14 Taylor Expansion by Real Part & Imaginary Part

Abstract

When a complex function $f(z)$ $(z = x+iy)$ can be expanded into Taylor series with real coefficients around real number a , the real part $u(x,y)$ and imaginary part $v(x,y)$ can each be expanded into a double Taylor series. And, the followings hold.

(1) With respect to y , $u(x,y)$ is an even function and $v(x,y)$ is an odd function.

(2) If $f(z)$ is an odd function, with respect to x , $u(x,y)$ is an odd function and $v(x,y)$ is an even function.

(3) If $f(z)$ is an even function, with respect to x , $u(x,y)$ is an even function and $v(x,y)$ is an odd function.

(4) When the Taylor series of $f(z)$ has a convergence circle, convergence regions of $u(x,y)$ and $v(x,y)$ each become inclined squares (**convergence diamonds**) inscribed in this convergence circle.

14.1 Lemma and Formulas

First, we prepare an important lemma

Lemma 14.1.0

When x, y are real numbers and r is a non-negative integer, the following expressions hold.

$$(x+iy)^r = \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} (-1)^s \binom{r}{2s} x^{r-2s} y^{2s} + i \sum_{s=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^s \binom{r}{2s+1} x^{r-2s-1} y^{2s+1} \tag{1.0}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

Proof

$$(x+iy)^1 = \sum_{s=0}^{1} {}_1C_s\, x^{1-s} i^s y^s$$

$$= {}_1C_0\, x^{1-0} y^0 + i\left({}_1C_1\, x^{1-1} y^1\right)$$

$$= \sum_{s=0}^{0} (-1)^s {}_1C_{2s}\, x^{1-2s} y^{2s} + i \sum_{s=0}^{0} (-1)^s {}_1C_{2s+1}\, x^{1-2s-1} y^{2s+1} \qquad \left(0 = \left\lceil \frac{1-1}{2} \right\rceil,\ 0 = \left\lfloor \frac{1-1}{2} \right\rfloor \right)$$

$$(x+iy)^2 = \sum_{s=0}^{2} {}_2C_s\, x^{2-s} i^s y^s$$

$$= {}_2C_0\, x^{2-0} y^0 - {}_2C_2\, x^{2-2} y^2 + i\left({}_2C_1\, x^{2-1} y^1\right)$$

$$= \sum_{s=0}^{1} (-1)^s {}_2C_{2s}\, x^{2-2s} y^{2s} + i \sum_{s=0}^{0} (-1)^s {}_2C_{2s+1}\, x^{2-2s-1} y^{2s+1} \qquad \left(1 = \left\lceil \frac{2-1}{2} \right\rceil,\ 0 = \left\lfloor \frac{2-1}{2} \right\rfloor \right)$$

$$(x+iy)^3 = \sum_{s=0}^{3} {}_3C_s\, x^{3-s} i^s y^s$$

$$= {}_3C_0\, x^{3-0} y^0 - {}_3C_2\, x^{3-2} y^2 + i\left({}_3C_1\, x^{3-1} y^1 - {}_3C_3\, x^{3-3} y^3\right)$$

$$= \sum_{s=0}^{1} (-1)^s {}_3C_{2s}\, x^{3-2s} y^{2s} + i \sum_{s=0}^{1} (-1)^s {}_3C_{2s+1}\, x^{3-2s-1} y^{2s+1} \qquad \left(1 = \left\lceil \frac{3-1}{2} \right\rceil,\ 1 = \left\lfloor \frac{3-1}{2} \right\rfloor \right)$$

$$(x+iy)^4 = \sum_{s=0}^{4} {}_4C_s\, x^{4-s} i^s y^s$$

$$= {}_4C_0\, x^{4-0} y^0 - {}_4C_2\, x^{4-2} y^2 - {}_4C_4\, x^{4-4} y^4 + i\left({}_4C_1\, x^{4-1} y^1 - {}_4C_3\, x^{4-3} y^3\right)$$

$$= \sum_{s=0}^{2} (-1)^s {}_4C_{2s}\, x^{4-2s} y^{2s} + i\sum_{s=0}^{1} (-1)^s {}_4C_{2s+1}\, x^{4-2s-1} y^{2s+1} \qquad \left(\begin{array}{l} 2 = \left\lceil \frac{4-1}{2} \right\rceil \\ 1 = \left\lfloor \frac{4-1}{2} \right\rfloor \end{array} \right)$$

Hereafter by induction, we obtain the desired expression.

Using this Lemma, we can expand complex functions into power series by real part and imaginary part.

Formula 14.1.1

Suppose that a complex function $f(z)$ $(z = x+iy)$ is expanded around a real number a into a Taylor series with real coefficients as follows.

$$f(z) = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} (z-a)^r \tag{1.1}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$

$$u(x,y) = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s} \tag{1.1u}$$

$$v(x,y) = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} \sum_{s=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1} \tag{1.1v}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

Proof

Replacing x with $x-a$ in Lemma 14.1.0 ,

$$\{(x-a)+iy\}^r = \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s}$$
$$+ i \sum_{s=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1}$$

Substituting this for (1.1) ,

$$f(x,y) = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s}$$
$$+ i \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} \sum_{s=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1}$$

Describing the real and imaginary parts as $u(x,y), v(x,y)$ respectively, we obtain the desired expressions.

The first few terms

The first few terms of $f_r(x,y)$ are as follows.

$$
\begin{aligned}
f_r(x,y) = & \frac{f^{(0)}(a)}{0!}\left\{\binom{0}{0}(x-a)^{0-0}y^0\right\} \\
& + \frac{f^{(1)}(c)}{1!}\left\{\binom{1}{0}(x-a)^{1-0}y^0\right\} \\
& + \frac{f^{(2)}(c)}{2!}\left\{\binom{2}{0}(x-a)^{2-0}y^0 - \binom{2}{2}(x-a)^{2-2}y^2\right\} \\
& + \frac{f^{(3)}(c)}{3!}\left\{\binom{3}{0}(x-a)^{3-0}y^0 - \binom{3}{2}(x-a)^{3-2}y^2\right\} \\
& + \frac{f^{(4)}(c)}{4!}\left\{\binom{4}{0}(x-a)^{4-0}y^0 - \binom{4}{2}(x-a)^{4-2}y^2 + \binom{4}{4}(x-a)^{4-4}y^4\right\} \\
& + \frac{f^{(5)}(c)}{5!}\left\{\binom{5}{0}(x-a)^{5-0}y^0 - \binom{5}{2}(x-a)^{5-2}y^2 + \binom{5}{4}(x-a)^{5-4}y^4\right\} \\
& + \\
& \vdots
\end{aligned}
$$

As seen from this example, This is a halfway formula that is difficult to call a series with respect to both x and y . So, we will rearrange it so that it is a series with respect to both x and y .

Formula 14.1.2

Suppose that a complex function $f(z)$ $(z=x+iy)$ is expanded around a real number a into a Taylor series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a)\frac{(z-a)^s}{s!} \tag{1.2}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$.

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v}$$

Where, $0^0 = 1$.

Proof

From Formula 14.1.1 ,

$$u(x,y) = \sum_{r=0}^{\infty}\frac{f^{(r)}(a)}{r!}\sum_{s=0}^{\left\lceil\frac{r-1}{2}\right\rceil}(-1)^s\binom{r}{2s}(x-a)^{r-2s}y^{2s} \tag{1.1u}$$

$$v(x,y) = \sum_{r=0}^{\infty}\frac{f^{(r)}(a)}{r!}\sum_{s=0}^{\left\lfloor\frac{r-1}{2}\right\rfloor}(-1)^s\binom{r}{2s+1}(x-a)^{r-2s-1}y^{2s+1} \tag{1.1v}$$

Here, let

$$A_r := \frac{f^{(r)}(a)}{r\,!} \qquad , \quad X := (x-a)$$

$$a_{rs} := (-1)^s \binom{r}{2s} \quad , \quad b_{rs} := (-1)^s \binom{r}{2s+1}$$

Then,

$$u(x,y) = \sum_{r=0}^{\infty} A_r \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} a_{rs} X^{r-2s} y^{2s} \tag{1.1u'}$$

$$v(x,y) = \sum_{r=0}^{\infty} A_r \sum_{s=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} b_{rs} X^{r-2s-1} y^{2s+1} \tag{1.1v'}$$

Rearrange (1.1u') as follows.

$$\begin{aligned}
u(x,y) &= \sum_{r=0}^{\infty} A_r \sum_{s=0}^{\left\lceil \frac{r-1}{2} \right\rceil} a_{rs} X^{r-2s} y^{2s} \\
&= A_0\left\{ a_{00} X^{0-0} y^0 \right\} \\
&\quad + A_1\left\{ a_{10} X^{1-0} y^0 \right\} \\
&\quad + A_2\left\{ a_{20} X^{2-0} y^0 + a_{21} X^{2-2} y^2 \right\} \\
&\quad + A_3\left\{ a_{30} X^{3-0} y^0 + a_{31} X^{3-2} y^2 \right\} \\
&\quad + A_4\left\{ a_{40} X^{4-0} y^0 + a_{41} X^{4-2} y^2 + a_{42} X^{4-4} y^4 \right\} \\
&\quad + A_5\left\{ a_{50} X^{5-0} y^0 + a_{51} X^{5-2} y^2 + a_{52} X^{5-4} y^4 \right\} \\
&\quad + \\
&\quad \vdots \\
&= \left\{ A_0 a_{00} X^{0-0} + A_1 a_{10} X^{1-0} + A_2 a_{20} X^{2-0} + A_3 a_{30} X^{3-0} + \cdots \right\} y^0 \\
&\quad + \left\{ A_2 a_{21} X^{2-2} + A_3 a_{31} X^{3-2} + A_4 a_{41} X^{4-2} + A_5 a_{51} X^{5-2} + \cdots \right\} y^2 \\
&\quad + \left\{ A_4 a_{42} X^{4-4} + A_5 a_{52} X^{5-4} + A_6 a_{62} X^{6-4} + A_7 a_{72} X^{7-4} + \cdots \right\} y^4 \\
&\quad + \\
&\quad \vdots \\
&= \left\{ \sum_{s=0}^{\infty} A_{0+s} a_{0+s,0} X^s \right\} y^0 + \left\{ \sum_{s=0}^{\infty} A_{2+s} a_{2+s,1} X^s \right\} y^2 + \left\{ \sum_{s=0}^{\infty} A_{4+s} a_{4+s,2} X^s \right\} y^4 + \cdots
\end{aligned}$$

i.e.

$$u(x,y) = \sum_{r=0}^{\infty} \left\{ \sum_{s=0}^{\infty} A_{2r+s} a_{2r+s,r} X^s \right\} y^{2r}$$

Since

$$A_r = \frac{f^{(r)}(a)}{r\,!} \qquad \Longrightarrow \qquad A_{2r+s} = \frac{f^{(2r+s)}(a)}{(2r+s)\,!}$$

$$a_{rs} = (-1)^s \binom{r}{2s} \implies a_{2r+s,r} = (-1)^r \binom{2r+s}{2r}$$

returning to the original symbol ,

$$u(x,y) = \sum_{r=0}^{\infty} \left\{ \sum_{s=0}^{\infty} \frac{f^{(2r+s)}(a)}{(2r+s)!} \binom{2r+s}{2r} (x-a)^s \right\} (-1)^r y^{2r}$$

Further,

$$\binom{2r+s}{2r} = \frac{(2r+s)!}{(2r)!\, s!}$$

Substituting this for the above,

$$u(x,y) = \sum_{r=0}^{\infty} \left\{ \sum_{s=0}^{\infty} \frac{f^{(2r+s)}(a)}{(2r+s)!} \frac{(2r+s)!}{(2r)!\, s!} (x-a)^s \right\} (-1)^r y^{2r}$$

i.e.

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f^{(2r+s)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u}$$

In the same way as above for (1.1v') , we obtain

$$v(x,y) = \sum_{r=0}^{\infty} \left\{ \sum_{s=0}^{\infty} \frac{f^{(2r+s)}(a)}{(2r+s)!} \binom{2r+s}{2r+1} (x-a)^{s-1} \right\} (-1)^r y^{2r+1}$$

Here, the first term in { } is

$$\frac{f^{(2r+0)}(a)}{(2r+0)!} \binom{2r+0}{2r+1} (x-a)^{0-1} = 0 \qquad for\ r = 0, 1, 2, \cdots$$

So,

$$\sum_{s=0}^{\infty} \frac{f^{(2r+s)}(a)}{(2r+s)!} \binom{2r+s}{2r+1} (x-a)^{s-1} = \sum_{s=1}^{\infty} \frac{f^{(2r+s)}(a)}{(2r+s)!} \binom{2r+s}{2r+1} (x-a)^{s-1}$$

$$= \sum_{s=0}^{\infty} \frac{f^{(2r+s+1)}(a)}{(2r+s+1)!} \binom{2r+s+1}{2r+1} (x-a)^s$$

Therefore,

$$v(x,y) = \sum_{r=0}^{\infty} \left\{ \sum_{s=0}^{\infty} \frac{f^{(2r+s+1)}(a)}{(2r+s+1)!} \binom{2r+s+1}{2r+1} (x-a)^s \right\} (-1)^r y^{2r+1}$$

Further,

$$\binom{2r+s+1}{2r+1} = \frac{(2r+s+1)!}{(2r+1)!\, s!}$$

Substituting this for the above,

$$v(x,y) = \sum_{r=0}^{\infty} \left\{ \sum_{s=0}^{\infty} \frac{f^{(2r+s+1)}(a)}{(2r+s+1)!} \frac{(2r+s+1)!}{(2r+1)!\, s!} (x-a)^s \right\} (-1)^r y^{2r+1}$$

i.e.

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f^{(2r+s+1)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v}$$

Note

With respect to y , the real part $u(x,y)$ is an even function and the imaginary part $v(x,y)$ is an odd function.

Cauchy–Riemann Equations

When (1.2u) and (1.2v) are partially differentiated with respect to x , y respectively, it is as follows.
These are the Cauchy–Riemann equations

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$\frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x} = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+2)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Maclaurin Expansion by Real and Imaginary Parts

The first few lines of Maclaurin series for $u(x,y)$, $v(x,y)$ are as follows.

$$\begin{aligned} u(x,y) = & \left\{ f^{(0)}(0)\frac{x^0}{0!} + f^{(1)}(0)\frac{x^1}{1!} + f^{(2)}(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots \right\} \frac{(-1)^0 y^0}{0!} \\ & + \left\{ f^{(2)}(0)\frac{x^0}{0!} + f^{(3)}(0)\frac{x^1}{1!} + f^{(4)}(0)\frac{x^2}{2!} + f^{(5)}(0)\frac{x^3}{3!} + \cdots \right\} \frac{(-1)^1 y^2}{2!} \\ & + \left\{ f^{(4)}(0)\frac{x^0}{0!} + f^{(5)}(0)\frac{x^1}{1!} + f^{(6)}(0)\frac{x^2}{2!} + f^{(7)}(0)\frac{x^3}{3!} + \cdots \right\} \frac{(-1)^2 y^4}{4!} \\ & + \left\{ f^{(6)}(0)\frac{x^0}{0!} + f^{(7)}(0)\frac{x^1}{1!} + f^{(8)}(0)\frac{x^2}{2!} + f^{(9)}(0)\frac{x^3}{3!} + \cdots \right\} \frac{(-1)^3 y^6}{6!} \\ & + \\ & \vdots \end{aligned}$$

$$\begin{aligned} v(x,y) = & \left\{ f^{(1)}(0)\frac{x^0}{0!} + f^{(2)}(0)\frac{x^1}{1!} + f^{(3)}(0)\frac{x^2}{2!} + f^{(4)}(0)\frac{x^3}{3!} + \cdots \right\} \frac{(-1)^0 y^1}{1!} \\ & + \left\{ f^{(3)}(0)\frac{x^0}{0!} + f^{(4)}(0)\frac{x^1}{1!} + f^{(5)}(0)\frac{x^2}{2!} + f^{(6)}(0)\frac{x^3}{3!} + \cdots \right\} \frac{(-1)^1 y^3}{3!} \\ & + \left\{ f^{(5)}(0)\frac{x^0}{0!} + f^{(6)}(0)\frac{x^1}{1!} + f^{(7)}(0)\frac{x^2}{2!} + f^{(8)}(0)\frac{x^3}{3!} + \cdots \right\} \frac{(-1)^2 y^5}{5!} \\ & + \left\{ f^{(7)}(0)\frac{x^0}{0!} + f^{(8)}(0)\frac{x^1}{1!} + f^{(9)}(0)\frac{x^2}{2!} + f^{(10)}(0)\frac{x^3}{3!} + \cdots \right\} \frac{(-1)^3 y^7}{7!} \\ & + \\ & \vdots \end{aligned}$$

A function whose Maclaurin series does not include even-order derivatives is called an odd function.
The following holds for the odd function.

Formula 14.1.2 ' (Odd Function)

Suppose a complex function $f(z)$ $(z=x+iy)$ is expanded into a Maclaurin series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(2s+1)}(0)\frac{z^{2s+1}}{(2s+1)!} \tag{1.2'}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$. Where, $0^0 = 1$.

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u'}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v'}$$

Note

With respect to x , the real part $u(x,y)$ is an odd function and the imaginary part $v(x,y)$ is an even function.

Example $f(z) = \sin z$

$$f^{(2s+1)}(0) = (-1)^s \qquad s=0,1,2,\cdots$$

$$f^{(2r+2s+1)}(0) = (-1)^{r+s} \qquad r,s=0,1,2,\cdots$$

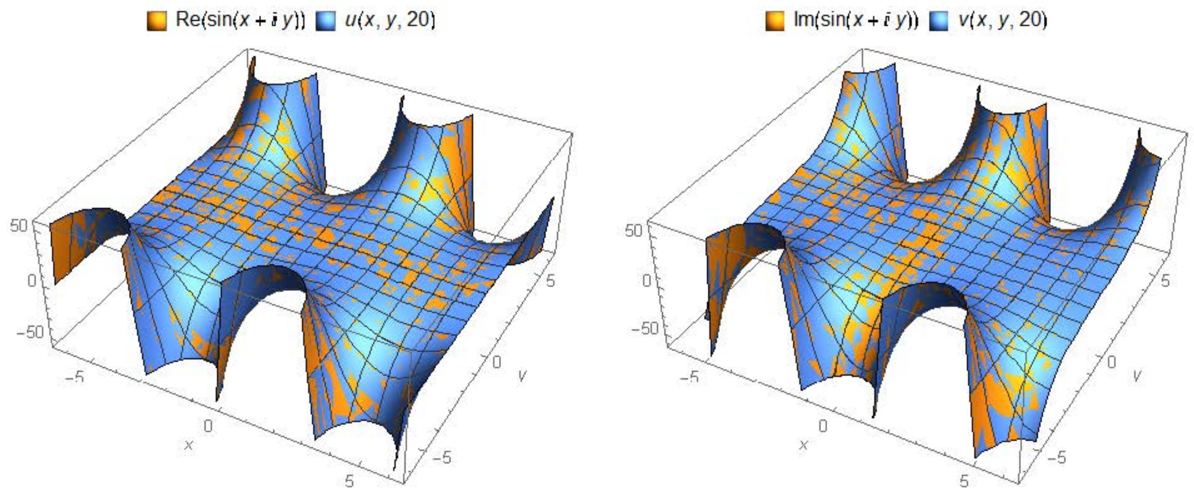
Substituting these for the above formula,

$$f(z) = \sum_{s=0}^{\infty}(-1)^s\frac{z^{2s+1}}{(2s+1)!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\frac{x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!}$$

Both sides of $u(x,y), v(x,y)$ are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. Both sides overlap exactly.



A function whose Maclaurin series does not include odd-order derivatives is called an even function. The following holds for the even function.

Formula 14.1.2 " (Even Function)

Suppose a complex function $f(z)$ $(z=x+iy)$ is expanded into a Maclaurin series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(2s)}(0)\frac{z^{2s}}{(2s)!} \tag{1.2"}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y)$, $v(x,y)$. Where, $0^0=1$.

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u"}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v"}$$

Note

With respect to x, the real part $u(x,y)$ is an even function and the imaginary part $v(x,y)$ is an odd function.

Example $f(z)=\cos z$

$$f^{(2s)}(0) = (-1)^s \qquad s=0,1,2,\cdots$$

$$f^{(2r+2s)}(0) = (-1)^{r+s} \qquad r,s=0,1,2,\cdots$$

$$f^{(2r+2s+2)}(0) = (-1)^{r+s+1} \qquad r,s=0,1,2,\cdots$$

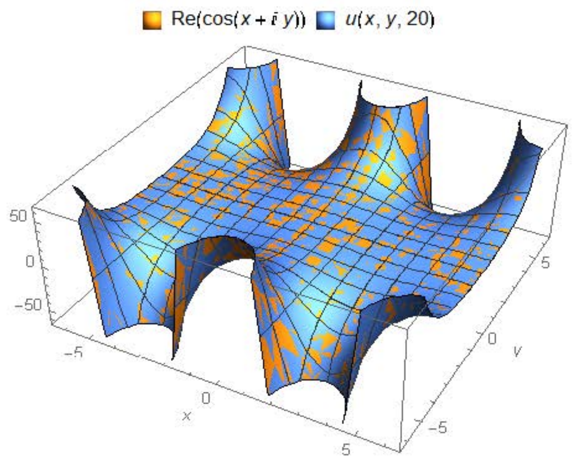
Substituting these for the above formula,

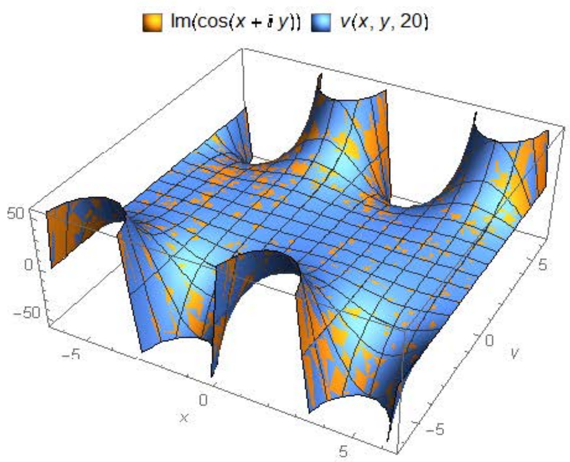
$$f(z) = \sum_{s=0}^{\infty}(-1)^s\frac{z^{2s}}{(2s)!}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\frac{x^{2s}}{(2s)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s+1}\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r+1}}{(2r+1)!}$$

Both sides of $u(x,y)$, $v(x,y)$ are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. Both sides overlap exactly.





Treatment of 0^0 in *Mathematica*

Formula manipulation soft ***Mathematica*** does not calculate 0^0 as *Indeterminate* . Since it is inconvenient, the following options are specified prior to calculation in this paper .

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Unprotect [Power] ;  Power [0,0] = 1 ;
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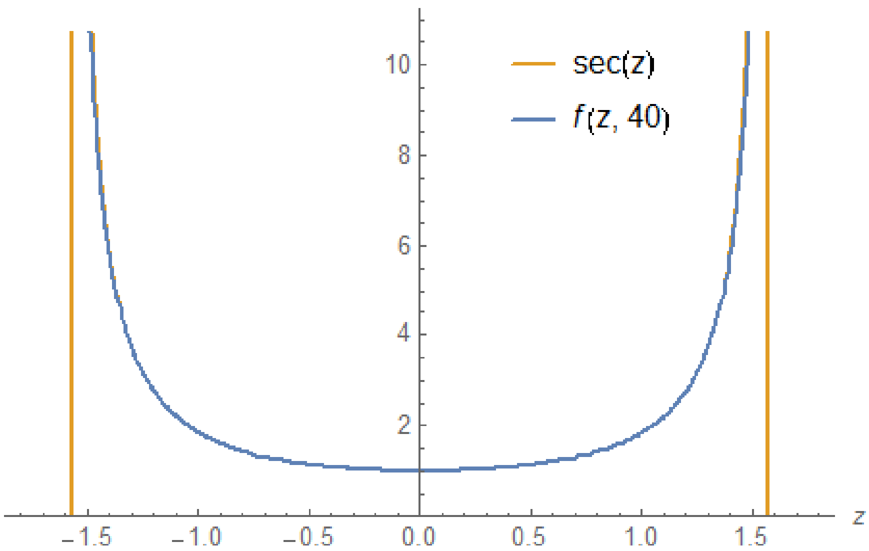
14.2 Example1: $\sec z$

Neither of the two examples in the previous section had a singularity. So, Formula 14.1.2 held over the whole complex plane. In this section and hereafter, we consider Taylor expansions of functions with singularities by real and imaginary parts. This section takes up $\sec z$ as the first.

When $f(z) = \sec z$, this is expanded into Maclaurin series using Euler numbers E_r $(r=0,1,2,\cdots)$ as follows. In addition, the first few of E_r are $1,0,-1,0,5,0,-61,0,1385,\cdots$.

$$f(z) = \sec z = \sum_{r=0}^{\infty} |E_r| \frac{z^r}{r!} \tag{2.1}$$

Both sides are drawn as follows. Orange is the left side and blue is the right side. Both sides overlap exactly. Singular points are observed at $z = \pm\pi/2$.



Expansion by real part and imaginary part by Formula 14.1.1

From (2.1),

$$f^{(r)}(0) = \sec^{(r)} 0 = |E_r| \qquad r=0,1,2,\cdots \tag{2.a}$$

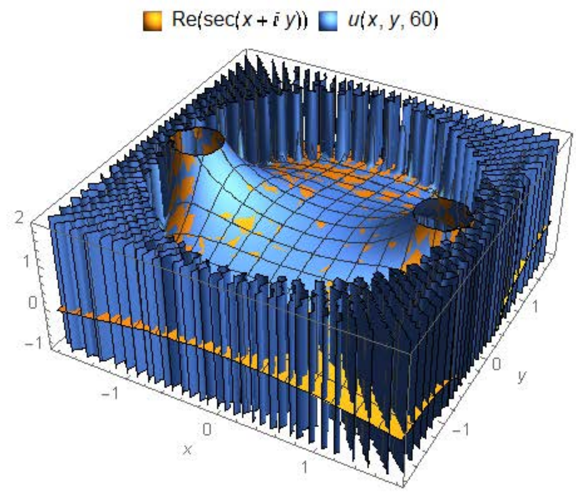
Substituting this for Formula 14.1.1,

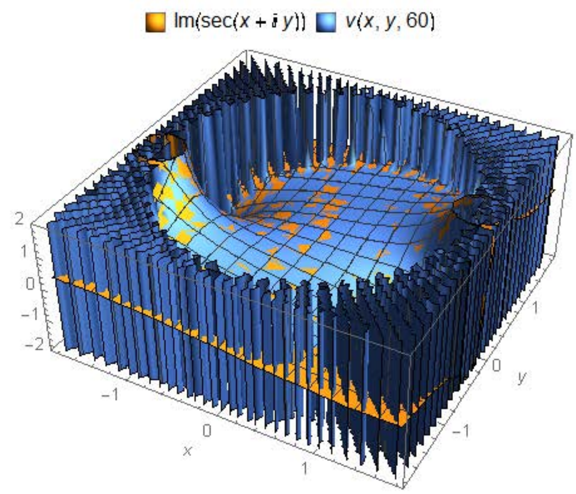
$$u(x,y) = \sum_{r=0}^{\infty} \frac{|E_r|}{r!} \sum_{s=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^s \binom{r}{2s} x^{r-2s} y^{2s}$$

$$v(x,y) = \sum_{r=0}^{\infty} \frac{|E_r|}{r!} \sum_{s=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^s \binom{r}{2s+1} x^{r-2s-1} y^{2s+1}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function, $\lfloor x \rfloor$ is the floor function.

Both sides are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. Convergence circles with radius $\pi/2$ are observed.





Expansion by real part and imaginary part by Formula 14.1.2"

Since $f(z)=sec\, z$ is an even function,

$$f^{(2s)}(0) = (-1)^s E_{2s} \qquad s=0,1,2,\cdots$$

$$f^{(2r+2s)}(0) = (-1)^{r+s} E_{2r+2s} \qquad r,s=0,1,2,\cdots$$

$$f^{(2r+2s+2)}(0) = (-1)^{r+s+1} E_{2r+2s+2} \qquad r,s=0,1,2,\cdots$$

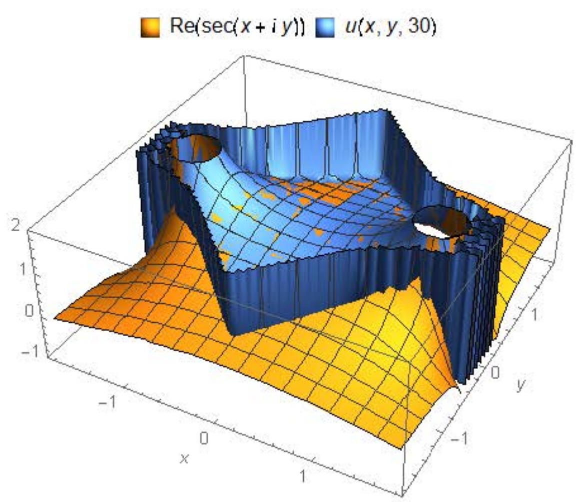
substituting these for Formula 14.1.2 " (1.2u") , (1.2u") ,

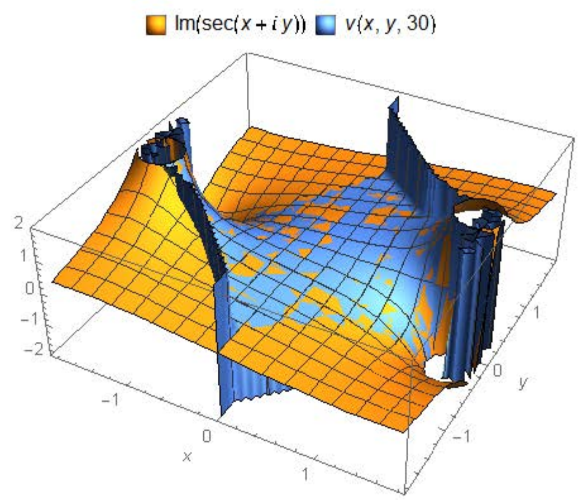
$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s E_{2r+2s} \frac{x^{2s}}{(2s)!} \frac{y^{2r}}{(2r)!} \tag{2.2u"}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{s+1} E_{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{y^{2r+1}}{(2r+1)!} \tag{2.2v"}$$

Where, $0^0 = 1$

Both sides are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side.





Surprisingly, these convergent circles are cut off at the four corners by hyperbolas. As the result of vaious numerical calculations, it turned out that both are series in the square, and are asymptotic expansions outside of the square. That is, **the convergence area is not a hexagon but a square**.

Convergence acceleration of (2,2u"), (2.2v")

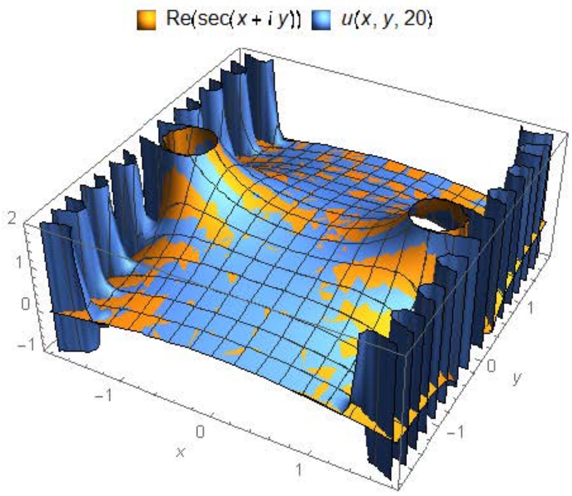
So, we try to accelerate these convergence by Euler transformation. We use parallel acceleration method. In addition, about the method, see **" 13 Convergence Acceleration of Multiple Series "** (A la carte) . It becomes as follows.

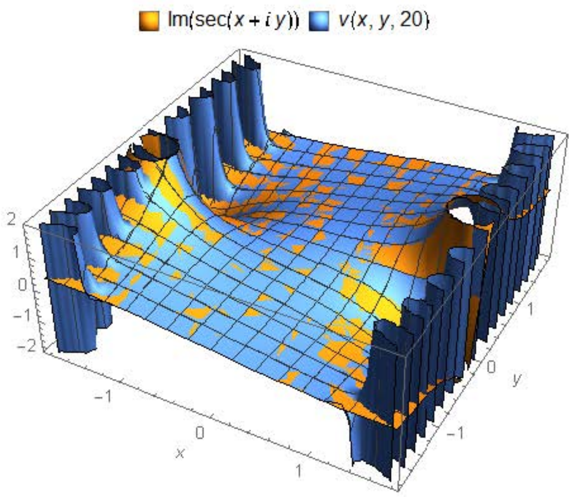
$$u(x,y) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k}\frac{1}{2^{k+1}}\binom{k}{r+s}(-1)^{s}E_{2r+2s}\frac{x^{2s}}{(2s)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k}\frac{1}{2^{k+1}}\binom{k}{r+s}(-1)^{s+1}E_{2r+2s+2}\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$

Both sides are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side.





By this acceleration, **the convergence circles have been recovered**. And as a result of numerical calculations, it turns out that the convergence areas are further asymptotically extended to the outside (mainly the vertical direction) of the convergence circles .

14.3 Example2: $1/(z^2+1)$

In this section, we consider the Taylor expansion by real part and imaginary part of the following fractional function. This is a typical non-entire function (with singularities).

$$f(z) = \frac{1}{z^2+1} \tag{3.0}$$

When $Re(z) \geqq 0$, the higher order derivative is given by the following. (See " 岩波 数学公式 I " p32,)

$$f^{(r)}(z) = \left(\frac{1}{z^2+1}\right)^{(r)} = \frac{(-1)^r r!}{(z^2+1)^{(r+1)/2}} \sin\{(r+1)\cot^{-1} z\} \qquad r=0,1,2,\cdots$$

Substituting real number a for this,

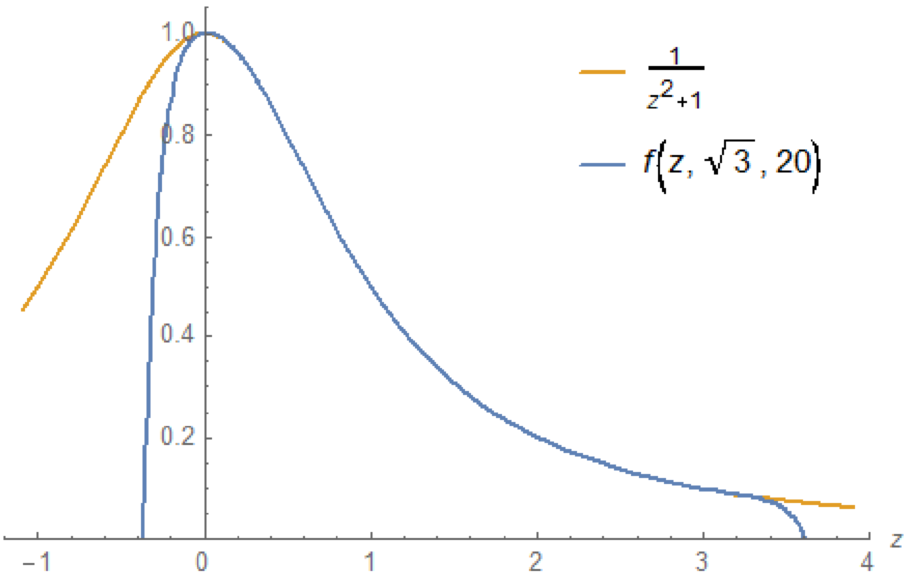
$$f^{(r)}(a) = \left(\frac{1}{z^2+1}\right)^{(r)}_{z=a} = \frac{(-1)^r r!}{(a^2+1)^{(r+1)/2}} \sin\{(r+1)\cot^{-1} a\} \qquad r=0,1,2,\cdots \tag{3.a}$$

Then, the Taylor expansion of $f(z)$ around a is

$$f(z) = \frac{1}{z^2+1} = \sum_{r=0}^{\infty} \frac{(-1)^r r!}{(a^2+1)^{(r+1)/2}} \sin\{(r+1)\cot^{-1} a\} \frac{(z-a)^r}{r!} \tag{3.1}$$

As is clear from (3.0) , this function has singular points at $z=\pm i$. The convergence radius R is the distance from a to $\pm i$. Since a is on the real axis, R is calculated as $R=\sqrt{a^2+1}$

When $a=\sqrt{3}$, both sides are drawn as follows. Orange is the left side and blue is the right side. The center point of the expansion is $\sqrt{3}$ and the convergence radius is 2 .



Expansion by real part and imaginary part by Formula 14.1.1

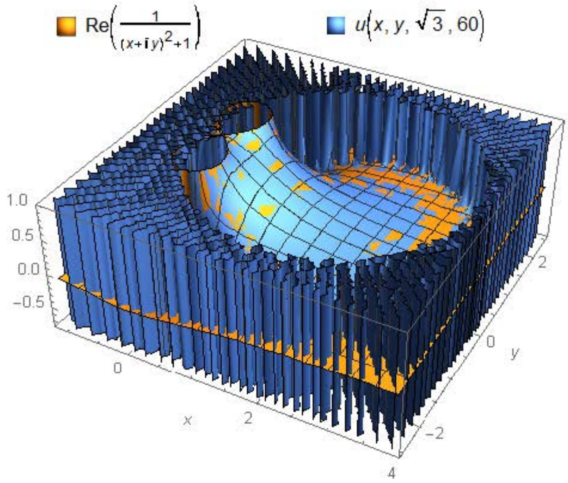
Substituting (3.a) for Formula 14.1.1 ,

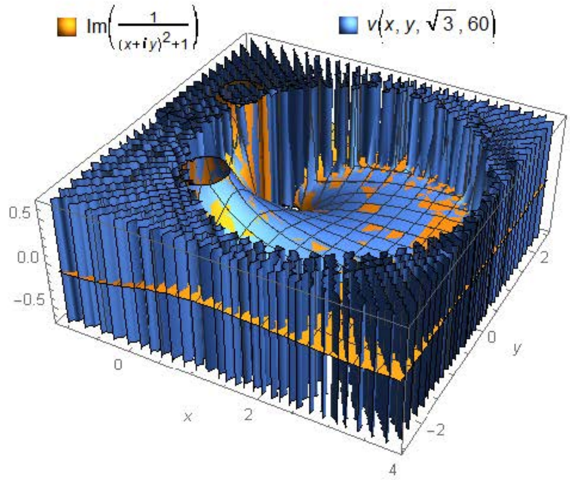
$$u(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r \sin\{(r+1)\cot^{-1}a\}}{(a^2+1)^{(r+1)/2}} \sum_{s=0}^{\lceil \frac{r-1}{2} \rceil} (-1)^s \binom{r}{2s} (x-a)^{r-2s} y^{2s}$$

$$v(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r \sin\{(r+1)\cot^{-1}a\}}{(a^2+1)^{(r+1)/2}} \sum_{s=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

When $a = \sqrt{3}$, both sides are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. Convergence circles with radius 2 are observed.





Expansion by real part and imaginary part by Formula 14.1.2

From (3.a) ,

$$f^{(2r+s)}(a) = \frac{(-1)^s (2r+s)!}{(a^2+1)^{(2r+s+1)/2}} \sin\{(2r+s+1)\cot^{-1}a\} \qquad r,s = 0,1,2,\cdots$$

$$f^{(2r+s+1)}(a) = -\frac{(-1)^s (2r+s+1)!}{(a^2+1)^{(2r+s+2)/2}} \sin\{(2r+s+2)\cot^{-1}a\} \qquad r,s = 0,1,2,\cdots$$

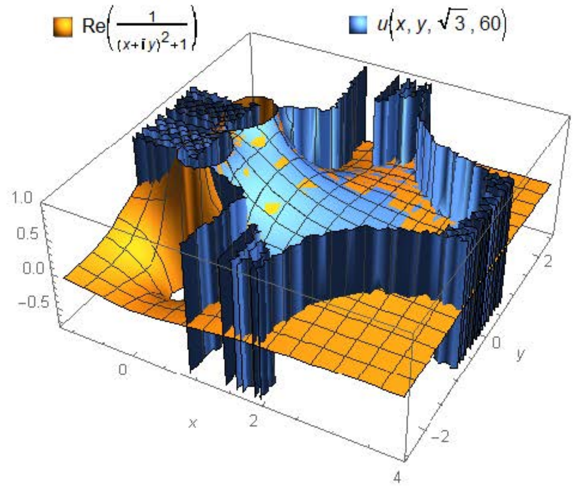
Substituting these for Formula 14.1.2 ,

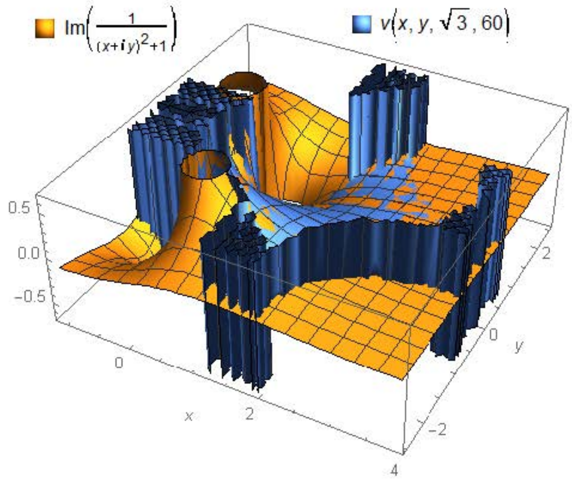
$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s (2r+s)!}{(a^2+1)^{(2r+s+1)/2}} \sin\{(2r+s+1)\cot^{-1}a\} \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!} \tag{3.2u}$$

$$v(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s (2r+s+1)!}{(a^2+1)^{(2r+s+2)/2}} \sin\{(2r+s+2)\cot^{-1}a\} \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{3.2v}$$

Where, $0^0 = 1$

When $a = \sqrt{3}$, both sides are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. As in the previous section, these convergence circles are cut off at the four corners by hyperbolas. Here, the convergence area is a square.





Convergence acceleration of (3,2u), (3.2v)

So, we try to accelerate these convergence by Euler transformation. We use parallel acceleration method. In addition, about the method, see **" 13 Convergence Acceleration of Multiple Series "** (A la carte) . It becoms as follows.

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k}\frac{1}{2^{k+1}}\binom{k}{r+s}$$

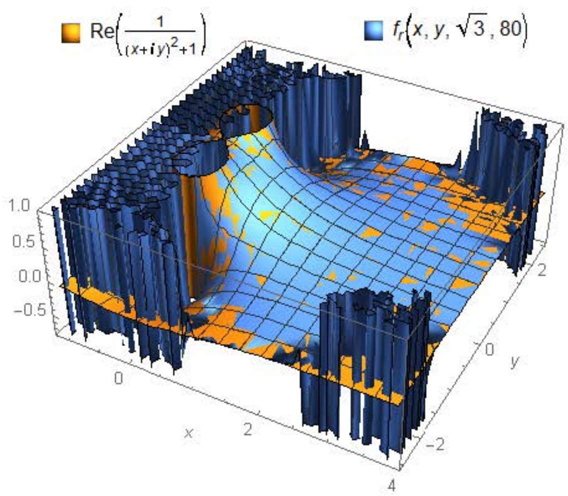
$$\times\frac{(-1)^s(2r+s)!}{\left(a^2+1\right)^{(2r+s+1)/2}}\sin\left\{(2r+s+1)\cot^{-1}a\right\}\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

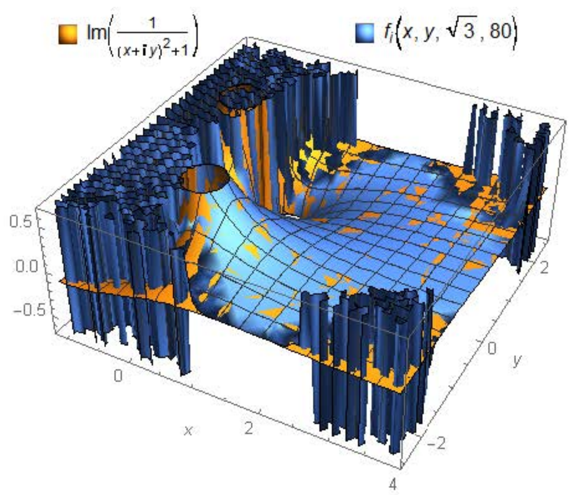
$$v(x,y) = -\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k}\frac{1}{2^{k+1}}\binom{k}{r+s}$$

$$\times\frac{(-1)^s(2r+s+1)!}{\left(a^2+1\right)^{(2r+s+2)/2}}\sin\left\{(2r+s+2)\cot^{-1}a\right\}\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$

When $a=\sqrt{3}$, both sides are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side.





These drawings took 10 hours each. This is no longer an acceleration. But analytical continuations have arized. That is, the convergence circles have been recovered, and the convergence areas are further asymptotically extended to the outside (mainly 3 directions) of the convergence circles.

14.4 Example3: $\log z$

In this section, we consider the Taylor expansion by real part and imaginary part of the logarithmic function. This function is known to have branch points (a kind of singularity).

When $f(z)=\log z$, from Formula 9.2.3 in " **09 Higher Derivative** " (Super Calculus),

$$f^{(r)}(z)=\begin{cases}\log z & r=0\\ (-1)^{r-1}(r-1)!\ z^{-r} & r=1,2,3,\cdots\end{cases}$$

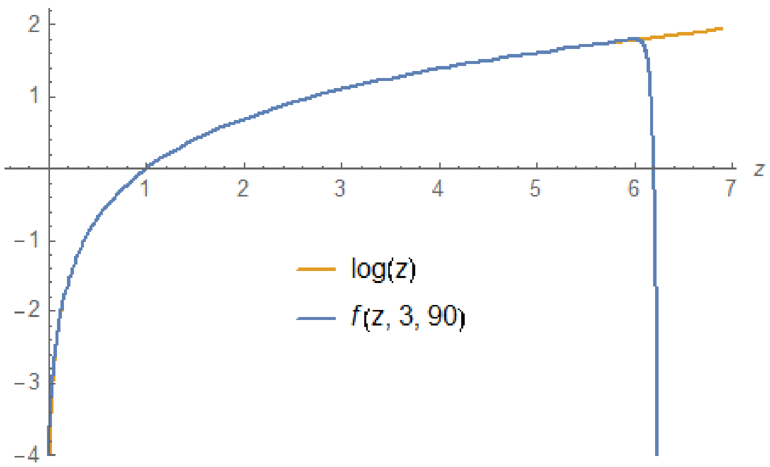
Substituting real number a for this,

$$f^{(r)}(a)=\begin{cases}\log a & r=0\\ (-1)^{r-1}(r-1)!\ a^{-r} & r=1,2,3,\cdots\end{cases} \tag{4.a}$$

Then, the Taylor expansion of $f(z)$ around a is

$$f(z)=\log z=\log a+\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{a^r r}(z-a)^r \tag{4.1}$$

When $a=3$, both sides are drawn as follows. Orange is the left side and blue is the right side. A singular point is observed at $z=0$. Since $a=3$, the convergence radius becoms 3.



Expansion by real part and imaginary part by Formula 14.1.1

Substituting (4.a) for Formula 14.1.1 ,

$$u(x,y)=\frac{\log a}{0!}+\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(r-1)!}{a^r r!}\sum_{s=0}^{\lceil\frac{r-1}{2}\rceil}(-1)^s\binom{r}{2s}(x-a)^{r-2s}y^{2s}$$

$$v(x,y)=\frac{\log a}{0!}\binom{0}{1}(x-a)^{-1}y^1$$

$$+\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(r-1)!}{a^r r!}\sum_{s=0}^{\lfloor\frac{r-1}{2}\rfloor}(-1)^s\binom{r}{2s+1}(x-a)^{r-2s-1}y^{2s+1}$$

i.e.

$$u(x,y)=\log a+\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{a^r r}\sum_{s=0}^{\lceil\frac{r-1}{2}\rceil}(-1)^s\binom{r}{2s}(x-a)^{r-2s}y^{2s}$$

$$v(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{a^r r} \sum_{s=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^s \binom{r}{2s+1} (x-a)^{r-2s-1} y^{2s+1}$$

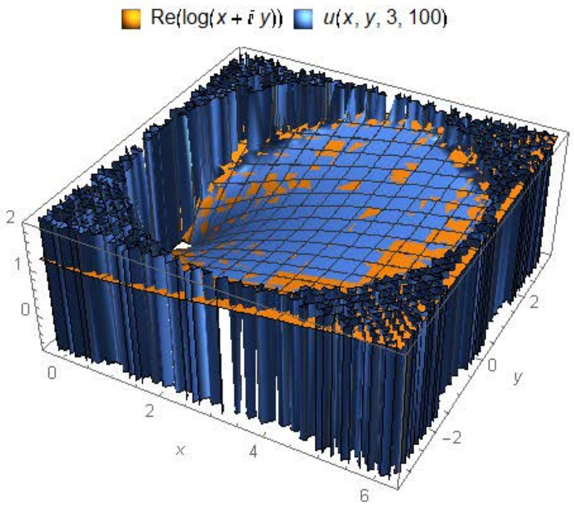
Changing the initial value of r from 1 to 0 ,

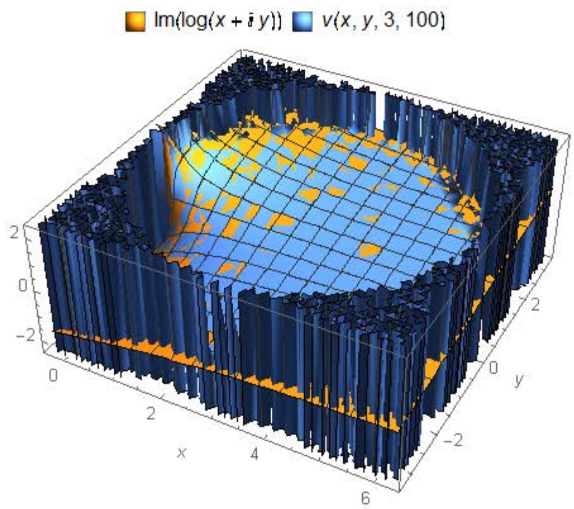
$$u(x,y) = \log a + \sum_{r=0}^{\infty} \frac{(-1)^r}{a^{r+1}(r+1)} \sum_{s=0}^{\lceil r/2 \rceil} (-1)^s \binom{r+1}{2s} (x-a)^{r-2s+1} y^{2s}$$

$$v(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r}{a^{r+1}(r+1)} \sum_{s=0}^{\lfloor r/2 \rfloor} (-1)^s \binom{r+1}{2s+1} (x-a)^{r-2s} y^{2s+1}$$

Where, $0^0 = 1$, $\lceil x \rceil$ is the ceiling function , $\lfloor x \rfloor$ is the floor function.

When $a = 3$, both sides are drawn as follows. The left is the real part and the right is the imaginary part.
In both figures, orange is the left side and blue is the right side. Convergence circles with radius 3 are observed.





Expansion by real part and imaginary part by Formula 14.1.2

From (4.a) ,

$$f^{(s)}(a) = (-1)^{s-1}(s-1)!\ a^{-s} \qquad s=1,2,3,\cdots$$

$$f^{(2r+s)}(a) = \begin{cases} \log a & 2r+s=0 \\ (-1)^{s-1}(2r+s-1)!\ a^{-2r-s} & 2r+s=1,2,3,\cdots \end{cases}$$

$$f^{(2r+s+1)}(a) = (-1)^s(2r+s)!\ a^{-2r-s-1} \qquad r,s = 0,1,2,\cdots$$

Next, Formula 14.1.2 (1.2u) is

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$= f^{(0)}(a) + \sum_{s=1}^{\infty} f^{(s)}(a) \frac{(x-a)^s}{s!} + \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a) \frac{(x-a)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

Substituting the above $f^{(s)}(a)$, $f^{(2r+s)}(a)$ for this,

$$u(x,y) = \log a + \sum_{s=1}^{\infty}(-1)^{s-1}\frac{(s-1)!}{a^s}\frac{(x-a)^s}{s!}$$

$$+\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(-1)^{s-1}\frac{(2r+s-1)!}{a^{2r+s}}\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

Changing the initial value of r from 1 to 0 in the 3rd term,

$$u(x,y) = \log a + \sum_{s=1}^{\infty}(-1)^{s-1}\frac{(s-1)!}{a^s}\frac{(x-a)^s}{s!}$$

$$+\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s-1}\frac{(2r+s+1)!}{a^{2r+s+2}}\frac{(x-a)^s}{s!}\frac{(-1)^{r+1} y^{2r+2}}{(2r+2)!}$$

i.e.

$$u(x,y) = \log a + \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{a^s}\frac{(x-a)^s}{s}$$

$$+\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(2r+s+1)!}{a^{2r+s+2}}\frac{(-1)^s(x-a)^s}{s!}\frac{(-1)^r y^{2r+2}}{(2r+2)!} \qquad (4.2u)$$

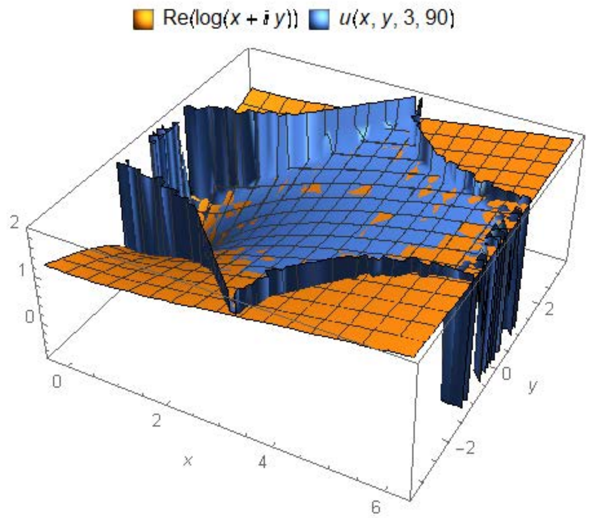
Next, Formula 14.1.2 (1.2v) is

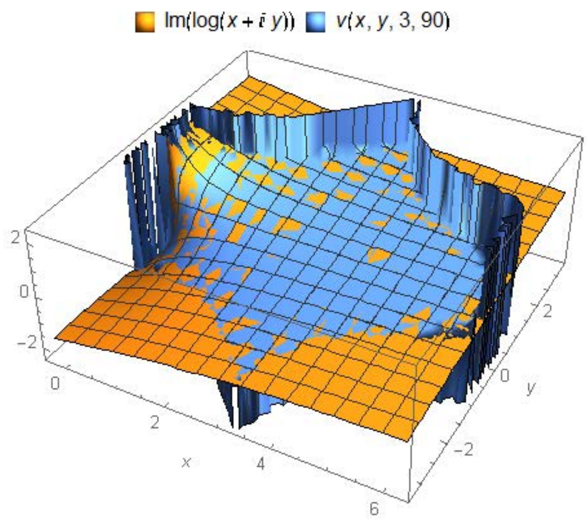
$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Substituting the above $f^{(2r+s+1)}(a)$ for this,

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(2r+s)!}{a^{2r+s+1}}\frac{(-1)^s(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad (4.2v)$$

When $a=3$, both sides of (4.2u), (4.2v) are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. As in the previous section, these convergence circles are cut off at the four corners by hyperbolas. Here, the convergence area is a square.





Convergence acceleration of (4,2r), (4.2i)

So, we try to accelerate these convergence by Euler transformation. We use parallel acceleration method. In addition, about the method, see **" 13 Convergence Acceleration of Multiple Series "** (A la carte) . It becomes as follows.

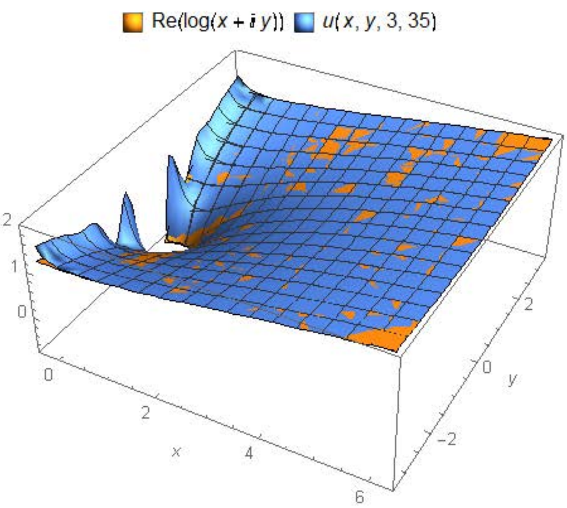
$$u(x,y) = \log a + \sum_{k=1}^{\infty}\sum_{s=1}^{k}\frac{1}{2^{k+1}}\binom{k}{s}\frac{(-1)^{s-1}}{a^s}\frac{(x-a)^s}{s}$$

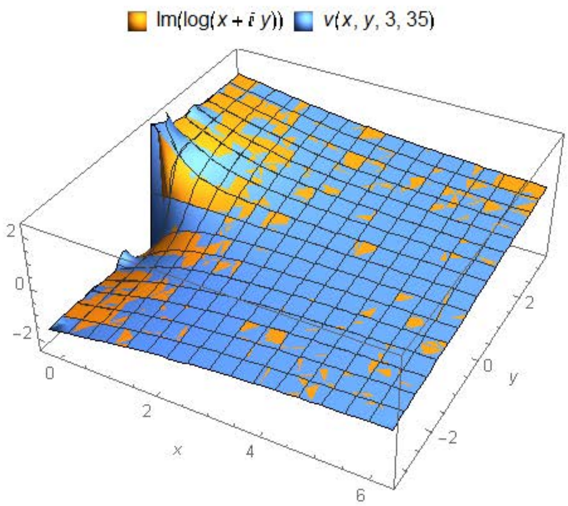
$$+\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k}\frac{1}{2^{k+1}}\binom{k}{r+s}\frac{(2r+s+1)!}{a^{2r+s+2}}\frac{(-1)^s(x-a)^s}{s!}\frac{(-1)^r y^{2r+2}}{(2r+2)!}$$

$$v(x,y) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k}\frac{1}{2^{k+1}}\binom{k}{r+s}\frac{(2r+s)!}{a^{2r+s+1}}\frac{(-1)^s(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$

When $a = 3$, both sides are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the left side and blue is the right side. It seems that the above hexagonal convergence areas are analytically continued to the half-plane.





Reference

Above, we showed several examples of Taylor series by real part and imaginary part. For other Taylor series and Laurent series by real and imaginary parts of elementary functions, see the following.

" **15 Taylor Series of Elementary Functions by Real & Imaginary Parts** "

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2020.02.16 Renewed
2023.11.10 Updated Sec.1

Kano Kono
Hiroshima, Japan

Alien's Mathematics