05 Termwise Higher Integral (Trigonometric, Hyperbolic)

In this chapter, for the function which second or more order integral cannot be expressed with the elementary functions among trigonometric functions and hyperbolic functions, we integrate the series expansion of these function termwise. Therefore, sin x, cos x, sinhx, cosh x mentioned in **" 4.3 Higher Integral of Elementary Functions "** are not treated here.

5.1 Termwise Higher Integral of tan x

Since both zeros of $\tan x$ and the primitive function $-\log\cos x$ are $x=0$, the latter seems to be a lineal primitive function with a fixed lower limit. Then, we assume zeros of the second or more order primitive function are also $x=0$.

5.1.0 Higher Integral of tan x

$$\int_0^x \tan x\,dx = -\log\cos x$$

$$\int_0^x\!\!\int_0^x \tan x\,dx^2 = \text{ non-elementary function}$$

5.1.1 Termwise Higher Integral of Taylor Series of tan x

Formula 5.1.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli Numbers, the following expressions hold for $|x|<\pi/2$.

$$\int_0^x \tan x\,dx = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k)!}x^{2k}$$

$$\int_0^x\!\!\int_0^x \tan x\,dx^2 = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+1)!}x^{2k+1}$$

$$\int_0^x\!\!\int_0^x\!\!\int_0^x \tan x\,dx^3 = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2)!}x^{2k+2}$$

$$\vdots$$

$$\int_0^x\!\cdots\!\int_0^x \tan x\,dx^n = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+n-1)!}x^{2k+n-1}$$

Proof

$\tan x$ can be expanded to Taylor series as follows.

$$\tan x = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-1)!}x^{2k-1} \qquad |x|<\frac{\pi}{2}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

(1) Termwise Higher Definite Integral of Taylor Series of tan x

Substituting $x=\pi/2$ for Formula 5.1.1 , we obtain the following expressions.

$$\int_0^{\frac{\pi}{2}}\!\!\int_0^x \tan x\,dx^2 = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1} \tag{1.2'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\int_0^x \tan x\, dx^3 = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2)\,!}\left(\frac{\pi}{2}\right)^{2k+2} \tag{1.3'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\cdots\int_0^x \tan x\, dx^4 = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+3)\,!}\left(\frac{\pi}{2}\right)^{2k+3} \tag{1.4'}$$

$$\vdots$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\cdots\int_0^x \tan x\, dx^n = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+n-1)\,!}\left(\frac{\pi}{2}\right)^{2k+n-1} \tag{1.n'}$$

(2) Taylor Series of the 1st order Integral of tan x

The1st order Integral of tan x is as follows.

$$\int_0^x \tan x\, dx = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k)\,!}x^{2k} = -\log\cos x \qquad |x|<\frac{\pi}{2}$$

From this, the following expression follows immediately.

$$\sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k)\,!}x^{2k} = -\log\cos x \qquad |x|<\frac{\pi}{2} \tag{1.t}$$

Substituting $x=1,1/2,\pi/4$ for this one by one, we obtain the following special values.

$$\sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k)\,!} = -\log\cos 1$$

$$\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k)\,!} = -\log\cos\frac{1}{2}$$

$$\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k)\,!}\left(\frac{\pi}{2}\right)^{2k} = -\log\cos\frac{\pi}{4}$$

This is used in **5.3.1** later.

5.1.2 Termwise Higher Integral of Fourier Series of tan x

Formula 5.1.2

Let $\eta(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^x}$ be a Dirichlet Eta Function and $\uparrow$ be a ceiling function, then the following expressions hold for $|x|<\pi/2$.

$$\int_0^x \tan x\, dx = \frac{1}{2^0}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^1}\sin\left(2kx-\frac{1\pi}{2}\right)+\frac{\eta(1)}{2^0}\frac{x^0}{0!}$$

$$\int_0^x\int_0^x \tan x\, dx^2 = \frac{1}{2^1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\sin\left(2kx-\frac{2\pi}{2}\right)+\frac{\eta(1)}{2^0}\frac{x^1}{1!}$$

$$\int_0^x\int_0^x\int_0^x \tan x\, dx^3 = \frac{1}{2^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}\sin\left(2kx-\frac{3\pi}{2}\right)+\frac{\eta(1)}{2^0}\frac{x^2}{2!}-\frac{\eta(3)}{2^2}\frac{x^0}{0!}$$

$$\int_0^x\cdots\int_0^x \tan x\, dx^4 = \frac{1}{2^3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^4}\sin\left(2kx-\frac{4\pi}{2}\right)+\frac{\eta(1)}{2^0}\frac{x^3}{3!}-\frac{\eta(3)}{2^2}\frac{x^1}{1!}$$

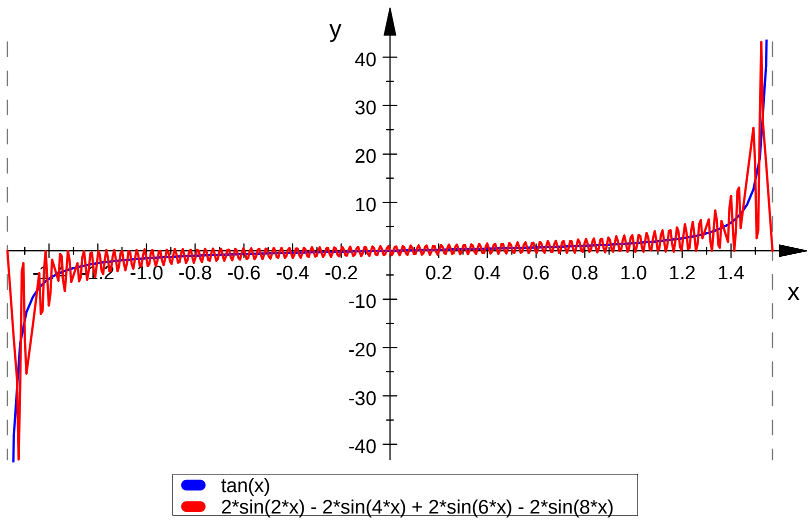
⋮

$$\int_0^x \cdots \int_0^x \tan x\, dx^n = \frac{1}{2^{n-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^n} \sin\left(2kx - \frac{n\pi}{2}\right)$$
$$+ \sum_{k=1}^{n/2\uparrow} (-1)^{k-1} \frac{\eta(2k-1)}{2^{2k-2}} \frac{x^{n+1-2k}}{(n+1-2k)!}$$

Proof

tan x can be expanded to Fourier series in a broad meaning.as follows.

$$\tan x = \frac{1}{i}\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = i\frac{1 - e^{2ix}}{1 + e^{2ix}} = i\left(1 - e^{2ix}\right)\left(1 - e^{2ix} + e^{4ix} - e^{6ix} + - \cdots\right)$$
$$= i - i\left(2e^{2ix} - 2e^{4ix} + 2e^{6ix} - 2e^{8ix} + - \cdots\right)$$
$$= 2\left(\sin 2x - \sin 4x + \sin 6x - \sin 8x + - \cdots\right)$$
$$-2i\left(\cos 2x - \cos 4x + \cos 6x - \cos 8x + - \cdots\right) + i \tag{2.0}$$



tan x is the locus of the median (central line) of this real number part. (See the figure) However calculus of the right side of (2.0) may be carried out, the real number does not turn into an imaginary number and the contrary does not exist either. Therefore, limiting the range of tan x and the higher order integral to the real number, we calculate only the real number part of the right side of (2.0) . Although the Riemann zeta $\zeta(2n)$ is obtained from calculation of the imaginary number part, since it is long, it is omitted in this section.

Now, integrate the real number part of both sides of (2.0) with respect to x from 0 to x, then

$$\int_0^x \tan x\, dx = \int_0^x 2\left(\sin 2x - \sin 4x + \sin 6x - \sin 8x + - \cdots\right) dx$$
$$= -\left[\frac{\cos 2x}{1} - \frac{\cos 4x}{2} + \frac{\cos 6x}{3} - \frac{\cos 8x}{4} + - \cdots\right]_0^x$$
$$= -\left(\frac{\cos 2x}{1} - \frac{\cos 4x}{2} + \frac{\cos 6x}{3} - \frac{\cos 8x}{4} + - \cdots\right) + \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots$$

i.e.

$$\int_0^x \tan x\, dx = -\frac{1}{2^0} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos 2kx}{k^1} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^1}$$

Here, let $-\cos x = \sin\left(x - \frac{\pi}{2}\right)$, $\eta(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^x}$, then we obtain

$$\int_0^x \tan x\,dx = \frac{1}{2^0}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^1}\sin\left(2kx - \frac{\pi}{2}\right) + \frac{\eta(1)}{2^0}\frac{x^0}{0!}$$

This is consistent with the real number part of the Fourier series of $-\log\cos x$.

Next, integrate both sides of this with respect to x from 0 to x, then

$$\int_0^x\int_0^x \tan x\,dx^2 = \int_0^x \left\{\frac{1}{2^0}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^1}\sin\left(2kx - \frac{\pi}{2}\right) - \frac{\eta(1)}{2^0}\frac{x^0}{0!}\right\}dx$$

$$= \left[\frac{1}{2^1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^1}\sin\left(2kx - \frac{2\pi}{2}\right) - \frac{\eta(1)}{2^0}\frac{x^1}{1!}\right]_0^x$$

$$= \frac{1}{2^1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^1}\sin\left(2kx - \frac{2\pi}{2}\right) + \frac{\eta(1)}{2^0}\frac{x^1}{1!}$$

Next, integrate both sides of this with respect to x from 0 to x, then

$$\int_0^x\int_0^x\int_0^x \tan x\,dx^3 = \left[\frac{1}{2^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^1}\sin\left(2kx - \frac{3\pi}{2}\right) + \frac{\eta(1)}{2^0}\frac{x^2}{2!}\right]_0^x$$

$$= \frac{1}{2^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^1}\sin\left(2kx - \frac{3\pi}{2}\right) + \frac{\eta(1)}{2^0}\frac{x^2}{2!} - \frac{1}{2^2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos 2k0}{k^3}$$

$$= \frac{1}{2^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^1}\sin\left(2kx - \frac{3\pi}{2}\right) + \frac{\eta(1)}{2^0}\frac{x^2}{2!} - \frac{\eta(3)}{2^2}\frac{x^0}{0!}$$

Hereafter, in a similar way we obtain the desired expressions.

(1) Termwise Higher Definite Integral of Fourier Series of tan x

Substituting $x = \pi/2$ for Formula 5.1.2 , we obtain the following expressions.

$$\int_0^{\frac{\pi}{2}}\int_0^x \tan x\,dx^2 = \frac{\eta(1)}{2^0 1!}\left(\frac{\pi}{2}\right)^1 \tag{2.2'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\int_0^x \tan x\,dx^3 = \frac{\eta(1)}{2^0 2!}\left(\frac{\pi}{2}\right)^2 - \frac{\eta(3)}{2^2 0!}\left(\frac{\pi}{2}\right)^0 - \frac{\zeta(3)}{2^2} \tag{2.3'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\cdots\int_0^x \tan x\,dx^4 = \frac{\eta(1)}{2^0 3!}\left(\frac{\pi}{2}\right)^3 - \frac{\eta(3)}{2^2 1!}\left(\frac{\pi}{2}\right)^1 \tag{2.4'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\cdots\int_0^x \tan x\,dx^5 = \frac{\eta(1)}{2^0 4!}\left(\frac{\pi}{2}\right)^4 - \frac{\eta(3)}{2^2 2!}\left(\frac{\pi}{2}\right)^2 + \frac{\eta(5)}{2^4 0!} + \frac{\zeta(5)}{2^4} \tag{2.5'}$$

$$\vdots$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\cdots\int_0^x \tan x\,dx^6 = \sum_{k=1}^{n/2\uparrow}\frac{(-1)^{k-1}\eta(2k-1)}{2^{2k-2}(n+1-2k)!}\left(\frac{\pi}{2}\right)^{n+1-2k} + \frac{\zeta(n)}{2^{n-1}}\sin\frac{n\pi}{2} \tag{2.n'}$$

(2) Fourier Series of the1st order Integral of tan x

The1st order Integral of tan x is as follows.

$$\int_0^x \tan x\, dx = -\frac{1}{2^0}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos 2kx}{k^1} + \frac{\log 2}{0!}x^0 = -\log\cos x$$

From this, the following expression follows immediately.

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos 2kx}{k^1} = \log(2\cos x) \qquad |x| < \frac{\pi}{2} \tag{2.f}$$

Substituting $x=1\,,1/2$ for this one by one, we obtain the following special values.

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos 2k}{k^1} = \log(2\cos 1)$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos k}{k^1} = \log\{2\cos(1/2)\}$$

5.1.3 Dirichlet Odd Eta

Comparing Taylor series of tan x and Fourier series of tan x, we obtain Dirichlet Odd Eta

Formula 5.1.3

When B_{2k} , $\eta(x)$, $\zeta(x)$ denote Bernoulli Numbers, Dirichlet Eta Function and Riemann Zeta Function respectively, the following expressions hold.

$$\eta(3) = -\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2)!}\pi^{2k+2} + \frac{\pi^2}{2!}\eta(1) - \zeta(3)$$

$$\eta(3) = -\frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+3)!}\pi^{2k+3} - \frac{\pi^3}{3!}\eta(1)\right\}$$

$$\eta(5) = \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+4)!}\pi^{2k+4} - \frac{\pi^4}{4!}\eta(1) + \frac{\pi^2}{2!}\eta(3) - \zeta(5)$$

$$\eta(5) = \frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+5)!}\pi^{2k+5} - \frac{\pi^5}{5!}\eta(1) + \frac{\pi^3}{3!}\eta(3)\right\}$$

$$\eta(7) = -\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+6)!}\pi^{2k+6} + \frac{\pi^6}{6!}\eta(1) - \frac{\pi^4}{4!}\eta(3) + \frac{\pi^2}{2!}\eta(5) - \zeta(7)$$

$$\eta(7) = -\frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+7)!}\pi^{2k+7} - \frac{\pi^7}{7!}\eta(1) + \frac{\pi^5}{5!}\eta(3) - \frac{\pi^3}{3!}\eta(5)\right\}$$

$$\vdots$$

$$\eta(2n+1) = (-1)^n\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2n)!}\pi^{2k+2n}$$

$$-(-1)^n\sum_{k=0}^{n-1}(-1)^k\frac{\pi^{2n-2k}}{(2n-2k)!}\eta(2k+1) - \zeta(2n+1)$$

$$\eta(2n+1) = \frac{(-1)^n}{\pi}\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2n+1)!}\pi^{2k+2n+1}$$

$$- \frac{(-1)^n}{\pi} \sum_{k=0}^{n-1} (-1)^k \frac{\pi^{2n+1-2k}}{(2n+1-2k)!} \eta(2k+1)$$

Proof

$$\int_0^{\frac{\pi}{2}} \int_0^x \int_0^x \tan x \, dx^3 = \frac{\eta(1)}{2^0 2!} \left(\frac{\pi}{2} \right)^2 - \frac{\eta(3)}{2^2 0!} \left(\frac{\pi}{2} \right)^0 - \frac{\zeta(3)}{2^2} \tag{2.3'}$$

$$\int_0^{\frac{\pi}{2}} \int_0^x \int_0^x \tan x \, dx^3 = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2)!} \left(\frac{\pi}{2} \right)^{2k+2} \tag{1.3'}$$

from these

$$\frac{\eta(1)}{2^0 2!} \left(\frac{\pi}{2} \right)^2 - \frac{\eta(3)}{2^2 0!} \left(\frac{\pi}{2} \right)^0 - \frac{\zeta(3)}{2^2} = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2)!} \left(\frac{\pi}{2} \right)^{2k+2}$$

$$\frac{\eta(1)}{2!} \pi^2 - \eta(3) - \frac{\zeta(3)}{2^2} = \sum_{k=1}^{\infty} \frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2)!} \pi^{2k+2}$$

i.e. $$\eta(3) = -\sum_{k=1}^{\infty} \frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2)!} \pi^{2k+2} + \frac{\pi^2}{2!} \eta(1) - \zeta(3)$$

Next

$$\int_0^{\frac{\pi}{2}} \int_0^x \cdots \int_0^x \tan x \, dx^4 = \frac{\eta(1)}{2^0 3!} \left(\frac{\pi}{2} \right)^3 - \frac{\eta(3)}{2^2 1!} \left(\frac{\pi}{2} \right)^1 \tag{2.4'}$$

$$\int_0^{\frac{\pi}{2}} \int_0^x \cdots \int_0^x \tan x \, dx^4 = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+3)!} \left(\frac{\pi}{2} \right)^{2k+3} \tag{1.4'}$$

from these

$$\frac{\eta(1)}{2^0 3!} \left(\frac{\pi}{2} \right)^3 - \frac{\eta(3)}{2^2 1!} \left(\frac{\pi}{2} \right)^1 = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+3)!} \left(\frac{\pi}{2} \right)^{2k+3}$$

$$\frac{\eta(1)}{3!} \pi^3 - \frac{\eta(3)}{1!} \pi^1 = \sum_{k=1}^{\infty} \frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+3)!} \pi^{2k+3}$$

i.e. $$\eta(3) = -\frac{1}{\pi} \left\{ \sum_{k=1}^{\infty} \frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+3)!} \pi^{2k+3} - \frac{\pi^3}{3!} \eta(1) \right\}$$

Hereafter, in a similar way we obtain the desired expressions.

5.1.4 Riemann Odd Zeta

Applying $\zeta(n) = \dfrac{2^{n-1}}{2^{n-1}-1} \eta(n)$ for Formula 5.1.3 , we obtain Riemann Zeta Function.

Formula 5.1.4

$$\zeta(3) = -\frac{2^2}{2^3-1} \left\{ \sum_{k=1}^{\infty} \frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2)!} \pi^{2k+2} - \frac{\pi^2}{2!} \log 2 \right\}$$

$$\zeta(3) = -\frac{2^2}{2^2-1}\frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+3)\,!}\pi^{2k+3} - \frac{\pi^3}{3\,!}\log 2\right\}$$

$$\zeta(5) = \frac{2^4}{2^5-1}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+4)\,!}\pi^{2k+4} - \frac{\pi^4}{4\,!}\log 2 + \frac{\pi^2}{2\,!}\frac{2^2-1}{2^2}\zeta(3)\right\}$$

$$\zeta(5) = \frac{2^4}{2^4-1}\frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+5)\,!}\pi^{2k+5} - \frac{\pi^5}{5\,!}\log 2 + \frac{\pi^3}{3\,!}\frac{2^2-1}{2^2}\zeta(3)\right\}$$

$$\zeta(7) = -\frac{2^6}{2^7-1}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+6)\,!}\pi^{2k+6} + \frac{\pi^6}{6\,!}\log 2\right\}$$

$$+ \frac{2^6}{2^7-1}\left\{-\frac{\pi^4}{4\,!}\frac{2^2-1}{2^2}\zeta(3) + \frac{\pi^2}{2\,!}\frac{2^4-1}{2^4}\zeta(5)\right\}$$

$$\zeta(7) = -\frac{2^6}{2^6-1}\frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+7)\,!}\pi^{2k+7} - \frac{\pi^7}{7\,!}\log 2\right\}$$

$$+ \frac{2^6}{2^6-1}\frac{1}{\pi}\left\{-\frac{\pi^5}{5\,!}\frac{2^2-1}{2^2}\zeta(3) + \frac{\pi^3}{3\,!}\frac{2^4-1}{2^4}\zeta(5)\right\}$$

$\vdots$

$$\zeta(2n+1) = (-1)^n\frac{2^{2n}}{2^{2n+1}-1}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2n)\,!}\pi^{2k+2n} - \frac{\pi^{2n}}{(2n)\,!}\log 2\right\}$$

$$- (-1)^n\frac{2^{2n}}{2^{2n+1}-1}\sum_{k=1}^{n-1}(-1)^k\frac{\pi^{2n-2k}}{(2n-2k)\,!}\frac{2^{2k-1}-1}{2^{2k-1}}\zeta(2k+1)$$

$$\zeta(2n+1) = \frac{(-1)^n}{\pi}\frac{2^{2n}}{2^{2n}-1}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+2n+1)\,!}\pi^{2k+2n+1} - \frac{\pi^{2n+1}}{(2n+1)\,!}\log 2\right\}$$

$$- \frac{(-1)^n}{\pi}\frac{2^{2n}}{2^{2n}-1}\sum_{k=1}^{n-1}(-1)^k\frac{\pi^{2n+1-2k}}{(2n+1-2k)\,!}\frac{2^{2k}-1}{2^{2k}}\zeta(2k+1)$$

5.2 Termwise Higher Integral of tanh x

Since both zeros of $\tanh x$ and the primitive function $\log\cosh x$ are $x=0$, the latter seems to be a lineal primitive function with a fixed lower limit. Then, we assume zeros of the second or more order primitive function are also $x=0$.

5.2.0 Higher Integral of tanh x

$$\int_0^x \tanh x\,dx = \log\cosh x$$

$$\int_0^x\!\!\int_0^x \tanh x\,dx^2 = \text{ non-elementary function}$$

5.2.1 Termwise Higher Integral of Taylor Series of tanh x

Formula 5.2.1

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli Numbers, the following expressions hold for $|x|<\pi/2$.

$$\int_0^x \tanh x\,dx = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k)!}x^{2k}$$

$$\int_0^x\!\!\int_0^x \tanh x\,dx^2 = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+1)!}x^{2k+1}$$

$$\int_0^x\!\!\int_0^x\!\!\int_0^x \tanh x\,dx^3 = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+2)!}x^{2k+2}$$

$$\vdots$$

$$\int_0^x\!\cdots\!\int_0^x \tanh x\,dx^n = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+n-1)!}x^{2k+n-1}$$

Proof

$\tanh x$ can be expanded to Taylor series as follows.

$$\tanh x = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-1)!}x^{2k-1} \qquad |x|<\frac{\pi}{2}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

(1) Termwise Higher Definite Integral of Taylor Series of tanh x

Substituting $x=1/2$ for Formula 5.2.1 , we obtain the following expressions.

$$\int_0^{\frac{1}{2}} \tanh x\,dx = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k)!}\left(\frac{1}{2}\right)^{2k} \tag{1.1'}$$

$$\int_0^{\frac{1}{2}}\!\!\int_0^x \tanh x\,dx^2 = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+1)!}\left(\frac{1}{2}\right)^{2k+1} \tag{1.2'}$$

$$\int_0^{\frac{1}{2}}\int_0^x\int_0^x \tanh x\, dx^3 = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+2)!}\left(\frac{1}{2}\right)^{2k+2} \tag{1.3'}$$

$$\vdots$$

$$\int_0^{\frac{1}{2}}\int_0^x\cdots\int_0^x \tanh x\, dx^n = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+n-1)!}\left(\frac{1}{2}\right)^{2k+n-1} \tag{1.n'}$$

(2) Taylor Series of the 1st order Integral of tanh x

The 1st order Integral of tanh x is as follows.

$$\int_0^x \tanh x dx = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k)!}x^{2k} = \log\cosh x \qquad |x| < \frac{\pi}{2}$$

From this, the following expression follows immediately.

$$\sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k)!}x^{2k} = \log\cosh x \qquad |x| < \frac{\pi}{2} \tag{1.t}$$

Substituting $x=1, 1/2, \pi/4$ for this one by one, we obtain the following special values.

$$\sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k)!} = \log\cosh 1$$

$$\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k)!} = \log\cosh\frac{1}{2}$$

$$\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k)!}\left(\frac{\pi}{2}\right)^{2k} = \log\cosh\frac{\pi}{4}$$

This is used in **5.4.1** later.

5.2.2 Termwise Higher Integral of Fourier Series of tanh x

Formula 5.2.2

Let $\eta(x)$ be a Dirichlet Eta Function, then the following expressions hold for $x>0$.

$$\int_0^x \tanh x dx = \frac{1}{2^0}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^1} + \frac{x^1}{1!} - \frac{\eta(1)}{2^0}\frac{x^0}{0!}$$

$$\int_0^x\int_0^x \tanh x dx^2 = -\frac{1}{2^1}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^2} + \frac{x^2}{2!} - \frac{\eta(1)}{2^0}\frac{x^1}{1!} + \frac{\eta(2)}{2^1}\frac{x^0}{0!}$$

$$\int_0^x\int_0^x\int_0^x \tanh x dx^3 = \frac{1}{2^2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^3} + \frac{x^3}{3!}$$

$$- \frac{\eta(1)}{2^0}\frac{x^2}{2!} + \frac{\eta(2)}{2^1}\frac{x^1}{1!} - \frac{\eta(3)}{2^2 0!}\frac{x^0}{0!}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \tanh x dx^n = \frac{(-1)^{n-1}}{2^{n-1}}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^n} + \frac{x^n}{n!}$$

$$-\sum_{k=0}^{n-1}(-1)^k \frac{\eta(k+1)}{2^k} \frac{x^{n-1-k}}{(n-1-k)!}$$

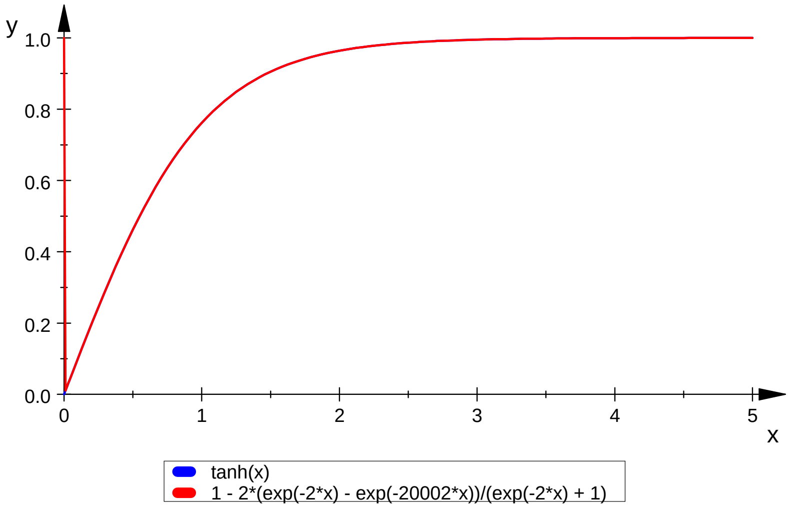
Proof

tanh x can be expanded to a Fourier series as follows. (See the following figure).

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1-e^{-2x}}{1+e^{-2x}} = \left(1-e^{-2x}\right)\left(1-e^{-2x}+e^{-4x}-e^{-6x}+-\cdots\right)$$

$$= 1-\left(2e^{-2x}-2e^{-4x}+2e^{-6x}-2e^{-8x}+-\cdots\right) \tag{2.0}$$

$$= 1-2(\cos 2ix - \cos 4ix + \cos 6ix - \cos 8ix +- \cdots)$$
$$- 2i(\sin 2ix - \sin 4ix + \sin 6ix - \sin 8ix +- \cdots)$$



Integrate both sides of (2.0) with respect to x from 0 to x, then

$$\int_0^x \tanh x\,dx = \int_0^x \left\{1-2\left(e^{-2x}-e^{-4x}+e^{-6x}-e^{-8x}+-\cdots\right)\right\}dx$$

$$= \left[x+\left(\frac{e^{-2x}}{1}-\frac{e^{-4x}}{2}+\frac{e^{-6x}}{3}-\frac{e^{-8x}}{4}+-\cdots\right)\right]_0^x$$

$$= x+\left(\frac{e^{-2x}}{1}-\frac{e^{-4x}}{2}+\frac{e^{-6x}}{3}-\frac{e^{-8x}}{4}+-\cdots\right)-\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+-\cdots\right)$$

i.e.

$$\int_0^x \tanh x\,dx = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^1} + x - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^1}$$

Here, let $\eta(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^x}$, then we obtain

$$\int_0^x \tanh x\,dx = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^1} + x - \eta(1)$$

Of course, this is also the Fourier series expansion of $\log\cosh x$.

Next, integrate both sides of this with respect to x from 0 to x, then

$$\int_0^x\int_0^x \tanh x\,dx^2 = \int_0^x \left\{ \frac{1}{2^0}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^1} + x - \eta(1) \right\} dx$$

$$= \left[-\frac{1}{2^1}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^2} + \frac{x^2}{2!} - \eta(1)\frac{x^1}{1!} \right]_0^x$$

$$= -\frac{1}{2^1}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^2} + \frac{x^2}{2!} - \frac{\eta(1)}{2^0}\frac{x^1}{1!} + \frac{\eta(2)}{2^1}\frac{x^0}{0!}$$

Hereafter, in a similar way we obtein the desired expressions.

(1) Termwise Higher Definite Integral of Fourier Series of tan x

Substituting $x = 1/2 > 0$ for Formula 5.2.2 , we obtain the following expressions.

$$\int_0^{\frac{1}{2}} \tanh x\,dx = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^1} + \frac{1}{1!}\frac{1}{2} - \frac{\eta(1)}{2^0 1!}\left(\frac{1}{2}\right)^0 \tag{2.1'}$$

$$\int_0^{\frac{1}{2}}\int_0^x \tanh x\,dx^2 = -\frac{1}{2^1}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^2} + \frac{1}{2!}\left(\frac{1}{2}\right)^2 - \frac{\eta(1)}{2^0 1!}\left(\frac{1}{2}\right)^1 + \frac{\eta(2)}{2^1 0!}\left(\frac{1}{2}\right)^0 \tag{2.2'}$$

$$\int_0^{\frac{1}{2}}\int_0^x\int_0^x \tanh x\,dx^3 = \frac{1}{2^2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^3} + \frac{1}{3!}\left(\frac{1}{2}\right)^3 - \frac{\eta(1)}{2^0 2!}\left(\frac{1}{2}\right)^2 + \frac{\eta(2)}{2^1 1!}\left(\frac{1}{2}\right)^1 - \frac{\eta(3)}{2^2 0!}\left(\frac{1}{2}\right)^0 \tag{2.3'}$$

$$\vdots$$

$$\int_0^{\frac{1}{2}}\int_0^x\cdots\int_0^x \tanh x\,dx^n = \frac{(-1)^{n-1}}{2^{n-1}}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^n} + \frac{1}{n!}\left(\frac{1}{2}\right)^n - \sum_{k=0}^{n-1}\frac{(-1)^k}{2^k}\frac{\eta(k+1)}{(n-1-k)!}\left(\frac{1}{2}\right)^{n-1-k} \tag{2.n'}$$

(2) Taylor Series of the 1st order Integral of tanh x

The 1st order Integral of tanh x is as follows.

$$\int_0^x \tanh x\,dx = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^1} + x - \log 2 = \log\cosh x \qquad x>0$$

From this, the following expression follows immediately.

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^1} = \log(2\cosh x) - x \qquad x>0 \tag{2.f}$$

Substituting $x=1\,,\ 1/2\ ,\ \pi/2$ for this one by one, we obtain the following special values.

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2k}}{k^1} = \log(2\cosh 1) - 1$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^1} = log\left(2cosh\frac{1}{2}\right) - \frac{1}{2}$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k\pi}}{k^1} = log\left(2cosh\frac{\pi}{2}\right) - \frac{\pi}{2}$$

5.2.3 Dirichlet Eta Function

Comparing Taylor series of tanh x and Fourier series of tanh x, we obtain Dirichlet Eta Function.

Formula 5.2.3

When B_{2k} , $\eta(x)$ denote Bernoulli Numbers and Dirichlet Eta Function, the following expressions hold.

$$0 = \sum_{k=1}^{\infty}(-1)^{k-1}e^{-k} + \sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{2k(2k-1)!} - \frac{1}{2}\frac{1}{0!}$$

$$\eta(1) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^1} - \sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{2k(2k)!} + \frac{1}{2}\frac{1}{1!}$$

$$\eta(2) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^2} + \sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{2k(2k+1)!} - \frac{1}{2}\frac{1}{2!} + \frac{1}{1!}\eta(1)$$

$$\eta(3) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^3} - \sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{2k(2k+2)!} + \frac{1}{2}\frac{1}{3!} - \frac{1}{2!}\eta(1) + \frac{1}{1!}\eta(2)$$

$$\eta(4) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^4} + \sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{2k(2k+3)!} - \frac{1}{2}\frac{1}{4!} + \frac{1}{3!}\eta(1) - \frac{1}{2!}\eta(2) + \frac{1}{1!}\eta(3)$$

$$\vdots$$

$$\eta(n) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^n} + (-1)^n\sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{2k(2k+n-1)!} - \frac{(-1)^n}{2}\frac{1}{n!} - \sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\eta(n-k)$$

Proof

$$tanh\frac{1}{2} = 1 - 2\left(e^{-1} - e^{-2} + e^{-3} - e^{-4} +- \cdots\right) \quad (2.0')$$

$$tanh\frac{1}{2} = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{2k(2k-1)!}\left(\frac{1}{2}\right)^{2k-1} \quad (1.0')$$

$$\therefore\ 0 = \sum_{k=1}^{\infty}(-1)^{k-1}e^{-k} + \sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{2k(2k-1)!} - \frac{1}{2}$$

$$\int_0^{\frac{1}{2}} tanh\,x\,dx = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^1} + \frac{1}{2} - \eta(1) \quad (2.1')$$

$$\int_0^{\frac{1}{2}} tanh\,x\,dx \quad = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k)\,!}\left(\frac{1}{2}\right)^{2k} \tag{1.1'}$$

$$\therefore \quad \eta(1) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^1} - \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k)\,!} + \frac{1}{2}$$

$$\int_0^{\frac{1}{2}}\int_0^{x} tanh\,x\,dx^2 = -\frac{1}{2^1}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^2} + \frac{1}{2!}\left(\frac{1}{2}\right)^2 - \frac{\eta(1)}{2^0 1!}\left(\frac{1}{2}\right)^1 + \frac{\eta(2)}{2^1 0!}\left(\frac{1}{2}\right)^0 \tag{2.2'}$$

$$\int_0^{\frac{1}{2}}\int_0^{x} tanh\,x\,dx^2 \quad = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+1)\,!}\left(\frac{1}{2}\right)^{2k+1} \tag{1.2'}$$

$$\therefore \quad \eta(2) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^2} - \frac{1}{2}\frac{1}{2!} + \frac{1}{1!}\eta(1) + \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+1)\,!}$$

Hereafter, in a similar way we obtain the desired expressions..

5.2.4 Riemann Zeta Function

Applying $\zeta(n) = \dfrac{2^{n-1}}{2^{n-1}-1}\eta(n)$ for Formula 5.2.3 , we obtain Riemann Zeta Function.

Formula 5.2.4

$$\zeta(2) = \frac{2^1}{2^1-1}\left\{\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^2} + \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+1)\,!} - \frac{1}{2\cdot 2!} + \frac{log\,2}{1!}\right\}$$

$$\zeta(3) = \frac{2^2}{2^2-1}\left\{\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^3} - \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+2)\,!} + \frac{1}{2\cdot 3!} - \frac{log\,2}{2!} + \frac{2^1-1}{2^1}\frac{\zeta(2)}{1!}\right\}$$

$$\zeta(4) = \frac{2^3}{2^3-1}\left\{\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^4} + \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+3)\,!} - \frac{1}{2\cdot 4!} + \frac{log\,2}{3!} + \frac{2^2-1}{2^2}\frac{\zeta(3)}{1!} - \frac{2^1-1}{2^1}\frac{\zeta(2)}{2!}\right\}$$

$$\vdots$$

$$\zeta(n) = \frac{2^{n-1}}{2^{n-1}-1}\left\{\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^n} + \sum_{j=1}^{n-2}(-1)^{j-1}\frac{2^{n-1-j}-1}{2^{n-1-j}}\frac{\zeta(n-j)}{j\,!}\right\} - \frac{(-2)^{n-1}}{2^{n-1}-1}\left\{\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+n-1)\,!} - \frac{1}{2\cdot n\,!} + \frac{log\,2}{(n-1)\,!}\right\}$$

5.3 Termwise Higher Integral of cot x

Since both zeros of $\cot x$ and the primitive function $\log\sin x$ are $x=\pi/2$, the latter seems to be a lineal primitive function with a fixed lower limit. Then, we assume zeros of the second or more order primitive function are also $x=\pi/2$.

5.3.0 Higher Integral of cot x

$$\int_{\frac{\pi}{2}}^{x}\cot x\,dx = \log\sin x$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\cot x\,dx^2 = \text{non-elementary function}$$

5.3.1 Termwise Higher Integral of Taylor Series of cot x

Formula 5.3.1

When B_{2k} denote Bernoulli Numbers and functions $f^{<n>}(x)$ on $0<x<\pi$ are as follows.

$$f^{<n>}(x)=\frac{x^n}{n!}\left(\log x-\sum_{k=1}^{n}\frac{1}{k}\right)-\sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k\,(2k+n)!}x^{2k+n}\qquad n=1,2,3,\cdots$$

the following expressions hold .

$$\int_{\frac{\pi}{2}}^{x}\cot x\,dx = f^{<0>}(x) = \log x-\sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k\,(2k)!}x^{2k}$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\cot x\,dx^2 = f^{<1>}(x)-\frac{1}{0!}\left(x-\frac{\pi}{2}\right)^0 f^{<1>}\left(\frac{\pi}{2}\right)$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\cot x\,dx^3 = f^{<2>}(x)-\frac{1}{0!}\left(x-\frac{\pi}{2}\right)^0 f^{<2>}\left(\frac{\pi}{2}\right)-\frac{1}{1!}\left(x-\frac{\pi}{2}\right)^1 f^{<1>}\left(\frac{\pi}{2}\right)$$

$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x}\cot x\,dx^4 = f^{<3>}(x)-\frac{1}{0!}\left(x-\frac{\pi}{2}\right)^0 f^{<3>}\left(\frac{\pi}{2}\right)-\frac{1}{1!}\left(x-\frac{\pi}{2}\right)^1 f^{<2>}\left(\frac{\pi}{2}\right)-\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 f^{<1>}\left(\frac{\pi}{2}\right)$$

$$\vdots$$

$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x}\cot x\,dx^n = f^{<n-1>}(x)-\sum_{k=0}^{n-2}\frac{1}{k!}\left(x-\frac{\pi}{2}\right)^k f^{<n-1-k>}\left(\frac{\pi}{2}\right)\qquad n=2,3,4,\cdots$$

where $f^{<0>}\left(\frac{\pi}{2}\right)=0$, $f^{<1>}\left(\frac{\pi}{2}\right)=-\frac{\pi}{2}\log 2$

Proof

$$x\cot x = 1+\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!}x^{2k} = 1+\sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{(2k)!}x^{2k}$$

From this

$$cot\,x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k-1)!} x^{2k-1}$$

Integrate the both sides with respect to x from $\pi/2$ to x, then

$$\int_{\frac{\pi}{2}}^{x} cot\,x\,dx = log\,x - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k)!} x^{2k} - \left\{ log\frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k)!} \left(\frac{\pi}{2}\right)^{2k} \right\}$$

Here

$$log\frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k)!} \left(\frac{\pi}{2}\right)^{2k} = 0$$

Because, the following equation is known (岩波数学公式Ⅱ p151)

$$log\,x - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k)!} x^{2k} = log\,sin\,x \qquad 0<x<\pi$$

Then, substituting $x=\pi/2$ for this, we obtain the above.

Thus,

$$\int_{\frac{\pi}{2}}^{x} cot\,x\,dx = log\,x - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k)!} x^{2k} = f^{<0>}(x)$$

Next, integrate the both sides with respect to x from $\pi/2$ to x, then

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} cot\,x\,dx^2 = \int_{\frac{\pi}{2}}^{x} f^{<0>}(x)\,dx = f^{<1>}(x) - f^{<1>}\left(\frac{\pi}{2}\right)$$

Furthermore, integrate the both sides with respect to x from $\pi/2$ to x, then

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} cot\,x\,dx^3 = \int_{\frac{\pi}{2}}^{x} \left\{ f^{<1>}(x) - f^{<1>}\left(\frac{\pi}{2}\right) \right\} dx$$

$$= f^{<2>}(x) - f^{<2>}\left(\frac{\pi}{2}\right) - \frac{1}{1!}\left(x-\frac{\pi}{2}\right)^{1} f^{<1>}\left(\frac{\pi}{2}\right)$$

Hereafter, in a similar way we obtein the desired expressions.

Although Formula 5.3.1 is a general expression anyway, if it is expand actually, it is very complicated. Then, this was devised and was made the same easy expression as tan x. It is as follows.

Formula 5.3.1'

When B_{2k} denote Bernoulli Numbers, the following expressions hold for $0<x<\pi$.

$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} cot\,x\,dx^n = -\sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k+n-1)!} \left(x-\frac{\pi}{2}\right)^{2k+n-1}$$

Proof

$$cot\,x = -tan\left(x-\frac{\pi}{2}\right)$$

$$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-1)\,!} x^{2k-1} \qquad |x| < \frac{\pi}{2}$$

From these

$$\cot x = -\sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-1)\,!} \left(x - \frac{\pi}{2}\right)^{2k-1} \qquad 0 < x < \pi$$

Integratingboth sides of this with respect to x from $\pi/2$ to x repeatedly, we obtain the desired expressions.

(1) Termwise Higher Definite Integral of Taylor Series of cot x

When

$$f^{<n>}(x) = \frac{x^n}{n\,!}\left(\log x - \sum_{k=1}^{n}\frac{1}{k}\right) - \sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k\,(2k+n)\,!}x^{2k+n} \qquad n = 1, 2, 3, \cdots$$

substituting $x=0$ for Formula 5.3.1 , we obtain the following expressions.

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x} \cot x\, dx^2 = -\frac{1}{0!}\left(\frac{\pi}{2}\right)^0 f^{<1>}\left(\frac{\pi}{2}\right) \tag{1.2'}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} \cot x\, dx^3 = -\frac{1}{0!}\left(\frac{\pi}{2}\right)^0 f^{<2>}\left(\frac{\pi}{2}\right) + \frac{1}{1!}\left(\frac{\pi}{2}\right)^1 f^{<1>}\left(\frac{\pi}{2}\right) \tag{1.3'}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} \cot x\, dx^4 = -\frac{1}{0!}\left(\frac{\pi}{2}\right)^0 f^{<3>}\left(\frac{\pi}{2}\right) + \frac{1}{1!}\left(\frac{\pi}{2}\right)^1 f^{<2>}\left(\frac{\pi}{2}\right) - \frac{1}{2!}\left(\frac{\pi}{2}\right)^2 f^{<1>}\left(\frac{\pi}{2}\right) \tag{1.4'}$$

$$\vdots$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} \cot x\, dx^n = \sum_{k=0}^{n-2}\frac{(-1)^{k+1}}{k\,!}\left(\frac{\pi}{2}\right)^k f^{<n-1-k>}\left(\frac{\pi}{2}\right) \tag{1.n'}$$

(2) Taylor Series of the 1st order Integral of cot x

The 1st order Integral of cot x is as follows.

$$\int_{\frac{\pi}{2}}^{x} \cot x\, dx = \log x - \sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k\,(2k)\,!}x^{2k} = \log\sin x \qquad 0 < x < \pi$$

From this, the following expression follows immediately.

$$\sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k\,(2k)\,!}x^{2k} = -\log\sin x + \log x \qquad 0 < x < \pi \tag{1.t}$$

Substituting $x=1\,,1/2\,,\pi/4$ for this one by one, we obtain the following special values.

$$\sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k\,(2k)\,!} = -\log\sin 1$$

$$\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k\,(2k)\,!} = -\log\sin\frac{1}{2} - \log 2$$

$$\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k\,(2k)\,!}\left(\frac{\pi}{2}\right)^{2k} = -\log\sin\frac{\pi}{4} + \log\frac{\pi}{4}$$

Moreover, from the last expressions of this and **5.1.1 (2)** , the next expression follows

$$\sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)\left|B_{2k}\right|}{2k\,(2k)\,!}\left(\frac{\pi}{2}\right)^{2k} = -log\frac{\pi}{4}$$

This is used in **5.5.1** later.

5.3.2 Termwise Higher Integral of Fourier Series of cot x

Formula 5.3.2

Let $\eta(x)$ be Dirichlet Eta Function and $\uparrow$ be the ceiling function, then the following expressions hold for $0<x<\pi$.

$$\int_{\frac{\pi}{2}}^{x} cot\,x\,dx = \frac{1}{2^0}\sum_{k=1}^{\infty}\frac{1}{k^1}sin\left(2kx-\frac{1\pi}{2}\right) - \frac{\eta(1)}{2^0 0\,!}\left(x-\frac{\pi}{2}\right)^0$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} cot\,x\,dx^2 = \frac{1}{2^1}\sum_{k=1}^{\infty}\frac{1}{k^2}sin\left(2kx-\frac{2\pi}{2}\right) - \frac{\eta(1)}{2^0 1\,!}\left(x-\frac{\pi}{2}\right)^1$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} cot\,x\,dx^3 = \frac{1}{2^2}\sum_{k=1}^{\infty}\frac{1}{k^3}sin\left(2kx-\frac{3\pi}{2}\right) - \frac{\eta(1)}{2^0 2\,!}\left(x-\frac{\pi}{2}\right)^2 + \frac{\eta(3)}{2^2 0\,!}\left(x-\frac{\pi}{2}\right)^0$$

$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} cot\,x\,dx^4 = \frac{1}{2^3}\sum_{k=1}^{\infty}\frac{1}{k^4}sin\left(2kx-\frac{4\pi}{2}\right) - \frac{\eta(1)}{2^0 3\,!}\left(x-\frac{\pi}{2}\right)^3 + \frac{\eta(3)}{2^2 1\,!}\left(x-\frac{\pi}{2}\right)^1$$

$$\vdots$$

$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} cot\,x\,dx^n = \frac{1}{2^{n-1}}\sum_{k=1}^{\infty}\frac{1}{k^n}sin\left(2kx-\frac{n\pi}{2}\right)$$

$$+\sum_{k=1}^{n/2\uparrow}\frac{(-1)^k}{2^{2k-2}}\frac{\eta(2k-1)}{(n+1-2k)\,!}\left(x-\frac{\pi}{2}\right)^{n+1-2k}$$

Proof

cot x can be expanded to Fourier series in a broad meaning.as follows.

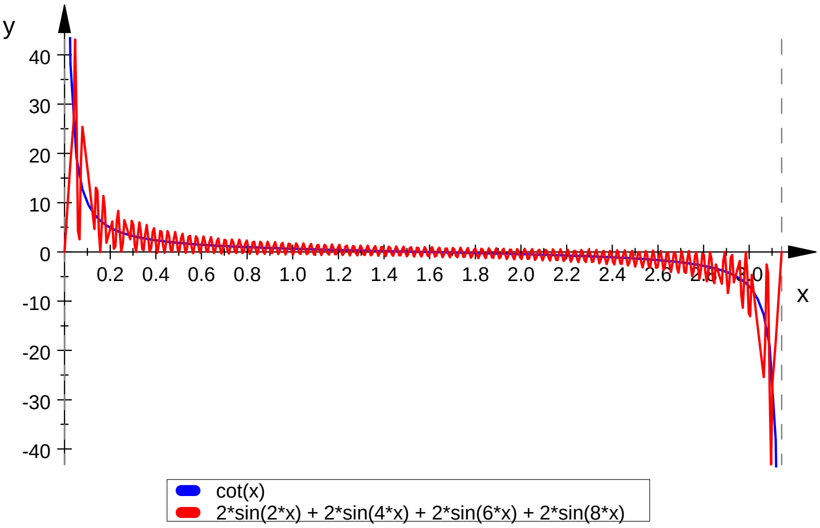
$$cot x = i\,\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} = -i\frac{1+e^{2ix}}{1-e^{2ix}} = -i\left(1+e^{2ix}\right)\left(1+e^{2ix}+e^{4ix}+e^{6ix}+\cdots\right)$$

$$= -i - i\left(2e^{2ix}+2e^{4ix}+2e^{6ix}+2e^{8ix}+\cdots\right)$$

$$= 2\left(sin\,2x+sin\,4x+sin\,6x+sin\,8x+\cdots\right)$$

$$-2i\left(cos\,2x+cos\,4x+cos\,6x+cos\,8x+\cdots\right) - i \qquad (2.0)$$

cot x is the locus of the median (central line) of this real number part. (See the figure of the following page.) However calculus of the right side of (2.0) may be carried out, the real number does not turn into an imaginary number and the contrary does not exist either. Therefore, limiting the range of cot x and the higher order integral to the real number, we calculate only the real number part of the right side of (2.0).
Although the Riemann zeta $\zeta(2n)$ is obtained from the calculation of the imaginary number part, since it is long, it is omitted in this section.



Now, integrate the real number part of both sides of (2.0) with respect to x from $\pi/2$ to x, then

$$\int_{\frac{\pi}{2}}^{x} \cot x\, dx = \int_{\frac{\pi}{2}}^{x} 2(\sin 2x + \sin 4x + \sin 6x * \sin 8x + \cdots)dx$$

$$= -\left[\frac{\cos 2x}{1} + \frac{\cos 4x}{2} + \frac{\cos 6x}{3} + \frac{\cos 8x}{4} + \cdots \right]_{\frac{\pi}{2}}^{x}$$

$$= -\left(\frac{\cos 2x}{1} + \frac{\cos 4x}{2} + \frac{\cos 6x}{3} + \frac{\cos 8x}{4} + \cdots \right) - \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots \right)$$

i.e.

$$\int_{\frac{\pi}{2}}^{x} \cot x\, dx = \log \sin x = -\frac{1}{2^0}\sum_{k=1}^{\infty}\frac{\cos 2kx}{k^1} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^1}$$

Here, let $-\cos x = \sin\left(x - \frac{\pi}{2}\right)$, $\eta(x) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^x}$, then we obtain

$$\int_{\frac{\pi}{2}}^{x} \cot x\, dx = \frac{1}{2^0}\sum_{k=1}^{\infty}\frac{1}{k^1}\sin\left(2kx - \frac{1\pi}{2}\right) - \frac{\eta(1)}{2^0 0!}\left(x - \frac{\pi}{2}\right)^0$$

This is consistent with the real number part of the Fourier series of $\log \sin x$.

Next, integrate both sides of this with respect to x from $\pi/2$ to x, then

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} \cot x\, dx^2 = \int_{\frac{\pi}{2}}^{x}\left\{ \frac{1}{2^0}\sum_{k=1}^{\infty}\frac{1}{k^1}\sin\left(2kx - \frac{1\pi}{2}\right) - \frac{\eta(1)}{2^0 0!}\left(x - \frac{\pi}{2}\right)^0 \right\}dx$$

$$= \left[\frac{1}{2^1}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\left(2kx - \frac{2\pi}{2}\right) - \frac{\eta(1)}{2^0 1!}\left(x - \frac{\pi}{2}\right)^1 \right]_{\frac{\pi}{2}}^{x}$$

$$= \frac{1}{2^1}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\left(2kx - \frac{2\pi}{2}\right) - \frac{\eta(1)}{2^0 1!}\left(x - \frac{\pi}{2}\right)^1$$

Next, integrate both sides of this with respectto x from $\pi/2$ to x, then

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\cot x\,dx^3=\int_{\frac{\pi}{2}}^{x}\left\{\frac{1}{2^1}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\left(2kx-\frac{2\pi}{2}\right)-\frac{\eta(1)}{2^0 1!}\left(x-\frac{\pi}{2}\right)^1\right\}dx$$

$$=\left[\frac{1}{2^2}\sum_{k=1}^{\infty}\frac{1}{k^3}\sin\left(2kx-\frac{3\pi}{2}\right)-\frac{\eta(1)}{2^0 2!}\left(x-\frac{\pi}{2}\right)^2\right]_{\frac{\pi}{2}}^{x}$$

$$=\frac{1}{2^2}\sum_{k=1}^{\infty}\frac{1}{k^3}\sin\left(2kx-\frac{3\pi}{2}\right)-\frac{\eta(1)}{2^0 2!}\left(x-\frac{\pi}{2}\right)^2-\frac{1}{2^2}\sum_{k=1}^{\infty}\frac{\cos 2k\pi}{k^3}$$

$$=\frac{1}{2^2}\sum_{k=1}^{\infty}\frac{1}{k^3}\sin\left(2kx-\frac{3\pi}{2}\right)-\frac{\eta(1)}{2^0 2!}\left(x-\frac{\pi}{2}\right)^2+\frac{\eta(3)}{2^2 2!}\left(x-\frac{\pi}{2}\right)^0$$

Hereafter, in a similar way we obtain the desired expressions.

(1) Termwise Higher Definite Integral of Fourier Series of cot x

Substituting $x=0$ for Formula 5.3.2 , we obtain the following expressions.

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cot x\,dx^2=\frac{\eta(1)}{2^0 1!}\left(\frac{\pi}{2}\right)^1 \tag{2.2'}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\cot x\,dx^3=-\frac{\eta(1)}{2^0 2!}\left(\frac{\pi}{2}\right)^2+\frac{\eta(3)}{2^2 0!}\left(\frac{\pi}{2}\right)^0+\frac{\zeta(3)}{2^2} \tag{2.3'}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x}\cot x\,dx^4=\frac{\eta(1)}{2^0 3!}\left(\frac{\pi}{2}\right)^3-\frac{\eta(3)}{2^2 1!}\left(\frac{\pi}{2}\right)^1 \tag{2.4'}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x}\cot x\,dx^5=-\frac{\eta(1)}{2^0 4!}\left(\frac{\pi}{2}\right)^4+\frac{\eta(3)}{2^2 2!}\left(\frac{\pi}{2}\right)^2-\frac{\eta(5)}{2^4 0!}-\frac{\zeta(5)}{2^4} \tag{2.5'}$$

$\vdots$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x}\cot x\,dx^n=(-1)^n\sum_{k=1}^{n/2\uparrow}\frac{(-1)^{k-1}\eta(2k-1)}{2^{2k-2}(n+1-2k)!}\left(\frac{\pi}{2}\right)^{n+1-2k}-\frac{\zeta(n)}{2^{n-1}}\sin\frac{n\pi}{2} \tag{2.n'}$$

(2) Fourier Series of the 1st order Integral of cot x

The 1st order Integral of cot x is as follows.

$$\int_0^x \cot x\,dx=-\frac{1}{2^0}\sum_{k=1}^{\infty}\frac{\cos 2kx}{k^1}-\log 2=\log\sin x \qquad 0<x<\pi$$

From this, the following expression follows immediately.

$$\sum_{k=1}^{\infty}\frac{\cos 2kx}{k^1}=-\log(2\sin x) \qquad 0<x<\pi \tag{2.f}$$

Substituting $x=1\,,1/2$ for this one by one, we obtain the following special values.

$$\sum_{k=1}^{\infty}\frac{\cos 2k}{k^1}=-\log(2\sin 1)$$

$$\sum_{k=1}^{\infty}\frac{\cos k}{k^1} = -\log\{2\sin(1/2)\}$$

5.3.3 Dirichlet Odd Eta & Riemann Odd Zeta

Comparing Taylor series of cot x and Fourier series of cot x, we obtain Dirichlet Odd Eta and Riemann Odd Zeta.

Formula 5.3.3

When B_{2k} , $\eta(x)$, $\zeta(x)$ denote Bernoulli Numbers, Dirichlet Eta Function and Riemann Zeta Function respectively, the following expressions hold.

$$\log\pi = 1+\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+1)!}\pi^{2k}$$

$$\eta(3) = -\frac{\pi^2}{2!}\left(\log\pi-\sum_{k=1}^{2}\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+2)!}\pi^{2k+2}-\zeta(3)$$

$$\zeta(3) = -\frac{1}{\pi}\left\{\frac{\pi^3}{3!}\left(\log\pi-\sum_{k=1}^{3}\frac{1}{k}\right)-\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+3)!}\pi^{2k+3}\right\}$$

$$\eta(5) = \frac{\pi^4}{4!}\left(\log\pi-\sum_{k=1}^{4}\frac{1}{k}\right)-\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+4)!}\pi^{2k+4}+\frac{\pi^2}{2!}\zeta(3)-\zeta(5)$$

$$\zeta(5) = \frac{1}{\pi}\left\{\frac{\pi^5}{5!}\left(\log\pi-\sum_{k=1}^{5}\frac{1}{k}\right)-\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+5)!}\pi^{2k+5}+\frac{\pi^3}{3!}\zeta(3)\right\}$$

$$\eta(7)= -\frac{\pi^6}{6!}\left(\log\pi-\sum_{k=1}^{6}\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+6)!}\pi^{2k+6}-\frac{\pi^4}{4!}\zeta(3)+\frac{\pi^2}{2!}\zeta(5)-\zeta(7)$$

$$\zeta(7)= -\frac{1}{\pi}\left\{\frac{\pi^7}{7!}\left(\log\pi-\sum_{k=1}^{7}\frac{1}{k}\right)-\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+7)!}\pi^{2k+7}+\frac{\pi^5}{5!}\zeta(3)-\frac{\pi^3}{3!}\zeta(5)\right\}$$

$$\vdots$$

$$\eta(2n+1)= (-1)^n\left\{\frac{\pi^{2n}}{(2n)!}\left(\log\pi-\sum_{k=1}^{2n}\frac{1}{k}\right)-\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+2n)!}\pi^{2k+2n}\right\}$$
$$+(-1)^n\sum_{k=1}^{n-1}(-1)^{k-1}\frac{\pi^{2n-2k}}{(2n-2k)!}\zeta(2k+1)-\zeta(2n+1)$$

$$\zeta(2n+1)= (-1)^n\frac{1}{\pi}\left\{\frac{\pi^{2n+1}}{(2n+1)!}\left(\log\pi-\sum_{k=1}^{2n+1}\frac{1}{k}\right)-\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+2n+1)!}\pi^{2k+2n+1}\right\}$$
$$+(-1)^n\frac{1}{\pi}\sum_{k=1}^{n-1}(-1)^{k-1}\frac{\pi^{2n-2k+1}}{(2n-2k+1)!}\zeta(2k+1)$$

Proof

From Fourier series (2.2') and Taylor series (1.2')

$$\frac{\log 2}{2^0 1!}\left(\frac{\pi}{2}\right)^1 = -\frac{1}{0!}\left(\frac{\pi}{2}\right)^0 f^{<1>}\left(\frac{\pi}{2}\right) \tag{2.2''}$$

$$= -\frac{1}{1!}\left(\frac{\pi}{2}\right)^1\left(\log\frac{\pi}{2}-1\right)-\sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}$$

Hence we obtain

$$\log\pi = 1 + \sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k\,(2k+1)\,!}\pi^{2k}$$

Next, substitute (2.2") for Taylor series (1.3') , then

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\cot x\,dx^3 = -f^{<2>}\left(\frac{\pi}{2}\right) + \frac{1}{1\,!}\left(\frac{\pi}{2}\right)^1 f^{<1>}\left(\frac{\pi}{2}\right)$$

$$= -f^{<2>}\left(\frac{\pi}{2}\right) - \frac{\log 2}{1\,!}\left(\frac{\pi}{2}\right)^2$$

From this and Fourier series (2.3') , we obtain

$$\frac{1}{2^2}\zeta(3) - \frac{\log 2}{2\,!}\left(\frac{\pi}{2}\right)^2 + \frac{1}{2^2}\frac{\eta(3)}{0\,!} = -f^{<2>}\left(\frac{\pi}{2}\right) - \frac{\log 2}{1\,!}\left(\frac{\pi}{2}\right)^2$$

i.e. $$\frac{1}{2^2}\zeta(3) + \frac{1}{2^2}\frac{\eta(3)}{0\,!} = -f^{<2>}\left(\frac{\pi}{2}\right) - \frac{\log 2}{2\,!}\left(\frac{\pi}{2}\right)^2 \qquad \text{(a)}$$

From (a), it is as follows.

$$\eta(3) = -2^2 f^{<2>}\left(\frac{\pi}{2}\right) - \frac{\pi^2}{2\,!}\log 2 - \zeta(3)$$

$$= -2^2\left\{\frac{1}{2\,!}\left(\frac{\pi}{2}\right)^2\left(\log\frac{\pi}{2} - \sum_{k=1}^{2}\frac{1}{k}\right) - \sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k\,(2k+2)\,!}\left(\frac{\pi}{2}\right)^{2k+2}\right\}$$

$$- \frac{\pi^2}{2\,!}\log 2 - \zeta(3)$$

$$= -\frac{\pi^2}{2\,!}\left(\log\pi - \sum_{k=1}^{2}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k\,(2k+2)\,!}\pi^{2k+2} - \zeta(3)$$

Next, substitute (2.2") and (a) for Taylor series (1.4') one by one, then

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x}\cot x\,dx^4 = -f^{<3>}\left(\frac{\pi}{2}\right) + \frac{1}{1\,!}\left(\frac{\pi}{2}\right)^1 f^{<2>}\left(\frac{\pi}{2}\right) - \frac{1}{2\,!}\left(\frac{\pi}{2}\right)^2 f^{<1>}\left(\frac{\pi}{2}\right)$$

$$= -f^{<3>}\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^1\left\{\frac{1}{2^2}\zeta(3) + \frac{1}{2^2}\frac{\eta(3)}{0\,!}\right\}$$

From this and Fourier series (2.4') , we obtain

$$\frac{\log 2}{3\,!}\left(\frac{\pi}{2}\right)^3 - \frac{1}{2^2}\frac{\eta(3)}{1\,!}\left(\frac{\pi}{2}\right)^1 = -f^{<3>}\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^1\left\{\frac{1}{2^2}\zeta(3) + \frac{1}{2^2}\frac{\eta(3)}{0\,!}\right\}$$

i.e. $$\frac{1}{2^2}\left(\frac{\pi}{2}\right)^1\zeta(3) = -f^{<3>}\left(\frac{\pi}{2}\right) - \frac{\log 2}{3\,!}\left(\frac{\pi}{2}\right)^3 \qquad \text{(b)}$$

Substitute $f^{<3>}\left(\frac{\pi}{2}\right)$ for (b), then

$$\frac{1}{2^2}\left(\frac{\pi}{2}\right)^1\zeta(3) = -f^{<3>}\left(\frac{\pi}{2}\right) - \frac{\log 2}{3\,!}\left(\frac{\pi}{2}\right)^3$$

$$= -\frac{1}{3\,!}\left(\frac{\pi}{2}\right)^3\left(\log\frac{\pi}{2} - \sum_{k=1}^{3}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k\,(2k+3)\,!}\left(\frac{\pi}{2}\right)^{2k+3} - \frac{\log 2}{3\,!}\left(\frac{\pi}{2}\right)^3$$

$$= -\frac{1}{3!}\left(\frac{\pi}{2}\right)^3\left(\log\pi - \sum_{k=1}^{3}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k+3)!}\left(\frac{\pi}{2}\right)^{2k+3}$$

From this, we obtain

$$\zeta(3) = -2^2\left(\frac{\pi}{2}\right)^{-1}\left\{\frac{1}{3!}\left(\frac{\pi}{2}\right)^3\left(\log\pi - \sum_{k=1}^{3}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k+3)!}\left(\frac{\pi}{2}\right)^{2k+3}\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{\pi^3}{3!}\left(\log\pi - \sum_{k=1}^{3}\frac{1}{k}\right) - \sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+3)!}\pi^{2k+3}\right\}$$

Hereafter, in a similar way we obtein the desired expressions.

5.3.4 Another set of Riemann Odd Zeta

Substituting $\eta(2n+1) = \frac{2^{2n}-1}{2^{2n}}\zeta(2n+1)$ for Formula 5.3.3 , we obtain another set of Riemann Odd Zeta.

Formula 5.3.4

$$\zeta(3) = -\frac{2^2}{2^3-1}\left\{\frac{\pi^2}{2!}\left(\log\pi - \sum_{k=1}^{2}\frac{1}{k}\right) - \sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+2)!}\pi^{2k+2}\right\}$$

$$\zeta(5) = \frac{2^4}{2^5-1}\left\{\frac{\pi^4}{4!}\left(\log\pi - \sum_{k=1}^{4}\frac{1}{k}\right) - \sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+4)!}\pi^{2k+4} + \frac{\pi^2}{2!}\zeta(3)\right\}$$

$$\zeta(7) = -\frac{2^6}{2^7-1}\left\{\frac{\pi^6}{6!}\left(\log\pi - \sum_{k=1}^{6}\frac{1}{k}\right) - \sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+6)!}\pi^{2k+6}\right\}$$

$$- \frac{2^6}{2^7-1}\left\{\frac{\pi^4}{4!}\zeta(3) - \frac{\pi^2}{2!}\zeta(5)\right\}$$

$$\vdots$$

$$\zeta(2n+1) = (-1)^n\frac{2^{2n}}{2^{2n+1}-1}\left\{\frac{\pi^{2n}}{(2n)!}\left(\log\pi - \sum_{k=1}^{2n}\frac{1}{k}\right) - \sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k(2k+2n)!}\pi^{2k+2n}\right\}$$

$$+ (-1)^n\frac{2^{2n}}{2^{2n+1}-1}\sum_{k=1}^{n-1}(-1)^{k-1}\frac{\pi^{2n-2k}}{(2n-2k)!}\zeta(2k+1) \qquad n=2,3,4,\cdots$$

5.4 Termwise Higher Integral of coth x

Since the zero of the first order integral $\log\sinh x$ of $\coth x$ is $x=\sinh^{-1}1=0.8813\cdots$, we assume zeros of the second or more order integral of $\coth x$ are also $x=0.8813\cdots$.
Where, since the zero of $\coth x$ is not $0.8813\cdots$, these seem to be a **collateral higher integral**.

5.4.0 Collateral Higher Integral of coth x

When $\rho=2\sinh^{-1}1\ (=1.762747174\cdots)$

$$\int_{\frac{\rho}{2}}^{x}\coth x\,dx = \log\sinh x$$

$$\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x}\coth x\,dx^2 = \text{non-elementary function}$$

5.4.1 Termwise Higher Integral of Taylor Series of coth x

Formula 5.4.1

Let $\rho=2\sinh^{-1}1\ (=1.762747174\cdots)$ and $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli Numbers and functions $f^{<n>}(x)$ on $x>0$ be as follows.

$$f^{<n>}(x)=\frac{x^n}{n!}\left(\log x-\sum_{k=1}^{n}\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{2k\,(2k+n)!}x^{2k+n} \qquad n=1,2,3,\cdots$$

Then the following expressions hold for $x>0$.

$$\int_{\frac{\rho}{2}}^{x}\coth x\,dx = f^{<0>}(x) = \log x+\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{2k\,(2k)!}x^{2k}$$

$$\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x}\coth x\,dx^2 = f^{<1>}(x)-\frac{1}{0!}\left(x-\frac{\rho}{2}\right)^0 f^{<1>}\left(\frac{\rho}{2}\right)$$

$$\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x}\coth x\,dx^3 = f^{<2>}(x)-\frac{1}{0!}\left(x-\frac{\rho}{2}\right)^0 f^{<2>}\left(\frac{\rho}{2}\right)-\frac{1}{1!}\left(x-\frac{\rho}{2}\right)^1 f^{<1>}\left(\frac{\rho}{2}\right)$$

$$\int_{\frac{\rho}{2}}^{x}\cdots\int_{\frac{\rho}{2}}^{x}\coth x\,dx^4 = f^{<3>}(x)-\frac{1}{0!}\left(x-\frac{\rho}{2}\right)^0 f^{<3>}\left(\frac{\rho}{2}\right)-\frac{1}{1!}\left(x-\frac{\rho}{2}\right)^1 f^{<2>}\left(\frac{\rho}{2}\right)-\frac{1}{2!}\left(x-\frac{\rho}{2}\right)^2 f^{<1>}\left(\frac{\rho}{2}\right)$$

$$\vdots$$

$$\int_{\frac{\rho}{2}}^{x}\cdots\int_{\frac{\rho}{2}}^{x}\coth x\,dx^n = f^{<n-1>}(x)-\sum_{k=0}^{n-2}\frac{1}{k!}\left(x-\frac{\rho}{2}\right)^k f^{<n-1-k>}\left(\frac{\rho}{2}\right)$$

Proof

$$x\coth x = 1+\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{(2k)!}x^{2k}$$

From this

$$coth\ x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k-1)!} x^{2k-1}$$

Integrate the both sides with respect to x from $\rho/2$ to x , then

$$\int_{\frac{\rho}{2}}^{x} coth\,x\,dx = log\ x + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k)!} x^{2k} - \left\{ log\frac{\rho}{2} + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k)!} \left(\frac{\rho}{2}\right)^{2k} \right\}$$

Here

$$log\frac{\rho}{2} + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k)!} \left(\frac{\rho}{2}\right)^{2k} = 0$$

Because, the following equation is known (岩波数学公式Ⅱ p150)

$$log\ x + \sum_{k=1}^{\infty} \frac{2^{2k-1}B_{2k}}{k(2k)!} x^{2k} = log\,sinh\,x \qquad x>0$$

Then, substituting $x=\rho/2$ for this, we obtain the above.
Thus,

$$\int_{\frac{\rho}{2}}^{x} coth\,x\,dx = log\ x + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k)!} x^{2k} = f^{<0>}(x)$$

Next, integrate the both sides with respect to x from $\rho/2$ to x, then

$$\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x} coth\,x\,dx^2 = \int_{\frac{\rho}{2}}^{x} f^{<0>}(x)dx = f^{<1>}(x) - f^{<1>}\left(\frac{\rho}{2}\right)$$

Furthermore, integrate the both sides with respect to x from $\rho/2$ to x, then

$$\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x} coth\,x\,dx^3 = \int_{\frac{\rho}{2}}^{x} \left\{ f^{<1>}(x) - f^{<1>}\left(\frac{\rho}{2}\right) \right\} dx$$

$$= f^{<2>}(x) - f^{<2>}\left(\frac{\rho}{2}\right) - \frac{1}{1!}\left(x-\frac{\rho}{2}\right)^1 f^{<1>}\left(\frac{\rho}{2}\right)$$

Hereafter, in a similar way we obtein the desired expressions.

(1) Termwise Higher Definite Integral of Taylor Series of coth x

When

$$f^{<n>}(x) = \frac{x^n}{n!}\left(log\,x - \sum_{k=1}^{n}\frac{1}{k} \right) + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k+n)!} x^{2k+n} \qquad n=1,2,3,\cdots$$

substituting $x=0$ for Formula 5.4.1 , we obtain the following expressions.

$$\int_{\frac{\rho}{2}}^{0}\int_{\frac{\rho}{2}}^{x} coth\,x\,dx^2 = -\frac{1}{0!}\left(\frac{\rho}{2}\right)^0 f^{<1>}\left(\frac{\rho}{2}\right) \tag{1.2'}$$

$$\int_{\frac{\rho}{2}}^{0}\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x} coth\,x\,dx^3 = -\frac{1}{0!}\left(\frac{\rho}{2}\right)^0 f^{<2>}\left(\frac{\rho}{2}\right) + \frac{1}{1!}\left(\frac{\rho}{2}\right)^1 f^{<1>}\left(\frac{\rho}{2}\right) \tag{1.3'}$$

$$\int_{\frac{\rho}{2}}^{0}\int_{\frac{\rho}{2}}^{x}\cdots\int_{\frac{\rho}{2}}^{x} \coth x\, dx^4 = -\frac{1}{0!}\left(\frac{\rho}{2}\right)^0 f^{<3>}\left(\frac{\rho}{2}\right) + \frac{1}{1!}\left(\frac{\rho}{2}\right)^1 f^{<2>}\left(\frac{\rho}{2}\right)$$

$$-\frac{1}{2!}\left(\frac{\rho}{2}\right)^2 f^{<1>}\left(\frac{\rho}{2}\right) \qquad (1.4')$$

$\vdots$

$$\int_{\frac{\rho}{2}}^{0}\int_{\frac{\rho}{2}}^{x}\cdots\int_{\frac{\rho}{2}}^{x} \coth x\, dx^n = -\sum_{k=0}^{n-2}\frac{(-1)^k}{k!}\left(\frac{\rho}{2}\right)^k f^{<n-1-k>}\left(\frac{\rho}{2}\right) \qquad (1.n')$$

(2) Taylor Series of the 1st order Integral of coth x

The 1st order Integral of coth x is as follows.

$$\int_{\frac{\rho}{2}}^{x} \coth x\, dx = \log x + \sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{2k(2k)!}x^{2k} = \log\sinh x \qquad 0<x<\pi$$

From this, the following expression follows immediately.

$$\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{2k(2k)!}x^{2k} = \log\sinh x - \log x \qquad 0<x<\pi \qquad (1.t)$$

Substituting $x=1, 1/2, \pi/4$ for this one by one, we obtain the following special values.

$$\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{2k(2k)!} = \log\sinh 1$$

$$\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k)!} = \log\sinh\frac{1}{2} + \log 2$$

$$\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k)!}\left(\frac{\pi}{2}\right)^{2k} = \log\sinh\frac{\pi}{4} - \log\frac{\pi}{4}$$

Moreover, from the last expressions of this and **5.2.1 (2)** , the next expression follows

$$\sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)B_{2k}}{2k(2k)!}\left(\frac{\pi}{2}\right)^{2k} = \log\coth\frac{\pi}{4} + \log\frac{\pi}{4}$$

5.4.2 Termwise Higher Integral of Fourier Series of coth x

Formula 5.4.2

Let $\rho = 2\sinh^{-1}1\ (=1.762747174\cdots)$, then the following expressions hold for $x>0$.

$$\int_{\frac{\rho}{2}}^{x} \coth x\, dx = \log\sinh x = -\frac{1}{2^0}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^1} + \left(x-\frac{\rho}{2}\right) + \frac{\rho}{2} - \log 2$$

$$\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x} \coth x\, dx^2 = \frac{1}{2^1}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^2} + \frac{1}{2!}\left(x-\frac{\rho}{2}\right)^2 + \frac{\rho/2-\log 2}{1!}\left(x-\frac{\rho}{2}\right)$$

$$-\frac{1}{2^1}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^2}$$

$$\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x} \coth x\, dx^3 = -\frac{1}{2^2}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^3} + \frac{1}{3!}\left(x-\frac{\rho}{2}\right)^3 + \frac{\rho/2-\log 2}{2!}\left(x-\frac{\rho}{2}\right)^2$$

$$-\frac{1}{2^1}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^2}\left(x-\frac{\rho}{2}\right) + \frac{1}{2^2}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^3}$$

$$\int_{\frac{\rho}{2}}^{x}\cdots\int_{\frac{\rho}{2}}^{x} \coth x\, dx^4 = \frac{1}{2^3}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^4} + \frac{1}{4!}\left(x-\frac{\rho}{2}\right)^4 + \frac{\rho/2-\log 2}{3!}\left(x-\frac{\rho}{2}\right)^3$$

$$-\frac{1}{2^1}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^2}\frac{1}{2!}\left(x-\frac{\rho}{2}\right)^2 + \frac{1}{2^2}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^3}\left(x-\frac{\rho}{2}\right) - \frac{1}{2^3}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^4}$$

$$\vdots$$

$$\int_{\frac{\rho}{2}}^{x}\cdots\int_{\frac{\rho}{2}}^{x} \coth x\, dx^n = \frac{(-1)^n}{2^{n-1}}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^n} + \frac{1}{n!}\left(x-\frac{\rho}{2}\right)^n + \frac{\rho/2-\log 2}{(n-1)!}\left(x-\frac{\rho}{2}\right)^{n-1}$$

$$-\sum_{j=0}^{n-2}\frac{(-1)^{n-j}}{2^{n-1-j}}\frac{1}{j!}\left(x-\frac{\rho}{2}\right)^j\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^{n-j}}$$

Proof

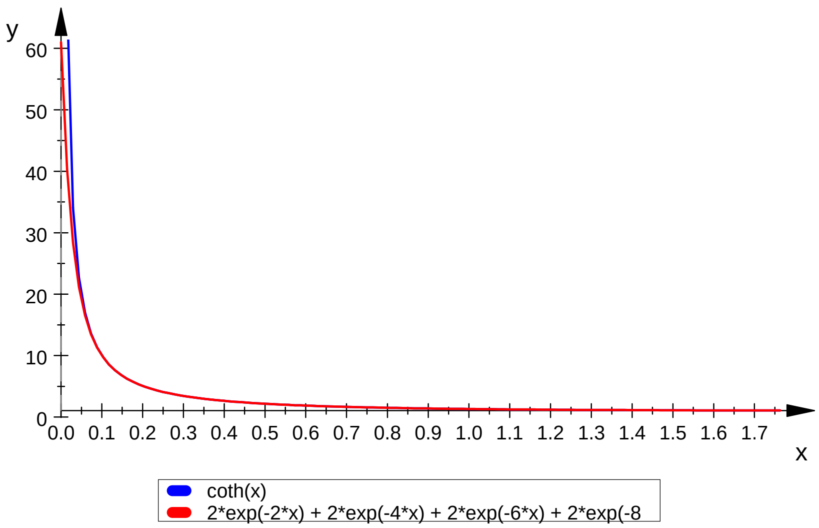
coth x can be expanded to a Fourier series as follows. (See the following figure).

$$\coth x = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{1+e^{-2x}}{1-e^{-2x}} = \left(1+e^{-2x}\right)\left(1+e^{-2x}+e^{-4x}+e^{-6x}+\cdots\right)$$

$$= 1+\left(2e^{-2x}+2e^{-4x}+2e^{-6x}+2e^{-8x}+-\cdots\right) \tag{2.0}$$

$$= 1+2(\cos 2ix+\cos 4ix+\cos 6ix+\cos 8ix+\cdots)$$

$$+2i(\sin 2ix+\sin 4ix+\sin 6ix+\sin 8ix+\cdots)$$



Now, let $\rho/2 = \sinh^{-1} 1\ (=0.881373587\cdots)$. Integrate the both sides of (2.0) with respect to x from $\rho/2$ to x, then

$$\int_{\frac{\rho}{2}}^{x} coth\,x\,dx = \int_{\frac{\rho}{2}}^{x}\left\{2\left(e^{-2x}+e^{-4x}+e^{-6x}+e^{-8x}+\cdots\right)+1\right\}dx$$

$$= -\left[\frac{e^{-2x}}{1}+\frac{e^{-4x}}{2}+\frac{e^{-6x}}{3}+\frac{e^{-8x}}{4}+\cdots-\left(x-\frac{\rho}{2}\right)\right]_{\frac{\rho}{2}}^{x}$$

$$= -\left\{\frac{e^{-2x}}{1}+\frac{e^{-4x}}{2}+\frac{e^{-6x}}{3}+\frac{e^{-8x}}{4}+\cdots-\left(x-\frac{\rho}{2}\right)\right\}$$

$$+\frac{e^{-\rho}}{1}+\frac{e^{-2\rho}}{2}+\frac{e^{-3\rho}}{3}+\frac{e^{-4\rho}}{4}+\cdots$$

i.e.

$$\int_{\frac{\rho}{2}}^{x} coth\,x\,dx = -\frac{1}{2^0}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^1}+\left(x-\frac{\rho}{2}\right)+\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^1}$$

Here, the following equation holds,

$$\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^1} = \frac{\rho}{2} - log\,2 \qquad \text{(w)}$$

Because,

$$log(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k} \qquad -1\leqq x<1$$

From this,

$$\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k} = -log\left(1-e^{-2x}\right) \qquad x>0$$

On the other hand,

$$1 = sech\,x = \frac{e^x-e^{-x}}{2} = \frac{e^x\left(1-e^{-2x}\right)}{2}$$

From this,

$$\frac{1}{1-e^{-2x}} = \frac{e^x}{2} \Rightarrow log\frac{1}{1-e^{-2x}} = log\frac{e^x}{2} \Rightarrow -log\left(1-e^{-2x}\right) = x - log\,2$$

Then,

$$\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k} = -log\left(1-e^{-2x}\right) = x - log\,2$$

Substituting $x = sech^{-1}1 = \frac{\rho}{2}$ for this, we obtain (w).

Thus, using (w), we obtain

$$\int_{\frac{\rho}{2}}^{x} coth\,x\,dx = log\,sinh\,x = -\frac{1}{2^0}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^1}+\left(x-\frac{\rho}{2}\right)+\frac{\rho}{2} - log\,2$$

This is consistent with the real number part of the Fourier series of $log\,sinh\,x$.

Next, integrate both sides of this with respect to x from $\rho/2$ to x, then

$$\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x} \coth x\, dx^2 = \int_{\frac{\rho}{2}}^{x}\left\{-\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^1} + \left(x-\frac{\rho}{2}\right) + \left(\frac{\rho}{2} - \log 2\right)\right\} dx$$

$$= \left[\frac{1}{2^1}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^2} + \frac{1}{2!}\left(x-\frac{\rho}{2}\right)^2 + \frac{\rho/2-\log 2}{1!}\left(x-\frac{\rho}{2}\right)\right]_{\frac{\rho}{2}}^{x}$$

$$= \frac{1}{2^1}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^2} + \frac{1}{2!}\left(x-\frac{\rho}{2}\right)^2 + \frac{\rho/2-\log 2}{1!}\left(x-\frac{\rho}{2}\right) - \frac{1}{2^1}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^2}$$

Hereafter, in a similar way we obtein the desired expressions.

(1) Termwise Higher Definite Integral of Fourier Series of coth x

Substituting $x=0$ for Formula 5.4.2 , we obtain the following expressions.

$$\int_{\frac{\rho}{2}}^{0}\int_{\frac{\rho}{2}}^{x} \coth x\, dx^2 = \frac{\zeta(2)}{2^1} - \frac{1}{2!}\left(\frac{\rho}{2}\right)^2 + \frac{\log 2}{1!}\left(\frac{\rho}{2}\right) - \frac{1}{2^1 0!}\sum_{s=1}^{\infty}\frac{e^{-s\rho}}{s^2} \tag{2.2'}$$

$$\int_{\frac{\rho}{2}}^{0}\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x} \coth x\, dx^3 = -\frac{\zeta(3)}{2^2} + \frac{2}{3!}\left(\frac{\rho}{2}\right)^3 - \frac{\log 2}{2!}\left(\frac{\rho}{2}\right)^2 + \frac{1}{2^1 1!}\left(\frac{\rho}{2}\right)\sum_{s=1}^{\infty}\frac{e^{-s\rho}}{s^2} + \frac{1}{2^2 0!}\sum_{s=1}^{\infty}\frac{e^{-s\rho}}{s^3} \tag{2.3'}$$

$$\int_{\frac{\rho}{2}}^{0}\int_{\frac{\rho}{2}}^{x}\cdots\int_{\frac{\rho}{2}}^{x} \coth x\, dx^4 = \frac{\zeta(4)}{2^3} - \frac{3}{4!}\left(\frac{\rho}{2}\right)^4 + \frac{\log 2}{3!}\left(\frac{\rho}{2}\right)^3 - \frac{1}{2^1 2!}\left(\frac{\rho}{2}\right)^2\sum_{s=1}^{\infty}\frac{e^{-s\rho}}{s^2} - \frac{1}{2^2 1!}\left(\frac{\rho}{2}\right)\sum_{s=1}^{\infty}\frac{e^{-s\rho}}{s^3} - \frac{1}{2^3 0!}\sum_{s=1}^{\infty}\frac{e^{-s\rho}}{s^4} \tag{2.4'}$$

$\vdots$

$$\int_{\frac{\rho}{2}}^{0}\int_{\frac{\rho}{2}}^{x}\cdots\int_{\frac{\rho}{2}}^{x} \coth x\, dx^n = (-1)^n\left\{\frac{\zeta(n)}{2^{n-1}} - \frac{n-1}{n!}\left(\frac{\rho}{2}\right)^n + \frac{\log 2}{(n-1)!}\left(\frac{\rho}{2}\right)^{n-1}\right\} - (-1)^n\sum_{j=0}^{n-2}\frac{1}{2^{n-1-r}r!}\left(\frac{\rho}{2}\right)^r\sum_{s=1}^{\infty}\frac{e^{-s\rho}}{s^{n-r}} \tag{2.n'}$$

(2) Fourier Series of the 1st order Integral of coth x

The 1st order Integral of coth x is as follows.

$$\int_{\frac{\rho}{2}}^{x} \coth x\, dx = -\frac{1}{2^0}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^1} + \left(x-\frac{\rho}{2}\right) + \frac{\rho}{2} - \log 2 = \log\sinh x \qquad x>0$$

From this, the following expression follows immediately.

$$\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^1} = x - \log(2\sinh x) \qquad x>0 \tag{2.f}$$

Substituting $x=1\,,\ 1/2\,,\ \pi/2$ for this one by one, we obtain the following special values.

$$\sum_{k=1}^{\infty}\frac{e^{-2k}}{k^1} = 1 - \log(2\sinh 1)$$

$$\sum_{k=1}^{\infty}\frac{e^{-k}}{k^1} = \frac{1}{2} - \log\left(2\sinh\frac{1}{2}\right)$$

$$\sum_{k=1}^{\infty}\frac{e^{-k\pi}}{k^1} = \frac{\pi}{2} - \log\left(2\sinh\frac{\pi}{2}\right)$$

5.4.3 Riemann Zeta Function

Comparing Taylor series of coth x and Fourier series of coth x, we obtain Riemann Zeta Function.

Formula 5.4.3

When $\rho = 2\sinh^{-1}1\ (=1.762747174\cdots)$ and B_{2k} are Bernoulli Numbers, the following expressions hold.

$$\zeta(2) = \sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^2} - \left\{\frac{\rho}{1!}\left(\log\rho - \sum_{k=1}^{1}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{B_{2k}\,\rho^{2k+1}}{2k\,(2k+1)\,!} - \frac{1}{2}\frac{\rho^2}{2!}\right\}$$

$$\zeta(3) = \sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^3} + \frac{\rho^2}{2!}\left(\log\rho - \sum_{k=1}^{2}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{B_{2k}\,\rho^{2k+2}}{2k\,(2k+2)\,!} - \frac{1}{2}\frac{\rho^3}{3!}$$
$$+ \frac{\rho}{1!}\zeta(2)$$

$$\zeta(4) = \sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^4} - \left\{\frac{\rho^3}{3!}\left(\log\rho - \sum_{k=1}^{3}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{B_{2k}\,\rho^{2k+3}}{2k\,(2k+3)\,!} - \frac{1}{2}\frac{\rho^4}{4!}\right\}$$
$$+ \frac{\rho}{1!}\zeta(3) - \frac{\rho^2}{2!}\zeta(2)$$

$\vdots$

$$\zeta(n) = \sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^n} + \sum_{j=1}^{n-2}(-1)^{j-1}\frac{\rho^j}{j\,!}\zeta(n-j)$$
$$+ (-1)^{n-1}\left\{\frac{\rho^{n-1}}{(n-1)\,!}\left(\log\rho - \sum_{k=1}^{n-1}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{B_{2k}\,\rho^{2k+n-1}}{2k\,(2k+n-1)!} - \frac{1}{2}\frac{\rho^n}{n!}\right\}$$

Proof

From Fourier series (2.2') and Taylor series (1.2')

$$f^{<1>}\left(\frac{\rho}{2}\right) = -\frac{1}{2^1}\zeta(2) + \frac{1}{2!}\left(\frac{\rho}{2}\right)^2 - \frac{\log 2}{1!}\left(\frac{\rho}{2}\right) + \frac{1}{2^1}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^2} \qquad \text{(a)}$$

From this

$$\zeta(2) = -2f^{<1>}\left(\frac{\rho}{2}\right) + \frac{2}{2!}\left(\frac{\rho}{2}\right)^2 - \frac{\rho\log 2}{1!} + \sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^2}$$

$$= -2\left\{\frac{1}{1!}\left(\frac{\rho}{2}\right)^{1}\left(\log\frac{\rho}{2}-\sum_{k=1}^{1}\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{2k\,(2k+1)!}\left(\frac{\rho}{2}\right)^{2k+1}\right\}$$

$$+\frac{2}{2!}\left(\frac{\rho}{2}\right)^{2}-\frac{\rho\log 2}{1!}+\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^{2}}$$

$$= -\left\{\frac{\rho}{1!}\left(\log\rho-\sum_{k=1}^{1}\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{B_{2k}\,\rho^{2k+1}}{2k\,(2k+1)!}\right\}+\frac{1}{2}\frac{\rho^{2}}{2!}+\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^{2}}$$

Next, substitute (a) for Taylor series (1.3') , then

$$\int_{\frac{\rho}{2}}^{0}\int_{\frac{\rho}{2}}^{x}\int_{\frac{\rho}{2}}^{x}\coth x\,dx^{3} = -f^{<2>}\left(\frac{\rho}{2}\right) \tag{1.3'}$$

$$-\frac{1}{2^{1}}\left(\frac{\rho}{2}\right)\zeta(2)+\frac{1}{2!}\left(\frac{\rho}{2}\right)^{3}-\frac{\log 2}{1!}\left(\frac{\rho}{2}\right)^{2}+\frac{1}{2^{1}}\left(\frac{\rho}{2}\right)\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^{2}}$$

From this and Fourier series (2.3') , we obtain

$$f^{<2>}\left(\frac{\rho}{2}\right) = -\frac{1}{2^{1}}\left(\frac{\rho}{2}\right)\zeta(2)+\frac{1}{3!}\left(\frac{\rho}{2}\right)^{3}-\frac{\log 2}{2!}\left(\frac{\rho}{2}\right)^{2}$$

$$+\frac{1}{2^{2}}\zeta(3)-\frac{1}{2^{2}}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^{3}} \tag{b}$$

From this,

$$\zeta(3) = 2^{2}f^{<2>}\left(\frac{\rho}{2}\right)+2\left(\frac{\rho}{2}\right)\zeta(2)-\frac{2^{2}}{3!}\left(\frac{\rho}{2}\right)^{3}+\frac{\rho^{2}\log 2}{2!}+\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^{3}}$$

$$= 2^{2}\left\{\frac{1}{2!}\left(\frac{\rho}{2}\right)^{2}\left(\log\frac{\rho}{2}-\sum_{k=1}^{2}\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{2k\,(2k+2)!}\left(\frac{\rho}{2}\right)^{2k+2}\right\}$$

$$+\rho\zeta(2)-\frac{2^{2}}{3!}\left(\frac{\rho}{2}\right)^{3}+\frac{\rho^{2}\log 2}{2!}+\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^{3}}$$

$$= \frac{\rho^{2}}{2!}\left(\log\rho-\sum_{k=1}^{2}\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{\rho^{2k+2}B_{2k}}{2k\,(2k+2)!}-\frac{1}{2}\frac{\rho^{3}}{3!}+\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^{3}}+\rho\zeta(2)$$

Next, substitute (a), (b) for Taylor series (1.4') , then

$$\int_{\frac{\rho}{2}}^{0}\int_{\frac{\rho}{2}}^{x}\cdots\int_{\frac{\rho}{2}}^{x}\coth x\,dx^{4} = -f^{<3>}\left(\frac{\rho}{2}\right)+\frac{1}{1!}\left(\frac{\rho}{2}\right)^{1}f^{<2>}\left(\frac{\rho}{2}\right)$$

$$-\frac{1}{2!}\left(\frac{\rho}{2}\right)^{2}f^{<1>}\left(\frac{\rho}{2}\right) \tag{1.4')'}$$

$$= -f^{<3>}\left(\frac{\rho}{2}\right)-\frac{1}{2^{2}}\left(\frac{\rho}{2}\right)\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^{3}}-\frac{1}{2\cdot 2!}\left(\frac{\rho}{2}\right)^{2}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^{2}}$$

$$-\frac{1}{2\cdot 2!}\left(\frac{\rho}{2}\right)^{2}\zeta(2)+\frac{1}{2^{2}}\left(\frac{\rho}{2}\right)\zeta(3)-\frac{1}{2\cdot 3!}\left(\frac{\rho}{2}\right)^{4}$$

From this and Fourier series (2.4') , we obtain

$$f^{<3>}\left(\frac{\rho}{2}\right) = -\frac{1}{2\cdot 2!}\left(\frac{\rho}{2}\right)^2\zeta(2) + \frac{1}{2^2}\left(\frac{\rho}{2}\right)\zeta(3)$$

$$-\frac{1}{2^3}\zeta(4) + \frac{1}{4!}\left(\frac{\rho}{2}\right)^4 - \frac{\log 2}{3!}\left(\frac{\rho}{2}\right)^3 + \frac{1}{2^3}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^4} \quad \text{(c)}$$

From this,

$$\zeta(4) = -2^3\left\{\frac{1}{3!}\left(\frac{\rho}{2}\right)^3\left(\log\frac{\rho}{2} - \sum_{k=1}^{3}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{2k\,(2k+3)\,!}\left(\frac{\rho}{2}\right)^{2k+3}\right\}$$

$$-\frac{\log 2}{3!}\left(\frac{\rho}{2}\right)^3 + \frac{2^3}{4!}\left(\frac{\rho}{2}\right)^4$$

$$+\frac{2^3}{2^3}\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^4} - \frac{2^3}{2\cdot 2!}\left(\frac{\rho}{2}\right)^2\zeta(2) + \frac{2^3}{2^2}\left(\frac{\rho}{2}\right)\zeta(3)$$

$$= -\left\{\frac{\rho^3}{3!}\left(\log\rho - \sum_{k=1}^{3}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{B_{2k}\,\rho^{2k+3}}{2k\,(2k+3)\,!} - \frac{\rho^4}{2\cdot 4!}\right\}$$

$$+\sum_{k=1}^{\infty}\frac{e^{-k\rho}}{k^4} - \frac{\rho^2}{2!}\zeta(2) + \rho\zeta(3)$$

Hereafter, by induction we obtain $\zeta(n)$ for natural number $n>2$.

5.5 Termwise Higher Integral of csc x

Since the zero of the first order integral $log\,tan\,(x/2)$ of $csc\,x$ is $x=\pi/2$, we assume zeros of the second or more order integral are also $x=\pi/2$. Where, since the zero of $csc\,x$ is $\mp\infty i$, these seem to be a **collateral higher integral**.

5.5.0 Collateral Higher Integral of csc x

$$\int_{\frac{\pi}{2}}^{x} csc\,x\,dx = log\,tan\frac{x}{2} \qquad \left\{= \frac{1}{2}log\frac{1-cos\,x}{1+cos\,x}\right\}$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^2 = \text{non-elementary function}$$

5.5.1 Termwise Higher Integral of Taylor Series of csc x

Formula 5.5.1

When B_{2k} denote Bernoulli Numbers and functions $f^{<n>}(x)$ on $0<x<\pi$ are as follows

$$f^{<n>}(x) = \frac{x^n}{n!}\left(log\,x - \sum_{k=1}^{n}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k+n)!}x^{2k+n} \qquad n=1,2,3,\cdots$$

the following expressions hold for $x>0$.

$$\int_{\frac{\pi}{2}}^{x} csc\,x\,dx = f^{<0>}(x) - \frac{1}{0!}\left(x-\frac{\pi}{2}\right)^0 f^{<0>}\left(\frac{\pi}{2}\right)$$

$$= log\,x + \sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k)!}x^{2k} - log\,2$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^2 = f^{<1>}(x) - \frac{1}{0!}\left(x-\frac{\pi}{2}\right)^0 f^{<1>}\left(\frac{\pi}{2}\right) - \frac{1}{1!}\left(x-\frac{\pi}{2}\right)^1 f^{<0>}\left(\frac{\pi}{2}\right)$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^3 = f^{<2>}(x) - \frac{1}{0!}\left(x-\frac{\pi}{2}\right)^0 f^{<2>}\left(\frac{\pi}{2}\right) - \frac{1}{1!}\left(x-\frac{\pi}{2}\right)^1 f^{<1>}\left(\frac{\pi}{2}\right) - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 f^{<0>}\left(\frac{\pi}{2}\right)$$

$$\vdots$$

$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^n = f^{<n-1>}(x) - \sum_{k=0}^{n-1}\frac{1}{k!}\left(x-\frac{\pi}{2}\right)^k f^{<n-1-k>}\left(\frac{\pi}{2}\right)$$

Where $f^{<0>}\left(\frac{\pi}{2}\right) = log\,2$

Proof

$$\int_{\frac{\pi}{2}}^{x} csc\,x\,dx = f^{<0>}(x) - f^{<0>}\left(\frac{\pi}{2}\right) = \left[log\,x + \sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k)!}x^{2k}\right]_{\frac{\pi}{2}}^{x}$$

$$= \log x + \sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k)!} x^{2k} - \left\{ \log\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k)!} \left(\frac{\pi}{2}\right)^{2k} \right\}$$

$$= \log x + \sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k)!} x^{2k} - \left\{ \log\frac{\pi}{2} - \log\frac{\pi}{4} \right\}$$

$$\left\{ \because \quad \sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k)!} \left(\frac{\pi}{2}\right)^{2k} = -\log\frac{\pi}{4} \qquad \textbf{5.3.1 (2)} \right\}$$

$$= \log x + \sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k)!} x^{2k} - \log 2$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} \csc x\, dx^2 = \int_{\frac{\pi}{2}}^{x} \left\{ f^{<0>}(x) - f^{<0>}\left(\frac{\pi}{2}\right) \right\} dx$$

$$= \left[f^{<1>}(x) - \frac{1}{1!}\left(x-\frac{\pi}{2}\right)^{1} f^{<0>}\left(\frac{\pi}{2}\right) \right]_{\frac{\pi}{2}}^{x}$$

$$= f^{<1>}(x) - f^{<1>}\left(\frac{\pi}{2}\right) - \frac{1}{1!}\left(x-\frac{\pi}{2}\right)^{1} f^{<0>}\left(\frac{\pi}{2}\right)$$

Hereafter, in a similar way we obtain the desired expressions..

Although Formula 5.5.1 is a general expression anyway, if it is expand actually, it is very complicated. Then, this was devised and was made the same easy expression as sec x. It is as follows.

Formula 5.5.1'

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots$ are Euler Numbers,the following expressions hold for $0 < x < \pi$.

$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} \csc x\, dx^n = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k+n)!} \left(x-\frac{\pi}{2}\right)^{2k+n}$$

Proof

$$\csc x = \frac{1}{\sin x} = \frac{1}{\cos\left(x-\frac{\pi}{2}\right)} = \sec\left(x-\frac{\pi}{2}\right)$$

$$\sec x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} x^{2k} \qquad |x| < \frac{\pi}{2}$$

From these

$$\csc x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} \left(x-\frac{\pi}{2}\right)^{2k} \qquad 0 < x < \pi$$

Integrating both sides of this with respect to x from $\pi/2$ to x repeatedly, we obtain the desired expressions.

(1) Termwise Higher Definite Integral of Taylor Series of csc x

Substituting $x=0$ for Formula 5.5.1 , we obtain the following expressions.

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x} \csc x\, dx^2 = -\frac{1}{0!}\left(\frac{\pi}{2}\right)^0 f^{<1>}\left(\frac{\pi}{2}\right) + \frac{1}{1!}\left(\frac{\pi}{2}\right)^1 f^{<0>}\left(\frac{\pi}{2}\right) \tag{1.2'}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} \csc x\, dx^3 = -\frac{1}{0!}\left(\frac{\pi}{2}\right)^0 f^{<2>}\left(\frac{\pi}{2}\right) + \frac{1}{1!}\left(\frac{\pi}{2}\right)^1 f^{<1>}\left(\frac{\pi}{2}\right) - \frac{1}{2!}\left(\frac{\pi}{2}\right)^2 f^{<0>}\left(\frac{\pi}{2}\right) \tag{1.3'}$$

$$\vdots$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} \csc x\, dx^n = -\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<n-1-k>}\left(\frac{\pi}{2}\right) \tag{1.n'}$$

Where $f^{<0>}\left(\frac{\pi}{2}\right) = \log 2$

(2) Taylor Series of the 1st order Integral of csc x

The 1st order Integral of csc x is as follows.

$$\int_{\frac{\pi}{2}}^{x} \csc x\, dx = \log x + \sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k)!}x^{2k} - \log 2 = \log\tan\frac{x}{2} \qquad 0<x<\pi$$

From this, the following expression follows immediately.

$$\sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k)!}x^{2k} = \log\tan\frac{x}{2} - \log\frac{x}{2} \qquad 0<x<\pi \tag{1.t}$$

Substituting $x=1$ for this , we obtain the following special value.

$$\sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k)!} = \log\tan\frac{1}{2} + \log 2$$

5.5.2 Termwise Higher Integral of Fourier Series of csc x

Formula 5.5.2

Let $\beta(n) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^n}$ be Dirichlet Beta Function and $\downarrow$ be floor function. Then the following expressions hold for $0<x<\pi$.

$$\int_{\frac{\pi}{2}}^{x} \csc x\, dx = 2\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left\{(2k+1)x - \frac{1\pi}{2}\right\}$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} \csc x\, dx^2 = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\sin\left\{(2k+1)x - \frac{2\pi}{2}\right\} + \frac{2\beta(2)}{0!}\left(x-\frac{\pi}{2}\right)^0$$

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} \csc x\, dx^3 = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\sin\left\{(2k+1)x - \frac{3\pi}{2}\right\} + \frac{2\beta(2)}{1!}\left(x-\frac{\pi}{2}\right)^1$$

$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} \csc x\, dx^4 = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}\sin\left\{(2k+1)x-\frac{4\pi}{2}\right\}$$

$$+\frac{2\beta(2)}{2!}\left(x-\frac{\pi}{2}\right)^2-\frac{2\beta(4)}{0!}\left(x-\frac{\pi}{2}\right)^0$$

$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} \csc x\, dx^5 = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^5}\sin\left\{(2k+1)x-\frac{5\pi}{2}\right\}$$

$$+\frac{2\beta(2)}{3!}\left(x-\frac{\pi}{2}\right)^3-\frac{2\beta(4)}{1!}\left(x-\frac{\pi}{2}\right)^1$$

$$\vdots$$

$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} \csc x\, dx^n = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^n}\sin\left\{(2k+1)x-\frac{n\pi}{2}\right\}$$

$$+\sum_{k=1}^{n/2\downarrow}(-1)^{k-1}\frac{2\beta(2k)}{(n-2k)!}\left(x-\frac{\pi}{2}\right)^{n-2k}$$

Proof

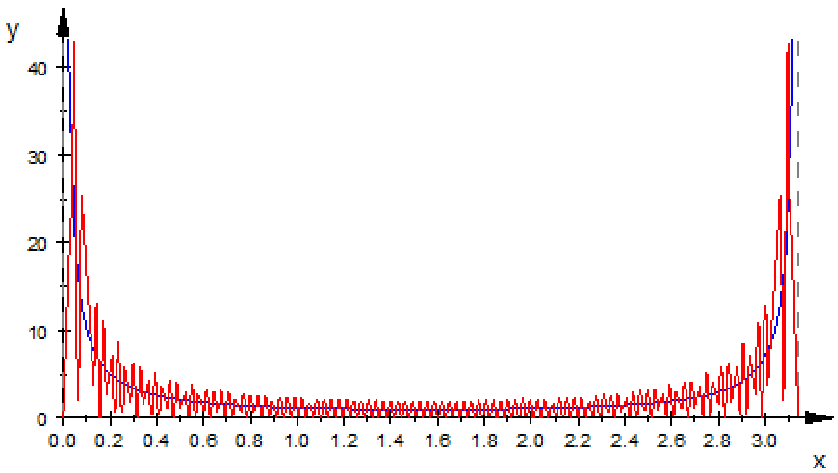
csc x can be expanded to Fourier series in a broad meaning.as follows.

$$\csc x = \frac{2i}{e^{ix}-e^{-ix}} = -\frac{2ie^{ix}}{1-e^{2ix}} = -2ie^{ix}\left(1+e^{2ix}+e^{4ix}+e^{6ix}+\cdots\right)$$

$$= -2i\left(e^{ix}+e^{3ix}+e^{5ix}+e^{7ix}+\cdots\right)$$

$$= 2\left(\sin x+\sin 3x+\sin 5x++\cdots\right)-2i\left(\cos x+\cos 3x+\cos 5x+\cdots\right) \tag{2.0}$$

csc x is the locus of the median (central line) of this real number part. (See the figure)



However, calculus of the right side of (2.0) may be carried out, the real number does not turn into an imaginary number and the contrary does not exist, either. Therefore, limiting the range of csc x and the higher order integral to the real number, we calculate only the real number part of the right side of (2.0). Although Dirichlet Odd Beta is obtained from calculation of the imaginary number part, since it is long, it is omitted in this section

Now, integrate the real number part of both sides of (2.0) with respect to x from $\pi/2$ to x. Then

$$\int_{\frac{\pi}{2}}^{x} csc\,x\,dx = \int_{\frac{\pi}{2}}^{x} 2(sin\,x + sin\,3x + sin\,5x + sin\,7x + \cdots)dx$$

$$= -2\left[\frac{cos\,1x}{1} + \frac{cos\,3x}{3} + \frac{cos\,5x}{5} + \frac{cos\,7x}{7} + \cdots \right]_{\frac{\pi}{2}}^{x}$$

$$= -2\left(\frac{cos\,1x}{1} + \frac{cos\,3x}{3} + \frac{cos\,5x}{5} + \frac{cos\,8x}{7} + \cdots \right)$$

i.e.

$$\int_{\frac{\pi}{2}}^{x} csc\,x\,dx = -2\sum_{k=0}^{\infty} \frac{cos\{(2k+1)x\}}{2k+1}$$

Here, let $-cos\,x = sin\left(x - \frac{\pi}{2}\right)$. Then using this we obtain

$$\int_{\frac{\pi}{2}}^{x} csc\,x\,dx = 2\sum_{k=0}^{\infty} \frac{1}{2k+1} sin\left\{ (2k+1)x - \frac{1\pi}{2} \right\}$$

This is consistent with the real number part of the Fourier series of $log\,tan(x/2)$.

Next, integrate both sides of this with respect to x from $\pi/2$ to x, then

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^2 = \int_{\frac{\pi}{2}}^{x} \left\{ 2\sum_{k=0}^{\infty} \frac{1}{2k+1} sin\left\{ (2k+1)x - \frac{1\pi}{2} \right\} \right\} dx$$

$$= \left[2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} sin\left\{ (2k+1)x - \frac{2\pi}{2} \right\} \right]_{\frac{\pi}{2}}^{x}$$

$$= 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} sin\left\{ (2k+1)x - \frac{2\pi}{2} \right\} + 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

Here, let $\beta(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n}$, then we obtain

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^2 = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} sin\left\{ (2k+1)x - \frac{2\pi}{2} \right\} + 2\beta(2)$$

Next, integrate both sides of this with respect to x from $\pi/2$ to x, then

$$\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^3 = \int_{\frac{\pi}{2}}^{x} \left\{ 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} sin\left\{ (2k+1)x - \frac{2\pi}{2} \right\} + 2\beta(2) \right\} dx$$

$$= \left[2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} sin\left\{ (2k+1)x - \frac{3\pi}{2} \right\} + 2\beta(2)\left(x - \frac{\pi}{2}\right) \right]_{\frac{\pi}{2}}^{x}$$

$$= 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} sin\left\{ (2k+1)x - \frac{3\pi}{2} \right\} + 2\beta(2)\left(x - \frac{\pi}{2}\right)$$

Hereafter, in a similar way we obtain the desired expression..

(1) Termwise Higher Definite Integral of Fourier Series of csc x

Substituting $x=0$ for Formula 5.5.2 , we obtain the following expressions.

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^2 = \frac{2\beta(2)}{0!}\left(\frac{\pi}{2}\right)^0 \tag{2.2'}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^3 = -\frac{2\beta(2)}{1!}\left(\frac{\pi}{2}\right)^1 + \frac{2^3-1}{2^2}\zeta(3) \tag{2.3'}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^4 = \frac{2\beta(2)}{2!}\left(\frac{\pi}{2}\right)^2 - \frac{2\beta(4)}{0!}\left(\frac{\pi}{2}\right)^0 \tag{2.4'}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^5 = -\frac{2\beta(2)}{3!}\left(\frac{\pi}{2}\right)^3 + \frac{2\beta(4)}{1!}\left(\frac{\pi}{2}\right)^1 - \frac{2^5-1}{2^4}\zeta(5) \tag{2.5'}$$

$$\vdots$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^n = (-1)^n\sum_{k=1}^{n/2\downarrow}(-1)^{k-1}\frac{2\beta(2k)}{(n-2k)!}\left(\frac{\pi}{2}\right)^{n-2k} - sin\frac{n\pi}{2}\cdot\frac{2^n-1}{2^{n-1}}\zeta(n) \tag{2.n'}$$

Proof

Since

$$sin\left\{(2k+1)\cdot 0 - \frac{n\pi}{2}\right\} = 0 \qquad for\ \ n=2,4,6,\cdots$$

$$sin\left\{(2k+1)\cdot 0 - \frac{n\pi}{2}\right\} = \pm 1 \qquad for\ \ n=3,5,7,\cdots\ ,$$

giving $x=0$ to Formula 5.5.2 and substituting these for them,

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^2 = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}sin\left\{(2k+1)\cdot 0 - \frac{2\pi}{2}\right\} + 2\beta(2) = 2\beta(2)$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^3 = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}sin\left\{(2k+1)\cdot 0 - \frac{3\pi}{2}\right\} + \frac{2\beta(2)}{1!}\left(0-\frac{\pi}{2}\right)^1$$

$$= 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} - \frac{2\beta(2)}{1!}\left(\frac{\pi}{2}\right)^1 = \frac{2^3-1}{2^2}\zeta(3) - \frac{2\beta(2)}{1!}\left(\frac{\pi}{2}\right)^1$$

Hereafter, in a similar way we obtain the desired expressions..

(2) Fourier Series of the 1st order Integral of csc x

The 1st order Integral of csc x is as follows.

$$\int_{\frac{\pi}{2}}^{x} csc\,x\,dx = -2\sum_{k=0}^{\infty}\frac{cos\{(2k+1)x\}}{2k+1} = log\,tan\frac{x}{2} \qquad 0<x<\pi$$

From this, the following expression follows immediately.

$$\sum_{k=0}^{\infty}\frac{\cos\{(2k+1)x\}}{2k+1} = -\frac{1}{2}\log\tan\frac{x}{2} \qquad 0<x<\pi \tag{2.f}$$

Substituting $x=1$ for this, we obtain the following special values.

$$\sum_{k=0}^{\infty}\frac{\cos(2k+1)}{2k+1} = -\frac{1}{2}\log\tan\frac{1}{2}$$

5.5.3 Dirichlet Even Beta

Comparing Taylor series of csc x and Fourier series of csc x, we obtain Dirichlet Even Beta. Although the general expression by the Euler Number is known about Dirichlet Odd Beta, the general expression of Dirichlet Even Beta is not known. The following formula shows this by a little complicated Bernouilli series.

Formula 5.5.3

When B_{2k} , $\beta(x)$, $\zeta(x)$ denote Bernoulli Numbers, Dirichlet Beta Function, Riemann Zeta Function respectively, and functions $f^{<n>}(x)$ on $0<x<\pi$ are as follows

$$f^{<n>}(x)=\frac{x^n}{n!}\left(\log x-\sum_{k=1}^{n}\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k(2k+n)!}x^{2k+n} \qquad n=1,2,3,\cdots,$$

the following expressions hold.

$$\beta(2) = -\frac{1}{2}\sum_{k=0}^{1}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<1-k>}\left(\frac{\pi}{2}\right) = \frac{1}{2}\left\{\frac{\pi\log 2}{2} - f^{<1>}\left(\frac{\pi}{2}\right)\right\}$$

$$\beta(2) = \frac{1}{2}\sum_{k=0}^{2}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^{k-1} f^{<2-k>}\left(\frac{\pi}{2}\right) + \frac{2^3-1}{2^2}\frac{\zeta(3)}{\pi}$$

$$\beta(4) = \frac{1}{2}\sum_{k=0}^{3}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<3-k>}\left(\frac{\pi}{2}\right) + \frac{\beta(2)}{2!}\left(\frac{\pi}{2}\right)^2$$

$$\beta(4) = -\frac{1}{2}\sum_{k=0}^{4}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^{k-1} f^{<4-k>}\left(\frac{\pi}{2}\right) + \frac{\beta(2)}{3!}\left(\frac{\pi}{2}\right)^2 + \frac{2^5-1}{2^4}\frac{\zeta(5)}{\pi}$$

$$\beta(6) = -\frac{1}{2}\sum_{k=0}^{5}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<5-k>}\left(\frac{\pi}{2}\right) + \frac{\beta(4)}{2!}\left(\frac{\pi}{2}\right)^2 - \frac{\beta(2)}{4!}\left(\frac{\pi}{2}\right)^4$$

$$\beta(6) = \frac{1}{2}\sum_{k=0}^{6}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^{k-1} f^{<6-k>}\left(\frac{\pi}{2}\right) + \frac{\beta(4)}{3!}\left(\frac{\pi}{2}\right)^2 - \frac{\beta(2)}{5!}\left(\frac{\pi}{2}\right)^4 + \frac{2^7-1}{2^6}\frac{\zeta(7)}{\pi}$$

$$\vdots$$

$$\beta(2n) = \frac{(-1)^n}{2}\sum_{k=0}^{2n-1}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<2n-1-k>}\left(\frac{\pi}{2}\right) - \sum_{k=1}^{n-1}(-1)^k\frac{\beta(2n-2k)}{(2k)!}\left(\frac{\pi}{2}\right)^{2k}$$

$$\beta(2n) = -\frac{(-1)^n}{2}\sum_{k=0}^{2n}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^{k-1} f^{<2n-k>}\left(\frac{\pi}{2}\right) - \sum_{k=1}^{n-1}(-1)^k\frac{\beta(2n-2k)}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k} + \frac{2^{2n+1}-1}{2^{2n}}\frac{\zeta(2n+1)}{\pi}$$

Proof

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\csc x\,dx^2 = -\frac{1}{0!}\left(\frac{\pi}{2}\right)^0 f^{<1>}\left(\frac{\pi}{2}\right) + \frac{1}{1!}\left(\frac{\pi}{2}\right)^1 f^{<0>}\left(\frac{\pi}{2}\right) \tag{1.2'}$$

$$= -\sum_{k=0}^{1}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<1-k>}\left(\frac{\pi}{2}\right)$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^2 = \frac{2\beta(2)}{0!} \tag{2.2'}$$

$$\therefore\quad \beta(2) = -\frac{1}{2}\sum_{k=0}^{1}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<1-k>}\left(\frac{\pi}{2}\right) = \frac{1}{2}\left\{\frac{\pi\log 2}{2} - f^{<1>}\left(\frac{\pi}{2}\right)\right\}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^3 = -\sum_{k=0}^{3-1}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<3-1-k>}\left(\frac{\pi}{2}\right) \tag{1.3'}$$

$$\int_{\frac{\pi}{2}}^{0}\int_{\frac{\pi}{2}}^{x}\int_{\frac{\pi}{2}}^{x} csc\,x\,dx^3 = \frac{2^3-1}{2^2}\zeta(3) - \frac{2\beta(2)}{1!}\left(\frac{\pi}{2}\right)^1 \tag{2.3'}$$

$$\therefore\quad \beta(2) = \frac{1}{2}\sum_{k=0}^{3-1}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^{k-1} f^{<3-1-k>}\left(\frac{\pi}{2}\right) + \frac{2^3-1}{2^2}\frac{\zeta(3)}{\pi}$$

Hereafter, in a similar way we obtain the desired expressions.

5.5.4 Riemann Odd Zeta

Riemann Odd Zeta can be conversely obtained from the half of Formula 5.5.3 . The following formula shows the relation between Riemann Odd Zeta and Dirichlet Even Beta by a little complicated expression.

Formula 5.5.4

When B_{2k} , $\beta(x), \zeta(x)$ denote Bernoulli Numbers, Dirichlet Beta Function, Riemann Zeta Function respectively, and functions $f^{<n>}(x)$ on $0 < x < \pi$ are as follows

$$f^{<n>}(x) = \frac{x^n}{n!}\left(\log x - \sum_{k=1}^{n}\frac{1}{k}\right) + \sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k+n)!}x^{2k+n} \qquad n=1,2,3,\cdots,$$

the following expressions hold.

$$\zeta(3) = \frac{2^3}{2^3-1}\frac{\beta(2)}{1!}\left(\frac{\pi}{2}\right)^1 - \frac{2^2}{2^3-1}\sum_{k=0}^{2}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<2-k>}\left(\frac{\pi}{2}\right)$$

$$\zeta(5) = \frac{2^5}{2^5-1}\left\{\frac{\beta(4)}{1!}\left(\frac{\pi}{2}\right)^1 - \frac{\beta(2)}{3!}\left(\frac{\pi}{2}\right)^3\right\} + \frac{2^4}{2^5-1}\sum_{k=0}^{4}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<4-k>}\left(\frac{\pi}{2}\right)$$

$$\zeta(7) = \frac{2^7}{2^7-1}\left\{\frac{\beta(6)}{1!}\left(\frac{\pi}{2}\right)^1 - \frac{\beta(4)}{3!}\left(\frac{\pi}{2}\right)^3 + \frac{\beta(2)}{5!}\left(\frac{\pi}{2}\right)^5\right\} - \frac{2^6}{2^7-1}\sum_{k=0}^{6}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<6-k>}\left(\frac{\pi}{2}\right)$$

$$\vdots$$

$$\zeta(2n+1) = \frac{2^{2n+1}}{2^{2n+1}-1}\sum_{k=0}^{n-1}(-1)^k\frac{\beta(2n-2k)}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}$$

$$+(-1)^n\frac{2^{2n}}{2^{2n+1}-1}\sum_{k=0}^{2n}\frac{(-1)^k}{k!}\left(\frac{\pi}{2}\right)^k f^{<2n-k>}\left(\frac{\pi}{2}\right)$$

5.6 Termwise Higher Integral of csch x

Since both zeros of $csch\,x$ and the primitive function $\log\tanh(x/2)$ are $x=\pm\infty$, the latter seems to be a lineal primitive function with a fixed lower limit. On the other hand, the Fourier series of $csch\,x$ is known for $x>0$. Then, we assume zeros of the second or more order primitive function are also $x=\infty$.

5.6.0 Higher Integral of csch x

$$\int_{\infty}^{x} csch\,x\,dx \quad = \log\tanh\frac{x}{2}$$

$$\int_{\infty}^{x}\int_{\infty}^{x} csch\,x\,dx^2 = \text{ non-elementary function}$$

5.6.1 Termwise Higher Integral of Taylor Series of csch x

The Taylor series of $csch\,x$ is known as follows.

$$csch\,x = \frac{1}{x} - \sum_{k=1}^{\infty}\frac{\left(2^{2k}-2\right)B_{2k}}{2k\,(2k-1)\,!}\,x^{2k-1} \qquad 0<x<\pi$$

However, this can not be integrated with the lower limit $x=\infty$.

5.6.2 Termwise Higher Integral of Fourier Series of csch x

Formula 5.6.2

The following expressions hold for $x>0$.

$$\int_{\infty}^{x} csch\,x\,dx \quad = -2\sum_{k=0}^{\infty}\frac{e^{-(2k+1)x}}{2k+1}$$

$$\int_{\infty}^{x}\int_{\infty}^{x} csch\,x\,dx^2 \quad = \quad 2\sum_{k=0}^{\infty}\frac{e^{-(2k+1)x}}{(2k+1)^2}$$

$$\int_{\infty}^{x}\int_{\infty}^{x}\int_{\infty}^{x} csch\,x\,dx^3 = -2\sum_{k=0}^{\infty}\frac{e^{-(2k+1)x}}{(2k+1)^3}$$

$$\vdots$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} csch\,x\,dx^n = (-1)^n 2\sum_{k=0}^{\infty}\frac{e^{-(2k+1)x}}{(2k+1)^n}$$

Proof

csch x can be expanded to a Fourier series as follows. (See the figure of the following page.)

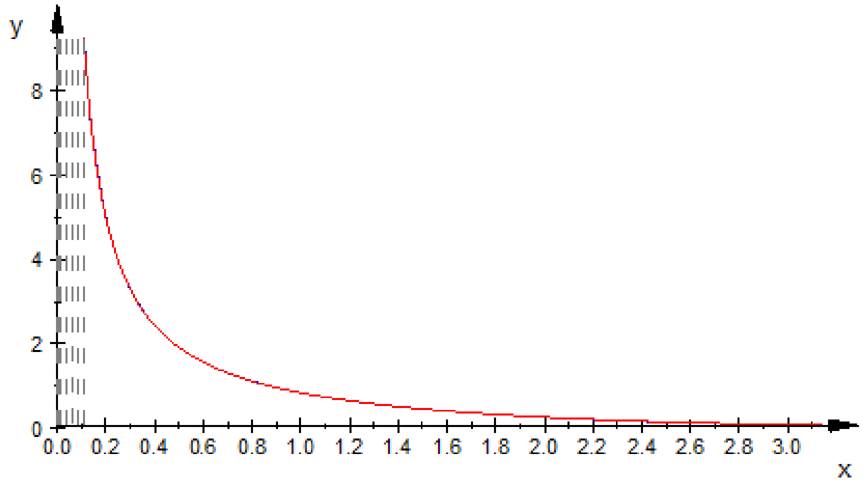
$$csch\,x = \frac{2}{e^{x}-e^{-x}} = \frac{2e^{-x}}{1-e^{-2x}} = 2e^{-x}\left(1+e^{-2x}+e^{-4x}+e^{-6x}+\cdots\right)$$

$$= 2\left(e^{-1x}+e^{-3x}+e^{-5x}+e^{-7x}+\cdots\right) \tag{2.0}$$

$$= 2(\cosh 1x+\cosh 3x+\cosh 5x+\cdots) - 2(\sinh 1x+\sinh 3x+\sinh 5x+\cdots)$$

$$= 2(\cosh ix+\cosh 3ix+\cosh 5ix+\cdots) - 2i(\sinh ix+\sinh 3ix+\sinh 5ix+\cdots)$$

$$\left(\because\quad \cosh x = \cos ix\,,\qquad \sinh x = -i\sin ix\right)$$



Integrate the both sides of (2.0) with respect to x from ∞ to x, then

$$\int_{\infty}^{x} csch\, x\, dx = \int_{\infty}^{x} 2\left(e^{-1x} + e^{-3x} + e^{-5x} + e^{-7x} + \cdots\right) dx$$

$$= -2\left[\frac{e^{-x}}{1} + \frac{e^{-3x}}{3} + \frac{e^{-5x}}{5} + \frac{e^{-7x}}{7} + \cdots \right]_{\infty}^{x}$$

i.e.

$$\int_{\infty}^{x} csch\, x\, dx = -2 \sum_{k=1}^{\infty} \frac{e^{-(2k+1)x}}{2k+1}$$

This is consistent with the real number part of the Fourier series of $\log \tanh \frac{x}{2}$.

Next, integrate both sides of this with respect to x from ∞ to x, then

$$\int_{\infty}^{x}\int_{\infty}^{x} csch\, x\, dx^2 = \int_{\infty}^{x} \left\{ -2 \sum_{k=0}^{\infty} \frac{e^{-(2k+1)x}}{2k+1} \right\} dx = \left[2 \sum_{k=0}^{\infty} \frac{e^{-(2k+1)x}}{(2k+1)^2} \right]_{\infty}^{x}$$

i.e.

$$\int_{\infty}^{x}\int_{\infty}^{x} csch\, x\, dx^2 = 2 \sum_{k=0}^{\infty} \frac{e^{-(2k+1)x}}{(2k+1)^2}$$

Hereafter, in a similar way we obtein the desired expression.

5.6.3 Dirichlet Lambda Function

Substituting $x=0$ for Formula 5.6.2 , we obtain Dirichlet Lambda Function immediately.

Formula 5.6.3

When $\lambda(n) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n}$, the following expressions hold.

$$\int_{\infty}^{0}\int_{\infty}^{x} csch\, x\, dx^2 = 2\lambda(2)$$

$$\int_{\infty}^{0}\int_{\infty}^{x}\int_{\infty}^{x} csch\, x\, dx^3 = -2\lambda(3)$$

$$\vdots$$

$$\int_{\infty}^{0}\int_{\infty}^{x}\cdots\int_{\infty}^{x} csch\, x\, dx^n = (-1)^n 2\lambda(n)$$

5.6.4 Riemann Zeta Function

Applying $\zeta(n) = \dfrac{2^n}{2^n - 1}\lambda(n)$ for Formula 5.6.3 , we obtain Riemann Zeta Function.

Formula 5.6.4

$$\zeta(n) = (-1)^n \frac{2^{n-1}}{2^n - 1}\int_{\infty}^{0}\int_{\infty}^{x}\cdots\int_{\infty}^{x} csch\, x\, dx^n$$

5.7 Termwise Higher Integral of sec x

Since the zero of the first order integral $\frac{1}{2}log\frac{1+sin x}{1-sin x}$ of $sec x$ is $x=0$, we assume the zeros of the second or more order integral are also $x=0$. Where, since the zero of $sec x$ is $\mp\infty i$, these seem to be a **collateral higher integral**.

5.7.0 Collateral Higher Integral of sec x

$$\int_0^x sec x\,dx = \frac{1}{2}log\frac{1+sin x}{1-sin x}$$

$$\int_0^x\int_0^x sec x\,dx^2 = \text{non-elementary function}$$

5.7.1 Termwise Higher Integral of Taylor Series of sec x

Formula 5.7.1

When $E_0=1, E_2=-1, E_4=5, E_6=-61, E_8=1385, \cdots$ are Euler Numbers, the following expressions hold for $|x|<\pi/2$.

$$\int_0^x sec x\,dx = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+1)!}x^{2k+1}$$

$$\int_0^x\int_0^x sec x\,dx^2 = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+2)!}x^{2k+2}$$

$$\int_0^x\int_0^x\int_0^x sec x\,dx^3 = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+3)!}x^{2k+3}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x sec x\,dx^n = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+n)!}x^{2k+n}$$

Proof

$sec x$ can be expanded to Taylor series as follows.

$$sec x = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k)!}x^{2k} \qquad |x|<\frac{\pi}{2}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

(1) Termwise Higher Definite Integral of Taylor Series of sec x

Substituting $x=\pi/2$ for Formula 5.7.1 , we obtain the following expressions.

$$\int_0^{\frac{\pi}{2}}\int_0^x sec x\,dx^2 = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+2)!}\left(\frac{\pi}{2}\right)^{2k+2} \tag{1.2'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\int_0^x sec x\,dx^3 = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+3)!}\left(\frac{\pi}{2}\right)^{2k+3} \tag{1.3'}$$

$$\vdots$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\cdots\int_0^x \sec x\, dx^n = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+n)!}\left(\frac{\pi}{2}\right)^{2k+n} \tag{1.n'}$$

(2) Taylor Series of the 1st order Integral of sec x

The 1st order Integral of sec x is as follows.

$$\int_0^x \sec x\, dx = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+1)!}x^{2k+1} = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} \qquad |x|<\frac{\pi}{2}$$

From this, the following expression follows immediately.

$$\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+1)!}x^{2k+1} = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} \qquad |x|<\frac{\pi}{2} \tag{1,t}$$

Substituting $x=1$ for this , we obtain the following special value.

$$\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+1)!} = \frac{1}{2}\log\frac{1+\sin 1}{1-\sin 1}$$

Reference : Euler Number , Tangent number, Bernoulli Number

Euler Number is defined as follows.

$$\operatorname{sech} x = \sum_{k=0}^{\infty}\frac{E_k}{k!}x^k \qquad |x| < \frac{\pi}{2}$$

As a result, $E_{2n-1} = 0 \quad n=1,2,3,\cdots$.

When Euler numbers , Tangent numbers and Bernoulli Numbers (all non-zero) are listed, it is as follows.

$$E_0=1,\ E_2=-1,\ E_4=5,\quad E_6=-61,\quad E_8=1385,\ E_{10}=-50521,\cdots$$

$$T_1=1,\ T_3=2,\ T_5=16,\quad T_7=272,\quad T_9=7936,\ T_{11}=353792,\cdots$$

$$B_0=1,\ B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\quad B_6=\frac{1}{42},\quad B_8=-\frac{1}{30},\ B_{10}=\frac{5}{66},\cdots$$

According to Mr. Sugimoto (http://homepage3.nift/y_sugi/gf/gf17.htm) , the following equations holds between Euler Numbers and Tangent Numbers.

$$E_{2n} = \sum_{k=1}^{n}(-1)^k\binom{2n}{2k-1}T_{2k-1} + 1 \qquad n=1,2,3,\cdots \tag{1}$$

On the other hand, between Tangent numbers and Bernoulli Numbers, the following equations hold.

$$T_{2n-1} = (-1)^{n-1}\frac{2^{2n}\left(2^{2n}-1\right)}{2n}B_{2n} \qquad n=1,2,3,\cdots \tag{2}$$

Therefore, the following equations have to hold between Euler Numbers and Bernoulli Numbers.

$$E_{2n} = 1 - \sum_{k=1}^{n}\frac{2^{2k}\left(2^{2k}-1\right)}{2k}B_{2k}\binom{2n}{2k-1} \qquad n=1,2,3,\cdots \tag{3}$$

In fact, for example,

$$E_6 = 1 - \left\{\frac{2^2\left(2^2-1\right)}{2}B_2\binom{2\cdot 3}{1} + \frac{2^4\left(2^4-1\right)}{4}B_4\binom{2\cdot 3}{3} + \frac{2^6\left(2^6-1\right)}{6}B_6\binom{2\cdot 3}{5}\right\}$$

$$= 1 - \left\{\frac{4\cdot 3}{2}\frac{1}{6}\binom{6}{1} - \frac{16\cdot 15}{4}\frac{1}{30}\binom{6}{3} + \frac{64\cdot 63}{6}\frac{1}{42}\binom{6}{5}\right\} = -61$$

On the contrary, the following equations between Bernoulli Numbers and Euler Numbers are shown by Mr. Nishimura. (http://www1.ocn.ne.jp/~oboetene/zeta.pdf).

$$B_{2n} = \frac{2n}{2^{2n}\left(2^{2n}-1\right)} \sum_{k=0}^{n-1} \binom{2n-1}{2k} E_{2k} \qquad n=1,2,3,\cdots \tag{4}$$

The following example shows the formula is right.

$$B_6 = \frac{2\cdot 3}{2^{2\cdot 3}\left(2^{2\cdot 3}-1\right)} \left\{ \binom{2\cdot 3-1}{0} E_0 + \binom{2\cdot 3-1}{2} E_2 + \binom{2\cdot 3-1}{4} E_4 \right\}$$

$$= \frac{2\cdot 3}{64\cdot 63} \left\{ \binom{5}{0} 1 - \binom{5}{2} 1 + \binom{5}{4} 5 \right\} = \frac{1}{42}$$

In addition, Mr. Nishimura expressed Even Zeta with the Euler numbers using this equation.

5.7.2 Termwise Higher Integral of Fourier Series of sec x

Formula 5.7.2

When E_{2k} , $\beta(n)$, $\downarrow$ denote Euler Numbers, Dirichlet Beta Function, floor function respectively, the following expressions hold for $|x| < \pi/2$.

$$\int_0^x sec\, x\, dx = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} cos\left\{ (2k+1)x - \frac{1\pi}{2} \right\}$$

$$\int_0^x\int_0^x sec\, x\, dx^2 = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} cos\left\{ (2k+1)x - \frac{2\pi}{2} \right\} + \frac{2\beta(2)}{0!} x^0$$

$$\int_0^x\int_0^x\int_0^x sec\, x\, dx^3 = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} cos\left\{ (2k+1)x - \frac{3\pi}{2} \right\} + \frac{2\beta(2)}{1!} x^1$$

$$\int_0^x\cdots\int_0^x sec\, x\, dx^4 = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^4} cos\left\{ (2k+1)x - \frac{4\pi}{2} \right\} + \frac{2\beta(2)}{2!} x^2 - \frac{2\beta(4)}{0!} x^0$$

$$\int_0^x\cdots\int_0^x sec\, x\, dx^5 = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^5} cos\left\{ (2k+1)x - \frac{5\pi}{2} \right\} + \frac{2\beta(2)}{3!} x^3 - \frac{2\beta(4)}{1!} x^1$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x sec\, x\, dx^n = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n} cos\left\{ (2k+1)x - \frac{n\pi}{2} \right\} + \sum_{k=1}^{n/2\downarrow} (-1)^{k-1} \frac{2\beta(2k)}{(n-2k)!} x^{n-2k}$$

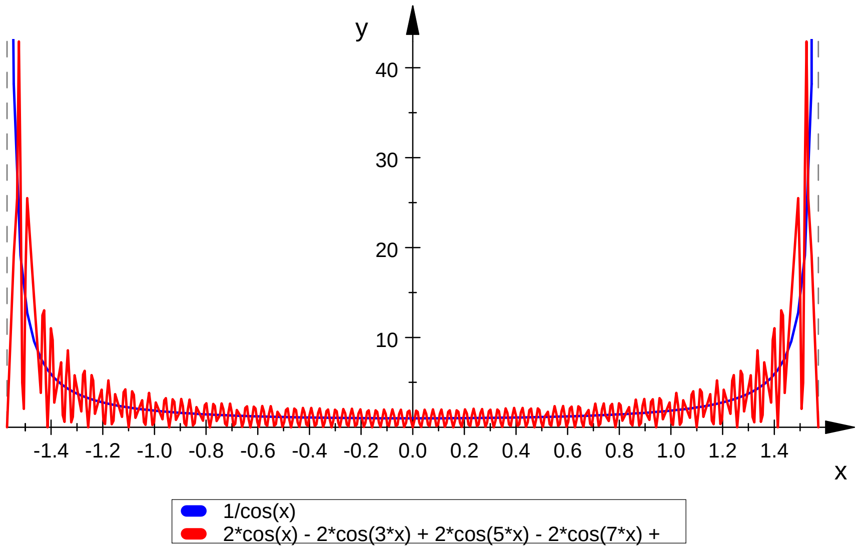
Proof

sec x can be expanded to Fourier series in a broad meaning.as follows.

$$sec x = \frac{2}{e^{ix}+e^{-ix}} = \frac{2e^{ix}}{1+e^{2ix}} = 2\left(e^{ix} - e^{3ix} + e^{5ix} - e^{7ix} +- \cdots \right)$$

$$= 2(cos x - cos 3x + cos 5x - cos 7x +- \cdots) + 2i(sin x - sin 3x + sin 5x - sin 7x +- \cdots) \tag{2.0}$$

sec x is the locus of the median (central line) of this real number part. (See the figure)



However calculus of the right side of (2.0) may be carried out, the real number does not turn into an imaginary number and the contrary does not exist, either. Therefore, limiting the range of sec x and the higher order integral to the real number, we calculate only the real number part of the right side of (2.0) .

Although the Dirichlet Odd Beta is obtained from calculation of the imaginary number part, since it is long, it is omitted in this section.

Now, integrate the real number part of both sides of (2.0) with respect to x from 0 to x, then

$$\int_0^x \sec x\,dx = \int_0^x 2(\cos x - \cos 3x + \cos 5x - \cos 7x + - \cdots)dx$$

$$= 2\left[\frac{\sin 1x}{1} - \frac{\sin 3x}{2} + \frac{\sin 5x}{3} - \frac{\sin 8x}{4} + - \cdots\right]_0^x$$

$$= 2\left(\frac{\sin 1x}{1} - \frac{\sin 3x}{2} + \frac{\sin 5x}{3} - \frac{\sin 8x}{4} + - \cdots\right)$$

i.e.

$$\int_0^x \sec x\,dx = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\sin\{(2k+1)x\}$$

Here, since $\sin x = \cos\left(x - \frac{\pi}{2}\right)$, we obtain

$$\int_0^x \sec x\,dx = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left\{(2k+1)x - \frac{1\pi}{2}\right\}$$

This is consistent with the real number part of the Fourier series of $\frac{1}{2}\log\frac{1+\sin x}{1-\sin x}$.

Next, integrate both sides of this with respect to x from 0 to x , then

$$\int_0^x\int_0^x \sec x\,dx^2 = \int_0^x\left\{2\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left\{(2k+1)x - \frac{1\pi}{2}\right\}\right\}dx$$

$$= \left[2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\cos\left\{(2k+1)x - \frac{2\pi}{2}\right\}\right]_0^x$$

$$= 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}cos\left\{(2k+1)x-\frac{2\pi}{2}\right\} - 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}cos(-\pi)$$

Here, let $\beta(n)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^n}$, then we obtain

$$\int_0^x\int_0^x sec\,x\,dx^2 = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}cos\left\{(2k+1)x-\frac{2\pi}{2}\right\}+\frac{2\beta(2)}{0!}x^0$$

Next, integrate both sides of this with respect to x from 0 to x, then

$$\int_0^x\int_0^x\int_0^x sec\,x\,dx^3 = \int_0^x\left\{2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}cos\left\{(2k+1)x-\frac{2\pi}{2}\right\}+\frac{2\beta(2)}{0!}x^0\right\}dx$$

$$= \left[2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}cos\left\{(2k+1)x-\frac{3\pi}{2}\right\}+\frac{2\beta(2)}{1!}x^1\right]_0^x$$

$$= 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}cos\left\{(2k+1)x-\frac{3\pi}{2}\right\}+\frac{2\beta(2)}{1!}x^1$$

Hereafter, in a similar way , we obtain the desired expressions.

(1) Termwise Higher Definite Integral of Fourier Series of sec x

Substituting $x=\pi/2$ for Formula 5.7.2 , we obtain the following expressions.

$$\int_0^{\frac{\pi}{2}}\int_0^x sec\,x\,dx^2 = \frac{2\beta(2)}{0!}\left(\frac{\pi}{2}\right)^0 \tag{2.2'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\int_0^x sec\,x\,dx^3 = \frac{2\beta(2)}{1!}\left(\frac{\pi}{2}\right)^1 - \frac{2^3-1}{2^2}\zeta(3) \tag{2.3'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\cdots\int_0^x sec\,x\,dx^4 = \frac{2\beta(2)}{2!}\left(\frac{\pi}{2}\right)^2 - \frac{2\beta(4)}{0!}\left(\frac{\pi}{2}\right)^0 \tag{2.4'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\cdots\int_0^x sec\,x\,dx^5 = \frac{2\beta(2)}{3!}\left(\frac{\pi}{2}\right)^3 - \frac{2\beta(4)}{1!}\left(\frac{\pi}{2}\right)^1 + \frac{2^5-1}{2^4}\zeta(5) \tag{2.5'}$$

$$\vdots$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\cdots\int_0^x sec\,x\,dx^n = \sum_{k=1}^{n/2\downarrow}(-1)^{k-1}\frac{2\beta(2k)}{(n-2k)!}\left(\frac{\pi}{2}\right)^{n-2k} + sin\frac{n\pi}{2}\cdot\frac{2^n-1}{2^{n-1}}\zeta(n) \tag{2.n'}$$

Proof

Since

$$cos\left\{(2k+1)\cdot\frac{\pi}{2}-\frac{n\pi}{2}\right\} = 0 \qquad for\quad n=2,4,6,\cdots$$

$$cos\left\{(2k+1)\cdot\frac{\pi}{2}-\frac{n\pi}{2}\right\} = \pm(-1)^{k+1} \qquad for\quad \begin{matrix}n=1,3,5,\cdots\\ k=0,1,2,\dots\end{matrix}$$,

giving $x=\pi/2$ to Formula 5.7.2 and substituting these for them,

$$\int_0^{\frac{\pi}{2}}\int_0^x \sec x\, dx^2 = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\cos\left\{(2k+1)\cdot\frac{\pi}{2}-\frac{2\pi}{2}\right\} + 2\beta(2) = 2\beta(2)$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\int_0^x \sec x\, dx^3 = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}\cos\left\{(2k+1)\cdot\frac{\pi}{2}-\frac{3\pi}{2}\right\} + \frac{2\beta(2)}{1!}\left(\frac{\pi}{2}\right)^1$$

$$= 2\sum_{k=0}^{\infty}\frac{(-1)^k(-1)^{k+1}}{(2k+1)^3} + \frac{2\beta(2)}{1!}\left(\frac{\pi}{2}\right)^1$$

$$= -\frac{2^3-1}{2^2}\zeta(3) + \frac{2\beta(2)}{1!}\left(\frac{\pi}{2}\right)^1$$

Hereafter, in a similar way we obtein the desired expressions.

(2) Fourier Series of the 1st order Integral of sec x

The 1st order Integral of sec x is as follows.

$$\int_0^x \sec x\, dx = 2\sum_{k=0}^{\infty}(-1)^k\frac{\sin\{(2k+1)x\}}{2k+1} = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} \qquad |x|<\frac{\pi}{2}$$

From this, the following expression follows immediately.

$$\sum_{k=0}^{\infty}(-1)^k\frac{\sin\{(2k+1)x\}}{2k+1} = \frac{1}{4}\log\frac{1+\sin x}{1-\sin x} \qquad |x|<\frac{\pi}{2} \qquad (2.f)$$

Substituting $x=1$ for this, we obtain the following special value.

$$\sum_{k=0}^{\infty}(-1)^k\frac{\sin(2k+1)}{2k+1} = \frac{1}{4}\log\frac{1+\sin 1}{1-\sin 1}$$

5.7.3 Dirichlet Even Beta

Comparing Taylor series of sec x and Fourier series of sec x, we obtain Dirichlet Even Beta. Although the general expression by the Euler Number is known about Dirichlet Odd Beta, the general expression of Dirichlet Even Beta is not known. The following formula shows this by the easy Euler series.

Formula 5.7.3

When E_{2k} , $\beta(x)$, $\zeta(x)$ denote Euler Numbers, Dirichlet Beta Function, Riemann Zeta Function respectively, the following expressions hold.

$$\beta(2) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+2)!}\left(\frac{\pi}{2}\right)^{2k+2}$$

$$\beta(2) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+3)!}\left(\frac{\pi}{2}\right)^{2k+2} + \frac{2^3-1}{2^2}\frac{\zeta(3)}{\pi}$$

$$\beta(4) = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+4)!}\left(\frac{\pi}{2}\right)^{2k+4} + \frac{\beta(2)}{2!}\left(\frac{\pi}{2}\right)^2$$

$$\beta(4) = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+5)!}\left(\frac{\pi}{2}\right)^{2k+4} + \frac{\beta(2)}{3!}\left(\frac{\pi}{2}\right)^2 + \frac{2^5-1}{2^4}\frac{\zeta(5)}{\pi}$$

$$\beta(6) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+6)!}\left(\frac{\pi}{2}\right)^{2k+6} + \frac{\beta(4)}{2!}\left(\frac{\pi}{2}\right)^2 - \frac{\beta(2)}{4!}\left(\frac{\pi}{2}\right)^4$$

$$\beta(6) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+7)!}\left(\frac{\pi}{2}\right)^{2k+6} + \frac{\beta(4)}{3!}\left(\frac{\pi}{2}\right)^{2} - \frac{\beta(2)}{5!}\left(\frac{\pi}{2}\right)^{4} + \frac{2^7-1}{2^6}\frac{\zeta(7)}{\pi}$$

⋮

$$\beta(2n) = \frac{(-1)^{n-1}}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+2n)!}\left(\frac{\pi}{2}\right)^{2k+2n} + \sum_{k=1}^{n}(-1)^{k-1}\frac{\beta(2n-2k)}{(2k)!}\left(\frac{\pi}{2}\right)^{2k}$$

$$\beta(2n) = \frac{(-1)^{n-1}}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+2n+1)!}\left(\frac{\pi}{2}\right)^{2k+2n+1} + \sum_{k=1}^{n}(-1)^{k-1}\frac{\beta(2n-2k)}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k} + \frac{2^{2n+1}-1}{2^{2n}}\frac{\zeta(2n+1)}{\pi}$$

Proof

$$\int_0^{\frac{\pi}{2}}\int_0^x sec\,x\,dx^2 = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+2)!}\left(\frac{\pi}{2}\right)^{2k+2} \tag{1.2'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x sec\,x\,dx^2 = \frac{2\beta(2)}{0!} \tag{2.2'}$$

$$\therefore\ \beta(2) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+2)!}\left(\frac{\pi}{2}\right)^{2k+2}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\int_0^x sec\,x\,dx^3 = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+3)!}\left(\frac{\pi}{2}\right)^{2k+3} \tag{1.3'}$$

$$\int_0^{\frac{\pi}{2}}\int_0^x\int_0^x sec\,x\,dx^3 = -\frac{2^3-1}{2^2}\zeta(3) + \frac{2\beta(2)}{1!}\left(\frac{\pi}{2}\right)^{1} \tag{2.3'}$$

$$\therefore\ \beta(2) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+3)!}\left(\frac{\pi}{2}\right)^{2k+2} + \frac{2^3-1}{2^2}\frac{\zeta(3)}{\pi}$$

Hereafter, in a similar way we obtein the desired expressions.

c.f.

Mr.Sugioka expressed Dirichlet Even Beta by definite Reamann Odd Zeta and infinite Reiamnn Even Zeta using his " Taylor System ". $\beta(4), \beta(6)$ which he calculated are as follows.

$$\beta(4) = \frac{2^2-1}{2^2}\zeta(3)\frac{1}{1!}\left(\frac{\pi}{2}\right)^{1} - \frac{\pi^3}{2^2}\left\{\frac{log\,2}{2^1\cdot 3!} - \frac{2^2-1}{2^2}\zeta(2)\frac{1!}{2^2\,5!} - \frac{2^4-1}{2^4}\zeta(4)\frac{3!}{2^4\,7!} - \frac{2^6-1}{2^6}\zeta(6)\frac{5!}{2^6\,9!} - \cdots\right\}$$

$$\beta(6) = \frac{2^4-1}{2^4}\zeta(5)\frac{1}{1!}\left(\frac{\pi}{2}\right)^{1} - \frac{2^2-1}{2^2}\zeta(3)\frac{1}{3!}\left(\frac{\pi}{2}\right)^{3} + \frac{\pi^5}{2^4}\left\{\frac{log\,2}{2^1\cdot 5!} - \frac{2^2-1}{2^2}\zeta(2)\frac{1!}{2^2\,7!} - \frac{2^4-1}{2^4}\zeta(4)\frac{3!}{2^4\,9!} - \frac{2^6-1}{2^6}\zeta(6)\frac{5!}{2^6\,11!} - \cdots\right\}$$

5.7.4 Riemann Odd Zeta

Riemann Odd Zeta can be conversely obtained from the half of Formula 5.7.3 . The following formula shows the relation between Riemann Odd Zeta and Dirichlet Even Beta by the easy expression.

Formula 5.7.4

When E_{2k} , $\beta(x), \zeta(x)$ denote Euler Numbers, Dirichlet Beta Function, Riemann Zeta Function respectively, the following expressions hold.

$$\zeta(3) = \frac{2^3}{2^3-1}\frac{\beta(2)}{1!}\left(\frac{\pi}{2}\right)^1 - \frac{2^2}{2^3-1}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+3)!}\left(\frac{\pi}{2}\right)^{2k+3}$$

$$\zeta(5) = \frac{2^5}{2^5-1}\left\{\frac{\beta(4)}{1!}\left(\frac{\pi}{2}\right)^1 - \frac{\beta(2)}{3!}\left(\frac{\pi}{2}\right)^3\right\} + \frac{2^4}{2^5-1}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+5)!}\left(\frac{\pi}{2}\right)^{2k+5}$$

$$\zeta(7) = \frac{2^7}{2^7-1}\left\{\frac{\beta(6)}{1!}\left(\frac{\pi}{2}\right)^1 - \frac{\beta(4)}{3!}\left(\frac{\pi}{2}\right)^3 + \frac{\beta(2)}{5!}\left(\frac{\pi}{2}\right)^5\right\} - \frac{2^6}{2^7-1}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+7)!}\left(\frac{\pi}{2}\right)^{2k+7}$$

$$\vdots$$

$$\zeta(2n+1) = \frac{2^{2n+1}}{2^{2n+1}-1}\sum_{k=0}^{n-1}(-1)^k\frac{\beta(2n-2k)}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1} + (-1)^n\frac{2^{2n}}{2^{2n+1}-1}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+2n+1)!}\left(\frac{\pi}{2}\right)^{2k+2n+1}$$

5.8 Termwise Higher Integral of sech x

Since both zeros of $sech\,x$ and the primitive function $2tan^{-1}(e^x)-\pi$ are $x=\infty$, the latter seems to be a lineal primitive function with a fixed lower limit. Then, we assume zeros of the second or more order primitive function are also $x=\infty$.

5.8.0 Higher Integral of sech x

$$\int_{\infty}^{x} sech\,x\,dx = 2tan^{-1}(e^x) - \pi$$

$$\int_{\infty}^{x}\int_{\infty}^{x} sech\,x\,dx^2 = \text{non-elementary function}$$

5.8.1 Termwise Higher Integral of Taylor Series of sech x

When E_{2k} denote Euler Numbers, the Taylor series of $sech\,x$ is known as follows.

$$sech\,x = \sum_{k=0}^{\infty}\frac{E_{2k}}{(2k)!}x^{2k} \qquad |x|<\frac{\pi}{2}$$

However, this can not be integrated with the lower limit $x=\infty$.

5.8.2 Termwise Higher Integral of Fourier Series of sech x

Formula 5.8.2

The following expressions hold for $x>0$.

$$\int_{\infty}^{x} sech\,x\,dx = -2\sum_{k=0}^{\infty}(-1)^k\frac{e^{-(2k+1)x}}{2k+1}$$

$$\int_{\infty}^{x}\int_{\infty}^{x} sech\,x\,dx^2 = 2\sum_{k=0}^{\infty}(-1)^k\frac{e^{-(2k+1)x}}{(2k+1)^2}$$

$$\int_{\infty}^{x}\int_{\infty}^{x}\int_{\infty}^{x} sech\,x\,dx^3 = -2\sum_{k=0}^{\infty}(-1)^k\frac{e^{-(2k+1)x}}{(2k+1)^3}$$

$$\vdots$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} sech\,x\,dx^n = (-1)^n 2\sum_{k=0}^{\infty}(-1)^k\frac{e^{-(2k+1)x}}{(2k+1)^n}$$

Proof

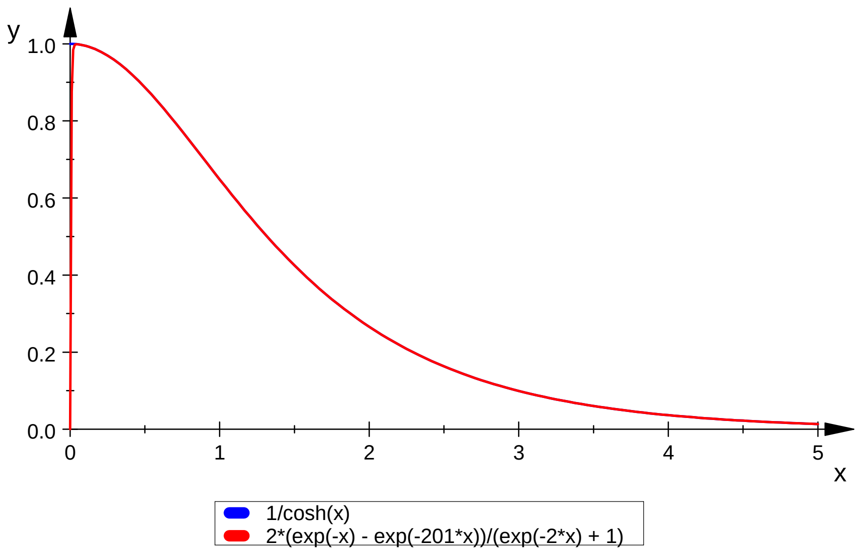
sech x can be expanded to a Fourier series as follows. (See the figure of the following page.)

$$sech\,x = \frac{2}{e^x+e^{-x}} = \frac{2e^{-x}}{1+e^{-2x}} = 2e^{-x}\left(1-e^{-2x}+e^{-4x}-e^{-6x}+-\cdots\right)$$

$$= 2\left(e^{-1x}-e^{-3x}+e^{-5x}-e^{-7x}+-\cdots\right) \tag{2.0}$$

$$= 2(cos\,1ix - cos\,3ix + cos\,5ix - cos\,7ix +- \cdots)$$

$$+2i(sin\,1ix - sin\,3ix + sin\,5ix - sin\,7ix +- \cdots)$$



Integrate the both sides of (2.0) with respect to x from ∞ to x, then

$$\int_\infty^x \operatorname{sech} x\,dx = \int_\infty^x 2\left(e^{-x} - e^{-3x} + e^{-5x} - e^{-7x} + - \cdots\right)dx$$

$$= -2\left[\frac{e^{-x}}{1} - \frac{e^{-3x}}{3} + \frac{e^{-5x}}{5} - \frac{e^{-7x}}{7} + - \cdots\right]_0^x$$

i.e.

$$\int_\infty^x \operatorname{sech} x\,dx = -2\sum_{k=0}^{\infty}(-1)^k \frac{e^{-(2k+1)x}}{2k+1}$$

This is consistent with the real number part of the Fourier series of $2\tan^{-1}\left(e^x\right) - \pi$.

Next, integrate both sides of this with respect to x from ∞ to x, then

$$\int_\infty^x\int_\infty^x \operatorname{sech} x\,dx^2 = \left[2\sum_{k=0}^{\infty}(-1)^k \frac{e^{-(2k+1)x}}{(2k+1)^2}\right]_\infty^x = 2\sum_{k=0}^{\infty}(-1)^k \frac{e^{-(2k+1)x}}{(2k+1)^2}$$

Hereafter, in a similar way we obtein the desired expressions.

5.8.3 Dirichlet Beta Function

Substituting $x=0$ for Formula 5.8.2 , we obtain Dirichlet Beta Function immediately.

Formula 5.8.3

When $\beta(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n}$ is the Dirichlet Beta Function, the following expressions hold.

$$\int_\infty^0 \operatorname{sech} x\,dx = -2\beta(1)$$

$$\int_\infty^0\int_\infty^x \operatorname{sech} x\,dx^2 = 2\beta(2)$$

$$\int_\infty^0\int_\infty^x\int_\infty^x \operatorname{sech} x\,dx^3 = -2\beta(3)$$

$$\vdots$$

$$\int_{\infty}^{0}\int_{\infty}^{x}\cdots\int_{\infty}^{x} sech\, x\, dx^n = (-1)^n 2\beta(n)$$

Note

$$\beta(1) = \frac{1}{1^1} - \frac{1}{3^1} + \frac{1}{5^1} - \frac{1}{7^1} + - \cdots = \frac{\pi}{4}$$ **: Madhava series**

$$\beta(2) = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + - \cdots = 0.9159655941\cdots$$ **: Catalan's constant**

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Alien's Mathematics