{(-1)^r(y-b)^{2r}}{(2r)!} - \cos\left(a+\frac{(2r+s+1)\pi}{2}\right)\sinh b\,\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{\cos\left(a+\frac{(2r+s)\pi}{2}\right)\sinh b\,\frac{(-1)^r(y-b)^{2r}}{(2r)!} + \sin\left(a+\frac{(2r+s+1)\pi}{2}\right)\cosh b\,\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Further ,

$$\sin\left(a+\frac{(2r+s)\pi}{2}\right) = (-1)^r\sin\left(a+\frac{s\pi}{2}\right)$$

$$\cos\left(a+\frac{(2r+s+1)\pi}{2}\right) = (-1)^r\cos\left(a+\frac{(s+1)\pi}{2}\right) = -(-1)^r\sin\left(a+\frac{s\pi}{2}\right)$$

So,

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{(-1)^r\sin\left(a+\frac{s\pi}{2}\right)\cosh b\,\frac{(-1)^r(y-b)^{2r}}{(2r)!} + (-1)^r\sin\left(a+\frac{s\pi}{2}\right)\sinh b\,\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sin\left(a+\frac{s\pi}{2}\right)\frac{(x-a)^s}{s!}\left\{\cosh b\,\frac{(y-b)^{2r}}{(2r)!} + \sinh b\,\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

i.e.

$$u(x,y) = \sum_{s=0}^{\infty}\sin\left(a+\frac{s\pi}{2}\right)\frac{(x-a)^s}{s!}\cdot\sum_{r=0}^{\infty}\left\{\cosh b\,\frac{(y-b)^{2r}}{(2r)!} + \sinh b\,\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Furthermore ,

$$\sin\left(a+\frac{(2r+s+1)\pi}{2}\right) = (-1)^r\sin\left(a+\frac{(s+1)\pi}{2}\right) = (-1)^r\cos\left(a+\frac{s\pi}{2}\right)$$

$$\cos\left(a+\frac{(2r+s)\pi}{2}\right) = (-1)^r\cos\left(a+\frac{s\pi}{2}\right)$$

So,

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{(-1)^r\cos\left(a+\frac{s\pi}{2}\right)\sinh b\,\frac{(-1)^r(y-b)^{2r}}{(2r)!} + (-1)^r\cos\left(a+\frac{s\pi}{2}\right)\cosh b\,\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

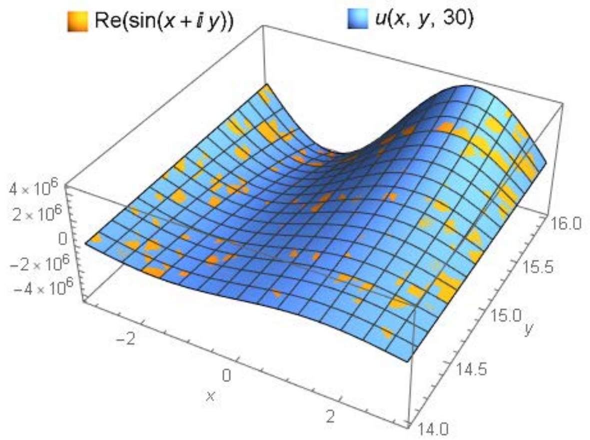
$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \cos\left(a+\frac{s\pi}{2}\right)\frac{(x-a)^s}{s!}\left\{\sinh b\,\frac{(y-b)^{2r}}{(2r)!}+\cosh b\,\frac{(y-b)^{2r+1}}{(2r+)!}\right\}$$

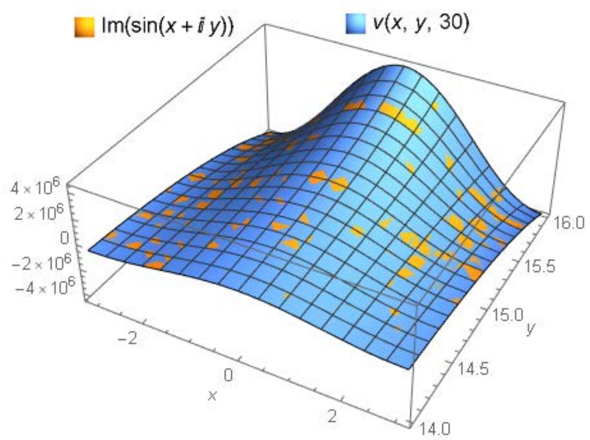
i.e.

$$v(x,y) = \sum_{s=0}^{\infty} \cos\left(a+\frac{s\pi}{2}\right)\frac{(x-a)^s}{s!}\cdot\sum_{r=0}^{\infty}\left\{\sinh b\,\frac{(y-b)^{2r}}{(2r)!}+\cosh b\,\frac{(y-b)^{2r+1}}{(2r+)!}\right\}$$

Expansion around $-1+i\,15$

When $a=-1\,,b=15$, the 3D figures of $u(x,y),v(x,y)$ are as follows. The left is $u(x,y)$ and the right is $v(x,y)$. In both figures, orange is a function and blue is a series. Both are overlapped exactly.





Note

These reduce to the followings respectively.

$$u(x,y) = \sin x\cdot\{\cosh b\,\cosh(y-b)+\sinh b\,\sinh(y-b)\} = \sin x\cdot\cosh y$$

$$v(x,y) = \cos x\cdot\{\sinh b\,\cosh(y-b)+\cosh b\,\sinh(y-b)\} = \cos x\cdot\sinh y$$

$$\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$$

21.2.2 Expansion around b on the vertical line $x=a$

Substituting $x=a$ for 21.2.1 ,

$$\sin(a+iy) = \sum_{s=0}^{\infty}\sin\left(a+ib+\frac{s\pi}{2}\right)\frac{\{(a-a)+i(y-b)\}^s}{s!}$$

$$u(a,y) = \sin a\sum_{r=0}^{\infty}\left\{\cosh b\,\frac{(y-b)^{2r}}{(2r)!}+\sinh b\,\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

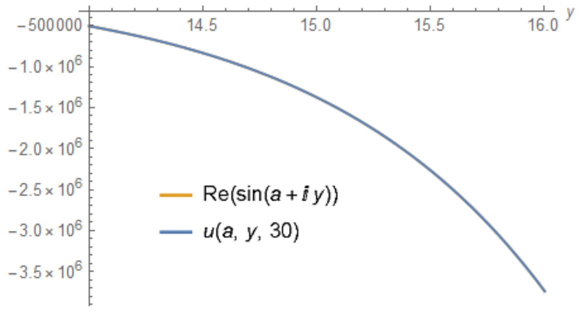
$$v(a,y) = \cos a\sum_{r=0}^{\infty}\left\{\sinh b\,\frac{(y-b)^{2r}}{(2r)!}+\cosh b\,\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

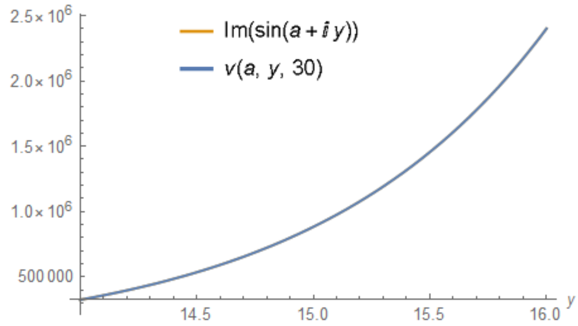
Where, $0^0=1$.

These are Taylor expansion around b on the vertical line $x=a$. Here, $u(a,y),v(a,y)$ are single series.

Expansion around 15 on the vertical line $x=-1$

When $a=-1\,,b=15$, the 2D figures of $u(a,y),v(a,y)$ are as follows. The left is $u(a,y)$ and the right is $v(a,y)$. In both figures, orange is a function and blue is a series. Both are overlapped exactly.





These are cut views at $x = -1$ in the 3D figures above.

21.3 Example 2: log z

In this section, $\log z$ is taken as an example of a function with a singularity. This is a bit complicated.

21.3.1 Expansion around $a+bi$

$$\log z = \log(a+ib) + \sum_{s=1}^{\infty}(-1)^{s-1}\frac{(s-1)!}{(a+ib)^s}\frac{\{z-(a+ib)\}^s}{s!}$$

$$u(x,y) = Re[\log(a+ib)] - Im\left[\frac{1}{a+ib}\right](y-b)$$

$$-\sum_{s=1}^{\infty}\frac{(-1)^s(x-a)^s}{s!}\left\{Re\left[\frac{(s-1)!}{(a+ib)^s}\right] + Im\left[\frac{s!}{(a+ib)^{s+1}}\right](y-b)\right\}$$

$$-\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s(x-a)^s}{s!}\left\{Re\left[\frac{(2r+s-1)!}{(a+ib)^{2r+s}}\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!}\right.$$

$$\left. + Im\left[\frac{(2r+s)!}{(a+ib)^{2r+s+1}}\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$v(x,y) = Im[\log(a+ib)] + Re\left[\frac{1}{a+ib}\right](y-b)$$

$$-\sum_{s=1}^{\infty}\frac{(-1)^s(x-a)^s}{s!}\left\{Im\left[\frac{(s-1)!}{(a+ib)^s}\right] - Re\left[\frac{s!}{(a+ib)^{s+1}}\right](y-b)\right\}$$

$$-\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s(x-a)^s}{s!}\left\{Im\left[\frac{(2r+s-1)!}{(a+ib)^{2r+s}}\right]\frac{(-1)^r(y-b)^{2r}}{(2r)!}\right.$$

$$\left. - Re\left[\frac{(2r+s)!}{(a+ib)^{2r+s+1}}\right]\frac{(-1)^r(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Where, $0^0 = 1$. $\quad |z| < \diamond$

Derivation

Higher order derivative of $f(z) = \log z$ is

$$f^{(s)}(z) = (-1)^{s-1}\frac{(s-1)!}{z^s} \qquad s=1,2,3,\cdots$$

From this,

$$f^{(s)}(a+ib) = (-1)^{s-1}\frac{(s-1)!}{(a+ib)^s} \qquad s=1,2,3,\cdots$$

$$f^{(2r+s)}(a+ib) = (-1)^{2r+s-1}\frac{(2r+s-1)!}{(a+ib)^{2r+s}} \qquad s=1,2,3,\cdots$$

$$f^{(2r+s+1)}(a+ib) = (-1)^{2r+s}\frac{(2r+s)!}{(a+ib)^{2r+s+1}} \qquad s=1,2,3,\cdots$$

Substituting these for Theorem 21.1.1 ,

$$\log z = \log(a+ib) + \sum_{s=1}^{\infty}(-1)^{s-1}\frac{(s-1)!}{(a+ib)^s}\frac{\{z-(a+ib)\}^s}{s!}$$

Here, formally,

$$log(a+ib) = (-1)^{-1}\frac{(-1)!}{(a+ib)^{0}}$$

This is used later.

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^{s}}{s!}\left\{Re\left[(-1)^{s-1}\frac{(2r+s-1)!}{(a+ib)^{2r+s}}\right]\frac{(-1)^{r}(y-b)^{2r}}{(2r)!}\right.$$

$$\left. - Im\left[(-1)^{s}\frac{(2r+s)!}{(a+ib)^{2r+s+1}}\right]\frac{(-1)^{r}(y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$= \frac{(x-a)^{0}}{0!}\left\{Re\left[(-1)^{-1}\frac{(-1)!}{(a+ib)^{0}}\right]\frac{(-1)^{0}(y-b)^{0}}{0!}\right.$$

$$\left. - Im\left[(-1)^{0}\frac{0!}{(a+ib)^{1}}\right]\frac{(-1)^{0}(y-b)^{1}}{1!}\right\}$$

$$+ \sum_{s=1}^{\infty}\frac{(x-a)^{s}}{s!}\left\{Re\left[(-1)^{s-1}\frac{(s-1)!}{(a+ib)^{s}}\right]\frac{(-1)^{0}(y-b)^{0}}{0!}\right.$$

$$\left. - Im\left[(-1)^{s}\frac{s!}{(a+ib)^{s+1}}\right]\frac{(-1)^{0}(y-b)^{1}}{1!}\right\}$$

$$+ \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^{s}}{s!}\left\{Re\left[(-1)^{s-1}\frac{(2r+s-1)!}{(a+ib)^{2r+s}}\right]\frac{(-1)^{r}(y-b)^{2r}}{(2r)!}\right.$$

$$\left. - Im\left[(-1)^{s}\frac{(2r+s)!}{(a+ib)^{2r+s+1}}\right]\frac{(-1)^{r}(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Here, replacing $(-1)^{-1}\frac{(-1)!}{(a+ib)^{0}}$ with $log(a+ib)$,

$$u(x,y) = Re[log(a+ib)] - Im\left[\frac{1}{a+ib}\right]\frac{(y-b)^{1}}{1!}$$

$$+ \sum_{s=1}^{\infty}\frac{(x-a)^{s}}{s!}\left\{Re\left[(-1)^{s-1}\frac{(s-1)!}{(a+ib)^{s}}\right] - Im\left[(-1)^{s}\frac{s!}{(a+ib)^{s+1}}\right]\frac{(y-b)^{1}}{1!}\right\}$$

$$+ \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^{s}}{s!}\left\{Re\left[(-1)^{s-1}\frac{(2r+s-1)!}{(a+ib)^{2r+s}}\right]\frac{(-1)^{r}(y-b)^{2r}}{(2r)!}\right.$$

$$\left. - Im\left[(-1)^{s}\frac{(2r+s)!}{(a+ib)^{2r+s+1}}\right]\frac{(-1)^{r}(y-b)^{2r+1}}{(2r+1)!}\right\}$$

i.e.

$$u(x,y) = Re[log(a+ib)] - Im\left[\frac{1}{a+ib}\right](y-b)$$

$$- \sum_{s=1}^{\infty}\frac{(-1)^{s}(x-a)^{s}}{s!}\left\{Re\left[\frac{(s-1)!}{(a+ib)^{s}}\right] + Im\left[\frac{s!}{(a+ib)^{s+1}}\right](y-b)\right\}$$

$$- \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s (x-a)^s}{s!}\left\{ Re\left[\frac{(2r+s-1)!}{(a+ib)^{2r+s}}\right]\frac{(-1)^r (y-b)^{2r}}{(2r)!} + Im\left[\frac{(2r+s)!}{(a+ib)^{2r+s+1}}\right]\frac{(-1)^r (y-b)^{2r+1}}{(2r+1)!}\right\}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{ Re\left[(-1)^{2r+s}\frac{(2r+s)!}{(a+ib)^{2r+s+1}}\right]\frac{(-1)^r (y-b)^{2r+1}}{(2r+1)!} + Im\left[(-1)^{2r+s-1}\frac{(2r+s-1)!}{(a+ib)^{2r+s}}\right]\frac{(-1)^r (y-b)^{2r}}{(2r)!}\right\}$$

$$= \frac{(x-a)^0}{0!}\left\{ Re\left[(-1)^0\frac{0!}{(a+ib)^1}\right]\frac{(-1)^0 (y-b)^1}{1!} + Im\left[(-1)^{-1}\frac{(-1)!}{(a+ib)^0}\right]\frac{(-1)^0 (y-b)^0}{0!}\right\}$$

$$+ \sum_{s=1}^{\infty}\frac{(x-a)^s}{s!}\left\{ Re\left[(-1)^s\frac{s!}{(a+ib)^{s+1}}\right]\frac{(-1)^0 (y-b)^1}{1!} + Im\left[(-1)^{s-1}\frac{(s-1)!}{(a+ib)^s}\right]\frac{(-1)^0 (y-b)^0}{0!}\right\}$$

$$+ \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{ Re\left[(-1)^{2r+s}\frac{(2r+s)!}{(a+ib)^{2r+s+1}}\right]\frac{(-1)^r (y-b)^{2r+1}}{(2r+1)!} + Im\left[(-1)^{2r+s-1}\frac{(2r+s-1)!}{(a+ib)^{2r+s}}\right]\frac{(-1)^r (y-b)^{2r}}{(2r)!}\right\}$$

$$= Re\left[\frac{1}{a+ib}\right](y-b) + Im[\log(a+ib)]$$

$$+ \sum_{s=1}^{\infty}\frac{(x-a)^s}{s!}\left\{ Re\left[(-1)^s\frac{s!}{(a+ib)^{s+1}}\right]\frac{(-1)^0 (y-b)^1}{1!} + Im\left[(-1)^{s-1}\frac{(s-1)!}{(a+ib)^s}\right]\frac{(-1)^0 (y-b)^0}{0!}\right\}$$

$$+ \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^s}{s!}\left\{ Re\left[(-1)^{2r+s}\frac{(2r+s)!}{(a+ib)^{2r+s+1}}\right]\frac{(-1)^r (y-b)^{2r+1}}{(2r+1)!} + Im\left[(-1)^{2r+s-1}\frac{(2r+s-1)!}{(a+ib)^{2r+s}}\right]\frac{(-1)^r (y-b)^{2r}}{(2r)!}\right\}$$

i.e.

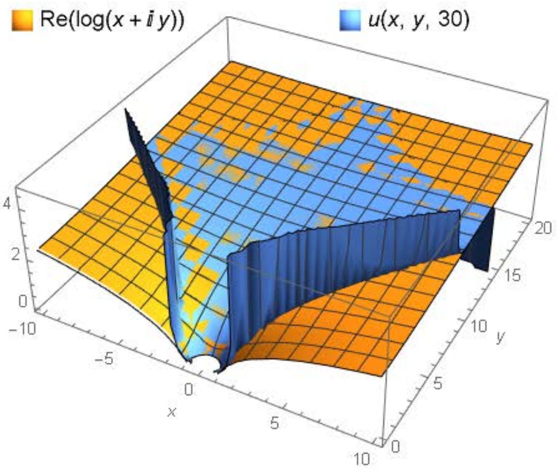
$$v(x,y) = Im[\log(a+ib)] + Re\left[\frac{1}{a+ib}\right](y-b)$$

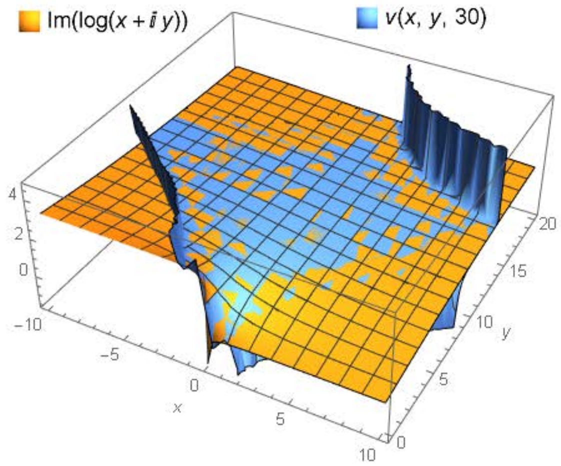
$$- \sum_{s=1}^{\infty}\frac{(-1)^s (x-a)^s}{s!}\left\{ Im\left[\frac{(s-1)!}{(a+ib)^s}\right] - Re\left[\frac{s!}{(a+ib)^{s+1}}\right](y-b)\right\}$$

$$- \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (x-a)^s}{s!} \left\{ Im\left[\frac{(2r+s-1)!}{(a+ib)^{2r+s}} \right] \frac{(-1)^r (y-b)^{2r}}{(2r)!} - Re\left[\frac{(2r+s)!}{(a+ib)^{2r+s+1}} \right] \frac{(-1)^r (y-b)^{2r+1}}{(2r+1)!} \right\}$$

Expansion around $0+i\,10$

When $a=0$, $b=10$, the 3D figures of $u(x,y), v(x,y)$ are as follows. The left is $u(x,y)$ and the right is $v(x,y)$. In both figures, orange is a function and blue is a series.





The origin $(0,0)$ is the singularity of this function. Therefore, these convergent areas are inside a square inscribed in the circle with a radius of 10 centered on the point $(0,10)$. Though $b=10$ in this figure, as b becomes larger, the square extends to the left, right and upward.

Verification

Taylor series are expanded up to 4 terms around $0+10i$. And, the function values and the series values at $1+9i$ are compared. The results are as follows.

```
Clear[a, b]; a = 0; b = 10;
N[{Re[Log[1 + ⅈ 9]], u[1, 9, 4]}]        N[{Im[Log[1 + ⅈ 9]], v[1, 9, 4]}]
          {2.20336, 2.20336}                          {1.46014, 1.46014}
```

The function value and the series value are completely the same in the calculation of only 4 terms. Considering that the center point of expansion and the destination point are close to each other, this convergence speed is surprising as a series of logarithmic function. Moreover, it is expected that the convergence speed will increase as the center point and the destination point move upward.

21.3.2 Expansion around b on the vertical line $x=a$

Substituting $x=a$ for 21.3.1,

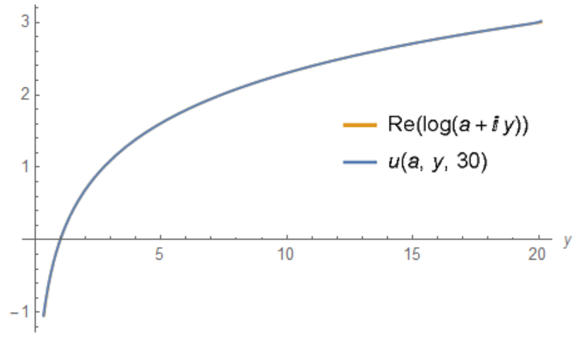
$$u(a,y) = Re[\log(a+ib)] - Im\left[\frac{1}{a+ib}\right](y-b)$$
$$- \sum_{r=1}^{\infty} (-1)^r \left\{ Re\left[\frac{(2r-1)!}{(a+ib)^{2r}} \right] \frac{(y-b)^{2r}}{(2r)!} + Im\left[\frac{(2r)!}{(a+ib)^{2r+1}} \right] \frac{(y-b)^{2r+1}}{(2r+1)!} \right\}$$

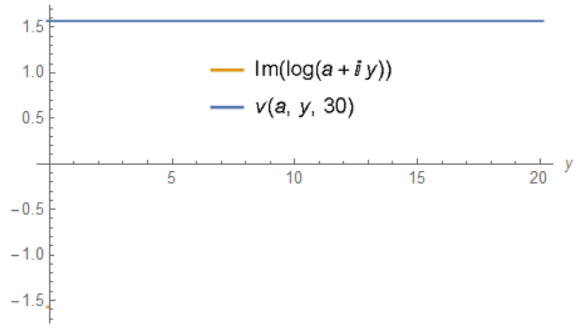
$$v(a,y) = Im[\log(a+ib)] + Re\left[\frac{1}{a+ib}\right](y-b)$$
$$- \sum_{r=1}^{\infty}(-1)^r\left\{ Im\left[\frac{(2r-1)!}{(a+ib)^{2r}}\right]\frac{(y-b)^{2r}}{(2r)!} - Re\left[\frac{(2r)!}{(a+ib)^{2r+1}}\right]\frac{(y-b)^{2r+1}}{(2r+1)!}\right\}$$

Where, $0^0 = 1$ $\quad |z-10| < 10$.

Expansion around 10 on the vertical line $x = 0$

When $a = 0$, $b = 10$, the 2D figures of $u(a,y), v(a,y)$ are as follows. The left is $u(a,y)$ and the right is $v(a,y)$. In both figures, orange is a function and blue is a series. Both are overlapped exactly.





21.4 Example 3: Dirichlet Eta Function

In this section, we take up Dirichlet Eta Function $\eta(z)$, which is defined by the following series.

$$\eta(z) = \sum_{s=1}^{\infty} (-1)^{s-1} e^{-z\log s} = \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + - \cdots$$

This function can be expanded to the Taylor series around any point on the complex plane. But of particular interest is the expansion on the critical line $x=1/2$ and on the boundary $x=1$ of the critical strip .

21.4.1 Expansion around $a+bi$

$$\eta(z) = \sum_{s=0}^{\infty} \eta^{(s)}(a+ib) \frac{\{z-(a+ib)\}^s}{s!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(x-a)^s}{s!} \left\{ Re\left[\eta^{(2r+s)}(a+ib)\right] \frac{(-1)^r (y-b)^{2r}}{(2r)!} - Im\left[\eta^{(2r+s+1)}(a+ib)\right] \frac{(-1)^r (y-b)^{2r+1}}{(2r+1)!} \right\}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(x-a)^s}{s!} \left\{ Im\left[\eta^{(2r+s)}(a+ib)\right] \frac{(-1)^r (y-b)^{2r}}{(2r)!} + Re\left[\eta^{(2r+s+1)}(a+ib)\right] \frac{(-1)^r (y-b)^{2r+1}}{(2r+1)!} \right\}$$

Where, $\eta^{(s)}(z) = \dfrac{z^{-s}}{\Gamma(1-s)} + (-1)^{-s} \displaystyle\sum_{k=2}^{\infty}\sum_{t=2}^{k} \frac{(-1)^{t-1}}{2^{k+1}} \binom{k}{t} \frac{\log^s t}{t^z}$, $0^0 = 1$.

Derivation

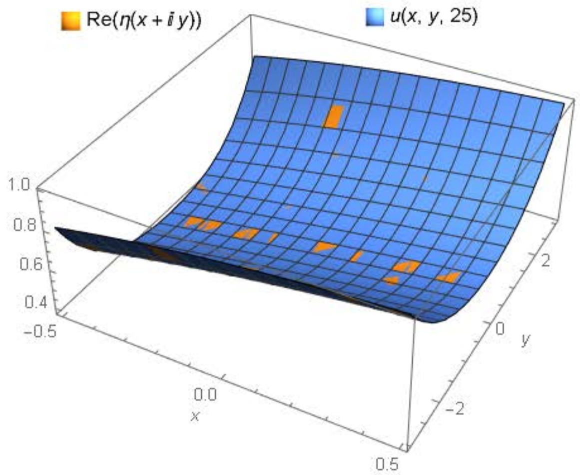
$\eta(z)$, $u(x,y)$, $v(x,y)$ are obtained immediately from Theorem.21.1.1 . $\eta^{(s)}(z)$ is obtained from **Formula 26.3.2h** in " **26 Higher and Super Calculus of Zeta Function etc** " (**SuperCalculus**)

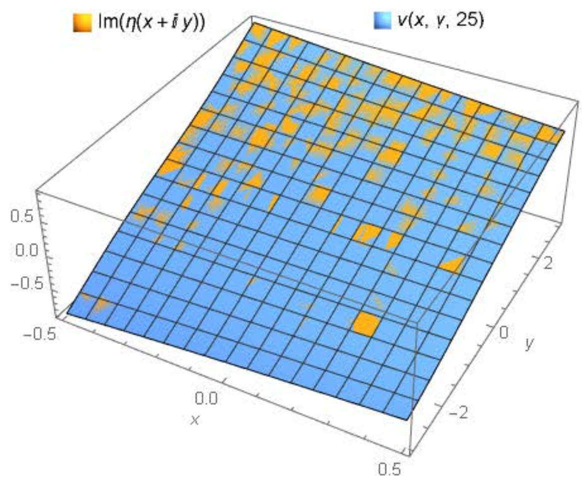
Expansion around $0.3+i\,2$

First, it is verification. Taylor series are expanded up to 25 terms around $0.3+i\,2$. And, the function values and the series values at $-0.3-i\,2$ are compared. The results are as follows. Both fully match.

```
Clear[a, b]; a = 0.3; b = 2;
N[{Re[DirichletEta[-0.3 - 2 i]], u[-0.3, -2, 25]}]
                    {0.574, 0.574}
N[{Im[DirichletEta[-0.3 - 2 i]], v[-0.3, -2, 25]}]
                    {-0.528098, -0.528098}
```

Next, we try to draw these. When $a=0.3$, $b=2$, the 3D figures of $u(x,y)$, $v(x,y)$ are as follows. The left is $u(x,y)$ and the right is $v(x,y)$. In both figures, orange is a function and blue is a series. Both are overlapped exactly.





21.4.2 Expansion around b on the vertical line $x = a$

Applying Theorem 21.1.2 for Dirichlet Eta Function $\eta(z)$,

$$\eta(a+iy) = \sum_{s=0}^{\infty} \frac{\eta^{(s)}(a+ib)}{s\,!}\{(a-a)+i(y-b)\}^s$$

$$u(a,y) = \sum_{r=0}^{\infty}(-1)^r\left\{Re\left[\frac{\eta^{(2r)}(a+ib)}{(2r)\,!}\right](y-b)^{2r} - Im\left[\frac{\eta^{(2r+1)}(a+ib)}{(2r+1)\,!}\right](y-b)^{2r+1}\right\}$$

$$v(a,y) = \sum_{r=0}^{\infty}(-1)^r\left\{Im\left[\frac{\eta^{(2r)}(a+ib)}{(2r)\,!}\right](y-b)^{2r} + Re\left[\frac{\eta^{(2r+1)}(a+ib)}{(2r+1)\,!}\right](y-b)^{2r+1}\right\}$$

Where, $0^0 = 1$.

The problem here is the high-order differential coefficient $\eta^{(s)}(a+ib)$. we used a half-double series to draw the above two 3D figures, but it took 80 minutes. In Taylor expansion around 10000 , drawing is almost impossible, aside from calculation. So, I devised the following expressions.

$$\eta(z) = \sum_{s=0}^{\infty} c_{ab}(s)\{x+iy-(a+ib)\}^s$$

$$u(a,y) = \sum_{r=0}^{\infty}(-1)^r\left\{Re[c_{ab}(2r)](y-b)^{2r} - Im[c_{ab}(2r+1)](y-b)^{2r+1}\right\}$$

$$v(a,y) = \sum_{r=0}^{\infty}(-1)^r\left\{Im[c_{ab}(2r)](y-b)^{2r} + Re[c_{ab}(2r+1)](y-b)^{2r+1}\right\}$$

Where, $0^0 = 1$.

These expressions are used in the following procedure. .

(i) Use formula manipulation to expand $\eta(z)$ to the Taylor series around $a+ib$.

(ii) The coefficients are stored in the array $c_{ab}(s)$ $s=0,1,2,\cdots,n$

(iii) The coefficients are taken out of the array and draw and calculate $u(a,y)$, $v(a,y)$.

Expansion around 10000 on the critical line $x = 1/2$

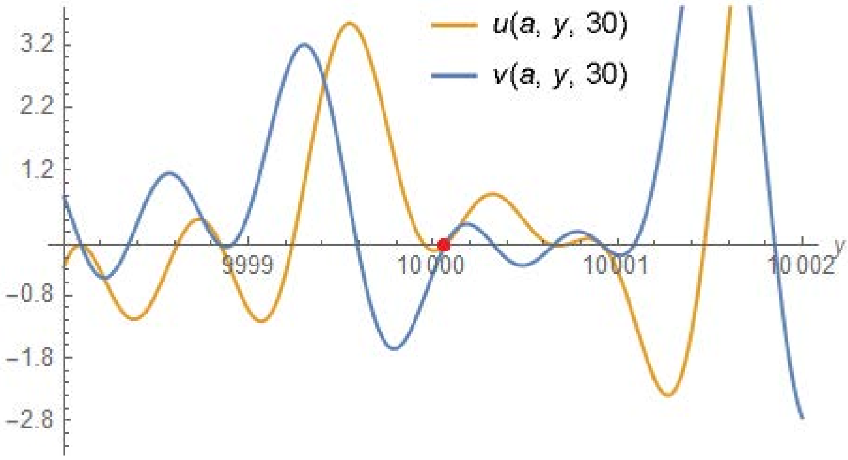
```
Array[cab, 101];
```

```
Clear[a, b]; a = 1 / 2; b = 10000;
N[Series[DirichletEta[z], {z, a + ⅈ b, 100}]];
Table[c_ab[n] = SeriesCoefficient[%, n], {n, 0, 100}];
```

$$u[a\_, y\_, m\_] := \sum_{r=0}^{m} (-1)^r \left(\mathrm{Re}[c_{ab}[2r]]\,(y-b)^{2r} - \mathrm{Im}[c_{ab}[2r+1]]\,(y-b)^{2r+1}\right)$$

$$v[a\_, y\_, m\_] := \sum_{r=0}^{m} (-1)^r \left(\mathrm{Im}[c_{ab}[2r]]\,(y-b)^{2r} + \mathrm{Re}[c_{ab}[2r+1]]\,(y-b)^{2r+1}\right)$$

```
Plot[{u[a, y, 30], v[a, y, 30]}, {y, 9998, 10002},
 AxesLabel → Automatic, PlotLegends → "Expressions", ClippingStyle → None,
 PlotRange → {-3.8, 3.8}, PlotStyle → {ColorData[97, 2], ColorData[97, 1]}]
```



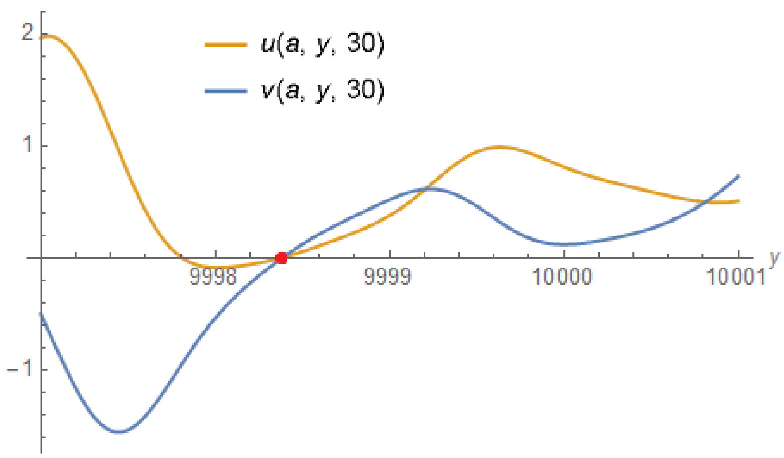
Five non-trivial zeros are observed in this interval, but the zeros near $y = 10000$ are as follows.

```
SetPrecision[FindRoot[u[a, y, 5], {y, 10000.1}], 15]
{y → 10000.0653454145}
SetPrecision[FindRoot[v[a, y, 5], {y, 10000.1}], 15]
{y → 10000.0653454145}
SetPrecision[Im[ZetaZero[10143]], 14]
10000.0653454145
```

The 10143 th non-trivial zero point is obtained by calculation of only 5 terms.

Expansion around 10000 on the boundary $x=1$ of the critical strip

When drawing in a similar way,



The zeros near $y = 9998$ are as follows.

```
SetPrecision[FindRoot[u[a, y, 27], {y, 9998.2}], 12]
{y → 9998.38647287}

SetPrecision[FindRoot[v[a, y, 27], {y, 9998.2}], 12]
{y → 9998.38647287}

N[2206 π / Log[2], 12]
9998.38647287
```

The 1103 th $\eta(z)$ -specific zero point is obtained by calculation of 27 terms.

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2022.04.15 Renewed

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