06 Termwise Higher Integral (Inv-Trigonometric, Inv-Hyperbolic)

The 2nd or more order integrals of Inverse Trigonometric Functions and Inverse Hyperbolic Functions were all shown in **4.4** and **4.5** . However, as for these, the integration lower limits and the function forms were complicated. On the contrary, integrating term by term the Taylor series of these functions, we can obtain the simple general forms. Naturally, many of these are the collateral higher integrals. However, it cannot be said that the lineal higher integral is useful and the collateral one is useless. Simple is the best ! A simple infinite series is superior to the complicated definite terms function.

6.1 Termwise Higher Integral of arctan x

Formula 6.1.1

The following expression holds for $|x| < 1$.

$$\int_0^x \cdots \int_0^x tan^{-1}x\, dx^n = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k+n+1)!} x^{2k+n+1}$$

Proof

$$tan^{-1}x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k+1)!} x^{2k+1}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

The lineal higher integral of this was as follows as shown in Formula 4.4.1 (**4.4**).

$$\int_0^x \cdots \int_0^x tan^{-1}x\, dx^n = \frac{tan^{-1}x}{n!} \sum_{k=0}^{n/2\downarrow} (-1)^k {}_nC_{n-2k}\, x^{n-2k} + \frac{log(1+x^2)}{2 \cdot n!} \sum_{k=1}^{n/2\uparrow} (-1)^k {}_nC_{n+1-2k}\, x^{n+1-2k} - \frac{1}{n!} \sum_{r=1}^{n/2\downarrow} (-1)^r {}_nC_{n+1-2r} \{ \psi(1+n) - \psi(2r) \} x^{n+1-2r}$$

Therefore, the termwise higher integral above is a **lineal higher integral**. And matching both, we obtain the following formula.

Formula 6.1.1'

The following expression holds for $|x| \leqq 1$.

$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k+n+1)!} x^{2k+n} = \frac{tan^{-1}x}{n!} \sum_{k=0}^{n/2\downarrow} (-1)^k {}_nC_{n-2k}\, x^{n-2k} + \frac{log(1+x^2)}{2 \cdot n!} \sum_{k=1}^{n/2\uparrow} (-1)^k {}_nC_{n+1-2k}\, x^{n+1-2k} - \frac{1}{n!} \sum_{r=1}^{n/2\downarrow} (-1)^r {}_nC_{n+1-2r} \{ \psi(1+n) - \psi(2r) \} x^{n+1-2r}$$

Substituting $x=1$ for this, we obtain the following special values.

$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k+2)!} = \frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} - \frac{1}{7 \cdot 8} + - \cdots = \frac{\pi}{4} - \frac{log\,2}{2}$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k+3)!} = \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} - \frac{1}{7 \cdot 8 \cdot 9} + - \cdots = \frac{1}{2} - \frac{log2}{2}$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2k+4)!}=\frac{1}{1\cdot 2\cdot 3\cdot 4}-\frac{1}{3\cdot 4\cdot 5\cdot 6}+-\cdots=\frac{5}{12}-\frac{\pi}{12}-\frac{\log 2}{6}$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2k+5)!}=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{1}{3\cdot 4\cdot 5\cdot 6\cdot 7}+-\cdots=\frac{5}{36}-\frac{\pi}{24}$$

$$\vdots$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2k+n+1)!}=\frac{\pi}{4n!}\sum_{k=0}^{n/2\downarrow}(-1)^k{}_nC_{n-2k}+\frac{\log 2}{2\cdot n!}\sum_{k=1}^{n/2\uparrow}(-1)^k{}_nC_{n+1-2k}$$
$$-\frac{1}{n!}\sum_{r=1}^{n/2\downarrow}(-1)^r{}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\} \quad (1.n')$$

Note

The following formula is known about these series.

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)(2k+2)\cdots(2k+m)}=\frac{1}{(m-1)!}\int_0^1\frac{(1-t)^{m-1}}{1+t^2}dt$$

This is easily drawn from Formula 6.1.1 using Riemann-Liouville Integral.
And the following expression is also drawn easily..

$$\int_0^1\frac{(1-x)^n}{1+x^2}dx=\frac{\pi}{4}\sum_{k=0}^{n/2\downarrow}(-1)^k{}_nC_{n-2k}+\frac{\log 2}{2}\sum_{k=1}^{n/2\uparrow}(-1)^k{}_nC_{n+1-2k}$$
$$-\sum_{r=1}^{n/2\downarrow}(-1)^r{}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}$$

Higher order Gregory Series

In the special values, a noteworthy series is the 4th. i.e.

$$\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2k+5)!}=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{1}{3\cdot 4\cdot 5\cdot 6\cdot 7}+-\cdots=\frac{5}{36}-\frac{\pi}{24}$$

Remembering that the 0th series was Gregory Series

$$\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2k+1)!}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-\cdots=\frac{\pi}{4}$$

we can find out such a series occurs at four cycles. Then, let us derive such a series.

First, from (1.n')

$$\frac{\pi}{4n!}\sum_{k=0}^{n/2\downarrow}(-1)^k{}_nC_{n-2k}-\frac{1}{n!}\sum_{r=1}^{n/2\downarrow}(-1)^r{}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}$$
$$+\frac{\log 2}{2\cdot n!}\sum_{k=1}^{n/2\uparrow}(-1)^k{}_nC_{n+1-2k}=\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2k+n+1)!}$$

Especially, when n is the multiple of 4, since $\sum_{k=1}^{n/2\uparrow}(-1)^{k-1}{}_nC_{n+1-2k}=0$

$$\frac{\pi}{4n!}\sum_{k=0}^{n/2\downarrow}(-1)^k{}_nC_{n-2k}-\frac{1}{n!}\sum_{r=1}^{n/2\downarrow}(-1)^r{}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}$$
$$=\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2k+n+1)!}$$

Replacing n with 4n, we obtain the following equation.

$$\frac{\pi}{4(4n)!}\sum_{k=0}^{2n}(-1)^k{}_{4n}C_{4n-2k}-\frac{1}{(4n)!}\sum_{r=1}^{2n}(-1)^r{}_{4n}C_{4n+1-2r}\{\psi(1+4n)-\psi(2r)\}$$
$$=\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2k+4n+1)!}$$

This is the one having to call *Higher order Gregory Series* . When n=2,3, it is as followes.

$$\frac{\pi}{10080}-\frac{109}{325800}=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}-\frac{1}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}+-\cdots$$

$$\frac{87217}{829870272000}-\frac{\pi}{29937600}=\frac{1}{1\cdot 2\cdot \cdots \cdot 12\cdot 13}-\frac{1}{3\cdot 4\cdot \cdots \cdot 14\cdot 15}+-\cdots$$

6.2 Termwise Higher Integral of arccot x

Formula 6.2.1

The following expression holds for $0 < x \leqq 1$.

$$\int_0^x \cdots \int_0^x \cot^{-1} x \, dx^n = \frac{\pi}{2} \frac{x^n}{n!} - \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k+n+1)!} x^{2k+n+1}$$

Proof

$$\cot^{-1} x = \frac{\pi}{2} - \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1} = \frac{\pi}{2} - \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(2k+1)!} x^{2k+1}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

The lineal higher integral of this was as follows as shown in Formula 4.4.1 (**4.4**).

$$\int_0^x \cdots \int_0^x \cot^{-1} x \, dx^n = \frac{x^n}{n!} \cot^{-1} x - \frac{\tan^{-1} x}{n!} \sum_{k=1}^{n/2\downarrow} (-1)^k {}_nC_{n-2k} x^{n-2k}$$

$$- \frac{\log\left(1+x^2\right)}{2 \cdot n!} \sum_{k=1}^{n/2\uparrow} (-1)^k {}_nC_{n+1-2k} x^{n+1-2k}$$

$$+ \frac{1}{n!} \sum_{r=1}^{n/2\downarrow} (-1)^r {}_nC_{n+1-2r} \{ \psi(1+n) - \psi(2r) \} x^{n+1-2r}$$

Therefore, the termwise higher integral above is a **lineal higher integral**.

6.3 Termwise Higher Integral of arcsin x

Formula 6.3.1

The following expression holds for $|x| < 1$.

$$\int_0^x \cdots \int_0^x \sin^{-1}x \, dx^n = \sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+n+1)!} x^{2k+n+1}$$ **Collateral**

Proof

$$\sin^{-1}x = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!\ (2k+1)} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(2k-1)!!\,(2k)!}{(2k)!!\ (2k+1)!} x^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+1)!} x^{2k+1} \quad \{\because \ (2k)! = (2k)!! \cdot (2k-1)!!\}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

The collateral higher integral of this was as follows as shown in Formula 4.4.2' (**4.4**).

$$\int_0^x \cdots \int_0^x \sin^{-1}x \, dx^n = \sum_{r=0}^{n/2\downarrow} \frac{x^{n-2r}}{(2r)!!^2 (n-2r)!} \sin^{-1}x$$

$$+ \sum_{r=1}^{n/2\uparrow} \sum_{s=0}^{n-2r+1} (-1)^s {}_{n-2r+1}C_s \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r+1}}{(n-2r+1)!} \sqrt{1-x^2}$$

$$+ \sum_{r=1}^{n/2\uparrow} \frac{x^{n-2r+1}}{(2r-1)!!^2 (n-2r+1)!}$$

Therefore, the termwise higher integral above is a **collateral higher integral**.
And matching both, we obtain the following formula.

Formula 6.3.1'

The following expression holds for $|x| \leqq 1$.

$$\sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+n+1)!} x^{2k+n+1} = \sum_{r=0}^{n/2\downarrow} \frac{x^{n-2r}}{(2r)!!^2 (n-2r)!} \sin^{-1}x$$

$$+ \sum_{r=1}^{n/2\uparrow} \sum_{s=0}^{n-2r+1} (-1)^s {}_{n-2r+1}C_s \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r+1}}{(n-2r+1)!} \sqrt{1-x^2}$$

$$+ \sum_{r=1}^{n/2\uparrow} \frac{x^{n-2r+1}}{(2r-1)!!^2 (n-2r+1)!}$$

Substituting $x=1$ for this, we obtain the following special values.

$$\sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+2)!} = \frac{\pi}{2} - 1$$

$$\sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+3)!} = \frac{3\pi}{8} - 1$$

$$\sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+4)!} = \frac{5\pi}{24} - \frac{11}{18}$$

⋮

$$\sum_{k=0}^{\infty}\frac{\{(2k-1)!!\}^2}{(2k+n+1)!} = \frac{{}_{2n-1}C_{n-1}}{(2n)!!}\pi - \sum_{k=1}^{n/2\uparrow}\frac{1}{(2k-1)!!^2(n-2k+1)!} \qquad (1.n')$$

Calculation of the circular constant by the double factorial series

We can calculate a circular constant using (1.n'). That is

$$\pi = \left\{ \sum_{j=0}^{\infty}\frac{(2j-1)!!^2}{(2j+n+1)!} + \sum_{k=1}^{n/2\uparrow}\frac{1}{(2k-1)!!^2(n-2k+1)!} \right\} \frac{(2n)!!}{{}_{2n-1}C_{n-1}}$$

Calculating formula

When cn , $c=0.85\sim1.00$ denotes the number of significant figures,

$$\pi(cn) = \left[\frac{1}{(n+1)!} + \sum_{k=1}^{n/2\uparrow}\left\{ \frac{(2k-1)!!^2}{(2k+n+1)!} + \frac{1}{(2k-1)!!^2(n-2k+1)!} \right\} \right] \frac{(2n)!!}{{}_{2n-1}C_{n-1}}$$

In this formula, the number of terms required for the number of significant figures cn is about $n/2$. And we can assume $c=1$ when large n is taken.

In the following example, supposing $n=1{,}700$ *and* $c=0.9$, we obtained the significant figure of 1,530 digits. And the number of terms required for this was 851.

Example $\pi(1530)$

```
• n:=1700: DIGITS:=floor(0.9*n): MAXDEPTH:=1000:
• f := sum((2*k-1)!!^2/(2*k+n+1)!+1/((2*k-1)!!^2*(n-2*k+1)!)
                                                , k=1..ceil(n/2)):
• pi := (1/(n+1)!+f)*(2*n)!!/binomial(2*n-1,n-1):
• float(pi)
```

```
3.14159265358979323846264338327950288419716939937510582097494459230781640628620\
89986280348253421170679821480865132823066470938446095505822317253594081284811174\
50284102701938521105559644622948954930381964428810975665933446128475648233786783\
16527120190914564856692346034861045432664821339360726024914127372458700660631558\
81748815209209628292540917153643678925903600113305305488204665213841469519415116\
09433057270365759591953092186117381932611793105118548074462379962749567351885752\
72489122793818301194912983367336244065664308602139494639522473719070217986094370\
27705392171762931767523846748184676694051320005681271452635608277857713427577896\
09173637178721468440901224953430146549585371050792279689258923542019956112129021\
96086403441815981362977477130996051870721134999999837297804995105973173281609631\
85950244594553469083026425223082533446850352619311881710100031378387528865875332\
08381420617177669147303598253490428755468731159562863882353787593751957781857780\
53217122680661300192787661119590921642019893809525720106548586327886593615338182\
79682303019520353018529689957736225994138912497217752834791315155748572424541506\
95950829533116861727855889075098381754637464939319255060400927701671139009848824\
01285836160356370766010471018194295559619894676783744944825537977472684710404753\
46462080466842590694912933136770289891521047521620569660240580381501935112533824\
30035587640247496473263914199272604269922796782354781636009341721641219924586315\
03028618297455570674983850549458858692699569092721079750930295532116534498720275\
596023648067
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6.4 Termwise Higher Integral of arccos x

Formula 6.4.1

The following expression holds for $|x| < 1$.

$$\int_0^x \cdots \int_0^x \cos^{-1} x \, dx^n = \frac{\pi}{2}\frac{x^n}{n!} - \sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+n+1)!} x^{2k+n+1}$$ **Collateral**

Proof

$$\cos^{-1} x = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!\ (2k+1)} x^{2k+1} = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k-1)!!\,(2k)!}{(2k)!!\ (2k+1)!} x^{2k+1}$$

$$= \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+1)!} x^{2k+1} \quad \{\because \quad (2k)! = (2k)!! \cdot (2k-1)!!\}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

The collateral higher integral of this was as follows as shown in Formula 4.4.2' (**4.4**).

$$\int_0^x \cdots \int_0^x \cos^{-1} x \, dx^n = \frac{x^n}{n!} \cos^{-1} x - \sum_{r=1}^{n/2\downarrow} \frac{x^{n-2r}}{(2r)!!^2 (n-2r)!} \sin^{-1} x$$

$$- \sum_{r=1}^{n/2\uparrow} \sum_{s=0}^{n-2r+1} (-1)^s {}_{n-2r+1}C_s \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r+1}}{(n-2r+1)!} \sqrt{1-x^2}$$

$$+ \sum_{r=1}^{n/2\uparrow} \frac{x^{n-2r+1}}{(2r-1)!!^2 (n-2r+1)!}$$

Therefore, the termwise higher integral above is a **collateral higher integral**.

6.5 Termwise Higher Integral of arctanh x

Formula 6.5.1

The following expression holds for $|x|<1$.

$$\int_0^x \cdots \int_0^x \tanh^{-1} x \, dx^n = \sum_{k=0}^{\infty} \frac{(2k)!}{(2k+n+1)!} x^{2k+n+1}$$

Proof

$$\tanh^{-1} x = \sum_{k=0}^{\infty} \frac{1}{2k+1} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(2k)!}{(2k+1)!} x^{2k+1}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

The lineal higher integral of this was as follows as shown in Formula 4.5.1 (**4.5**).

$$\int_0^x \cdots \int_0^x \tanh^{-1} x \, dx^n = \frac{\tanh^{-1} x}{n!} \sum_{k=0}^{n/2\downarrow} {}_nC_{n-2k} x^{n-2k}$$

$$+ \frac{\log(1-x^2)}{2 \cdot n!} \sum_{k=1}^{n/2\uparrow} {}_nC_{n+1-2k} x^{n+1-2k}$$

$$- \frac{1}{n!} \sum_{r=1}^{n/2\downarrow} {}_nC_{n+1-2r} \{\psi(1+n) - \psi(2r)\} x^{n+1-2r}$$

Therefore, the termwise higher integral above is a **lineal higher integral**.
And matching both, we obtain the following formula.

Formula 6.5.1'

The following expression holds for $|x| \leqq 1$.

$$\sum_{k=0}^{\infty} \frac{(2k)!}{(2k+n+1)!} x^{2k+n+1} = \frac{\tanh^{-1} x}{n!} \sum_{k=0}^{n/2\downarrow} {}_nC_{n-2k} x^{n-2k}$$

$$+ \frac{\log(1-x^2)}{2 \cdot n!} \sum_{k=1}^{n/2\uparrow} {}_nC_{n+1-2k} x^{n+1-2k}$$

$$- \frac{1}{n!} \sum_{r=1}^{n/2\downarrow} {}_nC_{n+1-2r} \{\psi(1+n) - \psi(2r)\} x^{n+1-2r}$$

Substituting $x=1$ for this, we obtain the following special values.

$$\sum_{k=0}^{\infty} \frac{(2k)!}{(2k+2)!} = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} + \cdots = \log 2$$

$$\sum_{k=0}^{\infty} \frac{(2k)!}{(2k+3)!} = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} + \frac{1}{7 \cdot 8 \cdot 9} + \cdots = \log 2 - \frac{1}{2}$$

$$\sum_{k=0}^{\infty} \frac{(2k)!}{(2k+4)!} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{1}{5 \cdot 6 \cdot 7 \cdot 8} + \cdots = \frac{2}{3} \log 2 - \frac{5}{12}$$

$$\vdots$$

$$\sum_{k=0}^{\infty} \frac{(2k)!}{(2k+n+1)!} = \frac{2^{n-1}}{n!} \log 2 - \frac{1}{n!} \sum_{r=1}^{n/2\downarrow} {}_nC_{n+1-2r} \{\psi(1+n) - \psi(2r)\}$$

Note

The following formula is known about these series.

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+2)\cdots(2k+m)} = \frac{1}{(m-1)!}\int_0^1 \frac{(1-t)^{m-1}}{1-t^2}dt$$

This is easily drawn from Formula 6.5.1 using Riemann-Liouville Integral.
And the following expression is also drawn easily..

$$\int_0^1 \frac{(1-x)^n}{1-x^2}dx = 2^{n-1}\log 2 - \sum_{r=1}^{n/2\downarrow} {}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}$$

6.6 Termwise Higher Integral of arcsinh x

Formula 6.6.1

The following expression holds for $|x| < 1$.

$$\int_0^x \cdots \int_0^x \sinh^{-1}x\, dx^n = \sum_{k=0}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{(2k+n+1)!} x^{2k+n+1} \qquad \textbf{Collateral}$$

Proof

$$\sinh^{-1}x = \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!\,(2k+1)} x^{2k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!\,(2k)!}{(2k)!!\,(2k+1)!} x^{2k+1}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{(2k+1)!} x^{2k+1} \qquad \{\because \ (2k)! = (2k)!!\cdot(2k-1)!!\}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

The collateral higher integral of this was as follows as shown in Formula 4.5.2' (**4.5**).

$$\int_0^x \cdots \int_0^x \sinh^{-1}x\, dx^n = \sum_{r=0}^{n/2\downarrow} \frac{(-1)^r x^{n-2r}}{(2r)!!^2 (n-2r)!} \sinh^{-1}x$$

$$+ \sum_{r=1}^{n/2\uparrow} \sum_{s=0}^{n-2r+1} (-1)^{r+s}\, {}_{n-2r+1}C_s \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r+1}}{(n-2r+1)!} \sqrt{1+x^2}$$

$$+ \sum_{r=1}^{n/2\uparrow} \frac{(-1)^{r-1} x^{n-2r+1}}{(2r-1)!!^2 (n-2r+1)!}$$

Therefore, the termwise higher integral above is a **collateral higher integral**.
And matching both, we obtain the following formula.

Formula 6.6.1'

The following expression holds for $|x| \leqq 1$.

$$\sum_{k=0}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{(2k+n+1)!} x^{2k+n+1} = \sum_{r=0}^{n/2\downarrow} \frac{(-1)^r x^{n-2r}}{(2r)!!^2 (n-2r)!} \sinh^{-1}x$$

$$+ \sum_{r=1}^{n/2\uparrow} \sum_{s=0}^{n-2r+1} (-1)^{r+s}\, {}_{n-2r+1}C_s \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r+1}}{(n-2r+1)!} \sqrt{1+x^2}$$

$$+ \sum_{r=1}^{n/2\uparrow} \frac{(-1)^{r-1} x^{n-2r+1}}{(2r-1)!!^2 (n-2r+1)!}$$

Substituting $x=1$ for this, we obtain the following special values.

$$\sum_{k=0}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{(2k+2)!} = \sinh^{-1}1 - \sqrt{2} + 1$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{(2k+3)!} = \frac{1}{4}\sinh^{-1}1 - \frac{3\sqrt{2}}{4} + 1$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{(2k+4)!} = -\frac{1}{12}\sinh^{-1}1 - \frac{7\sqrt{2}}{36} + \frac{7}{18}$$

⋮

$$\sum_{k=0}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{(2k+n+1)!} = \sum_{r=0}^{n/2\downarrow}\frac{(-1)^r}{(2r)!!^2(n-2r)!}\sinh^{-1}1$$

$$+\sum_{r=1}^{n/2\uparrow}\sum_{s=0}^{n-2r+1}(-1)^{r+s}{}_{n-2r+1}C_s\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{\sqrt{2}}{(n-2r+1)!}$$

$$+\sum_{r=1}^{n/2\uparrow}\frac{(-1)^{r-1}}{(2r-1)!!^2(n-2r+1)!}$$

6.7 Termwise Higher Integral of arcsech x

Formula 6.7.1

The following expression holds for $0<x<1$.

$$\int_0^x \cdots \int_0^x sech^{-1}x\,dx^n = \frac{x^n}{n!}\left(log\frac{2}{x}+\sum_{j=1}^{n}\frac{1}{j}\right) - \sum_{k=1}^{\infty}\frac{\{(2k-1)!!\}^2}{2k\,(2k+n)!}x^{2k+n}$$ **Collateral**

Proof

$$sech^{-1}x = log\frac{2}{x} - \sum_{k=1}^{\infty}\frac{(2k-1)!!}{(2k)!!\;2k}x^{2k} = log\frac{2}{x} - \sum_{k=1}^{\infty}\frac{(2k-1)!!\,(2k)!}{(2k)!!\;2k\,(2k)!}x^{2k}$$

$$= log\frac{2}{x} - \sum_{k=1}^{\infty}\frac{\{(2k-1)!!\}^2}{2k\,(2k)!}x^{2k} \qquad \{\because\ (2k)! = (2k)!!\cdot(2k-1)!!\}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

The collateral higher integral of this was as follows as shown in Formula 4.5.3' (**4.5**).

$$\int_0^x \cdots \int_0^x sech^{-1}x\,dx^n = \frac{x^n}{n!}sech^{-1}x + \sum_{r=0}^{(n-1)/2\downarrow}\frac{(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{x^{n-2r-1}}{(n-2r-1)!}sin^{-1}x$$

$$+\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^s\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{1-x^2}$$

$$-\sum_{r=1}^{n/2\downarrow}\frac{1}{2r(2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}$$

Therefore, the termwise higher integral above is a **collateral higher integral**.
And matching both, we obtain the following formula.

Formula 6.7.1'

The following expression holds for $0<x\leqq 1$.

$$\sum_{k=1}^{\infty}\frac{\{(2k-1)!!\}^2}{2k\,(2k+n)!}x^{2k+n} = \frac{x^n}{n!}\left(log\frac{2}{x}+\sum_{j=1}^{n}\frac{1}{j}\right) + \sum_{r=1}^{n/2\downarrow}\frac{1}{2r(2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}$$

$$-\frac{x^n}{n!}sech^{-1}x - \sum_{r=0}^{(n-1)/2\downarrow}\frac{(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{x^{n-2r-1}}{(n-2r-1)!}sin^{-1}x$$

$$-\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^s\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{1-x^2}$$

Substituting $x=1$ for this, we obtain the following special values.

$$\sum_{k=1}^{\infty}\frac{\{(2k-1)!!\}^2}{2k\,(2k+1)!} = \frac{1}{1!}(log\,2+1) - \frac{\pi}{2}$$

$$\sum_{k=1}^{\infty}\frac{\{(2k-1)!!\}^2}{2k\,(2k+2)!} = \frac{1}{2!}\left(log\,2+\frac{3}{2}\right) + \frac{1}{2} - \frac{\pi}{2}$$

$$\sum_{k=1}^{\infty}\frac{\{(2k-1)!!\}^2}{2k\,(2k+3)!} = \frac{1}{3!}\left(log\,2+\frac{11}{6}\right) + \frac{1}{2} - \frac{\pi}{2}\left(\frac{1}{2}+\frac{1}{12}\right)$$

$$\vdots$$

$$\sum_{k=1}^{\infty}\frac{\{(2k-1)!!\}^2}{2k\,(2k+n)!} = \frac{1}{n!}\left(\log 2+\sum_{j=1}^{n}\frac{1}{j}\right) + \sum_{r=1}^{n/2\downarrow}\frac{1}{2r(2r-1)!!^2}\frac{1}{(n-2r)!}$$

$$- \frac{\pi}{2}\sum_{r=0}^{(n-1)/2\downarrow}\frac{(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{1}{(n-2r-1)!}$$

6.8 Termwise Higher Integral of arccsch x

Formula 6.8.1

The following expression holds for $0< x<1$.

$$\int_0^x\cdots\int_0^x csch^{-1}x\,dx^n = \frac{x^n}{n!}\left(\log\frac{2}{x}+\sum_{j=1}^{n}\frac{1}{j}\right)-\sum_{k=1}^{\infty}(-1)^k\frac{\{(2k-1)!!\}^2}{2k\,(2k+n)!}x^{2k+n}$$ **Collateral**

Proof

$$\begin{aligned} csch^{-1}x &= \log\frac{2}{x}+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2k-1)!!}{(2k)!!\;2k}x^{2k} \\ &= \log\frac{2}{x}-\sum_{k=1}^{\infty}(-1)^{k}\frac{(2k-1)!!\,(2k)!}{(2k)!!\;2k\,(2k)!}x^{2k} \\ &= \log\frac{2}{x}-\sum_{k=1}^{\infty}(-1)^{k}\frac{\{(2k-1)!!\}^2}{2k\,(2k)!}x^{2k} \quad \{\because\ (2k)!=(2k)!!\cdot(2k-1)!!\} \end{aligned}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expression.

The collateral higher integral of this was as follows as shown in Formula 4.5.3' (**4.5**).

$$\begin{aligned}\int_0^x\cdots\int_0^x csch^{-1}x\,dx^n =& \frac{x^n}{n!}csch^{-1}x+\sum_{r=0}^{(n-1)/2\downarrow}\frac{(-1)^r(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{x^{n-2r-1}}{(n-2r-1)!}sinh^{-1}x \\ &+\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^{r+s}\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{x^2+1} \\ &-\sum_{r=1}^{n/2\downarrow}\frac{(-1)^r}{2r(2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\end{aligned}$$

Therefore, the termwise higher integral above is a **collateral higher integral**.

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Alien's Mathematics