15 Taylor Series of Elementary Functions by Real & Imaginary Parts

Legend

Formulas

" **14 Taylor Expansion by Real Part & Imaginary Part** " **Formula 14.1.2 , 2 ', 2 "** are used.

Formula 14.1.2 (Reprint)

Suppose that a complex function $f(z)$ $(z=x+iy)$ is expanded around a real number a into a Taylor series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(s)}(a)\frac{(z-a)^s}{s!} \tag{1.2}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$. Where, $0^0=1$.

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+s+1)}(a)\frac{(x-a)^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v}$$

Formula 14.1.2 ' (Odd Function) (Reprint)

Suppose a complex function $f(z)$ $(z=x+iy)$ is expanded into a Maclaurin series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(2s+1)}(0)\frac{z^{2s+1}}{(2s+1)!} \tag{1.2'}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$. Where, $0^0=1$.

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u'}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v'}$$

Formula 14.1.2 " (Even Function) (Reprint)

Suppose a complex function $f(z)$ $(z=x+iy)$ is expanded into a Maclaurin series with real coefficients as follows.

$$f(z) = \sum_{s=0}^{\infty} f^{(2s)}(0)\frac{z^{2s}}{(2s)!} \tag{1.2"}$$

Then, the following expressions hold for the real and imaginary parts $u(x,y), v(x,y)$. Where, $0^0=1$.

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!} \tag{1.2u"}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \tag{1.2v"}$$

Treatment of 0^0 in *Mathematica*

In this paper, formula manipulation soft ***Mathematica*** is used for drawing and calculation. The following options are specified prior to calculation .

Unprotect [Power] ; Power [0,0] = 1 ;

Symbols

In this paper, Bernoulli numbers and others are defined as follows.

Bernoulli and Euler numbers

$B,E \backslash n$	0	1	2	3	4	5	6	7	8	9	10	...
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	...
E_n	1	0	-1	0	5	0	-61	0	1385	0	-50521	...

Sign function

$$sign(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

Real and imaginary parts of complex function

The real and imaginary parts of complex function $f(z)$ $(z = x + iy)$ are denoted as follows.

$u(x,y)$: Real part

$v(x,y)$: Imaginary part

Method of verification

It was verified by 3D figures and numerical calculations using formula manipulation software ***Matehmatica***.
Here, the case of $z\,coth\,z$ is illustrated.

(1) Formulas

Maclaurin series of $z\,coth\,z$ by real and imaginary parts are described by *Mathematica* as follows.

```
Unprotect[Power]; Power[0, 0] = 1;
```

$B_{n\_} := \text{BernoulliB}[n]$

$$f[z\_, m\_] := \sum_{s=0}^{m} 2^{2s} B_{2s} \frac{z^{2s}}{(2s)!}$$

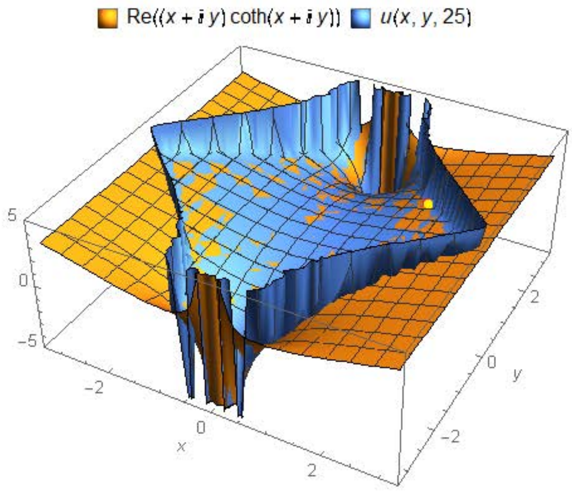
$$u[x\_, y\_, m\_] := \sum_{r=0}^{m} \sum_{s=0}^{m} 2^{2r+2s} B_{2r+2s} \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

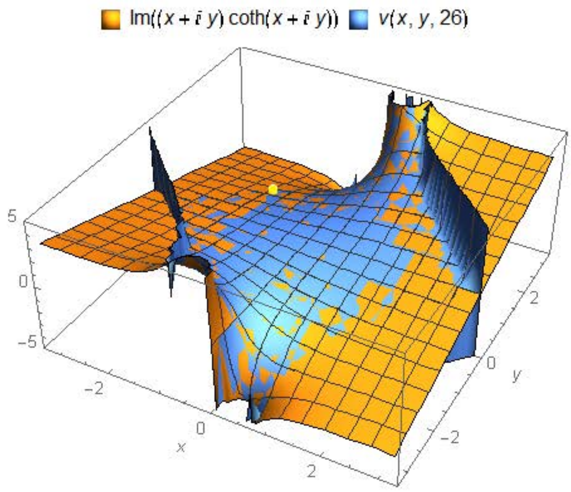
$$v[x\_, y\_, m\_] := \sum_{r=0}^{m} \sum_{s=0}^{m} 2^{2r+2s+2} B_{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

(2) 3D figures

3D figures are drawn by the following commands. The left is the real part and the right is the imaginary part.

```
Plot3D[{Re[(x + ⅈ y) Coth[x + ⅈ y]], u[x, y, 25]}, {x, -16/5, 16/5}, {y, -16/5, 16/5},
 PlotLegends → Placed["Expressions", Above], ClippingStyle → None,
 AxesLabel → Automatic, PlotRange → {-5, 5}]
Plot3D[{Im[(x + ⅈ y) Coth[x + ⅈ y]], v[x, y, 26]}, {x, -16/5, 16/5}, {y, -16/5, 16/5},
 PlotLegends → Placed["Expressions", Above], ClippingStyle → None,
 AxesLabel → Automatic, PlotRange → {-5, 5}]
```





In both figures, orange is the function and blue is the series. The overlapping of both is confirmed by the orange and blue spots.

(3) Numerical calculations

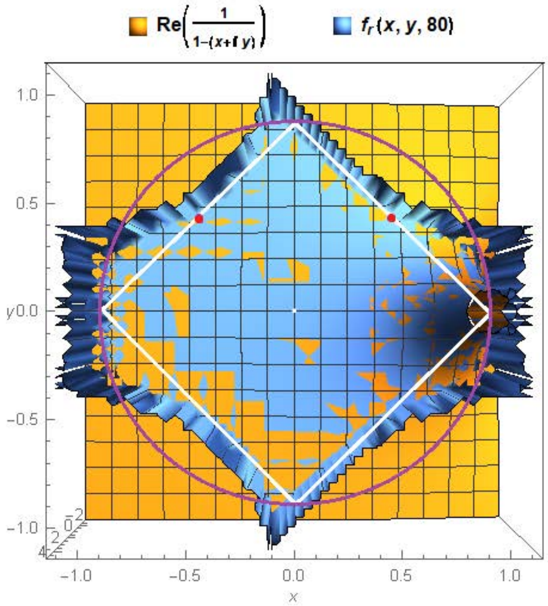
The coincidence of both sides is confirmed by calculating the function value near the midpoint (white point) of the hypotenuse.

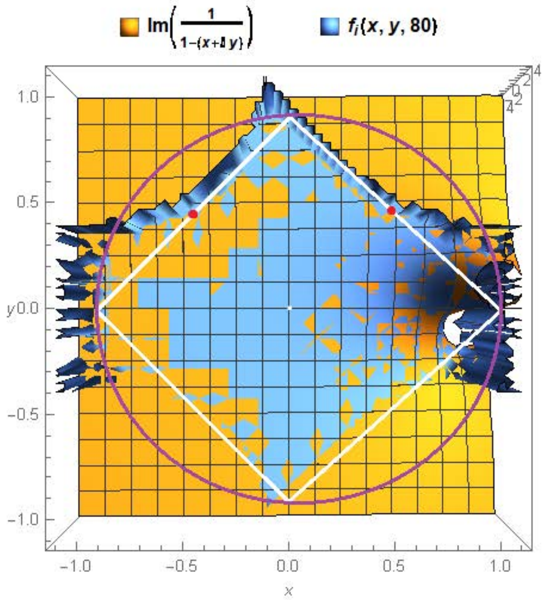
```
N[{Re[(1.56 + ⅈ 1.56) Coth[1.56 + ⅈ 1.56]], u[1.56, 1.56, 600]}]
{1.43081, 1.43081}
N[{Im[(-1.56 + ⅈ 1.56) Coth[-1.56 + ⅈ 1.56]], v[-1.56, 1.56, 600]}]
{-1.42535, -1.42535}
```

In the following sections, only these numerical calculation results are shown.

Convergence Square

The convergence region of the Taylor series of the complex function $f(z)$ is a circle, but the convergence regions of the Taylor series of $u(x,y), v(x,y)$ are squares inscribed in this circle. For example, the convergence regions of the series of $1/(1-z)$ are as follows. The left is the real part and the right is the imaginary part. In both figures, orange is the function and blue is the series.





The purple circle is the convergence circle of the series of $f(z)$, and the inscribed white squares are the convergence squares of the series of $u(x,y), v(x,y)$. Since the series converges in the square, it looks like spots. Spots can be seen outside the square, but the series becoms asymptotic expansion there.
In the following sections, this convergence square is described as ◇ .

15.1 Algebra Functions etc.

Geometric Series

$1/(1-z)$

$$\frac{1}{1-z} = \sum_{s=0}^{\infty} s!\,\frac{z^s}{s!} \qquad |z| < 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s)!\,\frac{x^s}{s!}\,\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1)!\,\frac{x^s}{s!}\,\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

|z| < ◇

Verification

$$N\left[\left\{Re\left[\frac{1}{1-\left(\frac{1}{2}+i\,\frac{1}{2}\right)}\right],\ u\left[\frac{1}{2},\ \frac{1}{2},\ 100\right]\right\}\right] \qquad N\left[\left\{Im\left[\frac{1}{1-\left(-\frac{1}{2}+i\,\frac{1}{2}\right)}\right],\ v\left[-\frac{1}{2},\ \frac{1}{2},\ 100\right]\right\}\right]$$

{1., 1.} {0.2, 0.2}

$1/(1+z)$

$$\frac{1}{1+z} = \sum_{s=0}^{\infty} s!\,\frac{(-1)^s z^s}{s!} \qquad |z| < 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s)!\,\frac{(-1)^s x^s}{s!}\,\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s+1)!\,\frac{(-1)^s x^s}{s!}\,\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

|z| < ◇

Verification

$$N\left[\left\{Re\left[\frac{1}{1+\left(\frac{1}{2}+i\,\frac{1}{2}\right)}\right],\ u\left[\frac{1}{2},\ \frac{1}{2},\ 70\right]\right\}\right] \qquad N\left[\left\{Im\left[\frac{1}{1+\left(-\frac{1}{2}+i\,\frac{1}{2}\right)}\right],\ v\left[-\frac{1}{2},\ \frac{1}{2},\ 70\right]\right\}\right]$$

{0.6, 0.6} {1., 1.}

Power Function z^p $(p,a \geqq 0)$

$$z^p = \sum_{s=0}^{\infty}\frac{\Gamma(1+p)}{\Gamma(1+p-s)}\,a^{p-s}\,\frac{(z-a)^s}{s!} \qquad |z| < |a|$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\Gamma(1+p)}{\Gamma(p-2r-s+1)}\,a^{p-2r-s}\,\frac{(x-a)^s}{s!}\,\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\Gamma(1+p)}{\Gamma(p-2r-s)}\,a^{p-2r-s-1}\,\frac{(x-a)^s}{s!}\,\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

|z| < ◇

Verification ($p=1/3$, $a=7/2$)

```
N[{Re[(21/4 + i 7/4)^(1/3)], u[21/4, 7/4, 1/3, 3.5, 70]}]
{ 1.75864 , 1.75864 }
N[{Im[(7/4 + i 7/4)^(1/3)], v[7/4, 7/4, 1/3, 3.5, 70]}]
{ 0.350091 , 0.350091 }
```

Exponential Function a^z $(a \geq 0)$

$$a^z = \sum_{s=0}^{\infty} \log^s a \frac{z^s}{s!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \log^{2r+s} a \frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \log^{2r+s+1} a \frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad |z| < \infty$$

Especially when $a = e$ $(=2.71828\cdots)$,

$$e^z = \sum_{s=0}^{\infty} \frac{z^s}{s!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{x^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{x^s}{s!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad |z| < \infty$$

Verification (When 3^z)

```
N[{Re[3^(2+i 5)], u[2, 5, 3, 30]}]
{ 6.33382 , 6.33382 }
N[{Im[3^(-2+i 5)], v[-2, 5, 3, 30]}]
{ -0.0789378 , -0.0789378 }
```

Logarithmic Functions

log z $(a \geq 0)$

$$\log z = \log a - \sum_{s=1}^{\infty} \frac{(s-1)!}{a^s}\frac{(-1)^s (z-a)^s}{s!} \qquad |z-a| \leq a,\ z \neq 0$$

$$u(x,y) = \log a - \sum_{s=1}^{\infty} \frac{(s-1)!}{a^s}\frac{(-1)^s (x-a)^s}{s!}$$

$$- \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} \frac{(2r+s-1)!}{a^{2r+s}}\frac{(-1)^s (x-a)^s}{s!}\frac{(-1)^r y^{2r}}{(2r)!} \qquad |z-a| \leq \diamond$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(2r+s)!}{a^{2r+s+1}}\frac{(-1)^s(x-a)^s}{s\,!}\frac{(-1)^r y^{2r+1}}{(2r+1)\,!} \qquad |z-a| \le \diamond$$

Verification (When $a=3$)

$$N\left[\left\{Re\left[Log\left[\frac{9}{2}+i\frac{3}{2}\right]\right],\ u\left[4.5,\frac{3}{2},3,90\right]\right\}\right]$$

{ 1.55676 , 1.55676 }

$$N\left[\left\{Im\left[Log\left[\frac{3}{2}+i\frac{3}{2}\right]\right],\ v\left[\frac{3}{2},\frac{3}{2},3,90\right]\right\}\right]$$

{ 0.785398 , 0.785398 }

log (1+*z*)

$$log(1+z) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}z^s}{s} \qquad |z| \le 1\,,\ z \ne -1$$

$$u(x,y) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}x^s}{s} - \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(2r+s-1)!\frac{(-1)^s x^s}{s\,!}\frac{(-1)^r y^{2r}}{(2r)\,!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s)!\frac{(-1)^s x^s}{s\,!}\frac{(-1)^r y^{2r+1}}{(2r+1)\,!} \qquad |z-a| \le \diamond$$

Verification

$$N\left[\left\{Re\left[Log\left[1+\frac{1}{2}+i\frac{1}{2}\right]\right],\ u\left[\frac{1}{2},\frac{1}{2},90\right]\right\}\right]$$

{ 0.458145 , 0.458145 }

$$N\left[\left\{Im\left[Log\left[1-\frac{1}{2}+i\frac{1}{2}\right]\right],\ v\left[-\frac{1}{2},\frac{1}{2},90\right]\right\}\right]$$

{ 0.785398 , 0.785398 }

log (1-*z*)

$$log(1-z) = -\sum_{s=1}^{\infty}\frac{z^s}{s} \qquad |z| \le 1\,,\ z \ne 1$$

$$u(x,y) = -\sum_{s=1}^{\infty}\frac{x^s}{s} - \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(2r+s-1)!\frac{x^s}{s\,!}\frac{(-1)^r y^{2r}}{(2r)\,!}$$

$$v(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+s)!\frac{x^s}{s\,!}\frac{(-1)^r y^{2r+1}}{(2r+1)\,!} \qquad |z-a| \le \diamond$$

Verification

$$N\left[\left\{Re\left[Log\left[1-\left(\frac{1}{2}+i\frac{1}{2}\right)\right]\right],\ u\left[\frac{1}{2},\frac{1}{2},90\right]\right\}\right]$$

{ -0.346574 , -0.346574 }

$$N\left[\left\{Im\left[Log\left[1-\left(-\frac{1}{2}+i\frac{1}{2}\right)\right]\right],\ f_i\left[-\frac{1}{2},\frac{1}{2},90\right]\right\}\right]$$

{ -0.321751 , -0.321751 }

15.2 Trigonometric Functions

sin z

$$\sin z = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{x^{2s+1}}{(2s+1)!} \frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{x^{2s}}{(2s)!} \frac{y^{2r+1}}{(2r+1)!} \qquad |z| < \infty$$

Verification

```
N[{Re[Sin[7 + 𝕚 8]], u[7, 8, 15]}]
{ 979.225 , 979.225}
```

```
N[{Im[Sin[-7 + 𝕚 8]], v[-7, 8, 15]}]
{ 1123.68 , 1123.68}
```

cos z

$$\cos z = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s}}{(2s)!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{x^{2s}}{(2s)!} \frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{s+1} \frac{x^{2s+1}}{(2s+1)!} \frac{y^{2r+1}}{(2r+1)!} \qquad |z| < \infty$$

Verification

```
N[{Re[Cos[5 + 𝕚 6]], u[5, 6, 15]}]
{ 57.2191 , 57.2191 }
```

```
N[{Im[Cos[-5 + 𝕚 6]], v[-5, 6, 15]}]
{ -193.428 , -193.428 }
```

tan z

$$\tan z = \sum_{s=0}^{\infty} (-1)^s \frac{2^{2s+2}\left(2^{2s+2}-1\right) B_{2s+2}}{2s+2} \frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \frac{\pi}{2}$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{2^{2r+2s+2}\left(2^{2r+2s+2}-1\right) B_{2r+2s+2}}{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{2^{2r+2s+2}\left(2^{2r+2s+2}-1\right) B_{2r+2s+2}}{2r+2s+2} \frac{x^{2s}}{(2s)!} \frac{y^{2r+1}}{(2r+1)!}$$

$|z| <$ ◇

Verification

```
N[{Re[Tan[0.78 + 𝕚 0.78]], u[0.78, 0.78, 600]}]
{ 0.400734 , 0.400734 }

N[{Im[Tan[-0.78 + 𝕚 0.78]], v[-0.78, 0.78, 600]}]
{ 0.91146 , 0.91146 }
```

z cot z

$$z \cot z = \sum_{s=0}^{\infty} (-1)^s 2^{2s} B_{2s} \frac{z^{2s}}{(2s)!} \qquad |z| < \pi$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s 2^{2r+2s} B_{2r+2s} \frac{x^{2s}}{(2s)!} \frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{s+1} 2^{2r+2s+2} B_{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{y^{2r+1}}{(2r+1)!} \qquad |z| < \diamond$$

Verification

```
N[{Re[(1.56 + ⅈ 1.56) Cot[1.56 + ⅈ 1.56]], u[1.56, 1.56, 600]}]
{ 1.43081 , 1.43081 }
N[{Im[(-1.56 + ⅈ 1.56) Cot[-1.56 + ⅈ 1.56]], v[-1.56, 1.56, 600]}]
{ 1.42535 , 1.42535 }
```

sec z

$$\sec z = \sum_{s=0}^{\infty} (-1)^s E_{2s} \frac{z^{2s}}{(2s)!} \qquad |z| < \frac{\pi}{2}$$

$$u(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s E_{2r+2s} \frac{x^{2s}}{(2s)!} \frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{s+1} E_{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{y^{2r+1}}{(2r+1)!} \qquad |z| < \diamond$$

Verification

```
N[{Re[Sec[0.78 + ⅈ 0.78]], u[0.78, 0.78, 500]}]
{ 0.752112 , 0.752112 }
N[{Im[Sec[0.78 + ⅈ 0.78]], v[0.78, 0.78, 500]}]
{ 0.485637 , 0.485637 }
```

z csc z

$$z \csc z = 1 - \sum_{s=1}^{\infty} (-1)^s (2^{2s}-2) B_{2s} \frac{z^{2s}}{(2s)!} \qquad |z| < \pi$$

$$u(x,y) = 1 - \sum_{s=1}^{\infty} (-1)^s (2^{2s}-2) B_{2s} \frac{x^{2s}}{(2s)!}$$

$$- \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} (-1)^s (2^{2r+2s}-2) B_{2r+2s} \frac{x^{2s}}{(2s)!} \frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s (2^{2r+2s+2}-2) B_{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{y^{2r+1}}{(2r+1)!} \qquad |z| < \diamond$$

Verification

```
N[{Re[(1.56 + ⅈ 1.56) Csc[1.56 + ⅈ 1.56]], u[1.56, 1.56, 600]}]
{0.634079, 0.634079}
N[{Im[(-1.56 + ⅈ 1.56) Csc[-1.56 + ⅈ 1.56]], v[-1.56, 1.56, 600]}]
{-0.621668, -0.621668}
```

15.3 Hyperbolic Functions

sinh z

$$\sinh z = \sum_{s=0}^{\infty} \frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad |z| < \infty$$

Verification

```
N[{Re[Sinh[5 + i 7]], u[5, 7, 14]}]
{55.942 , 55.942}
```

```
N[{Im[Sinh[-5 + i 7]], v[-5, 7, 14]}]
{48.7549 , 48.7549}
```

cosh z

$$\cosh z = \sum_{s=0}^{\infty} \frac{z^{2s}}{(2s)!} \qquad |z| < \infty$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad |z| < \infty$$

Verification

```
N[{Re[Cosh[4 + i 5]], u[4, 5, 11]}]
{7.74631 , 7.74631}
```

```
N[{Im[Cosh[-4 + i 5]], v[-4, 5, 11]}]
{26.169 , 26.169}
```

tanh z

$$\tanh z = \sum_{s=0}^{\infty} \frac{2^{2s+2}\left(2^{2s+2}-1\right)B_{2s+2}}{2s+2} \frac{z^{2s+1}}{(2s+1)!} \qquad |x| < \frac{\pi}{2}$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{2^{2r+2s+2}\left(2^{2r+2s+2}-1\right)B_{2r+2s+2}}{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{2^{2r+2s+2}\left(2^{2r+2s+2}-1\right)B_{2r+2s+2}}{2r+2s+2} \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad |z| < \diamond$$

Verification

```
N[{Re[Tanh[0.78 + i 0.78]], u[0.78, 0.78, 600]}]
{0.91146 , 0.91146}
```

```
N[{Im[Tanh[-0.78 + i 0.78]], v[-0.78, 0.78, 600]}]
{0.400734 , 0.400734}
```

15.5 Inverse Hyperbolic Functions

$\sinh^{-1} z$

$$\sinh^{-1} z = \sum_{s=0}^{\infty} (-1)^s \{(2s-1)!!\}^2 \frac{z^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^s \{(2r+2s-1)!!\}^2 \frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^s \{(2r+2s-1)!!\}^2 \frac{x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!} \qquad |z| \le ◇$$

Verification

```
N[{Re[ArcSinh[1/2 + i 1/2]], u[1/2, 1/2, 1200]}]
{0.530638, 0.530638}
N[{Im[ArcSinh[-1/2 + i 1/2]], v[-1/2, 1/2, 1200]}]
{0.452278, 0.452279}
```

$\tanh^{-1} z$

$$\tanh^{-1} z = \sum_{s=0}^{\infty} (2s)! \frac{z^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (2r+2s)! \frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (2r+2s)! \frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad |z| \le ◇$$

Verification

```
N[{Re[ArcTanh[0.49 + i 0.49]], u[0.49, 0.49, 130]}]
{0.398247, 0.398247}
N[{Im[ArcTanh[0.49 + i 0.49]], v[0.49, 0.49, 130]}]
{0.54156, 0.54156}
```

Verification

```
N[{Re[(1.56 + ⅈ 1.56) Csch[1.56 + ⅈ 1.56]], u[1.56, 1.56, 600]}]
                                        {0.634079, 0.634079}
N[{Im[(-1.56 + ⅈ 1.56) Csch[-1.56 + ⅈ 1.56]], v[-1.56, 1.56, 600]}]
                                           {0.621668, 0.621668}
```

15.4 Inverse Trigonometric Functions

$\sin^{-1} z$

$$\sin^{-1} z = \sum_{s=0}^{\infty} \{(2s-1)!!\}^2 \frac{z^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \{(2r+2s-1)!!\}^2 \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \{(2r+2s-1)!!\}^2 \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad |z| \le \diamond$$

Verification

$$\text{N}\left[\left\{\text{Re}\left[\text{ArcSin}\left[\frac{1}{2} + i\frac{1}{2}\right]\right], \text{u}\left[\frac{1}{2}, \frac{1}{2}, 1000\right]\right\}\right]$$

{0.452278 , 0.452279}

$$\text{N}\left[\left\{\text{Im}\left[\text{ArcSin}\left[-\frac{1}{2} + i\frac{1}{2}\right]\right], \text{v}\left[-\frac{1}{2}, \frac{1}{2}, 1000\right]\right\}\right]$$

{0.530638 , 0.530638}

$\cos^{-1} z$

$$\cos^{-1} x = \frac{\pi}{2} - \sum_{s=0}^{\infty} \{(2s-1)!!\}^2 \frac{z^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = \frac{\pi}{2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \{(2r+2s-1)!!\}^2 \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \{(2r+2s-1)!!\}^2 \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!} \qquad |z| \le \diamond$$

Verification

$$\text{N}\left[\left\{\text{Re}\left[\text{ArcCos}\left[\frac{1}{2} + i\frac{1}{2}\right]\right], \text{u}\left[\frac{1}{2}, \frac{1}{2}, 800\right]\right\}\right]$$

{1.11852 , 1.11852}

$$\text{N}\left[\left\{\text{Im}\left[\text{ArcCos}\left[-\frac{1}{2} + i\frac{1}{2}\right]\right], \text{v}\left[-\frac{1}{2}, \frac{1}{2}, 800\right]\right\}\right]$$

{-0.530638 , -0.530639}

$\tan^{-1} z$

$$\tan^{-1} x = \sum_{s=0}^{\infty} (-1)^s (2s)! \frac{x^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2r+2s)!\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2r+2s)!\frac{x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!} \qquad |z| \le \diamond$$

Verification

```
N[{Re[ArcTan[0.49 + i 0.49]], u[0.49, 0.49, 130]}]
{0.54156 , 0.54156}
N[{Im[ArcTan[-0.49 + i 0.49]], v[-0.49, 0.49, 130]}]
{0.398247 , 0.398247}
```

$cot^{-1}z$

$$cot^{-1}z = sign\{Re(z)\}\frac{\pi}{2} - \sum_{s=0}^{\infty}(-1)^s(2s)!\frac{x^{2s+1}}{(2s+1)!} \qquad |z| \le 1$$

$$u(x,y) = sign(x)\frac{\pi}{2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2r+2s)!\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(2r+2s)!\frac{x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!} \qquad |z| \le \diamond$$

Verification

```
N[{Re[ArcCot[0.49 + i 0.49]], u[0.49, 0.49, 130]}]
{1.02924 , 1.02924}
N[{Im[ArcCot[-0.49 + i 0.49]], v[-0.49, 0.49, 130]}]
{-0.398247 , -0.398247}
```

15.5 Inverse Hyperbolic Functions

$sinh^{-1}z$

$$sinh^{-1}z = \sum_{s=0}^{\infty}(-1)^s\{(2s-1)!!\}^2\frac{z^{2s+1}}{(2s+1)!} \qquad |z|\le 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\{(2r+2s-1)!!\}^2\frac{x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\{(2r+2s-1)!!\}^2\frac{x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!}$$

$|z| \le \diamond$

Verification

$$N\left[\left\{Re\left[ArcSinh\left[\frac{1}{2}+i\,\frac{1}{2}\right]\right],\ u\left[\frac{1}{2},\ \frac{1}{2},\ 1200\right]\right\}\right]$$

{0.530638 , 0.530638}

$$N\left[\left\{Im\left[ArcSinh\left[-\frac{1}{2}+i\,\frac{1}{2}\right]\right],\ v\left[-\frac{1}{2},\ \frac{1}{2},\ 1200\right]\right\}\right]$$

{0.452278 , 0.452279}

$tanh^{-1}z$

$$tanh^{-1}z = \sum_{s=0}^{\infty}(2s)!\frac{z^{2s+1}}{(2s+1)!} \qquad |z|\le 1$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+2s)!\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(2r+2s)!\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

$|z| \le \diamond$

Verification

```
N[{Re[ArcTanh[0.49 + i 0.49]], u[0.49, 0.49, 130]}]
```

{0.398247 , 0.398247}

```
N[{Im[ArcTanh[0.49 + i 0.49]], v[0.49, 0.49, 130]}]
```

{0.54156 , 0.54156}

15.6 Laurent Series of cot etc. by Real & Imaginary Parts

cot z

$$\cot z = \frac{1}{z} + \sum_{s=0}^{\infty}(-1)^{s+1}\frac{2^{2s+2}}{2s+2}B_{2s+2}\frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \pi$$

$$u(x,y) = \frac{x}{x^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}}{2r+2s+2}B_{2r+2s+2}\frac{(-1)^s x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$|z| < \diamond$$

$$v(x,y) = -\frac{y}{x^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}}{2r+2s+2}B_{2r+2s+2}\frac{(-1)^s x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!}$$

Verification

```
N[{Re[Cot[1.56 + i 1.56]], u[1.56, 1.56, 70]}]
{0.00174896, 0.00174846}
N[{Im[Cot[-1.56 + i 1.56]], v[-1.56, 1.56, 70]}]
{-0.915438, -0.91544}
```

csc z

$$\csc z = \frac{1}{z} + \sum_{s=0}^{\infty}(-1)^{s}\frac{2^{2s+2}-2}{2s+2}B_{2s+2}\frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \pi$$

$$u(x,y) = \frac{x}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}-2}{2r+2s+2}B_{2r+2s+2}\frac{(-1)^s x^{2s+1}}{(2s+1)!}\frac{y^{2r}}{(2r)!}$$

$$|z| < \diamond$$

$$v(x,y) = -\frac{y}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}-2}{2r+2s+2}B_{2r+2s+2}\frac{(-1)^s x^{2s}}{(2s)!}\frac{y^{2r+1}}{(2r+1)!}$$

Verification

```
N[{Re[Csc[1.56 + i 1.56]], u[1.56, 1.56, 70]}]
{0.402483, 0.402484}
N[{Im[Csc[-1.56 + i 1.56]], v[-1.56, 1.56, 70]}]
{-0.00397797, -0.00397574}
```

coth z

$$\coth z = \frac{1}{z} + \sum_{s=0}^{\infty}\frac{2^{2s+2}}{2s+2}B_{2s+2}\frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \pi$$

$$u(x,y) = \frac{x}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}}{2r+2s+2}B_{2r+2s+2}\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$|z| < \diamond$$

$$v(x,y) = -\frac{y}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{2^{2r+2s+2}}{2r+2s+2}B_{2r+2s+2}\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Verification

```
N[{Re[Coth[-1.5 + ⅈ 1.5]], u[-1.5, 1.5, 85]}]
{-0.905967, -0.905967}
N[{Im[Coth[-1.5 + ⅈ 1.5]], v[-1.5, 1.5, 85]}]
{-0.0127622, -0.0127622}
```

csch z

$$csch\, z = \frac{1}{z} - \sum_{s=0}^{\infty} \frac{2^{2s+2}-2}{2s+2} B_{2s+2} \frac{z^{2s+1}}{(2s+1)!} \qquad |z| < \pi$$

$$u(x,y) = \frac{x}{x^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{2^{2r+2s+2}-2}{2r+2s+2} B_{2r+2s+2} \frac{x^{2s+1}}{(2s+1)!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\frac{y}{x^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{2^{2r+2s+2}-2}{2r+2s+2} B_{2r+2s+2} \frac{x^{2s}}{(2s)!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Verification

```
N[{Re[Csch[-1.5 + ⅈ 1.5]], u[-1.5, 1.5, 70]}]
{-0.0272425, -0.0272436}
N[{Im[Csch[-1.5 + ⅈ 1.5]], v[-1.5, 1.5, 70]}]
{-0.424415, -0.424416}
```

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Alien's Mathematics