08 Taylor series and Maclaurin series

A holomorphic function $f(z)$ defined on a domain can be expanded into the Taylor series around a point a except a singular point. Also, $f(z)$ can be expanded into the Maclaurin series in the open disk with a radius from the origin O to the nearest singularity. When the two convergence circles share the origin O, the Taylor series results in the Maclaurin series. At this time, numerous equations are generated between the coefficients of both.

8.1 Case without a singular point

Theorem 8.1.1

When a function $f(z)$ is holomorphic in the whole domain D, the following expression holds for any $a \in D$.

$$\sum_{s=r}^{\infty} \frac{f^{(s)}(a)}{s\,!}(-1)^{s-r}{}_sC_r\,a^{s-r} = \frac{f^{(r)}(0)}{r\,!} \quad for\ \ r=0,1,2,\cdots$$

Proof

The function $f(z)$ can be expanded into the Maclaurin series in D, and the radius of convergence is $R_m = \infty$. Also, the function $f(z)$ can be expanded into the Taylor series around any $a \in D$ as follows.

$$f(z) = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r\,!}(z-a)^r$$

And this radius of convergence is also $R_t = \infty$. Here,

$$(z-a)^r = \sum_{s=0}^{r}(-1)^{r-s}{}_rC_s\,a^{r-s}z^s$$

Using this,

$$\sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r\,!}(z-a)^r = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r\,!}\sum_{s=0}^{r}(-1)^{r-s}{}_rC_s\,a^{r-s}z^s$$

$$= \sum_{r=0}^{\infty}\sum_{s=r}^{\infty} \frac{f^{(s)}(a)}{s\,!}(-1)^{s-r}{}_sC_r\,a^{s-r}z^r$$

Since this has to be equal to the Maclaurin series in D,

$$\sum_{r=0}^{\infty}\sum_{s=r}^{\infty} \frac{f^{(s)}(a)}{s\,!}(-1)^{s-r}{}_sC_r\,a^{s-r}z^r = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r\,!}z^r$$

Furthermore, since both series converge uniformly in D, these have to be same. If not so, it contradicts to the uniqueness of the series. Therefore,

$$\sum_{s=r}^{\infty} \frac{f^{(s)}(a)}{s\,!}(-1)^{s-r}{}_sC_r\,a^{s-r} = \frac{f^{(r)}(0)}{r\,!} \qquad \begin{cases} for\ any\ a \in D \\ r = 0,1,2,\cdots \end{cases}$$

Example 1 $f(z)=\cos z$

The Maclaurin series and the Taylor series around a are as follows.

$$f_m(z) = \sum_{r=0}^{\infty} \cos\frac{r\pi}{2}\cdot\frac{z^r}{r\,!}$$

$$f_t(z) = \sum_{r=0}^{\infty} \cos\left(a+\frac{r\pi}{2}\right)\cdot\frac{(z-a)^r}{r\,!}$$

$$= \sum_{r=0}^{\infty}\sum_{s=r}^{\infty}\frac{(-1)^{s-r}}{s!}\cos\left(a+\frac{s\pi}{2}\right){}_sC_r\,a^{s-r}z^r$$

These radius of convergence are $R_m = R_t = \infty$. Then, for any a , the following holds.

$$\sum_{s=r}^{\infty}\frac{(-1)^{s-r}}{s!}\cos\left(a+\frac{s\pi}{2}\right){}_sC_r\,a^{s-r} = \frac{1}{r!}\cos\frac{r\pi}{2} \qquad for\ r=0,1,2,\cdots$$

In fact, when $c_t(r,a)$ denotes the left side and $c_m(r)$ denotes the right side, each values of $c_t(100,a)$ at $a=\pi,\ -e,\ i$ are as follows.

```
N[{ct[100, π]  , ct[100, -e]   , ct[100, i]}]
{1.07151×10^-158, 1.07151×10^-158, 1.07151×10^-158}
```

Moreover, the following shows that the value of both sides are equal for $r=50, 51$ at $a=-i$.

```
N[{ct[50, -i]  , cm[50]}]
{-3.28795×10^-65 , -3.28795×10^-65}

N[{ct[51, -i]  , cm[51]}]
{0. + 0. i         , 0.}
```

Special values

When $r=0$,

$$\sum_{s=0}^{\infty}\frac{(-1)^{s}}{s!}a^s\cos\left(a+\frac{s\pi}{2}\right) = 1 \qquad for\ any\ a$$

i.e.

$$\frac{a^0\cos a}{0!}+\frac{a^1\sin a}{1!}-\frac{a^2\cos a}{2!}-\frac{a^3\sin a}{3!}++--\cdots = 1$$

e.g.

$$\frac{1}{0!}\left(\frac{\pi}{4}\right)^0+\frac{1}{1!}\left(\frac{\pi}{4}\right)^1-\frac{1}{2!}\left(\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(\frac{\pi}{4}\right)^3++--\cdots = \sqrt{2}$$

When $r=1$,

$$\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{(s-1)!}a^{s-1}\cos\left(a+\frac{s\pi}{2}\right) = 0 \qquad for\ any\ a$$

i.e.

$$\frac{a^1\cos a}{1!}+\frac{a^2\sin a}{2!}-\frac{a^3\cos a}{3!}-\frac{a^4\sin a}{4!}++--\cdots = \sin a$$

e.g.

$$\frac{1}{1!}\left(\frac{\pi}{4}\right)^1+\frac{1}{2!}\left(\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(\frac{\pi}{4}\right)^3-\frac{1}{4!}\left(\frac{\pi}{4}\right)^4++--\cdots = 1$$

Example 2 $\sinh z$

The Maclaurin series and the Taylor series around a are as follows.

$$f_m(z) = \sum_{r=0}^{\infty} \frac{1-(-1)^{-r}}{2} \frac{z^r}{r!}$$

$$f_t(z) = \sum_{r=0}^{\infty} \frac{e^a - (-1)^{-r} e^{-a}}{2} \frac{(z-a)^r}{r!}$$

From these,

$$\sum_{s=r}^{\infty} \frac{e^a - (-1)^{-s} e^{-a}}{2} \frac{(-1)^{s-r}}{s!} {}_sC_r a^{s-r} = \frac{1-(-1)^{-r}}{2} \frac{1}{r!} \qquad for\ \ r=0,1,2,\cdots$$

When $r=0$,

$$\sum_{s=0}^{\infty} (-1)^s \frac{e^a - (-1)^{-s} e^{-a}}{2} \frac{a^s}{s!} = 0 \qquad for\ any\ a$$

i.e.

$$\frac{a^0 \sinh a}{0!} - \frac{a^1 \cosh a}{1!} + \frac{a^2 \sinh a}{2!} - \frac{a^3 \cosh a}{3!} +-\cdots = 0$$

When $r=1$

$$\sum_{s=0}^{\infty} (-1)^s \frac{e^a - (-1)^{-s-1} e^{-a}}{2} \frac{a^s}{s!} = 1 \qquad for\ any\ a$$

i.e.

$$\frac{a^0 \cosh a}{0!} - \frac{a^1 \sinh a}{1!} + \frac{a^2 \cosh a}{2!} - \frac{a^3 \sinh a}{3!} +-\cdots = 1$$

Example 3 $(1+z)e^z$

The Maclaurin series and the Taylor series around a are as follows.

$$f_m(z) = \sum_{r=0}^{\infty} \frac{r+1}{r!} z^r$$

$$f_t(z) = e^a \sum_{r=0}^{\infty} \frac{r+1+a}{r!} (z-a)^r$$

From these,

$$e^a \sum_{s=r}^{\infty} \frac{s+1+a}{s!} (-1)^{s-r} {}_sC_r a^{s-r} = \frac{r+1}{r!} \qquad for\ \ r=0,1,2,\cdots$$

When $r=0$,

$$\sum_{s=0}^{\infty} \frac{(s+1-a)a^s}{s!} = e^a \qquad for\ any\ a$$

i.e.

$$\frac{(1-a)a^0}{0!} + \frac{(2-a)a^1}{1!} + \frac{(3-a)a^2}{2!} + \frac{(4-a)a^3}{3!} + \cdots = e^a$$

When $r=1$,

$$\sum_{s=0}^{\infty} \frac{(s+2-a)a^s}{s!} = 2e^a \qquad for\ any\ a$$

i.e.

$$\frac{(2-a)a^0}{0!}+\frac{(3-a)a^1}{1!}+\frac{(4-a)a^2}{2!}+\frac{(5-a)a^3}{3!}+\cdots=2e^a$$

Generally,

$$\sum_{s=0}^{\infty}\frac{(b-a+s)a^s}{s!}=be^a \qquad \textit{for any } a,b$$

i.e.

$$\frac{(b-a)a^0}{0!}+\frac{(b-a+1)a^1}{1!}+\frac{(b-a+2)a^2}{2!}+\frac{(b-a+3)a^3}{3!}+\cdots=be^a$$

In fact, calculation results giving various values to b,a are as follows.

```
fn[b_, a_] := Sum[((b - a + s) a^s)/s!, {s, 0, 100}]
```

$$\texttt{fn[b\_, a\_] :=} \sum_{s=0}^{100}\frac{(b-a+s)\,a^s}{s!}$$

```
N[{fn[7.1, 5], 7.1 e^5}]
{1053.73        , 1053.73 }

N[{fn[-0.1, π], -0.1 e^π}]
{-2.31407        , -2.31407 }

N[{fn[π, i]        , π e^i}]
{1.69741+2.64356 i , 1.69741+2.64356 i}
```

8.2 Case with a singular point

Theorem 8.2.1

When a function $f(z)$ is holomorphic in a domain D except a singular point p, the following expression holds for a s.t. $|a| < |p-a|$

$$\sum_{s=r}^{\infty} \frac{f^{(s)}(a)}{s!}(-1)^{s-r}{}_sC_r a^{s-r} = \frac{f^{(r)}(0)}{r!} \qquad for \ \ r=0,1,2,\cdots$$

Where, the singular point p is assumed to be nearest to the origin O and the point a.

Proof

The function $f(z)$ can be expanded into the Maclaurin series in a circle C_m of the radius $R_m = |p|$ as follows.

$$f(z) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} z^r \qquad\qquad |z| < R_m = |p|$$

Also, the function $f(z)$ can be expanded into the Taylor series around a except p as follows.

$$f(z) = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} (z-a)^r \qquad |z-a| < R_t = |p-a|$$

The center of the convergence circle C_t is a, and the radius is $R_t = |p-a|$.

Here,

$$(z-a)^r = \sum_{s=0}^{r} (-1)^{r-s}{}_rC_s a^{r-s} z^s.$$

Using this,

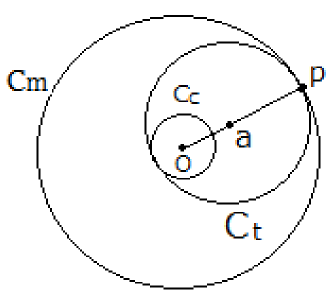
$$\sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!}(z-a)^r = \sum_{r=0}^{\infty}\sum_{s=r}^{\infty} \frac{f^{(s)}(a)}{s!}(-1)^{s-r}{}_sC_r a^{s-r} z^r$$

Since this has to be equal to the Maclaurin series in $C_m \cap C_t$,

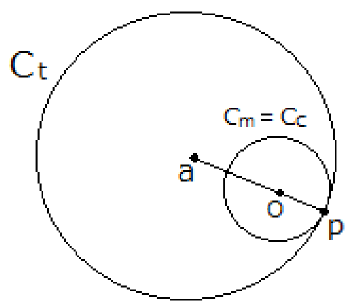
$$\sum_{r=0}^{\infty}\sum_{s=r}^{\infty} \frac{f^{(s)}(a)}{s!}(-1)^{s-r}{}_sC_r a^{s-r} z^r = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} z^r$$

When $|a| < |p-a|$, since the convergence circle C_t of the Taylor series includes the origin O, a circle C_c around the origin O exists touching the circle C_t internally. (The radius is $|p-a| - |a|$.)

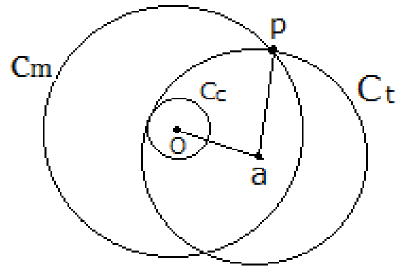
Case 1 : Cm ⊃ Ct



Case 2 : Cm ⊂ Ct



Case 3 : Cm ∩ Ct ≠ Φ



Since both series converge uniformly in C_c, these have to be same. If not so, it contradicts to the uniqueness of the series. Therefore,

$$\sum_{s=r}^{\infty}\frac{f^{(s)}(a)}{s\,!}(-1)^{s-r}{}_{s}C_{r}\,a^{s-r}=\frac{f^{(r)}(0)}{r\,!}\qquad\begin{cases}|a|<|p-a|\\ r=0,1,2,\cdots\end{cases}$$

Conditions for the theorem

Let $a=\alpha+i\beta\,,\ p=\mathrm{p}+i\mathrm{q}$. Then $|a|<|p-a|$ reduces to the following expression.

$$\mathrm{p}(\mathrm{p}-2\alpha)+\mathrm{q}(\mathrm{q}-2\beta)>0$$

Especially, when $\mathrm{q}=0$,

if $\mathrm{p}>0$ then $\alpha<\mathrm{p}/2$

if $\mathrm{p}<0$ then $\alpha>\mathrm{p}/2$

Especially, when $\mathrm{p}=0$,

if $\mathrm{q}>0$ then $\beta<\mathrm{p}/2$

if $\mathrm{q}<0$ then $\beta>\mathrm{p}/2$

8.3 Example of Case1 : $C_m \supset C_t$

This is a case where the convergence circle C_m of the Maclaurin series includes the convergence circle C_t of the Taylor series. We consider the following function as the example.

$$f(z) = tanh^{-1}z$$

The higher order derivative is given by the following expression. (See " 岩波 数学公式 I " p36,)

$$\left(tanh^{-1}z\right)^{(n)} = \frac{(n-1)!}{2}\left\{\frac{1}{(1-z)^n}+\frac{(-1)^{n-1}}{(1+z)^n}\right\} \qquad n=1,2,3,\cdots$$

The Maclaurin series is

$$f_m(z)= \sum_{r=0}^{\infty}\frac{f^{(r)}(0)}{r!}z^r = \frac{f^{(0)}(0)}{0!}z^0 + \sum_{r=1}^{\infty}\frac{f^{(r)}(0)}{r!}z^r$$

From this and the above,

$$f_m(z) = tanh^{-1}0 + \sum_{r=1}^{\infty}\frac{1+(-1)^{r-1}}{2r}z^r$$

On the other hand, the Taylor series around a is

$$\begin{aligned}f_t(z) &= \sum_{r=0}^{\infty}\sum_{s=r}^{\infty}\frac{f^{(s)}(a)}{s!}(-1)^{s-r}{}_sC_r a^{s-r}z^r \\ &= \sum_{s=0}^{\infty}\frac{f^{(s)}(a)}{s!}(-1)^s{}_sC_0 a^s z^0 + \sum_{r=1}^{\infty}\sum_{s=r}^{\infty}\frac{f^{(s)}(a)}{s!}(-1)^{s-r}{}_sC_r a^{s-r}z^r \\ &= f^{(0)}(a) + \sum_{s=1}^{\infty}\frac{f^{(s)}(a)}{s!}(-1)^s a^s + \sum_{r=1}^{\infty}\sum_{s=r}^{\infty}\frac{f^{(s)}(a)}{s!}(-1)^{s-r}{}_sC_r a^{s-r}z^r\end{aligned}$$

From this and the above,

$$\begin{aligned}f_t(z) &= tanh^{-1}a + \sum_{s=1}^{\infty}\frac{(-1)^s}{2s}\left\{\frac{1}{(1-a)^s}+\frac{(-1)^{s-1}}{(1+a)^s}\right\}a^s \\ &\quad + \sum_{r=1}^{\infty}\sum_{s=r}^{\infty}\frac{(-1)^{s-r}}{2s}\left\{\frac{1}{(1-a)^s}+\frac{(-1)^{s-1}}{(1+a)^s}\right\}{}_sC_r a^{s-r}z^r\end{aligned}$$

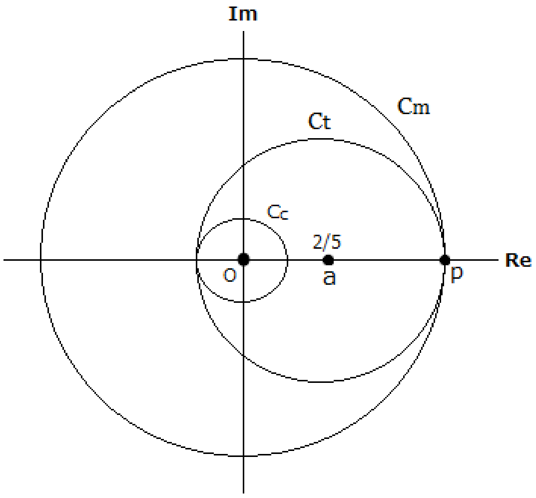
Then, these coefficients $c_m(r)$, $c_t(r,a)$ are as follows.

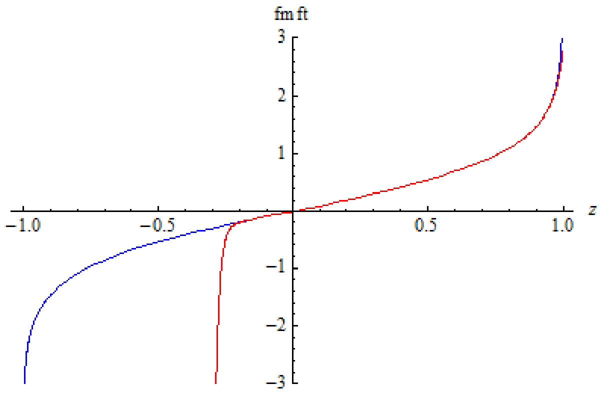
$$c_m(r) = \begin{cases} tanh^{-1}0 & r=0 \\ \dfrac{1+(-1)^{r-1}}{2r} & r>0 \end{cases} \tag{1.m}$$

$$c_t(r,a) = \begin{cases} tanh^{-1}a + \displaystyle\sum_{s=1}^{\infty}\frac{(-1)^s}{2s}\left\{\frac{1}{(1-a)^s}+\frac{(-1)^{s-1}}{(1+a)^s}\right\}a^s & r=0 \\ \displaystyle\sum_{s=r}^{\infty}\frac{(-1)^{s-r}}{2s}\left\{\frac{1}{(1-a)^s}+\frac{(-1)^{s-1}}{(1+a)^s}\right\}{}_sC_r a^{s-r} & r>0 \end{cases} \tag{1.t}$$

Since the $f(z)$ has the singular point at $z=\pm 1$, the convergence radius of the Maclaurin series is $R_m=1$ and the convergence radius of the Taylor series is $R_t=|1-a|$. Therefore, if $|Re(a)|<1/2$, $|a|<|1-a|$.

For example, if $a = 2/5 = 0.4$ then $|a|<|1-a|$.





Therefore, the following equations have to hold.

$$\tanh^{-1}a+\sum_{s=1}^{\infty}\frac{(-1)^s}{2s}\left\{\frac{1}{(1-a)^s}+\frac{(-1)^{s-1}}{(1+a)^s}\right\}a^s=\tanh^{-1}0 \qquad r=0$$

$$\sum_{s=r}^{\infty}\frac{(-1)^{s-r}}{2s}\left\{\frac{1}{(1-a)^s}+\frac{(-1)^{s-1}}{(1+a)^s}\right\}{}_sC_r\,a^{s-r}=\frac{1+(-1)^{r-1}}{2r} \qquad r=1,2,3,\cdots$$

In fact, when $a=2/5$, the result that the series of the left side was calculated to the 10000 th term for $r=0$ and $r=1,\cdots,4$ are as follows. We can see that even number terms of the Taylor series are almost 0.

r = 0

$$\mathrm{ct0}[a\_] := \mathrm{ArcTanh}[a]+\sum_{s=1}^{10000}\frac{(-1)^s}{2\,s}\left(\frac{1}{(1-a)^s}+\frac{(-1)^{s-1}}{(1+a)^s}\right)a^s$$

$$\mathrm{N}\left[\left\{\mathrm{ct0}\left[\frac{2}{5}\right],\ \mathrm{ArcTanh}[0]\right\}\right]$$

{0. , 0.}

r = 1,2,3,4

$$\mathrm{cm}[r\_] := \frac{1+(-1)^{r-1}}{2\,r}$$

$$\mathrm{ct}[r\_,\ a\_] := \sum_{s=r}^{10000}\frac{(-1)^{s-r}}{2\,s}\left(\frac{1}{(1-a)^s}+\frac{(-1)^{s-1}}{(1+a)^s}\right)\mathrm{Binomial}[s,\ r]\ a^{s-r}$$

N[Table[cm[r], {r, 1, 4}]]

{1. , 0. , 0.333333 , 0.}

$$\mathrm{N}\left[\mathrm{Table}\left[\mathrm{ct}\left[r,\ \frac{2}{5}\right],\ \{r,\ 1,\ 4\}\right]\right]$$

$$\{1.\ ,\ 7.643144817559070\times10^{-1758}\ ,\ 0.333333\ ,\ 3.979291991829457\times10^{-1750}\}$$

Special values

We obtain the following formula from the avobe example.

Formula 8.3.1

$$tanh^{-1}a = \sum_{s=1}^{\infty}\frac{1}{2s}\left\{\left(\frac{a}{1+a}\right)^{s} - (-1)^{s}\left(\frac{a}{1-a}\right)^{s}\right\} \qquad |Re(a)| < \frac{1}{2}$$

Proof

From (1.m) , (1.t) at $r=0$,

$$tanh^{-1}a + \sum_{s=1}^{\infty}\frac{(-1)^{s}}{2s}\left\{\frac{1}{(1-a)^{s}}+\frac{(-1)^{s-1}}{(1+a)^{s}}\right\}a^{s} = tanh^{-1}0 = 0$$

i.e.

$$tanh^{-1}a = -\sum_{s=1}^{\infty}\frac{(-1)^{s}}{2s}\left\{\left(\frac{a}{1-a}\right)^{s}+(-1)^{s-1}\left(\frac{a}{1+a}\right)^{s}\right\} \qquad |Re(a)| < \frac{1}{2}$$

Here, replacing a with $-a$,

$$tanh^{-1}(-a) = -\sum_{s=1}^{\infty}\frac{1}{2s}\left\{\left(\frac{a}{1+a}\right)^{s} - (-1)^{s}\left(\frac{a}{1-a}\right)^{s}\right\} \qquad |Re(a)| < \frac{1}{2}$$

Since this function is an odd function, $tanh^{-1}(-z) = -tanh^{-1}z$. Therefore,

$$tanh^{-1}a = \sum_{s=1}^{\infty}\frac{1}{2s}\left\{\left(\frac{a}{1+a}\right)^{s} - (-1)^{s}\left(\frac{a}{1-a}\right)^{s}\right\} \qquad |Re(a)| < \frac{1}{2}$$

Examples

$$tanh^{-1}\frac{1}{2} = \frac{1}{2}\left(\frac{1}{3^{1}}+1\right)+\frac{1}{4}\left(\frac{1}{3^{2}}-1\right)+\frac{1}{6}\left(\frac{1}{3^{3}}+1\right)+\frac{1}{8}\left(\frac{1}{3^{4}}-1\right)+\cdots$$

$$tanh^{-1}\frac{1}{3} = \frac{1}{2}\left(\frac{1}{4^{1}}+\frac{1}{2^{1}}\right)+\frac{1}{4}\left(\frac{1}{4^{2}}-\frac{1}{2^{2}}\right)+\frac{1}{6}\left(\frac{1}{4^{3}}+\frac{1}{2^{3}}\right)+\frac{1}{8}\left(\frac{1}{4^{4}}-\frac{1}{2^{4}}\right)+\cdots$$

8.4 Example of Case2 : $C_m \subset C_t$

This is a case where the convergence circle C_m of the Maclaurin series is included in the convergence circle C_t of the Taylor series. We consider the following function as the example.

$$f(z) = \frac{5}{5-z}$$

The Maclaurin series and the Taylor series around a are as follows.

$$f_m(z) = \sum_{r=0}^{\infty} \frac{1}{5^r} z^r$$

$$f_t(z) = \sum_{r=0}^{\infty} \frac{5}{(5-a)^{r+1}} (z-a)^r$$

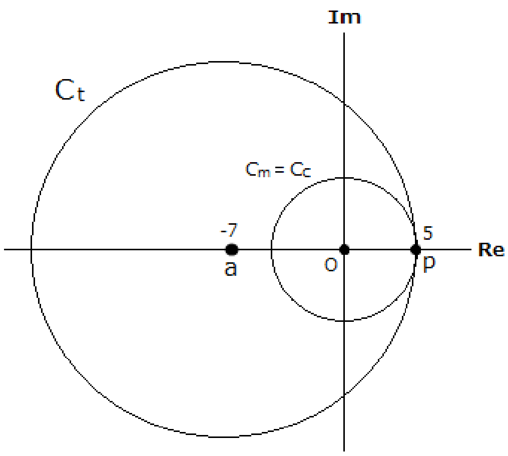
$$= \sum_{r=0}^{\infty} \sum_{s=r}^{\infty} \frac{5}{(5-a)^{s+1}} (-1)^{s-r} {}_sC_r a^{s-r} z^r$$

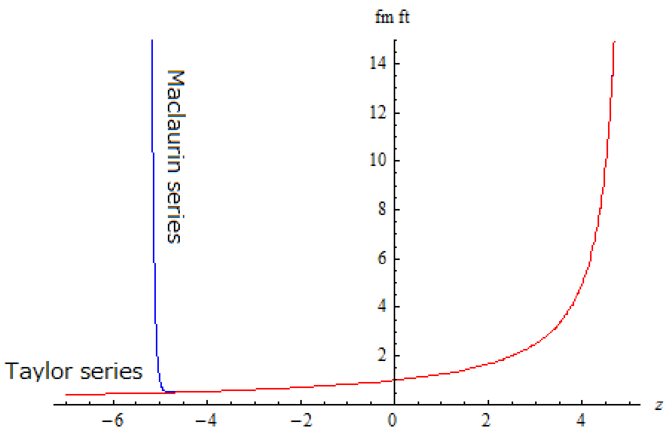
Then, these coefficients $c_m(r)$, $c_t(r,a)$ are as follows.

$$c_m(r) = \frac{1}{5^r} \tag{1.m}$$

$$c_t(r,a) = \sum_{s=r}^{\infty} \frac{5}{(5-a)^{s+1}} (-1)^{s-r} {}_sC_r a^{s-r} \tag{1.t}$$

Since the $f(z)$ has the singular point at $z=5$, the convergence radius of the Maclaurin series is $R_m=5$ and the convergence radius of the Taylor series is $R_t = |5-a|$. Therefore, if $Re(a) < 5/2$, $|a| < |p-a|$. For example, if $a = -7$ then $|a| < |5-a|$.





Therefore, the following equation has to hold.

$$\sum_{s=r}^{\infty} \frac{5}{(5-a)^{s+1}} (-1)^{s-r} {}_sC_r a^{s-r} = \frac{1}{5^r} \qquad \begin{cases} Re(a) < 5/2 \\ r = 0, 1, 2, \cdots \end{cases}$$

In fact, when $a = -7$, the result that the series of the left side was calculated to the 100 th term for $r = 0, 1, \cdots, 7$ are as follows. We can see that coefficients of the Taylor series and the Maclaurin series are consistent.

```
cm[r_] := 1/5^r
ct[r_, a_] := Sum[5/(5 - a)^(s+1) (-1)^(s-r) Binomial[s, r] a^(s-r), {s, r, 100}]
N[Table[cm[r], {r, 0, 7}]]
{1., 0.2, 0.04, 0.008, 0.0016, 0.00032, 0.000064, 0.0000128}
N[Table[ct[r, -7], {r, 0, 7}]]
{1., 0.2, 0.04, 0.008, 0.0016, 0.00032, 0.000064, 0.0000128}
```

Geometric Series with coefficients

From the avobe example, we obtain the following formula.

Formula 8.4.1

$$\sum_{s=r}^{\infty}(-1)^{-r}\frac{{}_sC_r}{x^s} = -\left(\frac{1}{1-x}\right)^r\frac{x}{1-x} \qquad \begin{cases}|x| > 1 \\ r = 0,1,2,\cdots\end{cases}$$

Especially, when $r=1$,

$$\sum_{s=1}^{\infty}\frac{s}{x^s} = \frac{x}{(1-x)^2} \qquad |x|>1$$

Proof

Replacing the convergence radius 5 with 1 in the avobe example, we obtain

$$\sum_{s=r}^{\infty}\frac{1}{(1-a)^{s+1}}(-1)^{s-r}{}_sC_r a^{s-r} = \frac{1}{1^r} \qquad \begin{cases}Re(a)<1/2 \\ r = 0,1,2\cdots\end{cases}$$

Transform this as follows.

$$\sum_{s=r}^{\infty}(-1)^{-r}{}_sC_r\left(\frac{a}{a-1}\right)^s = -a^r(a-1) \qquad \begin{cases}Re(a)<1/2 \\ r = 0,1,2,\cdots\end{cases}$$

Here, let

$$\frac{a}{a-1} = \frac{1}{x}$$

Then

$$a = \frac{1}{1-x} \quad , \qquad a-1 = \frac{x}{1-x}$$

Substituting these for theavobe,

$$\sum_{s=r}^{\infty}(-1)^{-r}{}_sC_r\frac{1}{x^s} = -\left(\frac{1}{1-x}\right)^r\frac{x}{1-x} \qquad \begin{cases}|x| > 1 \\ r = 0,1,2,\cdots\end{cases} \text{reduce}$$

As long as $|a|<1/2$, $|x|>1$ is guaranteed.

When $r=0$,

$$\sum_{s=0}^{\infty}\frac{1}{x^s} = -\frac{x}{1-x} \qquad |x|>1$$

This is the usual geometric series.

When $r=1$,

$$\sum_{s=1}^{\infty}\frac{s}{x^s} = \frac{x}{(1-x)^2} \qquad |x|>1$$

This is the geometric series with coefficients.

Example 1

$$\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots = \frac{2}{1^2}$$

$$\frac{1}{3^1}+\frac{2}{3^2}+\frac{3}{3^3}+\frac{4}{3^4}+\cdots = \frac{3}{2^2}$$

$$\vdots$$

Example 2

$$\frac{1}{2^1}-\frac{2}{2^2}+\frac{3}{2^3}-\frac{4}{2^4}+-\cdots = \frac{2}{3^2}$$

$$\frac{1}{3^1}-\frac{2}{3^2}+\frac{3}{3^3}-\frac{4}{3^4}+-\cdots = \frac{3}{4^2}$$

$$\vdots$$

Example 3

$$\frac{1}{1.3^1}+\frac{2}{1.3^2}+\frac{3}{1.3^3}+\frac{4}{1.3^4}+\cdots = \frac{1.3}{0.3^2}$$

$$\frac{1}{1.3^1}-\frac{2}{1.3^2}+\frac{3}{1.3^3}-\frac{4}{1.3^4}+-\cdots = \frac{1.3}{2.3^2}$$

$$\frac{1}{(1+i)^1}+\frac{2}{(1+i)^2}+\frac{3}{(1+i)^3}+\frac{4}{(1+i)^4}+\cdots = \frac{1+i}{i^2}$$

$$\frac{1}{(1+i)^1}-\frac{2}{(1+i)^2}+\frac{3}{(1+i)^3}-\frac{4}{(1+i)^4}+-\cdots = \frac{1+i}{(2+i)^2}$$

8.5 Example of Case3 : $C_m \cap C_t \neq \phi$

This is a case where the convergence circle C_m of the Maclaurin series and the convergence circle C_t of the Taylor series are overlapping partially. We consider the following function as the example.

$$f(z) = \frac{1}{1+z^2}$$

The higher order derivative is given by the following expression. (See " 岩波 数学公式 I " p32,)

$$\left(\frac{1}{1+z^2}\right)^{(n)} = n!(-1)^n\left(1+z^2\right)^{-\frac{n+1}{2}} \sin\left\{(n+1)\cot^{-1}z\right\} \qquad Re(z) \geqq 0$$

Substituting this for **Theorem 8.2.1** ,

$$c_m(r) = \frac{f^{(r)}(0)}{r!} = \frac{1}{r!}r!(-1)^r\left(1+0^2\right)^{-\frac{r+1}{2}} \sin\left\{(r+1)\cot^{-1}0\right\}$$

$$= (-1)^r \sin\left\{(r+1)\cot^{-1}0\right\}$$

$$c_t(r,a) = \sum_{s=r}^{\infty} \frac{f^{(s)}(a)}{s!}(-1)^{s-r}{}_sC_r a^{s-r}$$

$$= \sum_{s=r}^{\infty} \frac{1}{s!}s!(-1)^s\left(1+a^2\right)^{-\frac{s+1}{2}} \sin\left\{(s+1)\cot^{-1}a\right\}(-1)^{s-r}{}_sC_r a^{s-r}$$

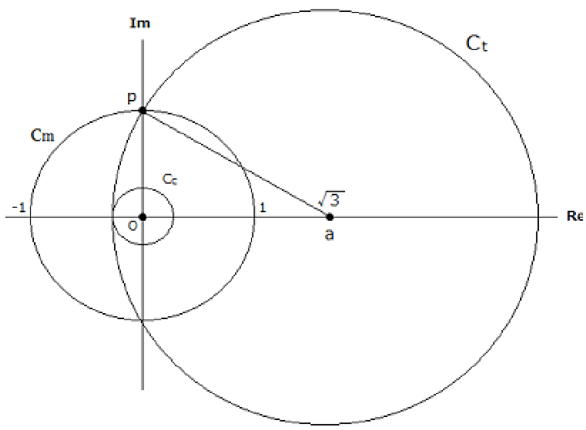
i.e.

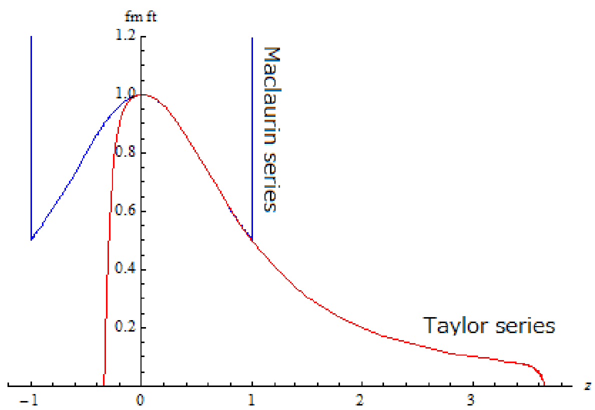
$$c_m(r) = (-1)^r \sin\frac{(r+1)\pi}{2} \tag{1.m}$$

$$c_t(r,a) = \sum_{s=r}^{\infty} (-1)^s\left(1+a^2\right)^{-\frac{s+1}{2}} \sin\left\{(s+1)\cot^{-1}a\right\}(-1)^{s-r}{}_sC_r a^{s-r} \tag{1.t}$$

Since the $f(z)$ has the singular point at $z = \pm i$, the convergence radius of the Maclaurin series is $R_m = 1$ and the convergence radius of the Taylor series is $R_t = |i-a|$. Therefore, if $|Im(a)| < 1/2$, $|a| < |i-a|$.

For example, if $= \sqrt{3} = 0.173\cdots$ then $|a| < |i-a|$.





Therefore, the following equation has to hold.

$$\sum_{s=r}^{\infty} (-1)^s\left(1+a^2\right)^{-\frac{s+1}{2}} \sin\left\{(s+1)\cot^{-1}a\right\}(-1)^{s-r}{}_sC_r a^{s-r} = (-1)^r \sin\frac{(r+1)\pi}{2}$$

In fact, when $a=\sqrt{3}$, the result that the series of the left side was calculated to the 10000 th term for $r=0,1,\cdots,3$ are as follows. We can see that odd number terms of the Taylor series are almost 0.

```
cm[r_] := (-1)^r Sin[(r+1) π / 2]

ct[r_, a_] := Sum[(-1)^s (1 + a^2)^(-(s+1)/2) Sin[(s+1) ArcCot[a]] (-1)^(s-r) Binomial[s, r] a^(s-r), {s, r, 10000}]

Table[cm[r], {r, 0, 3}]
{1, 0, -1, 0}

N[Table[ct[r, √3], {r, 0, 3}]]
{1. , -8.768078810462416×10^-622 , -1. , -4.871641133382931×10^-615 }
```

Alternating series of powers of $1/2$

From the avobe example, we obtain the following formula.

Formula 8.5.1

$$\frac{1}{2^0}+\frac{1}{2^1}-\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^6}+\frac{1}{2^7}-\frac{1}{2^9}-\frac{1}{2^{10}}++--\cdots=\frac{4}{3}$$

$$\frac{0}{2^0}+\frac{1}{2^1}-\frac{3}{2^3}-\frac{4}{2^4}+\frac{6}{2^6}+\frac{7}{2^7}-\frac{9}{2^9}-\frac{10}{2^{10}}++--\cdots=0$$

Proof

Substituting $a=1/\sqrt{3}$ for (1.t),

$$c_t\left(r,\frac{1}{\sqrt{3}}\right)=\sum_{s=r}^{\infty}(-1)^s\left(1+\frac{1}{3}\right)^{-\frac{s+1}{2}}\sin\left\{(s+1)\cot^{-1}\frac{1}{\sqrt{3}}\right\}(-1)^{s-r}{}_sC_r\left(\frac{1}{\sqrt{3}}\right)^{s-r}$$

$$=\sum_{s=r}^{\infty}(-1)^s\sin\frac{(s+1)\pi}{3}(-1)^{s-r}{}_sC_r\left(\frac{4}{3}\right)^{-\frac{s+1}{2}}\left(\frac{1}{\sqrt{3}}\right)^{s-r}$$

$$=\sum_{s=r}^{\infty}(-1)^s\sin\frac{(s+1)\pi}{3}(-1)^{s-r}{}_sC_r\frac{(\sqrt{3})^{s+1}}{2^{s+1}}\frac{(\sqrt{3})^r}{(\sqrt{3})^s}$$

$$=(\sqrt{3})^r\frac{\sqrt{3}}{2}\sum_{s=r}^{\infty}(-1)^s\sin\frac{(s+1)\pi}{3}(-1)^{s-r}\frac{{}_sC_r}{2^s}$$

Here, using Dirichlet character $\chi(m,j)_r$ $m:modulus, j:index$,

$$(-1)^s\sin\frac{(s+1)\pi}{3}=\frac{\sqrt{3}}{2}\chi(3,2)_{s+1}$$

Then

$$c_t\left(r,\frac{1}{\sqrt{3}}\right)=\frac{3}{4}(\sqrt{3})^r\sum_{s=r}^{\infty}(-1)^{s-r}\chi(3,2)_{s+1}\frac{{}_sC_r}{2^s}$$

$$c_m(r)=(-1)^r\sin\frac{(r+1)\pi}{2}$$

When $r=0$,

$$c_t\left(0,\frac{1}{\sqrt{3}}\right)=\frac{3}{4}\left(\sqrt{3}\right)^0\sum_{s=0}^{\infty}(-1)^{s-0}\chi(3,2)_{s+1}\frac{{}_sC_0}{2^s}$$

$$c_m(0)=(-1)^0 sin\frac{(0+1)\pi}{2}=1$$

From these,

$$\sum_{s=0}^{\infty}(-1)^s\chi(3,2)_{s+1}\frac{1}{2^s}=\frac{4}{3}$$

i.e.

$$\frac{1}{2^0}+\frac{1}{2^1}-\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^6}+\frac{1}{2^7}-\frac{1}{2^9}-\frac{1}{2^{10}}++--\cdots=\frac{4}{3}$$

When $r=1$,

$$c_t\left(1,\frac{1}{\sqrt{3}}\right)=\frac{3}{4}\left(\sqrt{3}\right)^1\sum_{s=1}^{\infty}(-1)^{s-1}\chi(3,2)_{s+1}\frac{{}_sC_1}{2^s}$$

$$c_m(1)=(-1)^1 sin\frac{(1+1)\pi}{2}=0$$

From these,

$$\sum_{s=1}^{\infty}(-1)^s\chi(3,2)_{s+1}\frac{s}{2^s}=0$$

i.e.

$$\frac{1}{2^1}-\frac{3}{2^3}-\frac{4}{2^4}+\frac{6}{2^6}+\frac{7}{2^7}-\frac{9}{2^9}-\frac{10}{2^{10}}+\frac{12}{2^{12}}+--+\cdots=0$$

Adding $0/2^0$ to the both sides, we obtain the desired expression.

8.6 Sum of Stieltjes constants

In this section, we prove the formula of sum of Stieltjes constants. It seems that this formula was discovered by ***O. Marichev*** by 2008. (http://mathworld.wolfram.com/StieltjesConstants.html) . Although I do not know how Dr.Marichev proved this, the proof performed in this section is a very good example of application of the **Theorem 8.1.1** .

Formula 8.6.1 (*O. Marichev*)

When γ_s are Stieltjes constants and $\zeta^{(n)}(0)$ is the n-th order differential coefficient of Riemann zeta function, The following expression holds.

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+n}}{s\,!} = (-1)^n\left\{n! + \zeta^{(n)}(0)\right\} \qquad n=0,1,2,\cdots \tag{1.n}$$

Proof

It is known that the function $f(z) = (z-1)\zeta(z)$ is expanded to a Taylor series as follows.

$$(z-1)\zeta(z) = 1 + \sum_{s=1}^{\infty}(-1)^{s-1}\frac{s\gamma_{s-1}}{s\,!}(z-1)^s \qquad \gamma_s : \textit{Stieltjes constant}$$

Then

$$\frac{f^{(s)}(1)}{s\,!} = \begin{cases} 1 & s=0 \\ \dfrac{(-1)^{s-1}s\gamma_{s-1}}{s\,!} & s>0 \end{cases} \tag{1.t}$$

Next, the Maclaurin expansion of $f(z)$ is as follows.

$$(z-1)\zeta(z) = \sum_{r=0}^{\infty}\frac{f^{(r)}(0)}{r\,!}z^r$$

Since $f(z)$ is a product of two functions, the higher order derivative is given by the following **Leibniz rule** .

$$\{g(z)h(z)\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r}g^{(n-r)}(z)h^{(r)}(z)$$

Here, put $g(z) = z-1$, $h(z) = \zeta(z)$, then

$$g^{(0)}(z) = z-1 \ , \ g^{(1)}(z) = 1 \ , \ g^{(s)}(z) = 0 \quad (s=2,3,4,\cdots)$$

Substituting these for the above,

$$f^{(n)}(z) = \{(z-1)\zeta(z)\}^{(n)} = \binom{n}{n-1}1\cdot\zeta^{(n-1)}(z) + \binom{n}{n}(z-1)\zeta^{(n)}(z)$$

i.e.

$$f^{(n)}(z) = \{(z-1)\zeta(z)\}^{(n)} = n\zeta^{(n-1)}(z) + (z-1)\zeta^{(n)}(z)$$

The differential coefficients at $z=0$ are

$$f^{(r)}(0) = r\zeta^{(r-1)}(0) - \zeta^{(r)}(0) \qquad r=0,1,2,\cdots$$

Then

$$\frac{f^{(r)}(0)}{r\,!} = \frac{r\zeta^{(r-1)}(0) - \zeta^{(r)}(0)}{r\,!} \qquad r=0,1,2,\cdots \tag{1.m}$$

Now, the function $f(z) = (z-1)\zeta(z)$ is holomorphic on the whole complex plane. So, **Theorem 8.1.1** holds and the following relation exists between the (1.t) and (1.m) .

$$\sum_{s=r}^{\infty}\frac{f^{(s)}(1)}{s\,!}(-1)^{s-r}{}_{s}C_{r}\,1^{s-r}=\frac{f^{(r)}(0)}{r\,!}\qquad for\ \ r=0,1,2,\cdots$$

Then, substituting (1.t) and (1.m) for the both sides one by one ($r=0,1,2,\cdots$), it is as follows.

When $\boldsymbol{r=0}$,

$$\sum_{s=0}^{\infty}\frac{f^{(s)}(1)}{s\,!}(-1)^{s-0}{}_{s}C_{0}\,1^{s-0}=1+\sum_{s=1}^{\infty}\frac{(-1)^{s-1}s\,\gamma_{s-1}}{s\,!}(-1)^{s-0}{}_{s}C_{0}1^{s-0}$$

$$=1-\sum_{s=1}^{\infty}\frac{\gamma_{s-1}}{(s-1)\,!}$$

i.e.

Left: $$\sum_{s=0}^{\infty}\frac{f^{(s)}(1)}{s\,!}(-1)^{s}{}_{s}C_{0}\,1^{s}=1-\sum_{s=0}^{\infty}\frac{\gamma_{s}}{s\,!}$$

Right: $$\frac{f^{(0)}(0)}{0\,!}=\frac{0-\zeta^{(0)}(0)}{0\,!}$$

From these,

$$\sum_{s=0}^{\infty}\frac{\gamma_s}{s\,!}=(-1)^{0}\left\{1!+\zeta^{(0)}(0)\right\}\tag{1.0}$$

When $\boldsymbol{r=1}$,

$$\sum_{s=1}^{\infty}\frac{f^{(s)}(1)}{s\,!}(-1)^{s-1}{}_{s}C_{1}\,1^{s-1}=\sum_{s=1}^{\infty}\frac{(-1)^{s-1}s\,\gamma_{s-1}}{s\,!}(-1)^{s-1}\frac{s\,!}{1!\,(s-1)!}$$

$$=\sum_{s=1}^{\infty}\frac{s\,\gamma_{s-1}}{1!\,(s-1)!}=\frac{1}{1!}\sum_{s=0}^{\infty}\frac{(s+1)\,\gamma_s}{s\,!}$$

$$=\frac{1}{1!}\left(\sum_{s=0}^{\infty}\frac{s\,\gamma_s}{s\,!}+\sum_{s=0}^{\infty}\frac{\gamma_s}{s\,!}\right)$$

Further,

$$\sum_{s=0}^{\infty}\frac{s\,\gamma_s}{s\,!}=\frac{0\gamma_0}{0\,!}+\frac{1\gamma_1}{1\,!}+\frac{2\gamma_2}{2\,!}+\frac{3\gamma_3}{3\,!}+\cdots=\frac{\gamma_1}{0\,!}+\frac{\gamma_2}{1\,!}+\frac{\gamma_3}{2\,!}+\cdots=\sum_{s=0}^{\infty}\frac{\gamma_{s+1}}{s\,!}$$

Using this,

Left: $$\sum_{s=1}^{\infty}\frac{f^{(s)}(1)}{s\,!}(-1)^{s-1}{}_{s}C_{1}\,1^{s-1}=\frac{1}{1!}\left(\sum_{s=0}^{\infty}\frac{\gamma_{s+1}}{s\,!}+\sum_{s=0}^{\infty}\frac{\gamma_s}{s\,!}\right)$$

Right: $$\frac{f^{(1)}(0)}{1\,!}=\frac{1\zeta^{(0)}(0)-\zeta^{(1)}(0)}{1\,!}$$

From these,

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+1}}{s\,!}+\sum_{s=0}^{\infty}\frac{\gamma_s}{s\,!}=\zeta^{(0)}(0)-\zeta^{(1)}(0)$$

Substituting (1.0) for this,

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+1}}{s\,!}+1!+\zeta^{(0)}(0)=\zeta^{(0)}(0)-\zeta^{(1)}(0)$$

From this,

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+1}}{s\,!} = (-1)^{1}\left\{1! + \zeta^{(1)}(0)\right\} \tag{1.1}$$

When $r = 2$,

$$\sum_{s=2}^{\infty}\frac{f^{(s)}(1)}{s\,!}(-1)^{s-2}{}_{s}C_{2}\,1^{s-2} = \sum_{s=2}^{\infty}\frac{(-1)^{s-1}s\,\gamma_{s-1}}{s\,!}(-1)^{s-2}\frac{s\,!}{2!\,(s-2)!}$$

$$= -\sum_{s=2}^{\infty}\frac{s\gamma_{s-1}}{2!\,(s-2)!} = -\frac{1}{2!}\sum_{s=0}^{\infty}\frac{(s+2)\,\gamma_{s+1}}{s\,!}$$

$$= -\frac{1}{2!}\left(\sum_{s=0}^{\infty}\frac{s\,\gamma_{s+1}}{s\,!} + 2\sum_{s=0}^{\infty}\frac{\gamma_{s+1}}{s\,!}\right)$$

Further,

$$\sum_{s=0}^{\infty}\frac{s\,\gamma_{s+1}}{s\,!} = \frac{0\gamma_1}{0\,!}+\frac{1\gamma_2}{1\,!}+\frac{2\gamma_3}{2\,!}+\frac{3\gamma_4}{3\,!}+\cdots = \frac{\gamma_2}{0\,!}+\frac{\gamma_3}{1\,!}+\frac{\gamma_4}{2\,!}+\cdots = \sum_{s=0}^{\infty}\frac{\gamma_{s+2}}{s\,!}$$

Using this,

Left: $$\sum_{s=2}^{\infty}\frac{f^{(s)}(1)}{s\,!}(-1)^{s-2}{}_{s}C_{2}\,1^{s-2} = -\frac{1}{2!}\left(\sum_{s=0}^{\infty}\frac{\gamma_{s+2}}{s\,!} + 2\sum_{s=0}^{\infty}\frac{\gamma_{s+1}}{s\,!}\right)$$

Right: $$\frac{f^{(2)}(0)}{2\,!} = \frac{2\zeta^{(1)}(0) - \zeta^{(2)}(0)}{2\,!}$$

From these,

$$-\left(\sum_{s=0}^{\infty}\frac{\gamma_{s+2}}{s\,!} + 2\sum_{s=0}^{\infty}\frac{\gamma_{s+1}}{s\,!}\right) = 2\zeta^{(1)}(0) - \zeta^{(2)}(0)$$

Substituting (1.1) for this,

$$-\left[\sum_{s=0}^{\infty}\frac{\gamma_{s+2}}{s\,!} - 2\left\{1! + \zeta^{(1)}(0)\right\}\right] = 2\zeta^{(1)}(0) - \zeta^{(2)}(0)$$

From this,

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+2}}{s\,!} = (-1)^{2}\left\{2\,! + \zeta^{(2)}(0)\right\} \tag{1.2}$$

When $r = 3$,

$$\sum_{s=3}^{\infty}\frac{f^{(s)}(1)}{s\,!}(-1)^{s-3}{}_{s}C_{3}\,1^{s-3} = \sum_{s=3}^{\infty}\frac{(-1)^{s-1}s\,\gamma_{s-1}}{s\,!}(-1)^{s-3}\frac{s\,!}{3!\,(s-3)!}$$

$$= \sum_{s=3}^{\infty}\frac{s\gamma_{s-1}}{3!\,(s-3)!} = \frac{1}{2!}\sum_{s=0}^{\infty}\frac{(s+2)\,\gamma_{s+1}}{s\,!}$$

$$= \frac{1}{3!}\left(\sum_{s=0}^{\infty}\frac{s\,\gamma_{s+2}}{s\,!} + 3\sum_{s=0}^{\infty}\frac{\gamma_{s+2}}{s\,!}\right)$$

Further,

$$\sum_{s=0}^{\infty}\frac{s\,\gamma_{s+2}}{s\,!} = \sum_{s=0}^{\infty}\frac{\gamma_{s+3}}{s\,!}$$

Using this,

Left: $$\sum_{s=3}^{\infty}\frac{f^{(s)}(1)}{s\,!}(-1)^{s-3}{}_sC_3\,1^{s-3}=\frac{1}{3!}\left(\sum_{s=0}^{\infty}\frac{\gamma_{s+3}}{s\,!}+3\sum_{s=0}^{\infty}\frac{\gamma_{s+2}}{s\,!}\right)$$

Right: $$\frac{f^{(3)}(0)}{3\,!}=\frac{3\zeta^{(2)}(0)-\zeta^{(3)}(0)}{3\,!}$$

From these,

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+3}}{s\,!}+3\sum_{s=0}^{\infty}\frac{\gamma_{s+2}}{s\,!}=3\zeta^{(2)}(0)-\zeta^{(3)}(0)$$

Substituting (1.2) for this,

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+3}}{s\,!}+3\left\{2\,!+\zeta^{(2)}(0)\right\}=3\zeta^{(2)}(0)-\zeta^{(3)}(0)$$

From this,

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+3}}{s\,!}=(-1)^3\left\{3\,!+\zeta^{(3)}(0)\right\} \tag{1.3}$$

Hereafter, by induction, we obtain the desired expression.

Special values

The following special values are known about Riemann zeta function.

$$\zeta^{(0)}(0)=-\frac{1}{2}\quad(=\zeta(0))\quad,\quad \zeta^{(1)}(0)=-\frac{\log 2\pi}{2}\quad(\textit{Glaisher-Kinkelin constant})$$

Therefore,

$$\sum_{s=0}^{\infty}\frac{\gamma_s}{s\,!}=(-1)^0\left\{1!+\zeta^{(0)}(0)\right\}=\frac{1}{2}$$

$$\sum_{s=0}^{\infty}\frac{\gamma_{s+1}}{s\,!}=(-1)^1\left\{1!+\zeta^{(1)}(0)\right\}=\frac{\log 2\pi}{2}-1=-0.0810614667\cdots$$

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Alien's Mathematics