10 Termwise Higher Derivative (Trigonometric, Hyperbolic)

In this chapter, for the function which the second or more order derivative is difficult to express with an easy unification notation among trigonometric functions and hyperbolic function, we differentiate the series expansion of these functions term by term. Therefore, sin x, cos x, sinh x, cosh x mentioned in "9.2 Higher Derivative of Elementary Functions " are not treated here.

10.1 Termwise Higher Derivative of \( \tan x \)

10.1.0 Higher Derivative of \( \tan x \)

According to Formula 9.2.6 (9.2), Higher Derivative of \( \tan x \) is expressed as follows.

\[
(tan x)^{(n)} = \sum_{r=0}^{n/2} n^r T_r (tan x)^{n+1-2r}
\]

(0.n)

Where, \(\uparrow\) is ceiling function and \(n^r T_r\) are coefficients as follows.

\[
\begin{align*}
T_0 &= 1 \\
T_1 &= 1 \\
T_2 &= 2 \\
T_3 &= 6 \\
T_4 &= 24 \\
T_5 &= 120 \\
T_6 &= 720 \\
\vdots
\end{align*}
\]

10.1.1 Termwise Higher Derivative of Taylor Series of \( \tan x \)

Formula 10.1.1

When \(B_0=1, B_2=1/6, B_4=-1/30, B_6=1/42\), \(\cdots\) are Bernoulli Numbers and \(\uparrow\) is ceiling function, the following expressions hold for \(|x| < \pi/2\).

\[
(tan x)^{(1)} = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k-1}) |B_{2k}|}{2k (2k-2)!} x^{2k-2}
\]

(1.1)

\[
(tan x)^{(2)} = \sum_{k=2}^{\infty} \frac{2^{2k} (2^{2k-1}) |B_{2k}|}{2k (2k-3)!} x^{2k-3}
\]

(1.2)

\[
(tan x)^{(3)} = \sum_{k=2}^{\infty} \frac{2^{2k} (2^{2k-1}) |B_{2k}|}{2k (2k-4)!} x^{2k-4}
\]

(1.3)

\[
(tan x)^{(4)} = \sum_{k=3}^{\infty} \frac{2^{2k} (2^{2k-1}) |B_{2k}|}{2k (2k-5)!} x^{2k-5}
\]

(1.4)

\[
(tan x)^{(n)} = \sum_{k=n+1}^{\infty} \frac{2^{2k} (2^{2k-1}) |B_{2k}|}{2k (2k-n-1)!} x^{2k-n-1}
\]

(1.n)
Proof

\[ \tan x \text{ is expanded to Taylor series as follows.} \]

\[ \tan x = \sum_{k=1}^{\infty} \frac{2^k (2^{2k-1}) |B_{2k}|}{2k (2k-1)!} x^{2k-1} \quad |x| < \frac{\pi}{2} \]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain (1.1) ~ (1.4).

Here, considering the relation between the derivative order \( n \) and the first term \( k_0 \) of \( \sum \), it is as follows.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_0 )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>...</td>
</tr>
</tbody>
</table>

Such a relation can be expressed by \( k_0 = \frac{n+1}{2} \) \( \uparrow \) using a ceiling function \( x \uparrow (\,= \lceil x \rceil) \). Then we obtain the desired expression.

Sum of Taylor series of the higher derivative of \( \tan x \)

From (0.\text{n}) and (1.\text{n}), we obtain the following expressions for \( |x| < \pi/2 \).

\[ \sum_{k=\frac{n+1}{2}}^{\infty} \frac{2^k (2^{2k-1}) |B_{2k}|}{2k (2k-n-1)!} x^{2k-n-1} = \sum_{r=0}^{n/2 + 1} T_r (\tan x)^{n+1-2r} \quad (1.1) \]

And giving \( x = \pi / 4 \) to this, we obtain the following special values.

\[ \sum_{k=1}^{\infty} \frac{2^k (2^{2k-1}) |B_{2k}|}{2k (2k-2)!} \left( \frac{\pi}{4} \right)^{2k-2} = \sum_{r=0}^{1/2 + 1} T_r = 2 \]
\[ \sum_{k=2}^{\infty} \frac{2^k (2^{2k-1}) |B_{2k}|}{2k (2k-3)!} \left( \frac{\pi}{4} \right)^{2k-3} = \sum_{r=0}^{2/2 + 1} 2 T_r = 4 \]
\[ \sum_{k=3}^{\infty} \frac{2^k (2^{2k-1}) |B_{2k}|}{2k (2k-4)!} \left( \frac{\pi}{4} \right)^{2k-4} = \sum_{r=0}^{3/2 + 1} 3 T_r = 16 \]
\[ \sum_{k=4}^{\infty} \frac{2^k (2^{2k-1}) |B_{2k}|}{2k (2k-5)!} \left( \frac{\pi}{4} \right)^{2k-5} = \sum_{r=0}^{4/2 + 1} 4 T_r = 80 \]

10.1.2 Termwise Higher Derivative of Fourier Series of \( \tan x \)

As seen in 5.1.2, \( \tan x \) is expanded to Fourier series in a broad meaning, as follows.

\[ \tan x = 2 (\sin 2x - \sin 4x + \sin 6x - \sin 8x + \cdots) - 2i (\cos 2x - \cos 4x + \cos 6x - \cos 8x + \cdots) + i \]

Although this does not hold as an equation, differentiating the real part of the both sides repeatedly, we obtain the following perfunctory expressions.

\[ (\tan x)^{(1)} = 2^2 (1 \cos 2x - 2 \cos 4x + 3 \cos 6x - 4 \cos 8x + \cdots) \]
\[ (\tan x)^{(2)} = -2^3 (1^2 \sin 2x - 2^2 \sin 4x + 3^2 \sin 6x - 4^2 \sin 8x + \cdots) \]
\[ (\tan x)^{(3)} = -2^4 (1^3 \cos 2x - 2^3 \cos 4x + 3^3 \cos 6x - 4^3 \cos 8x + \cdots) \]
\[(\tan x)^{(4)} = 2^5 (1^4 \sin 2x - 2^4 \sin 4x + 3^4 \sin 6x - 4^4 \sin 8x + \cdots)\]

\[(\tan x)^{(2n-1)} = (-1)^{n-1} 2^{2n} \sum_{k=1}^{\infty} (-1)^{k-1} k^{2n-1} \cos 2kx \quad (2.2n-1)\]

\[(\tan x)^{(2n)} = (-1)^n 2^{2n+1} \sum_{k=1}^{\infty} (-1)^{k-1} k^{2n} \sin 2kx \quad (2.2n)\]

Of course, these also do not hold as an equation. However, if the calculation is advanced assuming that these hold, interesting results are obtained as follows.

### 10.1.3 Dirichlet Odd Eta & Even Beta

**Formula 10.1.3**

When \( \eta(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^x} \), \( \beta(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n} \), \( B_{2k} \) are Bernoulli Numbers and \( n Tr \) are the coefficients mentioned in [10.1.0], the following expressions hold.

\[\eta(1-2n) = \frac{(-1)^{n-1}}{2^{2n-1}} \sum_{k=n}^{\infty} 2^k \left( \frac{2^k-1}{2k} \right) B_{2k} \left( \frac{\pi}{4} \right)^{2k-2n} \]

\[\beta(-2n) = \frac{(-1)^n}{2^{2n+1}} \sum_{k=n+1}^{\infty} 2^k \left( \frac{2^k-1}{2k} \right) B_{2k} \left( \frac{\pi}{4} \right)^{2k-2n-1} \]

**Proof**

From [0.\(n\), 1.\(n\), 2.2n-1) and (2.2n), we obtain the following perfunctory expressions.

\[\sum_{k=1}^{\infty} (-1)^{k-1} k^{2n-1} \cos 2kx = \frac{(-1)^{n-1}}{2^{2n}} \sum_{k=n}^{\infty} 2^k \left( \frac{2^k-1}{2k} \right) B_{2k} \left( \frac{\pi}{4} \right)^{2k-2n} x^{2k-2n}\]

\[\sum_{k=1}^{\infty} (-1)^{k-1} k^{2n} \sin 2kx = \frac{(-1)^n}{2^{2n+1}} \sum_{k=n+1}^{\infty} 2^k \left( \frac{2^k-1}{2k} \right) B_{2k} \left( \frac{\pi}{4} \right)^{2k-2n-1} x^{2k-2n-1}\]

If \( x = \pi / 4 \) is substituted for these, the left sides are as follows respectively.

\[\sum_{k=1}^{\infty} (-1)^{k-1} k^{2n-1} \cos \frac{k\pi}{2} = 2^{2n-1} \sum_{k=1}^{\infty} (-1)^{k-1} k^{2n-1} = 2^{2n-1} \eta(1-2n)\]

\[\sum_{k=1}^{\infty} (-1)^{k-1} k^{2n} \sin \frac{k\pi}{2} = \sum_{k=1}^{\infty} (-1)^{k-1} (2k-1)^2n = \beta(-2n)\]

Therefore, we obtain the desired expressions.

**Example1**

\[\eta(-1) = \frac{1}{2^3} \sum_{k=1}^{\infty} \frac{(2^k-1) |B_{2k}| \pi^{2k}}{2k (2k-2)!} = \frac{1}{2^3} \sum_{r=0}^{\infty} \frac{1}{2r+1} T_r = \frac{1}{4}\]
\[
\eta(-3) = -\frac{1}{2^7} \sum_{k=2}^{\infty} \frac{(2^{2k}-1) |B_{2k}| \pi^{2k}}{2k (2k-4)!} = -\frac{1}{2^7} \sum_{r=0}^{2} 3T_r = -\frac{1}{8}
\]

\[
\beta(-2) = -\frac{1}{2^3} \sum_{k=2}^{\infty} \frac{(2^{2k}-1) |B_{2k}| \pi^{2k}}{2k (2k-3)!} = -\frac{1}{2^3} \sum_{r=0}^{1} 2T_r = -\frac{1}{2}
\]

\[
\beta(-4) = \frac{1}{2^5} \sum_{k=3}^{\infty} \frac{(2^{2k}-1) |B_{2k}| \pi^{2k}}{2k (2k-5)!} = \frac{1}{2^5} \sum_{r=0}^{4} 4T_r = \frac{5}{2}
\]

**Example 2 \( \eta(-5), \beta(-6) \)**

\[
\text{Et} [\eta] := (1 - 2^{2\, n}) \zeta(1 - 2\, n)
\]

\[
\text{To} [\eta] := \frac{(-1)^{n-1}}{2^{4\, n-1}} \sum_{k=n+1}^{100} \frac{2^{2k} (2^{2k} - 1) \text{Abs}[\text{BernoulliB}[2\, k]]}{2k (2k - 2\, n)!} \left( \frac{\pi}{4} \right)^{2k-2\, n}
\]

\[
\text{Et}[\eta] = \frac{1}{4}
\]

\[
\text{To}[\eta] = 0.25
\]

\[
\text{Et}[\eta] := \text{DirichletL}[4, 2, -2\, n]
\]

\[
\text{Te} [\eta] := \frac{(-1)^{n}}{2^{4\, n+1}} \sum_{k=n+1}^{100} \frac{2^{2k} (2^{2k} - 1) \text{Abs}[\text{BernoulliB}[2\, k]]}{2k (2k - 2\, n - 1)!} \left( \frac{\pi}{4} \right)^{2k-2\, n-1}
\]

\[
\text{Et}[\eta] = \frac{31}{2}
\]

\[
\text{Te}[\eta] = -30.5
\]

**Note**

Using a relation \( \beta(-2n) = E_{2n} / 2 \) between Dirichlet Even Beta and Euler number, we obtain the following expression.

\[
E_{2n} = \frac{(-1)^{n}}{2^{2n}} \sum_{k=n+1}^{\infty} \frac{2^{2k} (2^{2k} - 1) |B_{2k}|}{2k (2k-2n-1)!} \left( \frac{\pi}{4} \right)^{2k-2n-1} = \frac{(-1)^{n}}{2^{2n}} \sum_{r=0}^{n} 2n T_r \quad (3.E)
\]
10.2 Termwise Higher Derivative of \( \tanh x \)

10.2.0 Higher Derivative of \( \tanh x \)

According to Formula 9.2.7 (9.2), Higher Derivative of \( \tanh x \) is expressed as follows.

\[
(tanh x)^{(n)} = (-1)^n \sum_{r=0}^{n/2} (-1)^r \binom{r}{n} T_r \cdot (tanh x)^{n+1-2r}
\]

Where, \( \binom{r}{n} \) is ceiling function and \( T_r \) are the same as the coefficients in 10.1.0.

10.2.1 Termwise Higher Derivative of Taylor Series of \( \tanh x \)

Formula 10.2.1

When \( B_0=1, B_2=1/6, B_4=-1/30, B_6=1/42, \ldots \) are Bernoulli Numbers and \( \binom{r}{n} \) is ceiling function, the following expressions hold for \(|x| < \pi/2\).

\[
(tanh x)^{(1)} = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k-1})B_{2k}}{2k(2k-2)!} x^{2k-2}
\]

\[
(tanh x)^{(2)} = \sum_{k=2}^{\infty} \frac{2^{2k}(2^{2k-1})B_{2k}}{2k(2k-3)!} x^{2k-3}
\]

\[
(tanh x)^{(n)} = \sum_{k=\frac{n+1}{2}}^{\infty} \frac{2^{2k}(2^{2k-1})B_{2k}}{2k(2k-n-1)!} x^{2k-n-1}
\]

Proof

\( \tanh x \) is expanded to Taylor series as follows.

\[
tanh x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k-1})B_{2k}}{2k(2k-1)!} x^{2k-1} \quad |x| < \frac{\pi}{2}
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain the desired expression. (The number of the first term of \( \sum \) is the same as it of \( \tanh x \).)

10.2.2 Termwise Higher Derivative of Fourier Series of \( \tanh x \)

Formula 10.2.2

The following expressions hold for \( x > 0 \).

\[
(tanh x)^{(1)} = 2^2(1^2 e^{-2x} - 2^1 e^{-4x} + 3^1 e^{-6x} - 4^1 e^{-8x} + \ldots)
\]

\[
(tanh x)^{(2)} = -2^3(1^2 e^{-2x} - 2^2 e^{-4x} + 3^2 e^{-6x} - 4^2 e^{-8x} + \ldots)
\]

\[
(tanh x)^{(n)} = (-1)^n \frac{2^{n+1}}{n+1} \sum_{k=1}^{n+1} (-1)^{k-1} k^n e^{-2kx}
\]

Proof

\( \tanh x \) is expanded to Fourier series for \( x > 0 \) as follows.

\[
tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} = (1 - e^{-2x})(1 - e^{-2x} + e^{-4x} - e^{-6x} + \ldots)
\]
\[= 1 - \left( 2e^{-2x} - 2e^{-4x} + 2e^{-6x} - 2e^{-8x} + \cdots \right) \]  
\[= 1 - 2 (\cos 2ix - \cos 4ix + \cos 6ix - \cos 8ix + \cdots) \]  
\[- 2i (\sin 2ix - \sin 4ix + \sin 6ix - \sin 8ix + \cdots) \]

Differentiating both sides of (2.0) with respect to \(x\) repeatedly, we obtain the desired expression.

### 10.2.3 Exponential series and Bernoulli series

Replacing \(x\) with \(x/2\) in (0.1.0), (1.1.0) and (2.1.0) we obtain the following formula.

**Formula 10.2.3**

When \(\uparrow\) is ceiling function, \(B_{2n}\) are Bernoulli Numbers and \(_nT_r\) are the coefficients mentioned in 10.1.0, the following expressions hold for \(0 < x < \pi\).

\[
\sum_{k=1}^{\infty} (-1)^{k-1} \frac{k^n}{e^{kx}} = (-1)^{n-1} \sum_{k=1}^{\infty} \frac{(2^{2k-1}) B_{2k}}{2k (2k - 1)!} x^{2k - n-1} \\
= - \frac{1}{2^{n+1}} \sum_{r=0}^{n/2} (-1)^r \_nT_r \left( \tanh \frac{x}{2} \right)^{n+1-2r}
\]

**Example**

\[
\frac{1^1}{e^x} - \frac{2^1}{e^{2x}} + \frac{3^1}{e^{3x}} - \frac{4^1}{e^{4x}} + \cdots = \sum_{k=1}^{\infty} \frac{(2^{2k-1}) B_{2k}}{2k (2k - 1)!} x^{2k - 2} = - \frac{1}{2^2} \left( \tanh^2 \frac{x}{2} - 1 \right)
\]

\[
\frac{1^2}{e^x} - \frac{2^2}{e^{2x}} + \frac{3^2}{e^{3x}} - \frac{4^2}{e^{4x}} + \cdots = - \sum_{k=2}^{\infty} \frac{(2^{2k-1}) B_{2k}}{2k (2k - 3)!} x^{2k - 3} = - \frac{1}{2^3} \left( 2 \tanh^3 \frac{x}{2} - 2 \tanh \frac{x}{2} \right)
\]

**Dirichlet Odd Eta (minus)**

Disregarding the convergence condition and substituting \(x = 0\) for Formula 10.2.3, we obtain Dirichlet Odd Eta (minus)

**Formula 10.2.3’**

Let \(\eta(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^x}\), and let Bernoulli number \(B_{2n}\) and tangent number \(_nT_r\) are as follows.

\[
B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42, \quad B_8 = -1/30, \quad B_{10} = 5/66, \ldots
\]

\[
T_1 = 1, \quad T_3 = 2, \quad T_5 = 16, \quad T_7 = 272, \quad T_9 = 7936, \ldots
\]

Then the following expressions hold.

\[
\eta(-2n+1) = \frac{(2^{2n}-1) B_{2n}}{2n} = -\frac{(-1)^n}{2^{2n}} T_{2n-1}
\]

\[
\eta(-2n) = 0
\]

**Proof**

Substitute \(x = 0\) for Formula 10.2.3. Then, the Fourier series is

\[
\sum_{k=1}^{\infty} (-1)^{k-1} \frac{k^n}{e^{k0}} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{-n}} = \eta(-n)
\]
The Taylor series and the polynomial are
\[
(-1)^{n-1} \sum_{k=\frac{n+1}{2}}^{\infty} \frac{(2^k-1)B_{2k}}{2k (2k-n-1)!} 0^{2k-n-1}
\]
when \( n = 2m - 1 \)

\[
= (-1)^{2m-1} \sum_{k=\frac{2m-1+1}{2}}^{\infty} \frac{(2^{2k-1})B_{2k}}{2k (2k-(2m-1)-1)!} 0^{2k-(2m-1)-1}
\]

\[
= \sum_{k=m}^{\infty} \frac{(2^{2k-1})B_{2k}}{2k (2k-2m)!} 0^{2k-2m} = \frac{(2^{2m-1})B_{2m}}{2m \cdot 0!} 0^0 = \frac{(2^{2m-1})B_{2m}}{2m}
\]

\[-\frac{1}{2^{n+1}} \sum_{r=0}^{\frac{n-1}{2}} (-1)^r nT_r \left( \tanh \frac{0}{2} \right)^{n+1-2r} \]

\[= -\frac{1}{2^m} \sum_{r=0}^{m} (-1)^r 2m \cdot 0^{2m-2r} = -\frac{1}{2^m} (-1)^m 2m \cdot 0^0\]

\[= -\frac{(-1)^m}{2^m} T_{2m-1} \quad (\because \quad 2m-1T_m = T_{2m-1})\]

when \( n = 2m \)

\[
(-1)^{n-1} \sum_{k=\frac{n+1}{2}}^{\infty} \frac{(2^k-1)B_{2k}}{2k (2k-n-1)!} 0^{2k-n-1}
\]

\[= (-1)^{2m-1} \sum_{k=\frac{2m+1+1}{2}}^{\infty} \frac{(2^{2k-1})B_{2k}}{2k (2k-2m-1)!} 0^{2k-2m-1} = 0
\]

\[-\frac{1}{2^{n+1}} \sum_{r=0}^{\frac{n-1}{2}} (-1)^r nT_r \left( \tanh \frac{0}{2} \right)^{n+1-2r} \]

\[= -\frac{1}{2^{2m+1}} \sum_{r=0}^{m} (-1)^r 2m \cdot 0^{2m+1-2r} = 0
\]

Replacing \( m \) with \( n \), we obtain \((3.2n-1')\) and \((3.2n')\).

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10.3 Termwise Higher Derivative of \( \cot x \)

10.3.0 Higher Derivative of \( \cot x \)

According to Formula 9.2.6 (9.2), Higher Derivative of \( \cot x \) is expressed as follows.

\[
(cotx)^{(n)} = (-1)^n \sum_{r=0}^{n/2} nT_r (cotx)^{n+1-2r}
\]  \( \text{(0,n)} \)

Where, ↑ is ceiling function and \( nT_r \) are the same as the coefficients in [10.1.0].

10.3.1 Termwise Higher Derivative of Taylor Series of \( \cot x \)

**Formula 10.3.1**

When \( B_0=1, B_2=1/6, B_4=-1/30, B_6=1/42, \ldots \) are Bernoulli Numbers and ↑ is ceiling function, the following expressions hold for \( 0 < x < \pi \).

\[
(cotx)^{(1)} = -\frac{1!}{x^2} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{2k (2k-2)!} x^{2k-2}
\]

\[
(cotx)^{(2)} = \frac{2!}{x^3} - \sum_{k=2}^{\infty} \frac{2^{2k} |B_{2k}|}{2k (2k-3)!} x^{2k-3}
\]

\[\vdots\]

\[
(cotx)^{(n)} = (-1)^n \frac{n!}{x^{n+1}} - \sum_{k=\frac{n+1}{2}}^{\infty} \frac{2^{2k} |B_{2k}|}{2k (2k-n-1)!} x^{2k-n-1}
\]  \( \text{(1,n)} \)

**Proof**

\[
x \cot x = 1 + \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k} \quad 0 < x < \pi
\]

From this

\[
\cot x = \frac{0!}{x^1} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{2k (2k-1)!} x^{2k-1}
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain the desired expression. (The number of the first term of \( \sum \) is the same as it of \( \tan x \).)

Formula 10.3.1 can also be expressed as follows.

**Formula 10.3.1’**

When \( B_{2k} \) denote Bernoulli Numbers and ↑ denotes ceiling function, the following expressions hold for \( \pi/2 < x < \pi \).

\[
(cotx)^{(n)} = -\sum_{k=\frac{n+1}{2}}^{\infty} \frac{2^{2k} (2^{2k-1}) |B_{2k}|}{2k (2k-n-1)!} x^{2k-n-1}
\]

**Proof**

\[
cot x = -\tan \left( x - \frac{\pi}{2} \right)
\]
\[ \tan x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1) B_{2k}}{2k (2k-1)!} x^{2k-1} \quad |x| < \frac{\pi}{2} \]

From these
\[ \cot x = -\sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1) B_{2k}}{2k (2k-1)!} \left( x - \frac{\pi}{2} \right)^{2k-1} \quad 0 < x < \pi \]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain the desired expression. (The number of the first term of \( \sum \) is the same as it of \( \tan x \).)

**Sum of Taylor series of the higher derivative of cot x**

From \([0,n]\) and \((1.n)\), we obtain the following expressions for \(0 < x < \pi\).

\[ \sum_{k=n+1,2}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-n-1)!} x^{2k} = (-1)^n \left\{ n! - x^{n+1} \sum_{r=0}^{n/2} nT_r (\cot x)^{n+1-2r} \right\} \quad (1.t) \]

And giving \( x = \pi/4 \) to this, we obtain the following special values.

\[ \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-2)!} \left( \frac{\pi}{4} \right)^{2k} = -1! + 2 \left( \frac{\pi}{4} \right)^2 \]
\[ \sum_{k=2}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-3)!} \left( \frac{\pi}{4} \right)^{2k} = 2! - 4 \left( \frac{\pi}{4} \right)^3 \]
\[ \sum_{k=2}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-4)!} \left( \frac{\pi}{4} \right)^{2k} = -3! + 16 \left( \frac{\pi}{4} \right)^4 \]
\[ \sum_{k=3}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-5)!} \left( \frac{\pi}{4} \right)^{2k} = 4! - 80 \left( \frac{\pi}{4} \right)^5 \]

10.3.2 Termwise Higher Derivative of Fourier Series of tan x

As seen in 5.3.2, \( \cot x \) is expanded to Fourier series in a broad meaning as follows.

\[ \cot x = 2 \left( \sin^2 x + \sin 4x + \sin 6x + \sin 8x + \cdots \right) - 2i \left( \cos^2 x + \cos 4x + \cos 6x + \cos 8x + \cdots \right) - i \]

Although this does not hold as an equation, differentiating the real part of the both sides repeatedly, we obtain the following perfunctory expressions..

\[ (\cot x)^{(1)} = 2^2 \left( 1 \cos 2x + 2 \cos 4x + 3 \cos 6x + 4 \cos 8x + \cdots \right) \]
\[ (\cot x)^{(2)} = 2^3 \left( 1^2 \sin^2 x + 2^2 \sin 4x + 3^2 \sin 6x + 4^2 \sin 8x + \cdots \right) \]
\[ (\cot x)^{(3)} = 2^4 \left( 1^3 \cos 2x + 2^3 \cos 4x + 3^3 \cos 6x + 4^3 \cos 8x + \cdots \right) \]
\[ (\cot x)^{(4)} = 2^5 \left( 1^4 \sin 2x + 2^4 \sin 4x + 3^4 \sin 6x + 4^4 \sin 8x + \cdots \right) \]

\( (\cot x)^{(2n-1)} = (-1)^{n-1} 2^{2n-1} \sum_{k=1}^{\infty} k^{2n-1} \cos 2kx \) \hspace{1cm} (2.2n-1)
\( (\cot x)^{(2n)} = (-1)^n 2^{2n+1} \sum_{k=1}^{\infty} k^{2n} \sin 2kx \) \hspace{1cm} (2.2n)
Of course, these also do not hold as an equation. However, if the calculation is advanced assuming that these hold, interesting results are obtained as follows.

10.3.3 Dirichlet Odd Eta & Even Beta

Formula 10.3.3

When \( \eta(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^x} \), \( \beta(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n} \), \( B_{2k} \) are Bernoulli Numbers and \( nT_r \) are the coefficients mentioned in 10.1.0, the following expressions hold.

\[
\eta(1-2n) = \frac{(-1)^{n-1}}{2^{4n-1}} \left( \frac{4}{\pi} \right)^{2n} \left\{ (2n-1) ! + \sum_{k=n}^{\infty} \frac{2^{2k} |B_{2k}|}{2k(2k-2n)!} \left( \frac{\pi}{4} \right)^{2k} \right\}
\]

\[
= \frac{(-1)^{n-1}}{2^{4n-1}} \sum_{r=0}^{n} 2n-1 T_r
\]

\[
\beta(-2n) = \frac{(-1)^n}{2^{2n+1}} \left( \frac{4}{\pi} \right)^{2n} \left\{ (2n) ! - \sum_{k=n+1}^{\infty} \frac{2^{2k} |B_{2k}|}{2k(2k-2n-1)!} \left( \frac{\pi}{4} \right)^{2k} \right\}
\]

\[
= (-1)^n \sum_{r=0}^{n} 2nT_r
\]

Proof

From \((0.n),(1.n),(2.2n-1),(2.2n)\), we obtain the following perfunctory expressions.

\[
\sum_{k=1}^{\infty} k^{2n-1} \cos 2kx = \frac{(-1)^n}{2^{2n}} \sum_{r=0}^{n} 2n-1 T_r (\cot x)^{2n-2r}
\]

\[
\sum_{k=1}^{\infty} k^{2n} \sin 2kx = \frac{(-1)^n}{2^{2n+1}} \sum_{r=0}^{n} 2n+1 T_r (\cot x)^{2n+1-2r}
\]

If \( x = \pi/4 \) is substituted for these, the left sides are as follows respectively.

\[
\sum_{k=1}^{\infty} k^{2n-1} \cos \frac{k\pi}{2} = -2^{2n-1} \sum_{k=1}^{\infty} (-1)^{k-1} k^{2n-1} = -2^{2n-1} \eta(1-2n)
\]

\[
\sum_{k=1}^{\infty} k^{2n} \sin \frac{k\pi}{2} = \sum_{k=1}^{\infty} (-1)^{k-1} (2k-1)^2n = \beta(-2n)
\]

Therefore, we obtain the desired expressions.

Example

\[
\eta(-3) = -\frac{1}{2^3} \left( \frac{4}{\pi} \right)^4 \left\{ 3! + \sum_{k=2}^{\infty} \frac{2^{2k} |B_{2k}|}{2k(2k-4)!} \left( \frac{\pi}{4} \right)^{2k} \right\} = -\frac{1}{2^3} \sum_{r=0}^{3} 3T_r = -\frac{1}{8}
\]

- 10 -
\[ \beta(-2) = -\frac{1}{2^3} \left( \frac{4}{\pi} \right)^2 \left\{ 2! - \sum_{k=2}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-3)!} \left( \frac{\pi}{4} \right)^{2k} \right\} = -\frac{1}{2^3} \sum_{r=0}^{\infty} T_r = -\frac{1}{2} \]

\[ \beta(-4) = \frac{1}{2^5} \left( \frac{4}{\pi} \right)^4 \left\{ 4! - \sum_{k=3}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-5)!} \left( \frac{\pi}{4} \right)^{2k} \right\} = \frac{1}{2^5} \sum_{r=0}^{n} 4T_r = \frac{5}{2} \]

### 10.3.4 Factorial and Bernoulli series

**Formula 10.3.4**

When \( B_{2k} \) denotes a Bernoulli Number, the following expressions hold.

\[(2n - 1)! = \sum_{k=n}^{\infty} \frac{2^{2k-1} B_{2k}}{2k (2k-2n)!} \left( \frac{\pi}{2} \right)^{2k-2n} \quad (4.1)\]

\[(2n)! = \sum_{k=n+1}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-2n-1)!} \left( \frac{\pi}{2} \right)^{2k} \quad (4.2)\]

**Proof**

Replacing \( n \) with \( 2n - 1 \) in Formula 10.3.1', we obtain

\[(\cot x)^{(2n-1)} = -\sum_{k=n}^{\infty} \frac{2^{2k}(2^{2k-1}) B_{2k}}{2k (2k-2n)!} \left( \frac{\pi}{4} \right)^{2k-2n} \]

Substituting \( x = \pi / 4 \) for this,

\[(\cot x)^{(2n-1)} \mid_{x=\pi/4} = -\sum_{k=n}^{\infty} \frac{2^{2k}(2^{2k-1}) B_{2k}}{2k (2k-2n)!} \left( \frac{\pi}{4} \right)^{2k-2n} \]

On the other hand, substituting \( x = \pi / 4 \) for \((2.2n-1)\),

\[(\cot x)^{(2n-1)} \mid_{x=\pi/4} = -(2n-1)! \left( \frac{4}{\pi} \right)^{2n} - \sum_{k=n}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-2n)!} \left( \frac{\pi}{4} \right)^{2k-2n} \]

Since these have to be equal, we obtain (4.1).

Next, replacing \( n \) with \( 2n \) in Formula 10.3.1',

\[(\cot x)^{(2n)} = -\sum_{k=n+1}^{\infty} \frac{2^{2k}(2^{2k-1}) B_{2k}}{2k (2k-2n-1)!} \left( \frac{\pi}{2} \right)^{2k-2n-1} \]

Substituting \( x = \pi / 4 \) for this, we obtain

\[(\cot x)^{(2n)} \mid_{x=\pi/4} = \sum_{k=n+1}^{\infty} \frac{2^{2k}(2^{2k-1}) B_{2k}}{2k (2k-2n-1)!} \left( \frac{\pi}{4} \right)^{2k-2n-1} \]

On the other hand, substituting \( x = \pi / 4 \) for \((2.2n)\),

\[(\cot x)^{(2n)} \mid_{x=\pi/4} = (2n)! \left( \frac{4}{\pi} \right)^{2n} - \sum_{k=n+1}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-2n-1)!} \left( \frac{\pi}{4} \right)^{2k-2n-1} \]

Since these have to be equal, we obtain (4.2).

**Proof of the pudding is in the eating.**

\[(2n-1)! = \sum_{k=n}^{\infty} \frac{(2^n - 2) \text{ Abs}[\text{BernoulliB}[2k]]}{2k (2k-2n)!} \left( \frac{\pi}{2} \right)^{2k-2n} \]
\[
\begin{align*}
\text{N}[\text{Fe}[1]] & \quad \text{N}[\text{Fe}[2]] & \quad \text{N}[\text{Fe}[3]] \\
1. & \quad 6. & \quad 120. \\
(2n)! & \\
\text{F}(n)_{\text{Fe}} := & \sum_{k=n+1}^{100} \frac{2^k \text{Abs}[\text{BernoulliB}[2,k]]}{2^k(2^k - 2n - 1)!} \left(\frac{\pi}{2}\right)^{2^k} \\
\text{N}[\text{Fe}[1]] & \quad \text{N}[\text{Fe}[2]] & \quad \text{N}[\text{Fe}[3]] \\
2. & \quad 24. & \quad 720.
\end{align*}
\]
10.4 Termwise Higher Derivative of \( \coth x \)

10.4.0 Higher Derivative of \( \coth x \)

According to Formula 9.2.7 (9.2), Higher Derivative of \( \coth x \) is expressed as follows.

\[
(coth x)^{(n)} = (-1)^n \frac{n}{2^n} \sum_{r=0}^{n/2} (-1)^r \frac{1}{n!} T_r (coth x)^{n+1-2r}
\]

(0.n)

Where, \( \uparrow \) is ceiling function and \( T_r \) are the same as the coefficients in [10.1.0].

10.4.1 Termwise Higher Derivative of Taylor Series of \( \coth x \)

Formula 10.4.1

When \( B_0 = 1, B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, \ldots \) are Bernoulli Numbers and \( \uparrow \) is ceiling function, the following expressions hold for \( 0 < x < \pi \).

\[
(coth x)^{(1)} = -\frac{1!}{x^2} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-2)!} x^{2k-2}
\]

\[
(coth x)^{(2)} = \frac{2!}{x^3} + \sum_{k=2}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-3)!} x^{2k-3}
\]

\vdots

\[
(coth x)^{(n)} = (-1)^n \frac{n!}{x^{n+1}} + \sum_{k=n+1}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-n-1)!} x^{2k-n-1}
\]

(1.n)

Proof

\( x \coth x = 1 + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k} \quad 0 < x < \pi \)

From this

\[
coth x = \frac{0!}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-1)!} x^{2k-1}
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain the desired expression. (The number of the first term of \( \sum \) is the same as it of \( \tan x \).)

10.4.2 Termwise Higher Derivative of Fourier Series of \( \coth x \)

Formula 10.4.2

The following expressions hold for \( x > 0 \).

\[
(coth x)^{(1)} = -2^2 \left( 1^1 e^{-2x} + 2^1 e^{-4x} + 3^1 e^{-6x} + 4^1 e^{-8x} + \cdots \right)
\]

\[
(coth x)^{(2)} = 2^3 \left( 1^2 e^{-2x} + 2^2 e^{-4x} + 3^2 e^{-6x} + 4^2 e^{-8x} + \cdots \right)
\]

\vdots

\[
(coth x)^{(n)} = (-1)^n 2^{n+1} \sum_{k=1}^{\infty} k^n e^{-2kx}
\]

(2.n)
Proof

coth x is expanded to Fourier series for $x > 0$ as follows.

$$
\text{coth } x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1 + e^{-2x}}{1 - e^{-2x}} = (1 + e^{-2x})(1 + e^{-4x} + e^{-6x} + \ldots)
$$

$$
= 1 + (2e^{-2x} + 2e^{-4x} + 2e^{-6x} + 2e^{-8x} + \ldots)
$$

$$
= 1 + 2 \left( \cos 2ix + \cos 4ix + \cos 6ix + \cos 8ix + \ldots \right)
$$

$$
+ 2i \left( \sin 2ix + \sin 4ix + \sin 6ix + \sin 8ix + \ldots \right)
$$

Differentiating both sides of (2.0) with respect to x repeatedly, we obtain the desired expression.

### 10.4.3 Exponential series and Bernoulli series

Replacing $x$ with $x/2$ in (0.n), (1.n) and (2.n), we obtain the following formula.

#### Formula 10.4.3

When $\uparrow$ is ceiling function, $B_{2n}$ are Bernoulli Numbers and $n T_r$ are the coefficients mentioned in 10.1.0, the following expressions hold for $0 < x < \pi$.

$$
\sum_{k=1}^\infty \frac{k^n}{e^k x} = \frac{1}{x^{n+1}} \left\{ n! + (-1)^n \sum_{k=\frac{n+1}{2}}^\infty \frac{B_{2k} x^{2k}}{2k (2k-n-1) !} \right\}
$$

$$
= \frac{1}{2^{n+1}} \sum_{r=0}^{n/2} (-1)^r n T_r \left( \coth \frac{x}{2} \right)^{n+1-2r}
$$

#### Example

$$
\frac{1}{e^1 x} + \frac{1}{e^2 x} + \frac{3}{e^3 x} + \frac{4}{e^4 x} + \ldots = \frac{1}{x^2} \left\{ 1! + \sum_{k=1}^\infty \frac{B_{2k} x^{2k}}{2k (2k-2) !} \right\} = \frac{1}{2^2} \left( \coth^2 \frac{x}{2} - 1 \right)
$$

$$
\frac{1}{e^1 x} + \frac{2}{e^2 x} + \frac{3}{e^3 x} + \frac{4}{e^4 x} + \ldots = \frac{1}{x^3} \left\{ 2! + \sum_{k=2}^\infty \frac{B_{2k} x^{2k}}{2k (2k-3) !} \right\} = \frac{1}{2^3} \left( 2 \coth^3 \frac{x}{2} - 2 \coth \frac{x}{2} \right)
$$

### Exponential series and Factorial

Given $x=\frac{1}{2}$ to Formula 10.4.3, we obtain the following special values.

#### Formula 10.4.3’

When $\uparrow$ is ceiling function, $B_{2n}$ are Bernoulli Numbers and $n T_r$ are the coefficients mentioned in 10.1.0, the following expressions hold.

$$
\sum_{k=1}^\infty \frac{k^n}{e^k} = n! + (-1)^n \sum_{k=\frac{n+1}{2}}^\infty \frac{B_{2k}}{2k (2k-n-1) !} = \frac{1}{2^{n+1}} \sum_{r=0}^{n/2} (-1)^r n T_r \left( \coth \frac{1}{2} \right)^{n+1-2r}
$$

#### Note

Mr. Sugimoto was foreseeing this formula. ( http://homepage3.nifty.com/y_sugi/sp/sp56.htm )

#### Example

$$
\frac{1}{e^1} + \frac{1}{e^2} + \frac{3}{e^3} + \frac{4}{e^4} + \ldots = 1! - \sum_{k=1}^\infty \frac{B_{2k}}{2k (2k-2) !} = \frac{1}{2^2} \left( \coth^2 \frac{1}{2} - 1 \right)
$$
\[
\frac{1^2}{e^1} + \frac{2^2}{e^2} + \frac{3^2}{e^3} + \frac{4^2}{e^4} + \cdots = 2! + \sum_{k=2}^{\infty} \frac{B_{2k}}{2k(2k-3)!} = \frac{1}{2^3} \left( 2 \coth^2 \frac{1}{2} - 2 \coth \frac{1}{2} \right)
\]
\[
\frac{1^3}{e^1} + \frac{2^3}{e^2} + \frac{3^3}{e^3} + \frac{4^3}{e^4} + \cdots = 3! - \sum_{k=2}^{\infty} \frac{B_{2k}}{2k(2k-4)!} = \frac{1}{2^4} \left( 6 \coth^4 \frac{1}{2} - 8 \coth^2 \frac{1}{2} + 2 \right)
\]

Therefore also, the following approximation formula is obtained.

**Formula 10.4.3"**

\[
\frac{1^p}{e^1} + \frac{2^p}{e^2} + \frac{3^p}{e^3} + \frac{4^p}{e^4} + \cdots \div \Gamma(1+p) \quad p > 0
\]

*(3.p)*

**Example \( \Gamma(1+2.5) \)**

\[
S[p_] := \sum_{k=1}^{100} \frac{k^p}{e^k}
\]

\[
S[2.5] \quad \text{Gamma [3.5]}
\]

\[
3.32641 \quad 3.32335
\]
10.5 Termwise Higher Derivative of $\csc x$

10.5.0 Higher Derivative of $\csc x$
According to Formula 9.2.8 (9.2), Higher Derivative of $\csc x$ is expressed as follows.

$$(\csc x)^{(n)} = \frac{(-1)^n}{\sin x} \sum_{r=0}^{n-1} n E_r (\cot x)^{n-2r}$$  \hspace{1cm} (0.1n)

Where, ↓ is floor function and $n E_r$ are coefficients as follows.

\[
\begin{array}{cccccccc}
0 E_0 & 1 \\
2 E_0 2 E_1 & 2 1 \\
3 E_0 3 E_1 3 E_3 & 6 5 \\
4 E_0 4 E_1 4 E_2 & 24 28 5 \\
5 E_0 5 E_1 5 E_2 5 E_4 & 120 180 61 \\
6 E_0 6 E_1 6 E_2 6 E_3 6 E_5 & 720 1320 662 61 \\
7 E_0 7 E_1 7 E_2 7 E_3 & 5040 10920 7266 1385 \\
\vdots & \vdots \\
\end{array}
\]

10.5.1 Termwise Higher Derivative of Taylor Series of $\csc x$

Formula 10.5.1
When $B_0=1$, $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$, are Bernoulli Numbers and ⌈ is ceiling function,
the following expressions hold for $0<x<\pi$.

$$(\csc x)^{(1)} = \frac{1!}{x^2} + \sum_{k=1}^{\infty} \frac{\left(2^{2k-2}\right) B_{2k}}{2k (2k-2)!} x^{2k-2}$$

$$(\csc x)^{(2)} = \frac{2!}{x^3} + \sum_{k=2}^{\infty} \frac{\left(2^{2k-2}\right) B_{2k}}{2k (2k-3)!} x^{2k-3}$$

\vdots

$$(\csc x)^{(n)} = (-1)^n \frac{n!}{x^{n+1}} + \sum_{k=\frac{n+1}{2}}^{\infty} \frac{\left(2^{2k-2}\right) B_{2k}}{2k (2k-n-1)!} x^{2k-n-1}$$ \hspace{1cm} (1.1n)

Proof

$$x \csc x = 1 + \sum_{k=1}^{\infty} \frac{\left(2^{2k-2}\right) B_{2k}}{(2k)!} x^{2k} \hspace{1cm} 0<x<\pi$$

From this

$$\csc x = \frac{0!}{x} + \sum_{k=1}^{\infty} \frac{\left(2^{2k-2}\right) B_{2k}}{2k (2k-1)!} x^{2k-1}$$

Differentiating both sides of this with respect to $x$ repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of $\tan x$.)

Formula 10.5.1 can also be expressed as follows.
Formula 10.5.1'
When $E_0=1$, $E_2=-1$, $E_4=5$, $E_6=-61$, $E_8=1385$, etc. are Euler Numbers and $\downarrow$ is floor function, the following expressions hold for $\pi/2 < x < \pi$.

\[
(csc x)^{(n)} = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k-n}
\]

proof
\[
csc x = \sec \left(x - \frac{\pi}{2}\right)
\]
\[
\sec x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} x^{2k} \quad |x| < \frac{\pi}{2}
\]

From these
\[
csc x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k} \quad 0 < x < \pi
\]

Differentiating both sides of this with respect to $x$ repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of $\sec x$.)

Sum of Taylor series of the higher derivative of $csc x$
From $[0,n]$ and $(1,n)$, we obtain the following expressions for $0 < x < \pi$.

\[
\sum_{k=0}^{\infty} \frac{(2^{2k-2}) |B_{2k}|}{2k (2k-n-1)!} x^{2k} = (-1)^{n-1} \left\{ n! - \frac{x^{n+1}}{\sin x} \sum_{r=0}^{n/2} \frac{n}{r}E_r (\cot x)^{n-2r} \right\}
\]  \hspace{1cm} (1.1)

And giving $x=\pi/2$ to this, we obtain the following special values.

\[
\sum_{k=1}^{\infty} \frac{(2^{2k-2}) |B_{2k}|}{2k (2k-2)!} \left(\frac{\pi}{2}\right)^{2k} = 1!
\]
\[
\sum_{k=2}^{\infty} \frac{(2^{2k-2}) |B_{2k}|}{2k (2k-3)!} \left(\frac{\pi}{2}\right)^{2k} = -2! + 1 \left(\frac{\pi}{2}\right)^3
\]
\[
\sum_{k=2}^{\infty} \frac{(2^{2k-2}) |B_{2k}|}{2k (2k-4)!} \left(\frac{\pi}{2}\right)^{2k} = 3!
\]
\[
\sum_{k=3}^{\infty} \frac{(2^{2k-2}) |B_{2k}|}{2k (2k-5)!} \left(\frac{\pi}{2}\right)^{2k} = -4! + 5 \left(\frac{\pi}{2}\right)^5
\]

10.5.2 Termwise Higher Derivative of Fourier Series of $csc x$
As seen in 5.5.2, $csc x$ is expanded to Fourier series in a broad meaning as follows.
\[
csc x = 2 (\sin x + \sin 3x + \sin 5x + \sin 7x + \cdots) - 2i (\cos x + \cos 3x + \cos 5x + \cos 7x + \cdots)
\]

Although this does not hold as an equation, differentiating the real part of the both sides repeatedly, we obtain the following perfunctory expressions...
(csc x) \((1)\) = 2 \((1 \cos x + 3 \cos 3x + 5 \cos 5x + 7 \cos 7x + \ldots)$$
(csc x) \((2)\) = -2 \((1^2 \sin x + 3^2 \sin 3x + 5^2 \sin 5x + 4^2 \sin 7x + \ldots)$$
(csc x) \((3)\) = -2 \((1^3 \cos x + 3^3 \cos 3x + 5^3 \cos 5x + 4^3 \cos 7x + \ldots)$$
(csc x) \((4)\) = 2 \((1^4 \sin x + 3^4 \sin 3x + 5^4 \sin 5x + 4^4 \sin 7x + \ldots)$$
\vdots$$
(csc x) \((2n-1)\) = (-1)^{n-1}2 \sum_{k=0}^{\infty} (2k+1) 2^{2n-1} \cos\{(2k+1)x\} \quad (2.2n-1)$$
(csc x) \((2n)\) = (-1)^n 2 \sum_{k=0}^{\infty} (2k+1) 2^n \sin\{(2k+1)x\} \quad (2.2n)$$

Of course, these also do not hold as an equation. However, if the calculation is advanced assuming that these hold, interesting results are obtained as follows.

10.5.3 Dirichlet Even Beta (minus)

Formula 10.5.3

Let \(\beta(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n}\), and let Bernoulli number \(B_{2n}\) and Euler number \(E_r\) are as follows.

\(B_0=1,\quad B_2=1/6,\quad B_4=-1/30,\quad B_6=1/42,\quad B_8=-1/30,\quad \ldots\)
\(E_0=1,\quad E_2=-1,\quad E_4=5,\quad E_6=-61,\quad E_8=1385,\quad \ldots\)

Then the following expressions hold.

\[\beta(-2n) = \frac{(-1)^n}{2} \left(\frac{2}{\pi}\right)^{2n+1} \left\{ (2n)! + \sum_{k=n+1}^{\infty} \frac{(2k-2)!}{2k(2k-2n-1)!} \left(\frac{\pi}{2}\right)^{2k} \right\} = \frac{E_{2n}}{2} \]

Proof

From \((0.n)\), \((1.n)\) and \((2.2n)\), we obtain the following perfunctory expression.

\[\sum_{k=0}^{\infty} (2k+1) 2^n \sin\{(2k+1)x\} = \frac{(-1)^n}{2} \left(\frac{2}{\pi}\right)^{2n+1} \left\{ (2n)! + \sum_{k=n+1}^{\infty} \frac{(2k-2)!}{2k(2k-2n-1)!} x^{2k} \right\} \]

If \(x=\pi/2\) is substituted for this, the left side is as follows.

\[\sum_{k=0}^{\infty} (2k+1) 2^n \sin\{(2k+1)x\} \cdot \frac{\pi}{2} = \sum_{k=0}^{\infty} (-1)^k (2k+1) 2^n = \beta(-2n) \]

Then,

\[\beta(-2n) = \frac{(-1)^n}{2} \left(\frac{2}{\pi}\right)^{2n+1} \left\{ (2n)! + \sum_{k=n+1}^{\infty} \frac{(2k-2)!}{2k(2k-2n-1)!} \left(\frac{\pi}{2}\right)^{2k} \right\} \]

\[= \frac{(-1)^n}{2} \sum_{r=0}^{\infty} 2n E_r 0^{2n-2r} = \frac{(-1)^n}{2} 2n E_n 0^0 \]

\[= \frac{E_{2n}}{2} \quad (\because (-1)^n 2n E_n = E_{2n}) \]
Example

\[ \beta(-2) = -\frac{1}{2} \left( \frac{2}{\pi} \right)^3 \left\{ 2! + \sum_{k=2}^{\infty} \frac{(2^{2k-2}) |B_{2k}|}{2k(2k-3)!} \left( \frac{\pi}{2} \right)^{2k} \right\} = \frac{E_2}{2} = -\frac{1}{2} \]

\[ \beta(-4) = \frac{1}{2} \left( \frac{2}{\pi} \right)^5 \left\{ 4! + \sum_{k=3}^{\infty} \frac{(2^{2k-2}) |B_{2k}|}{2k(2k-5)!} \left( \frac{\pi}{2} \right)^{2k} \right\} = \frac{E_4}{2} = \frac{5}{2} \]
10.6 Termwise Higher Derivative of \( \text{csch} \ x \)

10.6.0 Higher Derivative of \( \text{csch} \ x \)

According to Formula 9.2.9 (9.2), Higher Derivative of \( \text{csch} \ x \) is expressed as follows.

\[
(csch \ x)^{(n)} = \frac{(-1)^n n!}{\sinh x} \sum_{r=0}^{n} (-1)^r n E_r (\coth x)^{n-2r}
\]  
(0.1)

Where, \( \downarrow \) is floor function and \( n E_r \) are the same as the coefficients in 10.5.0.

10.6.1 Termwise Higher Derivative of Taylor Series of \( \text{csch} \ x \)

Formula 10.6.1

When \( B_0=1, \ B_2=1/6, \ B_4=-1/30, \ B_6=1/42, \cdots \) are Bernoulli Numbers and \( \uparrow \) is ceilling function, the following expressions hold for \( 0 < x < \pi \).

\[
(csch \ x)^{(1)} = -\frac{1!}{x^2} - \sum_{k=1}^{\infty} \frac{(2^{2k}-2)B_{2k}}{2k(2k-2)!} x^{2k-2}
\]

\[
(csch \ x)^{(2)} = -\frac{2!}{x^3} - \sum_{k=2}^{\infty} \frac{(2^{2k}-2)B_{2k}}{2k(2k-3)!} x^{2k-3}
\]

\vdots

\[
(csch \ x)^{(n)} = (-1)^n \frac{n!}{x^{n+1}} - \sum_{k=\frac{n+1}{2}}^{\infty} \frac{(2^{2k}-2)B_{2k}}{2k(2k-n-1)!} x^{2k-n-1}
\]  
(1.1)

Proof

\[
xcsch x = 1 - \sum_{k=1}^{\infty} \frac{(2^{2k}-2)B_{2k}}{(2k)!} x^{2k} \quad 0 < x < \pi
\]

From this

\[
csch x = \frac{0!}{x} - \sum_{k=1}^{\infty} \frac{(2^{2k}-2)B_{2k}}{2k(2k-1)!} x^{2k-1}
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain the desired expression. ( The number of the first term of \( \sum \) is the same as it of \( \tan x \).

10.6.2 Termwise Higher Derivative of Fourier Series of \( \text{csch} \ x \)

Formula 10.6.2

The following expressions hold for \( x > 0 \).

\[
(csch \ x)^{(1)} = -2 \left( 1e^{-1x} + 3e^{-3x} + 5e^{-5x} + 7e^{-7x} + \cdots \right)
\]

\[
(csch \ x)^{(2)} = 2 \left( 1^2 e^{-1x} + 3^2 e^{-3x} + 5^2 e^{-5x} + 4^2 e^{-7x} + \cdots \right)
\]

\vdots

\[
(csch \ x)^{(n)} = (-1)^n 2 \sum_{k=0}^{\infty} (2k+1)^n e^{-(2k+1)x}
\]  
(2.2)

Proof

\( \text{csch} \ x \) is expanded to Fourier series for \( x > 0 \) as follows.
\[ \text{csch } x = \frac{2}{e^x - e^{-x}} = \frac{2e^{-x}}{1 - e^{-2x}} = 2e^{-x}(1 + e^{-2x} + e^{-4x} + e^{-6x} + \ldots) \]

\[ = 2(e^{-1x} + e^{-3x} + e^{-5x} + e^{-7x} + \ldots) \]

\[ = 2(\cosh ix + \cosh 3ix + \cosh 5ix + \ldots) - 2i(\sinh ix + \sinh 3ix + \sinh 5ix + \ldots) \quad (2.0) \]

Differentiating both sides of (2.0) with respect to \( x \) repeatedly, we obtain the desired expression.

### 10.6.3 Exponential series and Bernoulli series

From\([0,n], (1,n)\) and \((2,n)\), we obtain the following formula.

**Formula 10.6.3**

When \( \downarrow, \uparrow \) are ceiling function and floor function respectively, \( B_{2n} \) are Bernoulli Numbers and \( \binom{n}{r} \) are the coefficients mentioned in 10.5.0, the following expressions hold for \( 0 < x < \pi \).

\[ \sum_{k=0}^{\infty} \frac{(2k+1)^n}{e^{(2k+1)x}} = \frac{1}{2x^{n+1}} \left( n! - (-1)^n \sum_{k=\frac{n+1}{2}}^{\infty} \binom{2k-2}{2k-2} B_{2k} x^{2k} \right) \]

\[ = \frac{1}{2 \sinh x} \sum_{r=0}^{\infty} (-1)^r \binom{2n}{2r} E_r (\text{csch } x)^{n-2r} \]

**Example**

\[ \frac{1^1}{e^{1x}} + \frac{3^1}{e^{3x}} + \frac{5^1}{e^{5x}} + \frac{7^1}{e^{7x}} + \ldots = \frac{1}{2} \left[ 1 + \sum_{k=1}^{\infty} \binom{2k-2}{2k} B_{2k} \frac{x^{2k}}{2k (2k-3)!} \right] = \frac{\coth^1 x}{2 \sinh x} \]

\[ \frac{1^2}{e^{1x}} + \frac{3^2}{e^{3x}} + \frac{5^2}{e^{5x}} + \frac{7^2}{e^{7x}} + \ldots = \frac{1}{2} \left[ 2 - \sum_{k=1}^{\infty} \binom{2k-2}{2k} B_{2k} \frac{x^{2k}}{2k (2k-3)!} \right] = \frac{2 \coth^2 x - \coth^0 x}{2 \sinh x} \]

**Exponential series and Factorial**

Giving \( x = 1 \) to Formula 10.6.3, we obtain the following special values.

**Formula 10.6.3’**

When \( \downarrow, \uparrow \) are ceiling function and floor function respectively, \( B_{2n} \) are Bernoulli Numbers and \( \binom{n}{r} \) are the coefficients mentioned in 10.5.0, the following expressions hold.

\[ \sum_{k=0}^{\infty} \frac{(2k+1)^n}{e^{(2k+1)x}} = \frac{1}{2} \left( n! - (-1)^n \sum_{k=\frac{n+1}{2}}^{\infty} \binom{2k-2}{2k} B_{2k} x^{2k} \right) = \frac{1}{2 \sinh x} \sum_{r=0}^{\infty} (-1)^r \binom{2n}{2r} E_r (\coth 1)^{n-2r} \]

**Example**

\[ \frac{1^1}{e^{1x}} + \frac{3^1}{e^{3x}} + \frac{5^1}{e^{5x}} + \frac{7^1}{e^{7x}} + \ldots = \frac{1}{2} \left[ 1 + \sum_{k=1}^{\infty} \binom{2k-2}{2k} B_{2k} \frac{x^{2k}}{2k (2k-3)!} \right] = \frac{\coth^1 1}{2 \sinh 1} \]

\[ \frac{1^2}{e^{1x}} + \frac{3^2}{e^{3x}} + \frac{5^2}{e^{5x}} + \frac{7^2}{e^{7x}} + \ldots = \frac{1}{2} \left[ 2 - \sum_{k=1}^{\infty} \binom{2k-2}{2k} B_{2k} \frac{x^{2k}}{2k (2k-3)!} \right] = \frac{2 \coth^2 1 - \coth^0 1}{2 \sinh 1} \]

Therefore also, this leads to the following expression conjointly with Formula 10.4.3’

\[ \frac{2^n}{e^2} + \frac{4^n}{e^4} + \frac{6^n}{e^6} + \frac{8^n}{e^8} + \ldots = \frac{n!}{2} + \left( -\frac{1}{2} \sum_{k=\frac{n+1}{2}}^{\infty} \frac{2^{2k} B_{2k}}{2k (2k-n-1)!} \right) \quad (3.e') \]
10.7 Termwise Higher Derivative of \( \sec x \)

10.7.0 Higher Derivative of \( \sec x \)
According to Formula 9.2.8 (9.2), Higher Derivative of \( \sec x \) is expressed as follows.

\[
(\sec x)^{(n)} = \frac{1}{\cos x} \sum_{r=0}^{n/2} n E_r (\tan x)^{n-2r}
\]  \[ (0.n) \]

Where, \( \downarrow \) is floor function and \( n E_r \) are the same as the coefficients in 10.5.0.

10.7.1 Termwise Higher Derivative of Taylor Series of \( \sec x \)

Formula 10.7.1
When \( E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \ldots \) are Euler Numbers and \( \downarrow \) is floor function, the following expressions hold for \( |x| < \pi/2 \).

\[
(\sec x)^{(1)} = \sum_{k=1}^{\infty} \frac{|E_{2k}|}{(2k-1)!} x^{2k-1}
\]  \[ (1.1) \]

\[
(\sec x)^{(2)} = \sum_{k=1}^{\infty} \frac{|E_{2k}|}{(2k-2)!} x^{2k-2}
\]  \[ (1.2) \]

\[
(\sec x)^{(3)} = \sum_{k=2}^{\infty} \frac{|E_{2k}|}{(2k-3)!} x^{2k-3}
\]  \[ (1.3) \]

\[
(\sec x)^{(4)} = \sum_{k=2}^{\infty} \frac{|E_{2k}|}{(2k-4)!} x^{2k-4}
\]  \[ (1.4) \]

\[
(\sec x)^{(n)} = \sum_{k=\frac{n+1}{2}}^{\infty} \frac{|E_{2k}|}{(2k-n)!} x^{2k-n}
\]  \[ (1.n) \]

Proof
\( \sec x \) is expanded to Taylor series as follows.

\[
\sec x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} x^{2k} \quad |x| < \frac{\pi}{2}
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain (1.1) ~ (1.4).

Here, considering the relation between the derivative order \( n \) and the first term \( n \) of \( \sum \), it is as follows.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_0 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>...</td>
</tr>
</tbody>
</table>

Such a relation can be expressed by \( k_0 = \frac{n+1}{2} \downarrow \) using a floor function \( x \downarrow (=\lfloor x \rfloor) \). Then we obtain the desired expression.

Sum of Taylor series of the higher derivative of \( \sec x \)
From (0.n), (1.n), we obtain the following expressions for \( |x| < \pi/2 \).

\[
\sum_{k=\frac{n+1}{2}}^{\infty} \frac{|E_{2k}|}{(2k-n)!} x^{2k-n} = \frac{1}{\cos x} \sum_{r=0}^{n/2} n E_r (\tan x)^{n-2r}
\]  \[ (1.t) \]
And giving \( x = \pi / 4 \) to this, we obtain the following special values.

\[
\sum_{k=1}^{\infty} \frac{|E_{2k}|}{(2k-1)!} \left( \frac{\pi}{4} \right)^{2k-1} = \sqrt{2} \sum_{r=0}^{0} r E_r = \sqrt{2} \cdot 1
\]

\[
\sum_{k=1}^{\infty} \frac{|E_{2k}|}{(2k-1)!} \left( \frac{\pi}{4} \right)^{2k-1} = \sqrt{2} \sum_{r=0}^{1} r E_r = \sqrt{2} \cdot 3
\]

\[
\sum_{k=2}^{\infty} \frac{|E_{2k}|}{(2k-3)!} \left( \frac{\pi}{4} \right)^{2k-3} = \sqrt{2} \sum_{r=0}^{1} r E_r = \sqrt{2} \cdot 11
\]

\[
\sum_{k=2}^{\infty} \frac{|E_{2k}|}{(2k-4)!} \left( \frac{\pi}{4} \right)^{2k-4} = \sqrt{2} \sum_{r=0}^{2} r E_r = \sqrt{2} \cdot 57
\]
10.8 Termwise Higher Derivative of sech x

10.8.0 Higher Derivative of sech x
According to Formula 9.2.9 (9.2), Higher Derivative of sech x is expressed as follows.

\[
(\text{sech } x)^{(n)} = \frac{(-1)^n n^{2k} \sum_{r=0}^{n} (-1)^r n \, E_r (\tanh x)^{n-2r}}{\cosh x}
\]

(0.n)

Where, ↓ is floor function and \( n \, E_r \) are the same as the coefficients in 10.5.0

10.8.1 Termwise Higher Derivative of Taylor Series of sech x

Formula 10.8.1
When \( E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \ldots \) are Euler Numbers and ↓ is floor function, the following expressions hold for \(|x| < \pi/2\).

\[
(\text{sech } x)^{(1)} = \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k-1)!} x^{2k-1}
\]

\[
(\text{sech } x)^{(2)} = \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k-2)!} x^{2k-2}
\]

\vdots

\[
(\text{sech } x)^{(n)} = \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k-n)!} x^{2k-n}
\]

(1.n)

Proof

sech x is expanded to Taylor series as follows.

\[
\text{sech } x = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k} \quad |x| < \frac{\pi}{2}
\]

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of \( \sum \) is the same as it of sec x.)

10.8.2 Termwise Higher Derivative of Fourier Series of sech x

Formula 10.8.2
The following expressions hold for \( x > 0 \).

\[
(\text{sech } x)^{(1)} = -2 \left( e^{-2x} - 3 e^{-3x} + 5 e^{-5x} - 7 e^{-7x} + \cdots \right)
\]

\[
(\text{sech } x)^{(2)} = 2 \left( 4 e^{-2x} - 3 e^{-3x} + 5 e^{-5x} - 4 e^{-7x} + \cdots \right)
\]

\vdots

\[
(\text{sech } x)^{(n)} = (-1)^n 2^{n+1} \sum_{k=0}^{\infty} (-1)^k (2k+1)^n e^{-(2k+1)x}
\]

(2.n)

Proof

sech x is expanded to Fourier series for \( x > 0 \) as follows.

\[
\text{sech } x = \frac{2}{e^x + e^{-x}} = \frac{2e^{-x}}{1 + e^{-2x}} = 2e^{-x} \left( 1 - e^{-2x} + e^{-4x} - e^{-6x} + \cdots \right)
\]
\[
\begin{align*}
= 2(e^{-1x} - e^{-3x} + e^{-5x} - e^{-7x} + \cdots) \\
= 2(cos 1ix - cos 3ix + cos 5ix - cos 7ix + \cdots) \\
+ 2i(sin 1ix - sin 3ix + sin 5ix - sin 7ix + \cdots)
\end{align*}
\]

Differentiating both sides of (2.0) with respect to \(x\) repeatedly, we obtain the desired expression.

### 10.8.3 Exponential series and Euler series

From \(0.n\), \(1.n\) and \(2.n\), we obtain the following formula.

**Formula 10.8.3**

When \(\downarrow\) is floor function, \(E_{2n}\) are Euler Numbers and \(nE_r\) are the coefficients mentioned in 10.5.0, the following expressions hold for \(0 < x < \pi/2\):

\[
\sum_{k=0}^{n} (-1)^k \frac{(2k+1)^n}{e^{(2k+1)x}} = \frac{(-1)^n}{2} \sum_{k=n+1}^{\infty} \frac{E_{2k}}{(2k-n)!} x^{2k-n} = \frac{1}{2 \cosh x} \sum_{r=0}^{n/4} (-1)^r nE_r (\tanh x)^{n-2r}
\]

**Example 1**

\[
\frac{1}{e^{1x}} - \frac{3}{e^{3x}} + \frac{5}{e^{5x}} - \frac{7}{e^{7x}} + \cdots = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k-1)!} x^{2k-1} = \frac{\tanh x}{2 \cosh x}
\]

**Example 2** \(n=3, x=1.4\)

\[
Pf[n, x] := \sum_{k=0}^{100} (-1)^k \frac{(2k+1)^n}{e^{(2k+1)x}}
\]

\[
Pf[n, x] := \frac{(-1)^n}{2} \sum_{k=\text{floor}[\frac{n}{4}]}^{300} \frac{E_{2k}}{(2k-n)!} x^{2k-n}
\]

\[
G3[x] := \frac{6 \tanh [x]^3 - 5 \tanh [x]}{2 \cosh [x]}
\]

\[
Pf[3, 1.4] \quad Pf[3, 1.4] \quad G3[1.4] \\
-0.0611079 \quad -0.0611079 \quad -0.0611079
\]

**Dirichlet Even Beta (minus)**

Giving \(x = 0\) to Formula 10.8.3, we obtain Dirichlet Even Beta (minus).

**Formula 10.8.3’**

When \(\beta(n) = \sum_{k=0}^{n} (-1)^k (2k+1)^n\) and \(E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots\) are Euler Numbers, the following expressions hold.

\[
\beta(-2n+1) = 0 \quad (3.2n-1')
\]
\[ \beta(-2n) = \frac{E_{2n}}{2} \]  

(3.2n')

Proof

Substitute \( x = 0 \) for Formula 10.8.3. Then, the Fourier series is

\[
\sum_{k=0}^{\infty} (-1)^k \left( \frac{2k+1}{e(2k+1)} \right)^n = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)^n} = \beta(-n)
\]

The Taylor series is

when \( n = 2m - 1 \)

\[
\frac{(-1)^n}{2} \sum_{k=\frac{n+1}{2}}^{\infty} \frac{E_{2k}}{(2k-n)!} 0^{2k-n}
\]

\[
= \frac{(-1)^{2m-1}}{2} \sum_{k=\frac{2m+1}{2}}^{\infty} \frac{E_{2k}}{(2k-(2m-1))!} 0^{2k-(2m-1)}
\]

\[
= \frac{1}{2} \sum_{k=m}^{\infty} \frac{E_{2k}}{(2k-2m+1)!} 0^{2k-2m+1} = 0
\]

when \( n = 2m \)

\[
\frac{(-1)^n}{2} \sum_{k=\frac{n+1}{2}}^{\infty} \frac{E_{2k}}{(2k-n)!} 0^{2k-n}
\]

\[
= \frac{(-1)^{2m}}{2} \sum_{k=\frac{2m}{2}}^{\infty} \frac{E_{2k}}{(2k-2m)!} 0^{2k-2m}
\]

\[
= \frac{1}{2} \sum_{k=m}^{\infty} \frac{E_{2k}}{(2k-2m)!} 0^{2k-2m} = \frac{1}{2} \frac{E_{2m}}{0!} 0^0
\]

\[
= \frac{E_{2m}}{2}
\]

Replacing \( m \) with \( n \), we obtain (3.2n-1') and (3.2n').

Note

These results are obtained also from the polynomial in Formula 10.8.3.