10 Termwise Higher Derivative (Trigonometric, Hyperbolic)

In this chapter, for the function which the second or more order derivative is difficult to express with an easy unification notation among trigonometric functions and hyperbolic function, we differentiate the series expansion of these functions term by term. Therefore, sin x, cosx, sinhx, cosh x mentioned in " **9.2** Higher Derivative of Elementary Functions " are not treated here.

10.1 Termwise Higher Derivative of tan x

10.1.0 Higher Derivative of tan x

Accordong to Formula 9.2.6 (**9.2**), Higher Derivative of tan x is expressed as follows.

$$(\tan x)^{(n)} = \sum_{r=0}^{n/2\uparrow} {}_nT_r (\tan x)^{n+1-2r} \tag{0.n}$$

Where, $\uparrow$ is celling function and ${}_nT_r$ are coefficients as follows.

$$\begin{array}{cccc} {}_0T_0 & & & \\ {}_1T_0 & {}_1T_1 & & \\ {}_2T_0 & {}_2T_1 & & \\ {}_3T_0 & {}_3T_1 & {}_3T_2 & \\ {}_4T_0 & {}_4T_1 & {}_4T_2 & \\ {}_5T_0 & {}_5T_1 & {}_5T_2 & {}_5T_3 \\ {}_6T_0 & {}_6T_1 & {}_6T_2 & {}_6T_3 \\ \vdots & & & \end{array} = \begin{array}{cccc} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & & \\ 6 & 8 & 2 & \\ 24 & 40 & 16 & \\ 120 & 240 & 136 & 16 \\ 720 & 1680 & 1232 & 272 \\ \vdots & & & \end{array}$$

10.1.1 Termwise Higher Derivative of Taylor Series of tan x

Formula 10.1.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli Numbers and $\uparrow$ is ceilling function, the following expressions hold for $|x|<\pi/2$.

$$(\tan x)^{(1)} = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-2)!} x^{2k-2} \tag{1.1}$$

$$(\tan x)^{(2)} = \sum_{k=2}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-3)!} x^{2k-3} \tag{1.2}$$

$$(\tan x)^{(3)} = \sum_{k=2}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-4)!} x^{2k-4} \tag{1.3}$$

$$(\tan x)^{(4)} = \sum_{k=3}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-5)!} x^{2k-5} \tag{1.4}$$

$$\vdots$$

$$(\tan x)^{(n)} = \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-n-1)!} x^{2k-n-1} \tag{1.n}$$

Proof

tan x is expanded to Tylor series as follows.

$$tan\,x = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-1)\,!} x^{2k-1} \qquad |x| < \frac{\pi}{2}$$

Differentiating both sides of this with respect to x repeatedly, we obtain (1.1) ~ (1.4).

Here, considering the relation between the derivative order n and the first term k_0 of $\sum$, it is as follows.

n	0	1	2	3	4	5	···
k_0	1	1	2	2	3	3	···

Such a relation can be expressed by $k_0 = \frac{n+1}{2}\uparrow$ using a ceiling function $x\uparrow\ (=\lceil x \rceil)$. Then we obtain the desired expression.

Sum of Taylor series of the higher derivative of tan x

From (0.n) and (1.n) , we obtain the following expressions for $|x|<\pi/2$.

$$\sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-n-1)!} x^{2k-n-1} = \sum_{r=0}^{n/2\uparrow} {}_nT_r(tan\,x)^{n+1-2r} \tag{1.t}$$

And giving $x=\pi/4$ to this, we obtain the following special values.

$$\sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-2)\,!}\left(\frac{\pi}{4}\right)^{2k-2} = \sum_{r=0}^{1/2\uparrow} {}_1T_r = 2$$

$$\sum_{k=2}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-3)\,!}\left(\frac{\pi}{4}\right)^{2k-3} = \sum_{r=0}^{2/2\uparrow} {}_2T_r = 4$$

$$\sum_{k=2}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-4)\,!}\left(\frac{\pi}{4}\right)^{2k-4} = \sum_{r=0}^{3/2\uparrow} {}_3T_r = 16$$

$$\sum_{k=3}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-5)\,!}\left(\frac{\pi}{4}\right)^{2k-5} = \sum_{r=0}^{4/2\uparrow} {}_4T_r = 80$$

$$\vdots$$

10.1.2 Termwise Higher Derivative of Fourier Series of tan x

As seen in **5.1.2** , tan x is expanded to Fourier series in a broad meaning.as follows.

$$\begin{aligned} tan\,x &= 2(sin\,2x - sin\,4x + sin\,6x - sin\,8x + - \cdots) \\ &\quad -2i(cos\,2x - cos\,4x + cos\,6x - cos\,8x + - \cdots) + i \end{aligned}$$

Although this does not hold as an equation, differentiating the real part of the both sides repeatedly, we obtain the following perfunctory expressions.

$$(tan\,x)^{(1)} = 2^2(1cos\,2x - 2cos\,4x + 3cos\,6x - 4cos\,8x + - \cdots)$$

$$(tan\,x)^{(2)} = -2^3\left(1^2 sin\,2x - 2^2 sin\,4x + 3^2 sin\,6x - 4^2 sin\,8x + - \cdots\right)$$

$$(tan\,x)^{(3)} = -2^4\left(1^3 cos\,2x - 2^3 cos\,4x + 3^3 cos\,6x - 4^3 cos\,8x + - \cdots\right)$$

$$(\tan x)^{(4)} = 2^5\left(1^4\sin 2x - 2^4\sin 4x + 3^4\sin 6x - 4^4\sin 8x + - \cdots\right)$$

$$\vdots$$

$$(\tan x)^{(2n-1)} = (-1)^{n-1}2^{2n}\sum_{k=1}^{\infty}(-1)^{k-1}k^{2n-1}\cos 2kx \tag{2.2n-1}$$

$$(\tan x)^{(2n)} = (-1)^{n}2^{2n+1}\sum_{k=1}^{\infty}(-1)^{k-1}k^{2n}\sin 2kx \tag{2.2n}$$

Of course, these also do not hold as an equation. However, if the calculation is advanced assuming that thiese hold, interesting results are obtained as follows.

10.1.3 Dirichlet Odd Eta & Even Beta

Formula 10.1.3

When $\eta(x) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^x}$, $\beta(n) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^n}$, B_{2k} are Bernoulli Numbers and ${}_nT_r$ are the coefficients mentioned in **10.1.0** , the following expressions hold.

$$\eta(1-2n) = \frac{(-1)^{n-1}}{2^{4n-1}}\sum_{k=n}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-2n)!}\left(\frac{\pi}{4}\right)^{2k-2n} = \frac{(-1)^{n-1}}{2^{4n-1}}\sum_{r=0}^{n}{}_{2n-1}T_r$$

$$\beta(-2n) = \frac{(-1)^{n}}{2^{2n+1}}\sum_{k=n+1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-2n-1)!}\left(\frac{\pi}{4}\right)^{2k-2n-1} = \frac{(-1)^{n}}{2^{2n+1}}\sum_{r=0}^{n}{}_{2n}T_r$$

Proof

From (0.n) , (1.n) , (2.2n-1) and (2.2n) , we obtain the following perfunctory expressions.

$$\sum_{k=1}^{\infty}(-1)^{k-1}k^{2n-1}\cos 2kx = \frac{(-1)^{n-1}}{2^{2n}}\sum_{k=n}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-2n)!}x^{2k-2n}$$

$$= \frac{(-1)^{n-1}}{2^{2n}}\sum_{r=0}^{n}{}_{2n-1}T_r(\tan x)^{2n-2r}$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}k^{2n}\sin 2kx = \frac{(-1)^{n}}{2^{2n+1}}\sum_{k=n+1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-2n-1)!}x^{2k-2n-1}$$

$$= \frac{(-1)^{n}}{2^{2n+1}}\sum_{r=0}^{n}{}_{2n}T_r(\tan x)^{2n+1-2r}$$

If $x=\pi/4$ is substituted for these, the left sides are as follows respectively.

$$\sum_{k=1}^{\infty}(-1)^{k-1}k^{2n-1}\cos\frac{k\pi}{2} = 2^{2n-1}\sum_{k=1}^{\infty}(-1)^{k-1}k^{2n-1} = 2^{2n-1}\eta(1-2n)$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}k^{2n}\sin k\frac{k\pi}{2} = \sum_{k=1}^{\infty}(-1)^{k-1}(2k-1)^{2n} = \beta(-2n)$$

Therefore, we obtain the desired expressions.

Example1

$$\eta(-1) = \frac{1}{2^3}\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|\pi^{2k}}{2k\,(2k-2)!} = \frac{1}{2^3}\sum_{r=0}^{1}{}_1T_r = \frac{1}{4}$$

$$\eta(-3) = -\frac{1}{2^7}\sum_{k=2}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|\,\pi^{2k}}{2k\,(2k-4)\,!} = -\frac{1}{2^7}\sum_{r=0}^{2}{}_3T_r = -\frac{1}{8}$$

$$\beta(-2) = -\frac{1}{2^3}\sum_{k=2}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|\,\pi^{2k}}{2k(2k-3)\,!} = -\frac{1}{2^3}\sum_{r=0}^{1}{}_2T_r = -\frac{1}{2}$$

$$\beta(-4) = \frac{1}{2^5}\sum_{k=3}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|\,\pi^{2k}}{2k(2k-5)\,!} = \frac{1}{2^5}\sum_{r=0}^{n}{}_4T_r = \frac{5}{2}$$

Example2 $\eta(-5), \beta(-6)$

```
Et[n_] := (1 - 2^(2 n)) Zeta[1 - 2 n]
```

$$\text{To}[n\_] := \frac{(-1)^{n-1}}{2^{4n-1}}\sum_{k=n}^{100}\frac{2^{2k}\left(2^{2k}-1\right)\text{Abs}[\text{BernoulliB}[2k]]}{2k\,(2k-2n)\,!}\left(\frac{\pi}{4}\right)^{2k-2n}$$

```
Et[3]      N[To[3]]
```

$\frac{1}{4}$ 0.25

```
Bt[n_] := DirichletL [4, 2, -2 n]
```

$$\text{Te}[n\_] := \frac{(-1)^{n}}{2^{2n+1}}\sum_{k=n+1}^{100}\frac{2^{2k}\left(2^{2k}-1\right)\text{Abs}[\text{BernoulliB}[2k]]}{2k\,(2k-2n-1)\,!}\left(\frac{\pi}{4}\right)^{2k-2n-1}$$

```
Bt[3]      N[Te[3]]
```

$-\frac{61}{2}$ -30.5

Note

Using a relation $\beta(-2n) = E_{2n}/2$ between Dirichlet Even Beta and Euler number, .we obtain the following expression.

$$E_{2n} = \frac{(-1)^n}{2^{2n}}\sum_{k=n+1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-2n-1)\,!}\left(\frac{\pi}{4}\right)^{2k-2n-1} = \frac{(-1)^n}{2^{2n}}\sum_{r=0}^{n}{}_{2n}T_r \qquad (3.E)$$

10.2 Termwise Higher Derivative of tanh x

10.2.0 Higher Derivative of tanh x

Accordong to Formula 9.2.7 (**9.2**), Higher Derivative of tanh x is expressed as follows.

$$(tanh x)^{(n)} = (-1)^n \sum_{r=0}^{n/2\uparrow} (-1)^r {}_nT_r (tanh x)^{n+1-2r} \tag{0.n}$$

Where, $\uparrow$ is celling function and ${}_nT_r$ are the same as the coefficients in **10.1.0** .

10.2.1 Termwise Higher Derivative of Taylor Series of tanh x

Formula 10.2.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli Numbers and $\uparrow$ is ceilling function, the following expressions hold for $|x|<\pi/2$.

$$(tanh x)^{(1)} = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-2)\,!} x^{2k-2}$$

$$(tanh x)^{(2)} = \sum_{k=2}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-3)\,!} x^{2k-3}$$

$$\vdots$$

$$(tanh x)^{(n)} = \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k(2k-n-1)!} x^{2k-n-1} \tag{1.n}$$

Proof

tanh x is expanded to Tylor series as follows.

$$tanh\ x = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-1)\,!} x^{2k-1} \qquad |x| < \frac{\pi}{2}$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of tan x.)

10.2.2 Termwise Higher Derivative of Fourier Series of tanh x

Formula 10.2.2

The following expressions hold for $x>0$.

$$(tanh x)^{(1)} = 2^2\left(1^1e^{-2x} - 2^1e^{-4x} + 3^1e^{-6x} - 4^1e^{-8x} +- \cdots\right)$$

$$(tanh x)^{(2)} = -2^3\left(1^2e^{-2x} - 2^2e^{-4x} + 3^2e^{-6x} - 4^2e^{-8x} +- \cdots\right)$$

$$\vdots$$

$$(tanh x)^{(n)} = (-1)^{n-1}2^{n+1}\sum_{k=1}^{\infty}(-1)^{k-1}k^n e^{-2kx} \tag{2.n}$$

Proof

tanh x is epanded to Fourier series for $x>0$ as follows.

$$tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1-e^{-2x}}{1+e^{-2x}} = \left(1-e^{-2x}\right)\left(1 - e^{-2x} + e^{-4x} - e^{-6x} +- \cdots\right)$$

$$= 1 - \left(2e^{-2x} - 2e^{-4x} + 2e^{-6x} - 2e^{-8x} + - \cdots\right) \tag{2.0}$$

$$= 1 - 2(\cos 2ix - \cos 4ix + \cos 6ix - \cos 8ix + - \cdots)$$

$$- 2i(\sin 2ix - \sin 4ix + \sin 6ix - \sin 8ix + - \cdots)$$

Differentiating both sides of (2.0) with respect to x repeatedly, we obtain the desired expression .

10.2.3 Exponential series and Bernouilli series

Replacing x with $x/2$ in (0.n) , (1.n) and (2.n) , we obtain the following formula.

Formula 10.2.3

When $\uparrow$ is ceilling function, B_{2n} are Bernoulli Numbers and ${}_nT_r$ are the coefficients mentioned in **10.1.0** , the following expressions hold. for $0<x<\pi$.

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k^n}{e^{kx}} = (-1)^{n-1}\sum_{k=\frac{n+1}{2}\uparrow}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k(2k-n-1)!}x^{2k-n-1}$$

$$= -\frac{1}{2^{n+1}}\sum_{r=0}^{n/2\uparrow}(-1)^r\,{}_nT_r\left(\tanh\frac{x}{2}\right)^{n+1-2r}$$

Example

$$\frac{1^1}{e^{1x}} - \frac{2^1}{e^{2x}} + \frac{3^1}{e^{3x}} - \frac{4^1}{e^{4x}} + - \cdots = \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-2)!}x^{2k-2} = -\frac{1}{2^2}\left(\tanh^2\frac{x}{2} - 1\right)$$

$$\frac{1^2}{e^{1x}} - \frac{2^2}{e^{2x}} + \frac{3^2}{e^{3x}} - \frac{4^2}{e^{4x}} + - \cdots = -\sum_{k=2}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-3)!}x^{2k-3} = -\frac{1}{2^3}\left(2\tanh^3\frac{x}{2} - 2\tanh\frac{x}{2}\right)$$

Dirichlet Odd Eta (minus)

Disregarding the convergence condition and substituting $x=0$ for Formula 10.2.3 , we obtain Dirichlet Odd Eta (minus)

Formula 10.2.3'

Let $\eta(x) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^x}$, and let Bernoulli number B_{2n} and tangent number T_r are as follows.

$$B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\ B_{10}=5/66,\ \cdots$$

$$T_1=1,\quad T_3=2,\quad T_5=16,\quad T_7=272,\quad T_9=7936,\ \cdots$$

Then the following expressions hold.

$$\eta(-2n+1) = \frac{\left(2^{2n}-1\right)B_{2n}}{2n} = -\frac{(-1)^n}{2^{2n}}T_{2n-1} \tag{3.2n-1'}$$

$$\eta(-2n) \quad = 0 \tag{3.2n'}$$

Proof

Sbstitute $x=0$ for Formula 10.2.3 . Then, the Fourier series is

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k^n}{e^{k0}} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{-n}} = \eta(-n)$$

The Taylor series and the polynomial are

when $n=2m-1$

$$(-1)^{n-1}\sum_{k=\frac{n+1}{2}\uparrow}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-n-1)\,!}0^{2k-n-1}$$

$$=(-1)^{2m-1-1}\sum_{k=\frac{2m-1+1}{2}\uparrow}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,\{2k-(2m-1)-1\}\,!}0^{2k-(2m-1)-1}$$

$$=\sum_{k=m}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-2m)\,!}0^{2k-2m}=\frac{\left(2^{2m}-1\right)B_{2m}}{2m\;0!}0^{0}=\frac{\left(2^{2m}-1\right)B_{2m}}{2m}$$

$$-\frac{1}{2^{n+1}}\sum_{r=0}^{n/2\uparrow}(-1)^{r}\,{}_{n}T_{r}\left(tanh\frac{0}{2}\right)^{n+1-2r}$$

$$=-\frac{1}{2^{2m}}\sum_{r=0}^{m}(-1)^{r}\;{}_{2m-1}T_{r}0^{2m-2r}=-\frac{1}{2^{2m}}(-1)^{m}\;{}_{2m-1}T_{m}0^{0}$$

$$=-\frac{(-1)^{m}}{2^{2m}}T_{2m-1}\qquad\left(\because\quad {}_{2m-1}T_{m}=T_{2m-1}\right)$$

when $n=2m$

$$(-1)^{n-1}\sum_{k=\frac{n+1}{2}\uparrow}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-n-1)\,!}0^{2k-n-1}$$

$$=(-1)^{2m-1}\sum_{k=\frac{2m+1}{2}\uparrow}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-2m-1)\,!}0^{2k-2m-1}=0$$

$$-\frac{1}{2^{n+1}}\sum_{r=0}^{n/2\uparrow}(-1)^{r}\,{}_{n}T_{r}\left(tanh\frac{0}{2}\right)^{n+1-2r}$$

$$=-\frac{1}{2^{2m+1}}\sum_{r=0}^{m}(-1)^{r}\;{}_{2m}T_{r}0^{2m+1-2r}=0$$

Replacing m with n , we obtain (3.2n-1') and (3.2n') .

10.3 Termwise Higher Derivative of cot x

10.3.0 Higher Derivative of cot x

Accordong to Formula 9.2.6 (**9.2**), Higher Derivative of cot x is expressed as follows.

$$(\cot x)^{(n)} = (-1)^n \sum_{r=0}^{n/2\uparrow} {}_nT_r (\cot x)^{n+1-2r} \qquad (0,n)$$

Where, ↑ is celling function and ${}_nT_r$ are the same as the coefficients in **10.1.0** .

10.3.1 Termwise Higher Derivative of Taylor Series of cot x

Formula 10.3.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli Numbers and ↑ is ceilling function, the following expressions hold for $0<x<\pi$.

$$(\cot x)^{(1)} = -\frac{1!}{x^2} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k-2)!} x^{2k-2}$$

$$(\cot x)^{(2)} = \frac{2!}{x^3} - \sum_{k=2}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k-3)!} x^{2k-3}$$

$$\vdots$$

$$(\cot x)^{(n)} = (-1)^n \frac{n!}{x^{n+1}} - \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k-n-1)!} x^{2k-n-1} \qquad (1.n)$$

Proof

$$x\cot x = 1 + \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k} \qquad 0<x<\pi$$

From this

$$\cot x = \frac{0!}{x^1} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k-1)!} x^{2k-1}$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of tan x.)

Formula 10.3.1 can also be expressed as follows.

Formula 10.3.1'

When B_{2k} denote Bernoulli Numbers and ↑ denotes ceilling function, the following expressions hold for $\pi/2<x<\pi$.

$$(\cot x)^{(n)} = -\sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-n-1)!} \left(x-\frac{\pi}{2}\right)^{2k-n-1}$$

Proof

$$\cot x = -\tan\left(x-\frac{\pi}{2}\right)$$

$$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k\,(2k-1)!} x^{2k-1} \qquad |x| < \frac{\pi}{2}$$

From these

$$\cot x = -\sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k\,(2k-1)!} \left(x-\frac{\pi}{2}\right)^{2k-1} \qquad 0 < x < \pi$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of tan x.)

Sum of Taylor series of the higher derivative of cot x

From (0,n) and (1.n) , we obtain the following expressions for $0 < x < \pi$.

$$\sum_{k=\lceil\frac{n+1}{2}\rceil}^{\infty} \frac{2^{2k}\left|B_{2k}\right|}{2k(2k-n-1)!} x^{2k} = (-1)^n \left\{ n! - x^{n+1} \sum_{r=0}^{\lceil n/2\rceil} {}_nT_r (\cot x)^{n+1-2r} \right\} \tag{1.t}$$

And giving $x=\pi/4$ to this, we obtain the following special values.

$$\sum_{k=1}^{\infty} \frac{2^{2k}\left|B_{2k}\right|}{2k(2k-2)!} \left(\frac{\pi}{4}\right)^{2k} = -1! + 2\left(\frac{\pi}{4}\right)^{2}$$

$$\sum_{k=2}^{\infty} \frac{2^{2k}\left|B_{2k}\right|}{2k(2k-3)!} \left(\frac{\pi}{4}\right)^{2k} = 2! - 4\left(\frac{\pi}{4}\right)^{3}$$

$$\sum_{k=2}^{\infty} \frac{2^{2k}\left|B_{2k}\right|}{2k(2k-4)!} \left(\frac{\pi}{4}\right)^{2k} = -3! + 16\left(\frac{\pi}{4}\right)^{4}$$

$$\sum_{k=3}^{\infty} \frac{2^{2k}\left|B_{2k}\right|}{2k(2k-5)!} \left(\frac{\pi}{4}\right)^{2k} = 4! - 80\left(\frac{\pi}{4}\right)^{5}$$

$$\vdots$$

10.3.2 Termwise Higher Derivative of Fourier Series of tan x

As seen in **5.3.2** , cot x is expanded to Fourier series in a broad meaning as follows.

$$\begin{aligned}\cot x &= 2(\sin 2x + \sin 4x + \sin 6x + \sin 8x + \cdots) \\ &\quad -2i(\cos 2x + \cos 4x + \cos 6x + \cos 8x + \cdots) - i\end{aligned}$$

Although this does not hold as an equation, differentiating the real part of the both sides repeatedly, we obtain the following perfunctory expressions..

$$(\cot x)^{(1)} = 2^2(1\cos 2x + 2\cos 4x + 3\cos 6x + 4\cos 8x + \cdots)$$

$$(\cot x)^{(2)} = -2^3(1^2\sin 2x + 2^2\sin 4x + 3^2\sin 6x + 4^2\sin 8x + \cdots)$$

$$(\cot x)^{(3)} = -2^4(1^3\cos 2x + 2^3\cos 4x + 3^3\cos 6x + 4^3\cos 8x + \cdots)$$

$$(\cot x)^{(4)} = 2^5(1^4\sin 2x + 2^4\sin 4x + 3^4\sin 6x + 4^4\sin 8x + \cdots)$$

$$\vdots$$

$$(\cot x)^{(2n-1)} = (-1)^{n-1} 2^{2n} \sum_{k=1}^{\infty} k^{2n-1} \cos 2kx \tag{2.2n-1}$$

$$(\cot x)^{(2n)} = (-1)^n 2^{2n+1} \sum_{k=1}^{\infty} k^{2n} \sin 2kx \tag{2.2n}$$

Of course, these also do not hold as an equation. However, if the calculation is advanced assuming that thiese hold, interesting results are obtained as follows.

10.3.3 Dirichlet Odd Eta & Even Beta

Formula 10.3.3

When $\eta(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^x}$, $\beta(n)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^n}$, B_{2k} are Bernoulli Numbers and ${}_nT_r$ are the coefficients mentioned in **10.1.0** , the following expressions hold.

$$\eta(1-2n)=\frac{(-1)^{n-1}}{2^{4n-1}}\left(\frac{4}{\pi}\right)^{2n}\left\{(2n-1)!+\sum_{k=n}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-2n)!}\left(\frac{\pi}{4}\right)^{2k}\right\}$$

$$=\frac{(-1)^{n-1}}{2^{4n-1}}\sum_{r=0}^{n}{}_{2n-1}T_r$$

$$\beta(-2n)=\frac{(-1)^n}{2^{2n+1}}\left(\frac{4}{\pi}\right)^{2n}\left\{(2n)!-\sum_{k=n+1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-2n-1)!}\left(\frac{\pi}{4}\right)^{2k}\right\}$$

$$=(-1)^n\sum_{r=0}^{n}{}_{2n}T_r$$

Proof

From (0.n) , (1.n) , (2.2n-1) and (2.2n) , we obtain the following perfunctory expressions.

$$\sum_{k=1}^{\infty}k^{2n-1}\cos 2kx=\frac{(-1)^n}{2^{2n}x^{2n}}\left\{(2n-1)!+\sum_{k=n}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-2n)!}x^{2k}\right\}$$

$$=\frac{(-1)^n}{2^{2n}}\sum_{r=0}^{n}{}_{2n-1}T_r(\cot x)^{2n-2r}$$

$$\sum_{k=1}^{\infty}k^{2n}\sin 2kx=\frac{(-1)^n}{2^{2n+1}x^{2n+1}}\left\{(2n)!-\sum_{k=n+1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-2n-1)!}x^{2k}\right\}$$

$$=(-1)^n\sum_{r=0}^{n}{}_{2n}T_r(\cot x)^{2n+1-2r}$$

If $x=\pi/4$ is substituted for these, the left sides are as follows respectively.

$$\sum_{k=1}^{\infty}k^{2n-1}\cos\frac{k\pi}{2}=-2^{2n-1}\sum_{k=1}^{\infty}(-1)^{k-1}k^{2n-1}=-2^{2n-1}\eta(1-2n)$$

$$\sum_{k=1}^{\infty}k^{2n}\sin k\frac{k\pi}{2}=\sum_{k=1}^{\infty}(-1)^{k-1}(2k-1)^{2n}=\beta(-2n)$$

Therefore, we obtain the desired expressions.

Example

$$\eta(-1)=\frac{1}{2^3}\left(\frac{4}{\pi}\right)^2\left\{1!+\sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-2)!}\left(\frac{\pi}{4}\right)^{2k}\right\}=\frac{1}{2^3}\sum_{r=0}^{1}{}_1T_r=\frac{1}{4}$$

$$\eta(-3)=-\frac{1}{2^7}\left(\frac{4}{\pi}\right)^4\left\{3!+\sum_{k=2}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-4)!}\left(\frac{\pi}{4}\right)^{2k}\right\}=-\frac{1}{2^7}\sum_{r=0}^{2}{}_3T_r=-\frac{1}{8}$$

$$\beta(-2) = -\frac{1}{2^3}\left(\frac{4}{\pi}\right)^2\left\{2! - \sum_{k=2}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-3)!}\left(\frac{\pi}{4}\right)^{2k}\right\} = -\frac{1}{2^3}\sum_{r=0}^{1}{}_2T_r = -\frac{1}{2}$$

$$\beta(-4) = \frac{1}{2^5}\left(\frac{4}{\pi}\right)^4\left\{4! - \sum_{k=3}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-5)!}\left(\frac{\pi}{4}\right)^{2k}\right\} = \frac{1}{2^5}\sum_{r=0}^{n}{}_4T_r = \frac{5}{2}$$

10.3.4 Factorial and Bernouilli series

Formula 10.3.4

When B_{2k} denotes a Bernoulli Number, the following expressions hold.

$$(2n-1)! = \sum_{k=n}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k(2k-2n)!}\left(\frac{\pi}{2}\right)^{2k} \tag{4.1}$$

$$(2n)! = \sum_{k=n+1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-2n-1)!}\left(\frac{\pi}{2}\right)^{2k} \tag{4.2}$$

Proof

Replacing n with $2n-1$ in Formula 10.3.1' , we obtain

$$(\cot x)^{(2n-1)} = -\sum_{k=n}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-2n)!}\left(x-\frac{\pi}{2}\right)^{2k-2n}$$

Substituting $x=\pi/4$ for this,

$$(\cot x)^{(2n-1)}\Big|_{x=\pi/4} = -\sum_{k=n}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-2n)!}\left(\frac{\pi}{4}\right)^{2k-2n}$$

On the othe hand, substituting $x=\pi/4$ for (2.2n-1) ,

$$(\cot x)^{(2n-1)}\Big|_{x=\pi/4} = -(2n-1)!\left(\frac{4}{\pi}\right)^{2n} - \sum_{k=n}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-2n)!}\left(\frac{\pi}{4}\right)^{2k-2n}$$

Since these have to be equal, we obtain (4.1).

Next, replacing n with $2n$ in Formula 10.3.1' ,

$$(\cot x)^{(2n)} = -\sum_{k=n+1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-2n-1)!}\left(x-\frac{\pi}{2}\right)^{2k-2n-1}$$

Substituting $x=\pi/4$ for this, we obtain

$$(\cot x)^{(2n)}\Big|_{x=\pi/4} = \sum_{k=n+1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-2n-1)!}\left(\frac{\pi}{4}\right)^{2k-2n-1}$$

On the othe hand, substituting $x=\pi/4$ for (2.2n) ,

$$(\cot x)^{(2n)}\Big|_{x=\pi/4} = (2n)!\left(\frac{4}{\pi}\right)^{2n+1} - \sum_{k=n+1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-2n-1)!}\left(\frac{\pi}{4}\right)^{2k-2n-1}$$

Since these have to be equal, we obtain (4.2).

Proof of the pudding is in the eating.

(2n-1)!

$$\text{Fo}[n\_] := \sum_{k=n}^{100}\frac{(2^{2k}-2)\ \text{Abs}[\text{BernoulliB}[2k]]}{2k\ (2k-2n)!}\left(\frac{\pi}{2}\right)^{2k}$$

```
N[Fo[1]]    N[Fo[2]]    N[Fo[3]]
1.          6.          120.
```

(2n)!

$$\texttt{Fe[n\_] :=} \sum_{k=n+1}^{100} \frac{2^{2k}\,\texttt{Abs[BernoulliB[2 k]]}}{2k\,(2k-2n-1)!}\left(\frac{\pi}{2}\right)^{2k}$$

```
N[Fe[1]]    N[Fe[2]]    N[Fe[3]]
2.          24.         720.
```

10.4 Termwise Higher Derivative of coth x

10.4.0 Higher Derivative of coth x

Accordong to Formula 9.2.7 (**9.2**), Higher Derivative of coth x is expressed as follows.

$$(\coth x)^{(n)} = (-1)^n \sum_{r=0}^{n/2\uparrow} (-1)^r \, {}_nT_r (\coth x)^{n+1-2r} \tag{0.n}$$

Where, ↑ is celling function and ${}_nT_r$ are the same as the coefficients in **10.1.0** .

10.4.1 Termwise Higher Derivative of Taylor Series of coth x

Formula 10.4.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli Numbers and ↑ is ceilling function, the following expressions hold for $0<x<\pi$.

$$(\coth x)^{(1)} = -\frac{1!}{x^2} + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k-2)!} x^{2k-2}$$

$$(\coth x)^{(2)} = \frac{2!}{x^3} + \sum_{k=2}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k-3)!} x^{2k-3}$$

$$\vdots$$

$$(\coth x)^{(n)} = (-1)^n \frac{n!}{x^{n+1}} + \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k-n-1)!} x^{2k-n-1} \tag{1.n}$$

Proof

$$x \coth x = 1 + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!} x^{2k} \qquad 0<x<\pi$$

From this

$$\coth x = \frac{0!}{x} + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k-1)!} x^{2k-1}$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of tan x.)

10.4.2 Termwise Higher Derivative of Fourier Series of coth x

Formula 10.4.2

The following expressions hold for $x>0$.

$$(\coth x)^{(1)} = -2^2\left(1^1 e^{-2x} + 2^1 e^{-4x} + 3^1 e^{-6x} + 4^1 e^{-8x} + \cdots\right)$$

$$(\coth x)^{(2)} = 2^3\left(1^2 e^{-2x} + 2^2 e^{-4x} + 3^2 e^{-6x} + 4^2 e^{-8x} + \cdots\right)$$

$$\vdots$$

$$(\coth x)^{(n)} = (-1)^n 2^{n+1} \sum_{k=1}^{\infty} k^n e^{-2kx} \tag{2.n}$$

Proof

coth x is expanded to Fourier series for $x>0$ as follows.

$$coth\,x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1+e^{-2x}}{1-e^{-2x}} = \left(1+e^{-2x}\right)\left(1+e^{-2x}+e^{-4x}+e^{-6x}+\cdots\right)$$

$$= 1+\left(2e^{-2x}+2e^{-4x}+2e^{-6x}+2e^{-8x}+-\cdots\right) \qquad (2.0)$$

$$= 1+2(cos\,2ix+cos\,4ix+cos\,6ix+cos\,8ix+\cdots)$$

$$+\,2i(sin\,2ix+sin\,4ix+sin\,6ix+sin\,8ix+\cdots)$$

Differentiating both sides of (2.0) with respect to x repeatedly, we obtain the desired expression.

10.4.3 Exponential series and Bernouilli series

Replacing x with $x/2$ in (0.n) , (1.n) and (2.n) , we obtain the following formula.

Formula 10.4.3

When $\uparrow$ is ceilling function, B_{2n} are Bernoulli Numbers and ${}_nT_r$ are the coefficients mentioned in **10.1.0** , the following expressions hold for $0<x<\pi$.

$$\sum_{k=1}^{\infty}\frac{k^n}{e^{kx}} = \frac{1}{x^{n+1}}\left\{n\,!+(-1)^n\sum_{k=\frac{n+1}{2}\uparrow}^{\infty}\frac{B_{2k}x^{2k}}{2k(2k-n-1)\,!}\right\}$$

$$= \frac{1}{2^{n+1}}\sum_{r=0}^{n/2\uparrow}(-1)^r\,{}_nT_r\left(coth\frac{x}{2}\right)^{n+1-2r}$$

Example

$$\frac{1^1}{e^{1x}}+\frac{2^1}{e^{2x}}+\frac{3^1}{e^{3x}}+\frac{4^1}{e^{4x}}+\cdots = \frac{1}{x^2}\left\{1!-\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{2k(2k-2)\,!}\right\} = \frac{1}{2^2}\left(coth^2\frac{x}{2}-1\right)$$

$$\frac{1^2}{e^{1x}}+\frac{2^2}{e^{2x}}+\frac{3^2}{e^{3x}}+\frac{4^2}{e^{4x}}+\cdots = \frac{1}{x^3}\left\{2!+\sum_{k=2}^{\infty}\frac{B_{2k}x^{2k}}{2k(2k-3)\,!}\right\} = \frac{1}{2^3}\left(2\,coth^3\frac{x}{2}-2\,coth\frac{x}{2}\right)$$

Exponential series and Factorial

Giving $x=1$ to Formula 10.4.3, we obtain the following special values.

Formula 10.4.3'

When $\uparrow$ is ceilling function, B_{2n} are Bernoulli Numbers and ${}_nT_r$ are the coefficients mentioned in **10.1.0** , the following expressions hold.

$$\sum_{k=1}^{\infty}\frac{k^n}{e^k} = n\,!+(-1)^n\sum_{k=\frac{n+1}{2}\uparrow}^{\infty}\frac{B_{2k}}{2k(2k-n-1)\,!} = \frac{1}{2^{n+1}}\sum_{r=0}^{n/2\uparrow}(-1)^r\,{}_nT_r\left(coth\frac{1}{2}\right)^{n+1-2r}$$

Note

Mr. Sugimoto was foreseeing this formula. (http://homepage3.nifty.com/y_sugi/sp/sp56.htm)

Example

$$\frac{1^1}{e^1}+\frac{2^1}{e^2}+\frac{3^1}{e^3}+\frac{4^1}{e^4}+\cdots = 1!-\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-2)\,!} = \frac{1}{2^2}\left(coth^2\frac{1}{2}-1\right)$$

$$\frac{1^2}{e^1}+\frac{2^2}{e^2}+\frac{3^2}{e^3}+\frac{4^2}{e^4}+\cdots = 2! + \sum_{k=2}^{\infty}\frac{B_{2k}}{2k(2k-3)!} = \frac{1}{2^3}\left(2coth^3\frac{1}{2}-2coth\frac{1}{2}\right)$$

$$\frac{1^3}{e^1}+\frac{2^3}{e^2}+\frac{3^3}{e^3}+\frac{4^3}{e^4}+\cdots = 3! - \sum_{k=2}^{\infty}\frac{B_{2k}}{2k(2k-4)!} = \frac{1}{2^4}\left(6coth^4\frac{1}{2}-8coth^2\frac{1}{2}+2\right)$$

Therefore also, the following approximation formula is obtained.

Formula 10.4.3"

$$\frac{1^p}{e^1}+\frac{2^p}{e^2}+\frac{3^p}{e^3}+\frac{4^p}{e^4}+\cdots \fallingdotseq \Gamma(1+p) \qquad p>0 \tag{3.p}$$

Example $\Gamma(1+2.5)$

```
s[p_] := Sum[k^p / e^k, {k, 1, 100}]
```

$$s[p\_] := \sum_{k=1}^{100}\frac{k^p}{e^k}$$

s[2.5]	Gamma[3.5]
3.32641	3.32335

10.5 Termwise Higher Derivative of csc x

10.5.0 Higher Derivative of csc x

Accordong to Formula 9.2.8 (**9.2**), Higher Derivative of csc x is expressed as follows.

$$(\csc x)^{(n)} = \frac{(-1)^n}{\sin x} \sum_{r=0}^{n/2\downarrow} {}_nE_r (\cot x)^{n-2r} \tag{0.n}$$

Where, $\downarrow$ is floor function and ${}_nE_r$ are coefficients as follows.

$$\begin{matrix} {}_0E_0 \\ {}_1E_0 \\ {}_2E_0 \; {}_2E_1 \\ {}_3E_0 \; {}_3E_1 \\ {}_4E_0 \; {}_4E_1 \; {}_4E_2 \\ {}_5E_0 \; {}_5E_1 \; {}_5E_2 \\ {}_6E_0 \; {}_6E_1 \; {}_6E_2 \; {}_6E_3 \\ {}_7E_0 \; {}_7E_1 \; {}_7E_2 \; {}_7E_3 \\ \vdots \end{matrix} = \begin{matrix} 1 \\ 1 \\ 2 \quad 1 \\ 6 \quad 5 \\ 24 \quad 28 \quad 5 \\ 120 \quad 180 \quad 61 \\ 720 \quad 1320 \quad 662 \quad 61 \\ 5040 \quad 10920 \quad 7266 \quad 1385 \\ \vdots \end{matrix}$$

10.5.1 Termwise Higher Derivative of Taylor Series of csc x

Formula 10.5.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli Numbers and $\uparrow$ is ceilling function, the following expressions hold for $0<x<\pi$.

$$(\csc x)^{(1)} = -\frac{1!}{x^2} + \sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-2)!} x^{2k-2}$$

$$(\csc x)^{(2)} = \frac{2!}{x^3} + \sum_{k=2}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-3)!} x^{2k-3}$$

$$\vdots$$

$$(\csc x)^{(n)} = (-1)^n \frac{n!}{x^{n+1}} + \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-n-1)!} x^{2k-n-1} \tag{1.n}$$

Proof

$$x\csc x = 1 + \sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{(2k)!} x^{2k} \qquad 0<x<\pi$$

From this

$$\csc x = \frac{0!}{x} + \sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-1)!} x^{2k-1}$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of tan x.)

Formula 10.5.1 can also be expressed as follows.

Formula 10.5.1'

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots$ are Euler Numbers and $\downarrow$ is floor function, the following expressions hold for $\pi/2 < x < \pi$.

$$(csc\,x)^{(n)} = \sum_{k=\frac{n+1}{2}\downarrow}^{\infty} \frac{|E_{2k}|}{(2k-n)!}\left(x-\frac{\pi}{2}\right)^{2k-n}$$

proof

$$csc\,x = sec\left(x-\frac{\pi}{2}\right)$$

$$sec\,x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!}x^{2k} \qquad |x| < \frac{\pi}{2}$$

From these

$$csc\,x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!}\left(x-\frac{\pi}{2}\right)^{2k} \qquad 0 < x < \pi$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of sec x.)

Sum of Taylor series of the higher derivative of csc x

From (0,n) and (1.n) , we obtainthe following expressions for $0 < x < \pi$.

$$\sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-n-1)!}\,x^{2k} = (-1)^{n-1}\left\{n! - \frac{x^{n+1}}{sin\,x}\sum_{r=0}^{n/2\downarrow} {}_{n}E_r (cot\,x)^{n-2r}\right\} \qquad (1.t)$$

And giving $x=\pi/2$ to this, we obtain the following special values.

$$\sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-2)!}\left(\frac{\pi}{2}\right)^{2k} = 1!$$

$$\sum_{k=2}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-3)!}\left(\frac{\pi}{2}\right)^{2k} = -2! + 1\left(\frac{\pi}{2}\right)^{3}$$

$$\sum_{k=2}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-4)!}\left(\frac{\pi}{2}\right)^{2k} = 3!$$

$$\sum_{k=3}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-5)!}\left(\frac{\pi}{2}\right)^{2k} = -4! + 5\left(\frac{\pi}{2}\right)^{5}$$

$$\vdots$$

10.5.2 Termwise Higher Derivative of Fourier Series of csc x

As seen in **5.5.2** , csc x is expanded to Fourier series in a broad meaning.as follows.

$$\begin{aligned} csc\,x &= 2(sin\,x + sin\,3x + sin\,5x + sin\,7x + \cdots) \\ &\quad -2i(cos\,x + cos\,3x + cos\,5x + cos\,7x + \cdots) \end{aligned}$$

Although this does not hold as an equation, differentiating the real part of the both sides repeatedly, we obtain the following perfunctory expressions..

$$(\csc x)^{(1)} = 2(1\cos x+3\cos 3x+5\cos 5x+7\cos 7x+-\cdots)$$
$$(\csc x)^{(2)} = -2\left(1^2\sin x+3^2\sin 3x+5^2\sin 5x+4^2\sin 7x+-\cdots\right)$$
$$(\csc x)^{(3)} = -2\left(1^3\cos x+3^3\cos 3x+5^3\cos 5x+4^3\cos 7x+-\cdots\right)$$
$$(\csc x)^{(4)} = 2\left(1^4\sin x+3^4\sin 3x+5^4\sin 5x+4^4\sin 7x+-\cdots\right)$$
$$\vdots$$
$$(\csc x)^{(2n-1)} = (-1)^{n-1}2\sum_{k=0}^{\infty}(2k+1)^{2n-1}\cos\{(2k+1)x\} \tag{2.2n-1}$$
$$(\csc x)^{(2n)} = (-1)^{n}2\sum_{k=0}^{\infty}(2k+1)^{2n}\sin\{(2k+1)x\} \tag{2.2n}$$

Of course, these also do not hold as an equation. However, if the calculation is advanced assuming that thiese hold, interesting results are obtained as follows.

10.5.3 Dirichlet Even Beta (minus)

Formula 10.5.3

Let $\beta(n) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^n}$, and let Bernoulli number B_{2n} and Euler number E_r are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$$
$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$$

Then the following expressions hold.

$$\beta(-2n) = \frac{(-1)^n}{2}\left(\frac{2}{\pi}\right)^{2n+1}\left\{(2n)!+\sum_{k=n+1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k(2k-2n-1)!}\left(\frac{\pi}{2}\right)^{2k}\right\} = \frac{E_{2n}}{2}$$

Proof

From (0.n) , (1.n) and (2.2n) , we obtain the following perfunctory expression.

$$\sum_{k=0}^{\infty}(2k+1)^{2n}\sin\{(2k+1)x\} = \frac{(-1)^n}{2x^{2n+1}}\left\{(2n)!+\sum_{k=n+1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-2n-1)!}\,x^{2k}\right\}$$
$$= \frac{(-1)^n}{2\sin x}\sum_{r=0}^{n}{}_{2n}E_r(\cot x)^{2n-2r}$$

If $x=\pi/2$ is substituted for this, the left side is as follows.

$$\sum_{k=0}^{\infty}(2k+1)^{2n}\sin\frac{(2k+1)\pi}{2} = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{2n} = \beta(-2n)$$

Then,

$$\beta(-2n) = \frac{(-1)^n}{2}\left(\frac{2}{\pi}\right)^{2n+1}\left\{(2n)!+\sum_{k=n+1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k(2k-2n-1)!}\left(\frac{\pi}{2}\right)^{2k}\right\}$$
$$= \frac{(-1)^n}{2}\sum_{r=0}^{n}{}_{2n}E_r\,0^{2n-2r} = \frac{(-1)^n}{2}\,{}_{2n}E_n0^0$$
$$= \frac{E_{2n}}{2} \qquad \left(\because\ (-1)^n\,{}_{2n}E_n = E_{2n}\right)$$

Example

$$\beta(-2) = -\frac{1}{2}\left(\frac{2}{\pi}\right)^3\left\{2! + \sum_{k=2}^{\infty}\frac{\left(2^{2k}-2\right)\left|B_{2k}\right|}{2k(2k-3)!}\left(\frac{\pi}{2}\right)^{2k}\right\} = \frac{E_2}{2} = -\frac{1}{2}$$

$$\beta(-4) = \frac{1}{2}\left(\frac{2}{\pi}\right)^5\left\{4! + \sum_{k=3}^{\infty}\frac{\left(2^{2k}-2\right)\left|B_{2k}\right|}{2k(2k-5)!}\left(\frac{\pi}{2}\right)^{2k}\right\} = \frac{E_4}{2} = \frac{5}{2}$$

10.6 Termwise Higher Derivative of csch x

10.6.0 Higher Derivative of csch x

Accordong to Formula 9.2.9 (**9.2**), Higher Derivative of csch x is expressed as follows.

$$(csch\,x)^{(n)} = \frac{(-1)^n}{sinh\,x}\sum_{r=0}^{n/2\downarrow}(-1)^r\,{}_nE_r(coth\,x)^{n-2r} \tag{0.n}$$

Where, $\downarrow$ is floor function and ${}_nE_r$ are the same as the coefficients in **10.5.0**.

10.6.1 Termwise Higher Derivative of Taylor Series of csch x

Formula 10.6.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli Numbers and $\uparrow$ is ceilling function, the following expressions hold for $0<x<\pi$.

$$(csch\,x)^{(1)} = -\frac{1!}{x^2} - \sum_{k=1}^{\infty}\frac{(2^{2k}-2)B_{2k}}{2k\,(2k-2)!}\,x^{2k-2}$$

$$(csch\,x)^{(2)} = \frac{2!}{x^3} - \sum_{k=2}^{\infty}\frac{(2^{2k}-2)B_{2k}}{2k\,(2k-3)!}\,x^{2k-3}$$

$$\vdots$$

$$(csch\,x)^{(n)} = (-1)^n\frac{n!}{x^{n+1}} - \sum_{k=\frac{n+1}{2}\uparrow}^{\infty}\frac{(2^{2k}-2)B_{2k}}{2k(2k-n-1)!}\,x^{2k-n-1} \tag{1.n}$$

Proof

$$x\,csch\,x = 1 - \sum_{k=1}^{\infty}\frac{(2^{2k}-2)B_{2k}}{(2k)!}\,x^{2k} \qquad 0<x<\pi$$

From this

$$csch\,x = \frac{0!}{x} - \sum_{k=1}^{\infty}\frac{(2^{2k}-2)B_{2k}}{2k\,(2k-1)!}\,x^{2k-1}$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of tan x.)

10.6.2 Termwise Higher Derivative of Fourier Series of csch x

Formula 10.6.2

The following expressions hold for $x>0$.

$$(csch\,x)^{(1)} = -2\left(1e^{-1x}+3e^{-3x}+5e^{-5x}+7e^{-7x}+\cdots\right)$$

$$(csch\,x)^{(2)} = 2\left(1^2e^{-1x}+3^2e^{-3x}+5^2e^{-5x}+4^2e^{-7x}+\cdots\right)$$

$$\vdots$$

$$(csch\,x)^{(n)} = (-1)^n 2\sum_{k=0}^{\infty}(2k+1)^n e^{-(2k+1)x} \tag{2.n}$$

Proof

csch x is expanded to Fourier series for $x>0$ as follows.

$$cschx = \frac{2}{e^{x}-e^{-x}} = \frac{2e^{-x}}{1-e^{-2x}} = 2e^{-x}\left(1+e^{-2x}+e^{-4x}+e^{-6x}+\cdots\right)$$
$$= 2\left(e^{-1x}+e^{-3x}+e^{-5x}+e^{-7x}+\cdots\right) \qquad (2.0)$$
$$= 2\left(cosh\,ix+cosh\,3ix+cosh\,5ix+\cdots\right)-2i\left(sinh\,ix+sinh\,3ix+sinh\,5ix+\cdots\right)$$

Differentiating both sides of (2.0) with respect to x repeatedly, we obtain the desired expression.

10.6.3 Exponential series and Bernouilli series

From (0.n) , (1.n) and (2.n) , we obtain the following formula.

Formula 10.6.3

When $\downarrow,\uparrow$ are ceilling function and floor function respectivly, B_{2n} are Bernoulli Numbers and ${}_{n}E_{r}$ are the coefficients mentioned in **10.5.0** , the following expressions hold for $0<x<\pi$.

$$\sum_{k=0}^{\infty}\frac{(2k+1)^{n}}{e^{(2k+1)x}} = \frac{1}{2x^{n+1}}\left\{n!-(-1)^{n}\sum_{k=\frac{n+1}{2}\uparrow}^{\infty}\frac{\left(2^{2k}-2\right)B_{2k}}{2k(2k-n-1)!}\,x^{2k}\right\}$$
$$= \frac{1}{2sinh\,x}\sum_{r=0}^{n/2\downarrow}(-1)^{r}\,{}_{n}E_{r}(csch\,x)^{n-2r}$$

Example

$$\frac{1^1}{e^{1x}}+\frac{3^1}{e^{3x}}+\frac{5^1}{e^{5x}}+\frac{7^1}{e^{7x}}+\cdots = \frac{1}{2x^2}\left\{1!+\sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)B_{2k}}{2k\,(2k-2)\,!}\,x^{2k}\right\} = \frac{coth^{1}x}{2sinh\,x}$$

$$\frac{1^2}{e^{1x}}+\frac{3^2}{e^{3x}}+\frac{5^2}{e^{5x}}+\frac{7^2}{e^{7x}}+\cdots = \frac{1}{2x^3}\left\{2!-\sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)B_{2k}}{2k\,(2k-3)\,!}\,x^{2k}\right\} = \frac{2coth^{2}x-coth^{0}x}{2sinh\,x}$$

Exponential series and Factorial

Giving $x=1$ to Formula 10.6.3, we obtain the following special values.

Formula 10.6.3'

When $\downarrow,\uparrow$ are ceilling function and floor function respectivly, B_{2n} are Bernoulli Numbers and ${}_{n}E_{r}$ are the coefficients mentioned in **10.5.0** , the following expressions hold.

$$\sum_{k=0}^{\infty}\frac{(2k+1)^{n}}{e^{(2k+1)}} = \frac{1}{2}\left\{n!-(-1)^{n}\sum_{k=\frac{n+1}{2}\uparrow}^{\infty}\frac{\left(2^{2k}-2\right)B_{2k}}{2k(2k-n-1)!}\right\} = \frac{1}{2sinh\,1}\sum_{r=0}^{n/2\downarrow}(-1)^{r}{}_{n}E_{r}(coth\,1)^{n-2r}$$

Example

$$\frac{1^1}{e^1}+\frac{3^1}{e^3}+\frac{5^1}{e^5}+\frac{7^1}{e^7}+\cdots = \frac{1}{2}\left\{1!+\sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)B_{2k}}{2k(2k-2)!}\right\} = \frac{coth^{1}1}{2sinh\,1}$$

$$\frac{1^2}{e^1}+\frac{3^2}{e^3}+\frac{5^2}{e^5}+\frac{7^2}{e^7}+\cdots = \frac{1}{2}\left\{2!-\sum_{k=2}^{\infty}\frac{\left(2^{2k}-2\right)B_{2k}}{2k(2k-3)!}\right\} = \frac{2coth^{2}1-coth^{0}1}{2sinh\,1}$$

Therefore also, this leads to the following expression conjointly with Formula 10.4.3'.

$$\frac{2^n}{e^2}+\frac{4^n}{e^4}+\frac{6^n}{e^6}+\frac{8^n}{e^8}+\cdots = \frac{n!}{2}+\frac{(-1)^n}{2}\sum_{k=\frac{n+1}{2}\uparrow}^{\infty}\frac{2^{2k}B_{2k}}{2k(2k-n-1)!} \qquad (3.e')$$

10.7 Termwise Higher Derivative of sec x

10.7.0 Higher Derivative of sec x

Accordong to Formula 9.2.8 (**9.2**), Higher Derivative of sec x is expressed as follows.

$$(\sec x)^{(n)} = \frac{1}{\cos x}\sum_{r=0}^{n/2\downarrow} {}_nE_r(\tan x)^{n-2r} \tag{0.n}$$

Where, $\downarrow$ is floor function and ${}_nE_r$ are the same as the coefficients in **10.5.0**.

10.7.1 Termwise Higher Derivative of Taylor Series of sec x

Formula 10.7.1

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots$ are Euler Numbers and $\downarrow$ is floor function, the following expressions hold for $|x|<\pi/2$

$$(\sec x)^{(1)} = \sum_{k=1}^{\infty}\frac{|E_{2k}|}{(2k-1)!}x^{2k-1} \tag{1.1}$$

$$(\sec x)^{(2)} = \sum_{k=1}^{\infty}\frac{|E_{2k}|}{(2k-2)!}x^{2k-2} \tag{1.2}$$

$$(\sec x)^{(3)} = \sum_{k=2}^{\infty}\frac{|E_{2k}|}{(2k-3)!}x^{2k-3} \tag{1.3}$$

$$(\sec x)^{(4)} = \sum_{k=2}^{\infty}\frac{|E_{2k}|}{(2k-4)!}x^{2k-4} \tag{1.4}$$

$$\vdots$$

$$(\sec x)^{(n)} = \sum_{k=\frac{n+1}{2}\downarrow}^{\infty}\frac{|E_{2k}|}{(2k-n)!}x^{2k-n} \tag{1.n}$$

Proof

sec x is expanded to Tylor series as follows.

$$\sec x = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k)!}x^{2k} \qquad |x|<\frac{\pi}{2}$$

Differentiating both sides of this with respect to x repeatedly, we obtain (1.1) ~ (1.4).

Here, considering the relation between the derivative order n and the first term n of $\sum$, it is as follows.

n	0	1	2	3	4	···
k_0	0	1	1	2	2	···

Such a relation can be expressed by $k_0 = \frac{n+1}{2}\downarrow$ using a floor function $x\downarrow \left(=\lfloor x\rfloor\right)$. Then we obtain the desired expression.

Sum of Taylor series of the higher derivative of sec x

From (0.n) , (1.n) , we obtain the following expressions for $|x|<\pi/2$.

$$\sum_{k=\frac{n+1}{2}\downarrow}^{\infty}\frac{|E_{2k}|}{(2k-n)!}x^{2k-n} = \frac{1}{\cos x}\sum_{r=0}^{n/2\downarrow} {}_nE_r(\tan x)^{n-2r} \tag{1.t}$$

And giving $x=\pi/4$ to this, we obtain the following special values.

$$\sum_{k=1}^{\infty}\frac{|E_{2k}|}{(2k-1)!}\left(\frac{\pi}{4}\right)^{2k-1}=\sqrt{2}\sum_{r=0}^{0}{}_{n}E_{r}=\sqrt{2}\cdot 1$$

$$\sum_{k=1}^{\infty}\frac{|E_{2k}|}{(2k-1)!}\left(\frac{\pi}{4}\right)^{2k-1}=\sqrt{2}\sum_{r=0}^{1}{}_{n}E_{r}=\sqrt{2}\cdot 3$$

$$\sum_{k=2}^{\infty}\frac{|E_{2k}|}{(2k-3)!}\left(\frac{\pi}{4}\right)^{2k-3}=\sqrt{2}\sum_{r=0}^{1}{}_{n}E_{r}=\sqrt{2}\cdot 11$$

$$\sum_{k=2}^{\infty}\frac{|E_{2k}|}{(2k-4)!}\left(\frac{\pi}{4}\right)^{2k-4}=\sqrt{2}\sum_{r=0}^{2}{}_{n}E_{r}=\sqrt{2}\cdot 57$$

$$\vdots$$

10.8 Termwise Higher Derivative of sech x

10.8.0 Higher Derivative of sech x

Accordong to Formula 9.2.9 (**9.2**), Higher Derivative of sech x is expressed as follows.

$$(\operatorname{sech} x)^{(n)} = \frac{(-1)^n}{\cosh x}\sum_{r=0}^{n/2\downarrow}(-1)^r {}_nE_r(\tanh x)^{n-2r} \tag{0.n}$$

Where, $\downarrow$ is floor function and ${}_nE_r$ are the same as the coefficients in **10.5.0**.

10.8.1 Termwise Higher Derivative of Taylor Series of sech x

Formula 10.8.1

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots$ are Euler Numbers and $\downarrow$ is floor function, the following expressions hold for $|x|<\pi/2$

$$(\operatorname{sech} x)^{(1)} = \sum_{k=1}^{\infty}\frac{E_{2k}}{(2k-1)!}x^{2k-1}$$

$$(\operatorname{sech} x)^{(2)} = \sum_{k=1}^{\infty}\frac{E_{2k}}{(2k-2)!}x^{2k-2}$$

$$\vdots$$

$$(\operatorname{sech} x)^{(n)} = \sum_{k=\frac{n+1}{2}\downarrow}^{\infty}\frac{E_{2k}}{(2k-n)!}x^{2k-n} \tag{1.n}$$

Proof

sech x is expanded to Tylor series as follows.

$$\operatorname{sech} x = \sum_{k=0}^{\infty}\frac{E_{2k}}{(2k)!}x^{2k} \qquad |x| < \frac{\pi}{2}$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of sec x.)

10.8.2 Termwise Higher Derivative of Fourier Series of sech x

Formula 10.8.2

The following expressions hold for $x>0$.

$$(\operatorname{sech} x)^{(1)} = -2\left(1e^{-1x} - 3e^{-3x} + 5e^{-5x} - 7e^{-7x} + - \cdots\right)$$

$$(\operatorname{sech} x)^{(2)} = 2\left(1^2e^{-1x} - 3^2e^{-3x} + 5^2e^{-5x} - 4^2e^{-7x} + \cdots\right)$$

$$\vdots$$

$$(\operatorname{sech} x)^{(n)} = (-1)^n 2\sum_{k=0}^{\infty}(-1)^k(2k+1)^n e^{-(2k+1)x} \tag{2.n}$$

Proof

sech x is expanded to Fourier series for $x>0$ as follows.

$$\operatorname{sech} x = \frac{2}{e^x+e^{-x}} = \frac{2e^{-x}}{1+e^{-2x}} = 2e^{-x}\left(1 - e^{-2x} + e^{-4x} - e^{-6x} + - \cdots\right)$$

$$= 2\left(e^{-1x} - e^{-3x} + e^{-5x} - e^{-7x} + - \cdots\right) \tag{2.0}$$

$$= 2(\cos 1ix - \cos 3ix + \cos 5ix - \cos 7ix + - \cdots)$$

$$+ 2i(\sin 1ix - \sin 3ix + \sin 5ix - \sin 7ix + - \cdots)$$

Differentiating both sides of (2.0) with respect to x repeatedly, we obtain the desired expression.

10.8.3 Exponential series and Euler series

From (0.n) , (1.n) and (2.n) , we obtain the following formula.

Formula 10.8.3

When $\downarrow$ is floor function, E_{2n} are Euler Numbers and ${}_nE_r$ are the coefficients mentioned in **10.5.0** , the following expressions hold for $0 < x < \pi/2$

$$\sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)^n}{e^{(2k+1)x}} = \frac{(-1)^n}{2} \sum_{k=\frac{n+1}{2}\downarrow}^{\infty} \frac{E_{2k}}{(2k-n)!} x^{2k-n}$$

$$= \frac{1}{2\cosh x} \sum_{r=0}^{n/2\downarrow} (-1)^r \, {}_nE_r (\tanh x)^{n-2r}$$

Example1

$$\frac{1^1}{e^{1x}} - \frac{3^1}{e^{3x}} + \frac{5^1}{e^{5x}} - \frac{7^1}{e^{7x}} + - \cdots = -\frac{1}{2}\sum_{k=1}^{\infty} \frac{E_{2k}}{(2k-1)!} x^{2k-1} = \frac{\tanh^1 x}{2\cosh x}$$

$$\frac{1^2}{e^{1x}} - \frac{3^2}{e^{3x}} + \frac{5^2}{e^{5x}} - \frac{7^2}{e^{7x}} + - \cdots = \frac{1}{2}\sum_{k=1}^{\infty} \frac{E_{2k}}{(2k-2)!} x^{2k-2} = \frac{2\tanh^2 x - \tanh^0 x}{2\cosh x}$$

Example2 $n=3, x=1.4$

```
Ff[n_, x_] := Sum[(-1)^k (2 k + 1)^n / E^((2 k+1) x), {k, 0, 100}]

Ft[n_, x_] := (-1)^n / 2 Sum[EulerE[2 k] / (2 k - n)! x^(2 k-n), {k, Floor[(n+1)/2], 300}]

G3[x_] := (6 Tanh[x]^3 - 5 Tanh[x]^1) / (2 Cosh[x])

Ff[3, 1.4]        Ft[3, 1.4]        G3[1.4]
-0.0611079        -0.0611079        -0.0611079
```

Dirichlet Even Beta (minus)

Giving $x=0$ to Formula 10.8.3 , we obtain Dirichlet Even Beta (minus).

Formula 10.8.3'

When $\beta(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n}$ and $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots$ are Euler Numbers, the following expressions hold.

$$\beta(-2n+1) = 0 \tag{3.2n-1'}$$

$$\beta(-2n) = \frac{E_{2n}}{2} \tag{3.2n'}$$

Proof

Sbstitute $x=0$ for Formula 10.8.3 . Then, the Fourier series is

$$\sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)^n}{e^{(2k+1)0}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{-n}} = \beta(-n)$$

The Taylor series is

when $n=2m-1$

$$\frac{(-1)^n}{2}\sum_{k=\frac{n+1}{2}\downarrow}^{\infty}\frac{E_{2k}}{(2k-n)!}0^{2k-n}$$

$$= \frac{(-1)^{2m-1}}{2}\sum_{k=\frac{2m-1+1}{2}\downarrow}^{\infty}\frac{E_{2k}}{\{2k-(2m-1)\}!}0^{2k-(2m-1)}$$

$$= -\frac{1}{2}\sum_{k=m}^{\infty}\frac{E_{2k}}{(2k-2m+1)!}0^{2k-2m+1} = 0$$

when $n=2m$

$$\frac{(-1)^n}{2}\sum_{k=\frac{n+1}{2}\downarrow}^{\infty}\frac{E_{2k}}{(2k-n)!}0^{2k-n} = \frac{(-1)^{2m}}{2}\sum_{k=\frac{2m+1}{2}\downarrow}^{\infty}\frac{E_{2k}}{(2k-2m)!}0^{2k-2m}$$

$$= \frac{1}{2}\sum_{k=m}^{\infty}\frac{E_{2k}}{(2k-2m)!}0^{2k-2m} = \frac{1}{2}\frac{E_{2m}}{0!}0^0$$

$$= \frac{E_{2m}}{2}$$

Replacing m with n , we obtain (3.2n-1') and (3.2n') .

Note

These results are obtained also from the polynomial in Formula 10.8.3 ..

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