11 Termwise Higher Derivative (Inv-Trigonometric, Inv-Hyperbolic)

11.1 Termwise Higher Derivative of Inverse Trigonometric Functions

11.1.1 Termwise Higher Derivative of arctan x , arccot x

Formula 11.1.1

When ↑ is ceilling function, the following expressions hold for $|x|<1$.

$$\left(tan^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{(2k+1-n)!} x^{2k+1-n} \tag{1.t}$$

$$\left(cot^{-1}x\right)^{(n)} = -\sum_{k=\frac{n-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{(2k+1-n)!} x^{2k+1-n} \tag{1.c}$$

Proof

arctan x is expanded to Tylor series as follows.

$$tan^{-1}x = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k+1)!} x^{2k+1} \qquad |x|<1$$

Differentiating both sides of this with respect to x repeatedly, we obtain the following.

$$\left(tan^{-1}x\right)^{(1)} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k)!} x^{2k}$$

$$\left(tan^{-1}x\right)^{(2)} = \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{(2k-1)!} x^{2k-1}$$

$$\left(tan^{-1}x\right)^{(3)} = \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{(2k-2)!} x^{2k-2}$$

$$\left(tan^{-1}x\right)^{(4)} = \sum_{k=2}^{\infty} (-1)^k \frac{(2k)!}{(2k-3)!} x^{2k-3}$$

$$\vdots$$

Here, considering the relation between the derivative order n and the first term k_0 of $\sum$, it is as follows.

n	0	1	2	3	4	5	···
k_0	0	0	1	1	2	2	···

Such a relation can be expressed by $k_0 = \frac{n-1}{2}\uparrow$ using a ceiling function $x\uparrow\ (=\lceil x \rceil)$. Then we obtain (1.t). And,

$$cot^{-1}x = \pm\frac{\pi}{2} - tan^{-1}x \qquad \begin{pmatrix} x \geqq 0 : + \\ x<0 : - \end{pmatrix}$$

Therefore, we obtain (1.c) immediately.

Sum of Taylor series of the higher derivative of arctan x

Formula 9.2.7 in "**9 Higher Derivative**" was as follows.

$$\left(tan^{-1}x\right)^{(n)} = (-1)^n \frac{(n-1)!}{\left(x^2+1\right)^n} \sum_{r=1}^{n/2\uparrow} (-1)^r {}_nC_{n+1-2r}\, x^{n+1-2r}$$

From (1.t) and this , the following equation follows for $|x|<1$.

$$\sum_{k=\lceil\frac{n-1}{2}\rceil}^{\infty}(-1)^k\frac{(2k)!}{(2k+1-n)!}x^{2k+1-n}=(-1)^n\frac{(n-1)!}{\left(x^2+1\right)^n}\sum_{r=1}^{\lceil n/2\rceil}(-1)^r{}_nC_{n+1-2r}x^{n+1-2r}$$

And giving $x=1$ to this without considering the convergence condition, we obtain the following special value.

$$\sum_{k=\lceil\frac{n-1}{2}\rceil}^{\infty}(-1)^k\frac{(2k)!}{(2k+1-n)!}=(-1)^n\frac{(n-1)!}{2^n}\sum_{r=1}^{\lceil n/2\rceil}(-1)^r{}_nC_{n+1-2r} \tag{1.t'}$$

Example

$$1-1+1-1+-\cdots=\frac{0!}{2^1}{}_1C_0=\frac{1}{2}$$

$$2-4+6-8+-\cdots=\frac{1!}{2^2}{}_2C_1=\frac{1}{2}$$

$$1\cdot2-3\cdot4+5\cdot6-7\cdot8+-\cdots=-\frac{2!}{2^3}\left({}_3C_2-{}_3C_0\right)=-\frac{1}{2}$$

$$2\cdot3\cdot4-4\cdot5\cdot6+6\cdot7\cdot8-8\cdot9\cdot10+-\cdots=-\frac{3!}{2^4}\left({}_4C_3-{}_4C_1\right)=0$$

11.1.2 Termwise Higher Derivative of arcsin x , arccos x

Formula 11.1.2

When $\uparrow$ is ceilling function, the following expressions hold for $|x|<1$.

$$\left(\sin^{-1}x\right)^{(n)}=\sum_{k=\lceil\frac{n-1}{2}\rceil}^{\infty}\frac{\{(2k-1)!!\}^2}{(2k+1-n)!}x^{2k+1-n} \tag{2.s}$$

$$\left(\cos^{-1}x\right)^{(n)}=-\sum_{k=\lceil\frac{n-1}{2}\rceil}^{\infty}\frac{\{(2k-1)!!\}^2}{(2k+1-n)!}x^{2k+1-n} \tag{2.c}$$

Proof

$$\sin^{-1}x=\sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!\,(2k+1)}x^{2k+1}=\sum_{k=0}^{\infty}\frac{(2k-1)!!\,(2k)!}{(2k)!!\,(2k+1)!}x^{2k+1}$$

$$=\sum_{k=0}^{\infty}\frac{\{(2k-1)!!\}^2}{(2k+1)!}x^{2k+1}\quad\left\{\because\quad(2k)!=(2k)!!\cdot(2k-1)!!\right\}$$

Differentiating both sides of this with respect to x repeatedly, we obtain (2.s). (The number of the first term of $\sum$ is the same as it of arctan x.)

(2.c) is obtained immediately from $\cos^{-1}x=\pi/2-\sin^{-1}x$.

Formula 11.1.2'

$$\sum_{r=0}^{2n-1}(-1)^r\binom{2n-1}{r}(2r-1)!!\,(4n-2r-3)!!=0 \tag{2.so}$$

$$\sum_{r=0}^{2n}(-1)^r\binom{2n}{r}(2r-1)!!\,(4n-2r-1)!! \quad = 2^{2n}\{(2n-1)!!\}^2 \qquad \text{(2.se)}$$

Proof

Formula 9.3.2' in "**9 Higher Derivative**" was as follows.

$$\left(\sin^{-1}x\right)^{(n)} = \frac{1}{2^{n-1}\sqrt{1-x^2}}\sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r}\frac{(2r-1)!!\,(2n-2r-3)!!}{(1+x)^r(1-x)^{n-1-r}}$$

From this,

$$\left(\sin^{-1}x\right)^{(2n)}\Big|_{x=0} = \frac{1}{2^{2n-1}}\sum_{r=0}^{2n-1}(-1)^r\binom{2n-1}{r}(2r-1)!!\,(4n-2r-3)!!$$

$$\left(\sin^{-1}x\right)^{(2n+1)}\Big|_{x=0} = \frac{1}{2^{2n}}\sum_{r=0}^{2n}(-1)^r\binom{2n}{r}(2r-1)!!\,(4n-2r-1)!!$$

From (2.s) ,

$$\left(\sin^{-1}x\right)^{(2n)}\Big|_{x=0} = \sum_{k=n}^{\infty}\frac{\{(2k-1)!!\}^2}{(2k+1-2n)!}0^{2k+1-2n} = 0$$

$$\left(\sin^{-1}x\right)^{(2n+1)}\Big|_{x=0} = \sum_{k=n}^{\infty}\frac{\{(2k-1)!!\}^2}{(2k+1-2n-1)!}0^{2k-2n} = \frac{\{(2n-1)!!\}^2}{0!}0^0$$

Thus, we obtain the desired expressions.

Example

$${}_3C_0(-1)!!\cdot 5!! - {}_3C_1 1!!\cdot 3!! + {}_3C_2 3!!\cdot 1!! - {}_3C_3 5!!\cdot(-1)!! \qquad = 0$$

$${}_4C_0(-1)!!\cdot 7!! - {}_4C_1 1!!\cdot 5!! + {}_4C_2 3!!\cdot 3!! - {}_4C_3 5!!\cdot 1!! + {}_4C_4 7!!\cdot(-1)!! = 144$$

11.1.3 Termwise Higher Derivative of arccsc x , arcsec x

Formula 11.1.3

The following expressions hold for $|x|>1$.

$$\left(\csc^{-1}x\right)^{(n)} = (-1)^n\sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!}\frac{(2k+n)!}{(2k+1)!}x^{-2k-n-1} \qquad \text{(3.c)}$$

$$\left(\sec^{-1}x\right)^{(n)} = (-1)^{n-1}\sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!}\frac{(2k+n)!}{(2k+1)!}x^{-2k-n-1} \qquad \text{(3.s)}$$

Proof

$$\csc^{-1}x = \sin^{-1}\frac{1}{x} = \sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!\,(2k+1)}x^{-2k-1} \qquad |x|>1$$

Differentiating both sides of this with respect to x repeatedly, we obtain (3.c).

(3.s) is obtained immediately from $\sec^{-1}x = \pi/2 - \csc^{-1}x$.

11.2 Termwise Higher Derivative of Inverse Hyperbolic Functions

11.2.1 Termwise Higher Derivative of arctanh x , arccoth x

Formula 11.2.1t

When ↑ is ceilling function, the following expressions hold for $|x|<1$.

$$\left(\tanh^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty} \frac{(2k)!}{(2k+1-n)!}x^{2k+1-n} \tag{1.t}$$

Proof

arctanh x is expanded to Tylor series as follows.

$$\tanh^{-1}x = \sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2k+1)!}x^{2k+1} \qquad |x|<1$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of arctan x.)

Formula 11.2.1c

The following expression holds for $|x|>1$.

$$\left(\coth^{-1}x\right)^{(n)} = (-1)^{(n)}\sum_{k=0}^{\infty}\frac{(2k+n)!}{(2k+1)!}x^{-2k-1-n} \tag{1.c}$$

Proof

arccoth x is expanded to Tylor series as follows.

$$\coth^{-1}x = \sum_{k=0}^{\infty}\frac{(2k)!}{(2k+1)!}x^{-2k-1} \qquad |x|>1$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression.

11.2.2 Termwise Higher Derivative of arcsinh x , arccosh x

Formula 11.2.2s

When ↑ is ceilling function, the following expression holds for $|x|<1$.

$$\left(\sinh^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{(2k+1-n)!}x^{2k+1-n} \tag{2.s}$$

Proof

arcsinh x is expanded to Tylor series for $|x|<1$ as follows.

$$\sinh^{-1}x = \sum_{k=0}^{\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!\,(2k+1)}x^{2k+1} = \sum_{k=0}^{\infty}(-1)^k\frac{(2k-1)!!\,(2k)!}{(2k)!!\,(2k+1)!}x^{2k+1}$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{(2k+1)!}x^{2k+1} \qquad \{\because\ (2k)! = (2k)!!\cdot(2k-1)!!\}$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression. (The number of the first term of $\sum$ is the same as it of arctan x.)

Sum of Taylor series of the higher derivative of arcsin x

Formula 9.4.2 in "**9 Higher Derivative**" was as follows.

$$\left(\sinh^{-1}x\right)^{(n)} = (-1)^{n-1}\sum_{r=0}^{n/2\downarrow}(-1)^r\binom{n-1}{n-1-2r}\frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(x^2+1\right)^{n-r-\frac{1}{2}}}$$

From (2.s) and this , the following equation follows for $|x|<1$.

$$\sum_{k=\frac{n-1}{2}\uparrow}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{(2k+1-n)!}x^{2k+1-n}$$

$$= (-1)^{n-1}\sum_{r=0}^{n/2\downarrow}(-1)^r\binom{n-1}{n-1-2r}\frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(x^2+1\right)^{n-r-\frac{1}{2}}}$$

And giving $x=1$ to this without considering the convergence condition, we obtain the following special value.

$$\sum_{k=\frac{n-1}{2}\uparrow}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{(2k+1-n)!} = (-1)^{n-1}\sum_{r=0}^{n/2\downarrow}(-1)^r\binom{n-1}{n-1-2r}\frac{(2r-1)!!\,(2n-3-2r)!!}{2^{n-r-\frac{1}{2}}}$$

Example

$$\frac{(-1)!!^2}{0!}-\frac{3!!^2}{2!}+\frac{5!!^2}{4!}-\frac{7!!^2}{6!}+-\cdots = \frac{\sqrt{2}}{2}$$

$$\frac{1!!^2}{1!}-\frac{3!!^2}{3!}+\frac{5!!^2}{5!}-\frac{7!!^2}{7!}+-\cdots = \frac{\sqrt{2}}{4}$$

$$\frac{1!!^2}{0!}-\frac{3!!^2}{2!}+\frac{5!!^2}{4!}-\frac{7!!^2}{6!}+-\cdots = -\frac{\sqrt{2}}{8}$$

$$\frac{3!!^2}{1!}-\frac{5!!^2}{3!}+\frac{7!!^2}{5!}-\frac{9!!^2}{7!}+-\cdots = \frac{3\sqrt{2}}{16}$$

Formula 11.2.2c

The following expressions hold for $|x|>1$.

$$\left(\cosh^{-1}x\right)^{(n)} = (-1)^{n-1}\frac{(n-1)!}{x^n} + (-1)^{n-1}\sum_{k=1}^{\infty}\frac{(2k+n-1)!}{\{(2k)!!\}^2}x^{-2k-n} \tag{2.c}$$

Proof

arccosh x is expanded to Tylor series for $|x|\geqq 1$ as follows.

$$\cosh^{-1}x = \log 2x - \sum_{k=1}^{\infty}\frac{(2k-1)!!}{(2k)!!\,2k}x^{-2k} = \log 2x - \sum_{k=1}^{\infty}\frac{(2k-1)!!\,(2k-1)!}{(2k)!!\,(2k)!}x^{-2k}$$

$$= \log 2x - \sum_{k=1}^{\infty}\frac{(2k-1)!}{\{(2k)!!\}^2}x^{-2k} \quad \{\because\ (2k)! = (2k)!!\cdot(2k-1)!!\}$$

Differentiating both sides of this with respect to x repeatedly, we obtain the desired expression.

11.2.3 Termwise Higher Derivative of arccsch x , arcsech x

Formula 11.2.3

When $\uparrow$ is ceilling function, the following expressions hold for $0<x<1$.

$$\left(csch^{-1}x\right)^{(n)} = (-1)^n\frac{(n-1)!}{x^n} - \sum_{k=\frac{n}{2}\uparrow}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{2k\,(2k-n)!}x^{2k-n} \tag{3.c}$$

$$\left(sech^{-1}x\right)^{(n)} = (-1)^n\frac{(n-1)!}{x^n} - \sum_{k=\frac{n}{2}\uparrow}^{\infty}\frac{\{(2k-1)!!\}^2}{2k\,(2k-n)!}x^{2k-n} \tag{3.s}$$

Proof

$$\begin{aligned} csch^{-1}x &= log\frac{2}{x} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2k-1)!!}{(2k)!!\;2k}x^{2k} \\ &= log\frac{2}{x} - \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!\,(2k)!}{(2k)!!\;2k\,(2k)!}x^{2k} \\ &= log\,2 - log\,x - \sum_{k=1}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{2k\,(2k)!}x^{2k} \quad \{\because (2k)!=(2k)!!\cdot(2k-1)!!\} \end{aligned}$$

Differentiating both sides of this with respect to x repeatedly, we obtain the following.

$$\left(csch^{-1}x\right)^{(1)} = -\frac{0!}{x^1} - \sum_{k=1}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{2k\,(2k-1)!}x^{2k-1}$$

$$\left(csch^{-1}x\right)^{(2)} = \frac{1!}{x^2} - \sum_{k=1}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{2k\,(2k-2)!}x^{2k-2}$$

$$\left(csch^{-1}x\right)^{(3)} = -\frac{2!}{x^3} - \sum_{k=2}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{2k\,(2k-3)!}x^{2k-3}$$

$$\left(csch^{-1}x\right)^{(4)} = \frac{3!}{x^4} - \sum_{k=2}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{2k\,(2k-4)!}x^{2k-4}$$

$$\vdots$$

Here, considering the relation between the derivative order n and the first term k_0 of $\sum$, it is as follows.

n	0	1	2	3	4	...
k_0	1	1	1	2	2	...

Such a relation can be expressed by $k_0 = \frac{n}{2}\uparrow$ using a ceiling function $x\uparrow\ (=\lceil x\rceil)$. Then we obtain (3.c) . Next,

$$\begin{aligned} sech^{-1}x &= log\frac{2}{x} + \sum_{k=1}^{\infty}\frac{(2k-1)!!}{(2k)!!\;2k}x^{2k} = log\frac{2}{x} - \sum_{k=1}^{\infty}\frac{(2k-1)!!\,(2k)!}{(2k)!!\;2k\,(2k)!}x^{2k} \\ &= log\,2 - log\,x - \sum_{k=1}^{\infty}\frac{\{(2k-1)!!\}^2}{2k\,(2k)!}x^{2k} \quad \{\because (2k)!=(2k)!!\cdot(2k-1)!!\} \end{aligned}$$

Differentiating both sides of this with respect to x repeatedly, we obtain (3.s) . (The number of the first term of $\sum$ is the same as it of arctan x.)

Sum of Taylor series of the higher derivative of arccsch x

Formula 9.4.2 in "**9 Higher Derivative**" was as follows.

$$\left(csch^{-1}x\right)^{(n)} = (-1)^{n-1}\sum_{r=1}^{n}\frac{(-1)^r {}_nA_r}{x^{n-1+2r}\left(x^{-2}+1\right)^{r-\frac{1}{2}}}$$

where ${}_nA_r$ are coefficients as follows.

$$\begin{matrix} {}_1A_1 \\ {}_2A_1\ {}_2A_2 \\ {}_3A_1\ {}_3A_2\ {}_3A_3 \\ {}_4A_1\ {}_4A_2\ {}_4A_3\ {}_4A_4 \\ {}_5A_1\ {}_5A_2\ {}_5A_3\ {}_5A_4\ {}_5A_5 \\ {}_6A_1\ {}_6A_2\ {}_6A_3\ {}_6A_4\ {}_6A_5\ {}_6A_6 \\ \vdots \end{matrix} = \begin{matrix} 1 \\ 2\quad 1 \\ 6\quad 7\quad 3 \\ 24\quad 48\quad 45\quad 15 \\ 120\quad 360\quad 549\quad 390\quad 105 \\ 720\quad 3000\quad 6570\quad 7425\quad 4200\quad 945 \\ \vdots \end{matrix}$$

From (3.s) and this , the following equation follows for $0<x<1$.

$$\sum_{k=\frac{n}{2}\uparrow}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{2k\,(2k-n)!}x^{2k-n} = \frac{(-1)^n}{x^n}\left\{(n-1)! + \sum_{r=1}^{n}\frac{(-1)^r {}_nA_r}{x^{2r-1}\left(x^{-2}+1\right)^{r-\frac{1}{2}}}\right\}$$

And giving $x=1$ to this without considering the convergence condition, we obtain the following special value.

$$\sum_{k=\frac{n}{2}\uparrow}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{2k\,(2k-n)!} = (-1)^n\left\{(n-1)! + \sqrt{2}\sum_{r=1}^{n}\frac{(-1)^r {}_nA_r}{2^r}\right\} \qquad (3.c')$$

Example

$$\frac{1!!^2}{2\cdot 1!} - \frac{3!!^2}{4\cdot 3!} + \frac{5!!^2}{6\cdot 5!} - \frac{7!!^2}{8\cdot 7!} + - \cdots = 0! - \frac{\sqrt{2}}{2}$$

$$\frac{1!!^2}{2\cdot 0!} - \frac{3!!^2}{4\cdot 2!} + \frac{5!!^2}{6\cdot 4!} - \frac{7!!^2}{8\cdot 6!} + - \cdots = -1! + \frac{3\sqrt{2}}{4}$$

$$\frac{3!!^2}{4\cdot 1!} - \frac{5!!^2}{6\cdot 3!} - \frac{7!!^2}{8\cdot 5!} - \frac{9!!^2}{10\cdot 7!} + - \cdots = -2! + \frac{13\sqrt{2}}{8}$$

$$\frac{3!!^2}{4\cdot 0!} - \frac{5!!^2}{6\cdot 2!} - \frac{7!!^2}{8\cdot 4!} - \frac{9!!^2}{10\cdot 6!} + - \cdots = 3! - \frac{75\sqrt{2}}{16}$$

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