11 Termwise Higher Derivative (Inv-Trigonometric, Inv-Hyperbolic)

11.1 Termwise Higher Derivative of Inverse Trigonometric Functions

11.1.1 Termwise Higher Derivative of \( \text{arctan} \ x \), \( \text{arccot} \ x \)

Formula 11.1.1
When \( \lceil \cdot \rceil \) is ceiling function, the following expressions hold for \( |x| < 1 \).

\[
\begin{align*}
(\tan^{-1} x)^{(n)} &= \sum_{k=\lceil \frac{n-1}{2} \rceil}^{\infty} (-1)^k \frac{(2k)!}{(2k+1-n)!} x^{2k+1-n} \\
(\cot^{-1} x)^{(n)} &= -\sum_{k=\lceil \frac{n-1}{2} \rceil}^{\infty} (-1)^k \frac{(2k)!}{(2k+1-n)!} x^{2k+1-n}
\end{align*}
\]  

(1.t) 

Proof
\( \text{arctan} \ x \) is expanded to Taylor series as follows.

\[
\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k+1)!} x^{2k+1} \quad |x| < 1
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain the following.

\[
\begin{align*}
(\tan^{-1} x)^{(1)} &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k+1)!} x^{2k} \\
(\tan^{-1} x)^{(2)} &= \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{(2k-1)!} x^{2k-1} \\
(\tan^{-1} x)^{(3)} &= \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{(2k-2)!} x^{2k-2} \\
(\tan^{-1} x)^{(4)} &= \sum_{k=2}^{\infty} (-1)^k \frac{(2k)!}{(2k-3)!} x^{2k-3}
\end{align*}
\]

Here, considering the relation between the derivative order \( n \) and the first term \( k_0 \) of \( \sum \), it is as follows.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_0 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>...</td>
</tr>
</tbody>
</table>

Such a relation can be expressed by \( k_0 = \frac{n-1}{2} \lceil \cdot \rceil \) using a ceiling function \( x \lceil \cdot \rceil (= \lceil x \rceil) \). Then we obtain (1.t). And,

\[
\cot^{-1} x = \pm \frac{\pi}{2} - \tan^{-1} x
\]

\( x \geq 0 : + \)

\( x < 0 : - \)

Therefore, we obtain (1.c) immediately.

Sum of Taylor series of the higher derivative of \( \text{arctan} \ x \)
Formula 9.2.7 in "9 Higher Derivative" was as follows.

\[
(\tan^{-1} x)^{(n)} = (-1)^n \frac{(n-1)!}{(x^2+1)^n} \sum_{r=1}^{n/2} (-1)^r \frac{n!}{(n+1-2r)!} \frac{x^{n+1-2r}}{r} C_{n+1-2r}\]

-1-
From (1.1) and this, the following equation follows for \( |x| < 1 \).

\[
\sum_{k=\frac{n-1}{2}}^{\infty} (-1)^{k} \frac{(2k)!}{(2k+1-n)!} x^{2k+1-n} = (-1)^{n} \frac{(n-1)!}{(x^2+1)^{n}} \sum_{r=1}^{\frac{n}{2}} (-1)^{r} C_{n+1-2r} x^{n+1-2r}
\]

And giving \( x=1 \) to this without considering the convergence condition, we obtain the following special value.

\[
\sum_{k=\frac{n-1}{2}}^{\infty} (-1)^{k} \frac{(2k)!}{(2k+1-n)!} = (-1)^{n} \frac{(n-1)!}{2^{n}} \sum_{r=1}^{\frac{n}{2}} (-1)^{r} C_{n+1-2r}
\] (1.1)

**Example**

\[
1 - 1 + 1 - 1 + \cdots = \frac{0!}{2^{1}} C_{0} = \frac{1}{2}
\]

\[
2 - 4 + 6 - 8 + \cdots = \frac{1!}{2^{2}} C_{1} = \frac{1}{2}
\]

\[
1 \cdot 2 - 3 \cdot 4 + 5 \cdot 6 - 7 \cdot 8 + \cdots = -\frac{2!}{2^{3}} (3 C_{2} - 3 C_{0}) = -\frac{1}{2}
\]

\[
2 \cdot 3 \cdot 4 - 4 \cdot 5 \cdot 6 + 6 \cdot 7 \cdot 8 - 8 \cdot 9 \cdot 10 + \cdots = -\frac{3!}{2^{3}} (4 C_{3} - 4 C_{1}) = 0
\]

**11.1.2 Termwise Higher Derivative of \( \arcsin x \), \( \arccos x \)**

**Formula 11.1.2**

When \( \uparrow \) is ceiling function, the following expressions hold for \( |x| < 1 \).

\[
\left( \sin^{-1} x \right)^{(n)} = \sum_{k=\frac{n-1}{2}}^{\infty} \frac{\{ (2k-1) !!! \}^{2}}{(2k+1-n)!} x^{2k+1-n} \quad (2.s)
\]

\[
\left( \cos^{-1} x \right)^{(n)} = -\sum_{k=\frac{n-1}{2}}^{\infty} \frac{\{ (2k-1) !!! \}^{2}}{(2k+1-n)!} x^{2k+1-n} \quad (2.c)
\]

**Proof**

\[
\sin^{-1} x = \sum_{k=0}^{\infty} \frac{(2k-1) !!!}{(2k) !!! (2k+1)} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(2k-1) !!!}{(2k) !!! (2k+1)} x^{2k+1}
\]

\[
= \sum_{k=0}^{\infty} \frac{\{ (2k-1) !!! \}^{2}}{(2k+1)!} x^{2k+1} \quad \because (2k)! = (2k) !!! \cdot (2k-1) !!!
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain (2.s). (The number of the first term of \( \sum \) is the same as it of \( \arctan x \).)

(2.c) is obtained immediately from \( \cos^{-1} x = \pi / 2 - \sin^{-1} x \).

**Formula 11.1.2**

\[
\sum_{r=0}^{2n-1} (-1)^{r} \binom{2n-1}{r} (2r-1) !!! (4n-2r-3) !!! = 0
\] (2.so)
\[
\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (2r-1)!! (4n-2r-1)!! = 2^{2n} \{ (2n-1)!! \}^2
\]  
(2.se)

**Proof**

Formula 9.3.2' in "9 Higher Derivative" was as follows.

\[
(\sin^{-1} x)^{(n)} = \frac{1}{2^{n-1} \sqrt{1-x^2}} \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} (2r-1)!! (2n-2r-3)!! \frac{1}{(1+x)^r (1-x)^{n-r}} 
\]

From this,

\[
(\sin^{-1} x)^{(2n)} \bigg|_{x=0} = \frac{1}{2^{2n-1}} \sum_{r=0}^{2n-1} (-1)^r \binom{2n-1}{r} (2r-1)!! (4n-2r-3)!! 
\]

\[
(\sin^{-1} x)^{(2n+1)} \bigg|_{x=0} = \frac{1}{2^{2n}} \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (2r-1)!! (4n-2r-1)!! 
\]

From [2.s],

\[
(\sin^{-1} x)^{(2n)} \bigg|_{x=0} = \sum_{k=n}^{\infty} \frac{\{ (2k-1)!! \}^2}{(2k+1-2n)!} 0^{2k+1-2n} = 0 
\]

\[
(\sin^{-1} x)^{(2n+1)} \bigg|_{x=0} = \sum_{k=n}^{\infty} \frac{\{ (2k-1)!! \}^2}{(2k+1-2n-1)!} 0^{2k+2-2n} = \frac{\{ (2n-1)!! \}^2}{0!} 0^0 
\]

Thus, we obtain the desired expressions.

**Example**

\[
\begin{align*}
3 \binom{0}{-1}!! \cdot 5!! & - 3 \binom{1}{1}!! \cdot 3!! + 3 \binom{2}{3}!! \cdot 1!! - 3 \binom{3}{5}!! \cdot (-1)!! \\
& = 0 \\
4 \binom{0}{-1}!! \cdot 7!! & - 4 \binom{1}{1}!! \cdot 5!! + 4 \binom{2}{3}!! \cdot 3!! - 4 \binom{3}{5}!! \cdot 1!! + 4 \binom{4}{7}!! \cdot (-1)!! \\
& = 144
\end{align*}
\]

**11.1.3 Termwise Higher Derivative of \( \arccsc x \), \( \text{arcsec} x \)**

**Formula 11.1.3**

The following expressions hold for \(|x| > 1\).

\[
(\csc^{-1} x)^{(n)} = (-1)^n \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!} \frac{(2k+n)!!}{(2k+1)!} x^{-2k-n-1} 
\]  
(3.c)

\[
(\sec^{-1} x)^{(n)} = (-1)^{n-1} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!} \frac{(2k+n)!!}{(2k+1)!} x^{-2k-n-1} 
\]  
(3.s)

**Proof**

\[
csc^{-1} x = \sin^{-1} \frac{1}{x} = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!! (2k+1)!} x^{-2k-1} \quad \text{for} \ |x| > 1 
\]

Differentiating both sides of this with respect to \(x\) repeatedly, we obtain (3.c).

(3.s) is obtained immediately from \( \sec^{-1} x = \pi/2 - \csc^{-1} x \).
11.2 Termwise Higher Derivative of Inverse Hyperbolic Functions

11.2.1 Termwise Higher Derivative of \( \text{arctanh} \ x \), \( \text{arccoth} \ x \)

Formula 11.2.1\( ^t \)

When \( \uparrow \) is ceiling function, the following expressions hold for \( |x| < 1 \).

\[
(tanh^{-1} x)^{(n)} = \sum_{k=\frac{n-1}{2}}^{\infty} \frac{(2k) !}{(2k+1-n) !} x^{2k+1-n}
\]

Proof

\( \text{arctanh} \ x \) is expanded to Taylor series as follows.

\[
tanh^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{(2k) !}{(2k+1) !} x^{2k+1} \quad |x| < 1
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain the desired expression. (The number of the first term of \( \sum \) is the same as it of \( \text{arctan} \ x \).)

Formula 11.2.1\( ^c \)

The following expression holds for \( |x| > 1 \).

\[
(coth^{-1} x)^{(n)} = (-1)^n \sum_{k=0}^{\infty} \frac{(2k+n) !}{(2k+1) !} x^{-2k-1-n}
\]

Proof

\( \text{arccoth} \ x \) is expanded to Taylor series as follows.

\[
coth^{-1} x = \sum_{k=0}^{\infty} \frac{(2k) !}{(2k+1) !} x^{-2k-1} \quad |x| > 1
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain the desired expression.

11.2.2 Termwise Higher Derivative of \( \text{arcsinh} \ x \), \( \text{arccosh} \ x \)

Formula 11.2.2\( ^s \)

When \( \uparrow \) is ceiling function, the following expression holds for \( |x| < 1 \).

\[
(sinh^{-1} x)^{(n)} = \sum_{k=\frac{n-1}{2}}^{\infty} (-1)^k \frac{(2k-1) !!}{(2k+1-n) !} x^{2k+1-n}
\]

Proof

\( \text{arcsinh} \ x \) is expanded to Taylor series for \( |x| < 1 \) as follows.

\[
sinh^{-1} x = \sum_{k=0}^{\infty} \frac{(2k-1) !!}{(2k) !} x^{2k} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1) !!}{(2k+1) !} x^{2k+1}
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1) !!}{(2k) !} x^{2k+1} \quad \{ \because (2k) ! = (2k) !! \cdot (2k-1) !! \}
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain the desired expression. (The number of the first term of \( \sum \) is the same as it of \( \text{arctan} \ x \).)
Sum of Taylor series of the higher derivative of \( \arcsin x \)

Formula 9.4.2 in "9 Higher Derivative" was as follows.

\[
(\sinh^{-1} x)^{(n)} = (-1)^{n-1} \frac{n}{2} \sum_{r=0}^{n/2} (-1)^r \binom{n-1}{n-1-2r} \left( \frac{(2r-1)!!}{(2k+1-n)!} \right) x^{2k+1-n}
\]

From (2.s) and this, the following equation follows for \( |x| < 1 \).

\[
\sum_{k=\frac{n-1}{2}}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k+1-n)!} x^{2k+1-n} = (-1)^{n-1} \sum_{r=0}^{n/2} (-1)^r \frac{n-1}{n-1-2r} \left( \frac{(2r-1)!!}{(2k+1-n)!} \right) x^{2k+1-n}
\]

And giving \( x=1 \) to this without considering the convergence condition, we obtain the following special value.

\[
\sum_{k=\frac{n-1}{2}}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k+1-n)!} = (-1)^{n-1} \sum_{r=0}^{n/2} (-1)^r \frac{n-1}{n-1-2r} \left( \frac{(2r-1)!!}{(2k+1-n)!} \right) x^{2k+1-n}
\]

Example

\[
\begin{align*}
\frac{(-1)!!}{0!} & = \frac{3!!}{2!} - \frac{5!!}{4!} + \cdots \\
\frac{1!!}{1!} & = \frac{3!!}{2!} - \frac{5!!}{4!} + \cdots \\
\frac{1!!}{0!} & = \frac{3!!}{2!} - \frac{5!!}{4!} + \cdots \\
\frac{3!!}{1!} & = \frac{5!!}{3!} - \frac{7!!}{5!} + \cdots
\end{align*}
\]

Formula 11.2.2c

The following expressions hold for \( |x| > 1 \).

\[
(\cosh^{-1} x)^{(n)} = (-1)^{n-1} \frac{(n-1)!}{x^n} + (-1)^{n-1} \sum_{k=1}^{\infty} \frac{(2k+n-1)!!}{(2k)!!} x^{-2k-n}
\]

Proof

\( \text{arccosh} x \) is expanded to Taylor series for \( |x| \geq 1 \) as follows.

\[
\cosh^{-1} x = \log 2x - \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} 2k x^{-2k} = \log 2x - \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} (2k-1)!! (2k) x^{-2k}
\]

\[
= \log 2x - \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} x^{-2k} \quad \therefore \quad (2k)!! = (2k)!! \cdot (2k-1)!!
\]

Differentiating both sides of this with respect to \( x \) repeatedly, we obtain the desired expression.
11.2.3 Termwise Higher Derivative of $\text{arccsch } x$, $\text{arcsech } x$

**Formula 11.2.3**

When $\uparrow$ is ceiling function, the following expressions hold for $0 < x < 1$.

\[
\begin{align*}
(\text{csch}^{-1} x)^{(n)} &= (-1)^n \frac{(n-1)!}{x^n} - \sum_{k=\frac{n}{2}}^{\infty} (-1)^k \frac{(2k-1)!!}{2k (2k-n)!} x^{2k-n} \quad (3.c) \\
(\text{sech}^{-1} x)^{(n)} &= (-1)^n \frac{(n-1)!}{x^n} - \sum_{k=\frac{n}{2}}^{\infty} \frac{(2k-1)!!}{2k (2k-n)!} x^{2k-n} \quad (3.s)
\end{align*}
\]

**Proof**

\[
\begin{align*}
\text{csch}^{-1} x &= \log \frac{2}{x} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(2k-1)!!}{(2k)!!} \frac{x^{2k}}{2k} \\
&= \log \frac{2}{x} - \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!} \frac{x^{2k}}{2k (2k-1)!!} \\
&= \log 2 - \log x - \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{2k (2k)!} x^{2k} \quad \{ \because (2k)! = (2k)!! \cdot (2k-1)!! \}
\end{align*}
\]

Differentiating both sides of this with respect to $x$ repeatedly, we obtain the following.

\[
\begin{align*}
(\text{csch}^{-1} x)^{(1)} &= -\frac{0!}{x^1} - \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{2k (2k-1)!!} x^{2k-1} \\
(\text{csch}^{-1} x)^{(2)} &= \frac{1!}{x^2} - \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{2k (2k-2)!!} x^{2k-2} \\
(\text{csch}^{-1} x)^{(3)} &= -\frac{2!}{x^3} - \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{2k (2k-3)!!} x^{2k-3} \\
(\text{csch}^{-1} x)^{(4)} &= \frac{3!}{x^4} - \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{2k (2k-4)!!} x^{2k-4} \\
&\vdots
\end{align*}
\]

Here, considering the relation between the derivative order $n$ and the first term $k_0$ of $\sum$, it is as follows.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>...</td>
</tr>
</tbody>
</table>

Such a relation can be expressed by $k_0 = \frac{n}{2} \uparrow$ using a ceiling function $x \uparrow (= \lfloor x \rfloor)$. Then we obtain (3.c). Next,

\[
\begin{align*}
\text{sech}^{-1} x &= \log \frac{2}{x} + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{x^{2k}}{2k} = \log \frac{2}{x} - \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{x^{2k}}{2k (2k)!!} \\
&= \log 2 - \log x - \sum_{k=1}^{\infty} \frac{(2k-1)!!}{2k (2k)!} x^{2k} \quad \{ \because (2k)! = (2k)!! \cdot (2k-1)!! \}
\end{align*}
\]

Differentiating both sides of this with respect to $x$ repeatedly, we obtain (3.s). (The number of the first term of $\sum$ is the same as it of $\text{arctan } x$.)
Sum of Taylor series of the higher derivative of \( \text{arccsch} \ x \)

Formula 9.4.2 in "9 Higher Derivative" was as follows.

\[
(csch^{-1}x)^{(n)} = (-1)^{n-1} \sum_{r=1}^{n} \frac{(-1)^r n A_r}{x^{n+2r}(x^{-2} + 1)^{r-\frac{1}{2}}}
\]

where \( n A_r \) are coefficients as follows.

\[
\begin{array}{cccccccc}
1 & A_1 & & & & & & \\
2 & A_1 & 2A_2 & & & & & \\
3 & A_1 & 3A_2 & 3A_3 & & & & \\
4 & A_1 & 4A_2 & 4A_3 & 4A_4 & & & \\
5 & A_1 & 5A_2 & 5A_3 & 5A_4 & 5A_5 & & \\
6 & A_1 & 6A_2 & 6A_3 & 6A_4 & 6A_5 & 6A_6 & \\
\end{array}
\]

From (3.s) and this, the following equation follows for \( 0 < x < 1 \).

\[
\sum_{k=n}^{\infty} (-1)^k \frac{(2k-1)!!}{2k(2k-n)!} x^{2k-n} = (-1)^n \left\{ (n-1)! + \sum_{r=1}^{n} \frac{(-1)^r n A_r}{x^{2r-1}(x^{-2} + 1)^{r-\frac{1}{2}}} \right\}
\]

And giving \( x = 1 \) to this without considering the convergence condition, we obtain the following special value.

\[
\sum_{k=n}^{\infty} (-1)^k \frac{(2k-1)!!}{2k(2k-n)!} = (-1)^n \left\{ (n-1)! + \sqrt{2} \sum_{r=1}^{n} \frac{(-1)^r n A_r}{2^r} \right\}
\] (3.c')

Example

\[
\begin{array}{cccccc}
1!!^2 & - & 3!!^2 & + & 5!!^2 & - \ldots \\
2 & \cdot & 1 & - & 4 & \cdot 3 & + & 6 & \cdot 5 & - \ldots & = & 0 & - \sqrt{2} \\
1!!^2 & - & 3!!^2 & + & 5!!^2 & - \ldots \\
2 & \cdot & 0 & - & 4 & \cdot 2 & + & 6 & \cdot 4 & - \ldots & = & -1 & + \frac{3\sqrt{2}}{4} \\
3!!^2 & - & 5!!^2 & + & 7!!^2 & - \ldots \\
4 & \cdot & 1 & - & 6 & \cdot 3 & + & 8 & \cdot 5 & - \ldots & = & -2 & + \frac{13\sqrt{2}}{8} \\
3!!^2 & - & 5!!^2 & + & 7!!^2 & - \ldots \\
4 & \cdot & 0 & - & 6 & \cdot 2 & + & 8 & \cdot 4 & - \ldots & = & 3 & - \frac{75\sqrt{2}}{16} \\
\end{array}
\]

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