13 Termwise Super Derivative

In this chapter, for the function whose super derivatives are difficult to be expressed with easy formulas, we differentiate the series expansion of these functions non integer times termwise. Therefore, $e^x, \log x, \sin x, \cos x, \sinh x, \cosh x$ mentioned in " **12 Super Derivative** " are not treated here.

13.1 Termwise Super Derivative of Trigonometric Functions & Hyperbolic Functions

Formula 13.1.1

Let $\Gamma(x)$ be gamma function, $\uparrow, \downarrow$ are ceilling function and floor function and let Bernoulli number B_{2n} and Euler number E_{2k} are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30, \cdots$$
$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$$

Then the following expressions hold for $p \geqq 0$ and $0< x < \pi/2$.

$$(\tan x)^{(p)} = \sum_{k=\frac{p+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,\Gamma(2k-p)} x^{2k-p-1}$$

$$(\tanh x)^{(p)} = \sum_{k=\frac{p+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,\Gamma(2k-p)} x^{2k-p-1}$$

$$(\sec x)^{(p)} = \sum_{k=\frac{p+1}{2}\downarrow}^{\infty} \frac{|E_{2k}|}{\Gamma(2k-p+1)} x^{2k-p} \qquad : collateral$$

$$(\mathrm{sech}\, x)^{(p)} = \sum_{k=\frac{p+1}{2}\downarrow}^{\infty} \frac{E_{2k}}{\Gamma(2k-p+1)} x^{2k-p} \qquad : collateral$$

Proof

There were the following formulas in " **10 Termwise Higher Derivative** "

Formula 10.1.1 $$(\tan x)^{(n)} = \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k-n-1)!} x^{2k-n-1}$$

Formula 10.2.1 $$(\tanh x)^{(n)} = \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-n-1)!} x^{2k-n-1}$$

Formula 10.7.1 $$(\sec x)^{(n)} = \sum_{k=\frac{n+1}{2}\downarrow}^{\infty} \frac{|E_{2k}|}{(2k-n)!} x^{2k-n}$$

Formula 10.8.1 $$(\mathrm{sech}\, x)^{(n)} = \sum_{k=\frac{n+1}{2}\downarrow}^{\infty} \frac{E_{2k}}{(2k-n)!} x^{2k-n}$$

In these formulas, replacing $m!$ with gamma function $\Gamma(1+m)$ and analytically continuing the index of the differentiation operator to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Since the termwise super integrals of arcsin x and arccos x were collateral, the super derivatives have to be collateral too. (This is the same in the following chapters.)

Example 1: 3/4 th order derivative of tan x

We calculated the the super differential coefficients on arbitrary one point $x = 0.4$ according to the formula and Riemann-Liouville differintegral. As the result, two values were almost corresponding. Moreover, in the figure, blue shows tan x, red shows the 3/4 th order derivative and green shows the 1 st order derivative.

Termwise super derivative of tan x

```
• tanp := (p,x)-> sum(2^(2*k)*(2^(2*k)-1)*abs(bernoulli(2*k))/((2*k
                *gamma(2*k-p))*x^(2*k-1-p),k=ceil((p+1)/2)..200
```

$$(p, x) \rightarrow \sum_{k=\left\lceil \frac{p+1}{2} \right\rceil}^{200} \frac{2^{2 \cdot k} \cdot \left(2^{2 \cdot k} - 1\right) \cdot |\mathrm{bernoulli}(2 \cdot k)|}{(2 \cdot k) \cdot \Gamma(2 \cdot k - p)} \cdot x^{2 \cdot k - 1 - p}$$

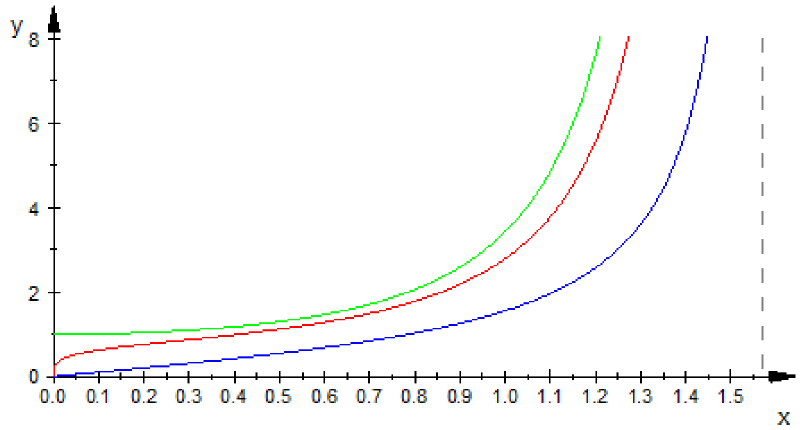
Riemann-Liouville differintegral

```
• p:=3/4: h:=10^-10:
• f := x-> 1/gamma(1-p)*int((x-t)^(1-p-1)*tan(t), t=0..x)
```

$$x \rightarrow \frac{1}{\Gamma(1-p)} \cdot \int_0^x (x-t)^{1-p-1} \cdot \tan(t) \, \mathrm{d}t$$

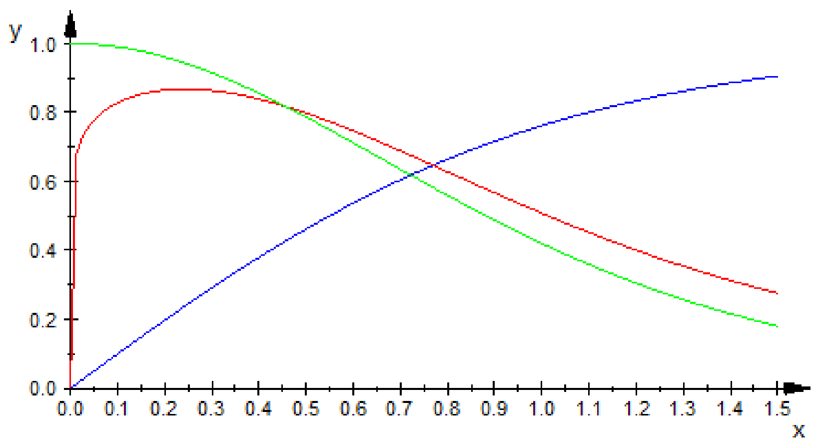
```
• float(tanp(3/4,0.4))          • float((f(0.4+h)-f(0.4))/h)
```

0.987302844   0.9873028437



Example 2: 9/10 th order derivative of tanh x

Only the figure is shown. Blue shows tanh x, red shows the 9/10 th order derivative and green shows the 1 st order derivative.



Example 3: 1/2 th order derivative of sec x

We calculated the the super differential coefficients on arbitrary one point $x=0.3$ according to the formula and Riemann-Liouville differintegral. As the result, two values were almost corresponding. Moreover, in the figure, blue shows sec x, red shows the 1/2 th order derivative and green shows the 1 st order derivative.

Termwize derivative of sec x

```
m = 200 ;
```

$$\text{secp}[p\_, x\_] := \sum_{k=\text{Floor}\left[\frac{p+1}{2}\right]}^{m} \frac{\text{Abs}[\text{EulerE}[2k]]}{\text{Gamma}[2k-p+1]} x^{2k-p}$$

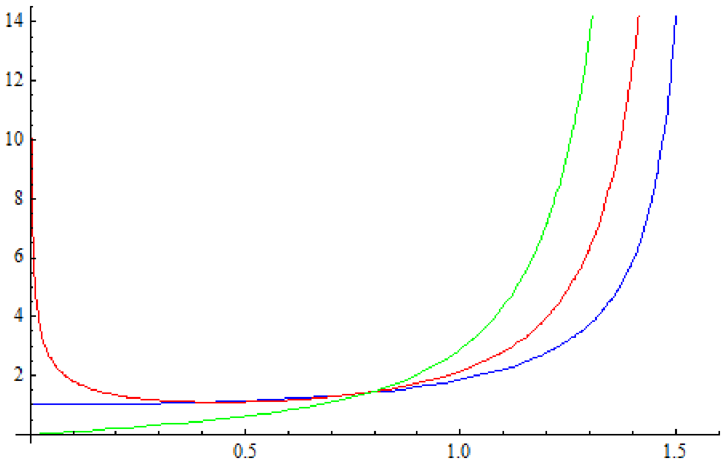
Riemann-Liouville differintegral

$$p = 1/2;\ h = 10^{-6};$$

$$f[x\_] := \frac{1}{\text{Gamma}[1-p]} \int_0^x (x-t)^{1-p-1}\,\text{Sec}[t]\,dt$$

$$\text{Rld} = \frac{f[0.3+h]-f[0.3]}{h};$$

```
N[secp [1 / 2, 0.3 ]]        N [Rld ]
1.16032                      1.16033
```



Formula 13.1.2

Let $\Gamma(x)$ be gamma function, $\uparrow, \downarrow$ are ceilling function and floor function and let Bernoulli number B_{2n} and Euler number E_{2k} are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30, \cdots$$
$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$$

Then the following expressions hold for $p \geqq 0$ and $\pi/2 < x < \pi$.

$$(\cot x)^{(p)} = -\sum_{k=\frac{p+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,\Gamma(2k-p)} \left(x-\frac{\pi}{2}\right)^{2k-p-1}$$

$$(\csc x)^{(p)} = \sum_{k=\frac{p+1}{2}\downarrow}^{\infty} \frac{|E_{2k}|}{\Gamma(2k-p+1)} \left(x-\frac{\pi}{2}\right)^{2k-p} \qquad : \textit{collateral}$$

Proof

There were the following formulas in " **10 Termwise Higher Derivative** "

Formula 10.3.1'
$$(\cot x)^{(n)} = -\sum_{k=\lceil\frac{n+1}{2}\rceil}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k(2k-n-1)!}\left(x-\frac{\pi}{2}\right)^{2k-n-1}$$

Formula 10.5.1'
$$(\csc x)^{(n)} = \sum_{k=\lfloor\frac{n+1}{2}\rfloor}^{\infty} \frac{\left|E_{2k}\right|}{(2k-n)!}\left(x-\frac{\pi}{2}\right)^{2k-n}$$

In these formulas, replacing $m!$ with gamma function $\Gamma(1+m)$ and analytically continuing the index o the differentiation operator to $[0,p]$ from $[1,n]$, we obtain the desired expressions.

Example 1: 3/4 th order derivative of cot x

We calculated the the super differential coefficients on arbitrary one point $x=1.7$ according to the formula and Riemann-Liouville differintegral. As the result, two values were almost corresponding. Moreover, in the figure, blue shows cot x, red shows the 3/4 th order derivative and green shows the 1 st order derivative.

Termwise super derivative of cot x

```
cotp := (p,x)-> -sum(2^(2*k)*(2^(2*k)-1)*abs(bernoulli(2*k))/((2*k)*gamma(2*k-p))*(x-PI/2)^(2*k-1-p),k=ceil((p+1)/2)..200)
```

$$(p, x) \rightarrow -\left(\sum_{k=\lceil\frac{p+1}{2}\rceil}^{200} \frac{2^{2\cdot k}\cdot\left(2^{2\cdot k}-1\right)\cdot\left|\mathrm{bernoulli}(2\cdot k)\right|}{(2\cdot k)\cdot\Gamma(2\cdot k-p)}\cdot\left(x-\frac{\pi}{2}\right)^{2\cdot k-1-p}\right)$$

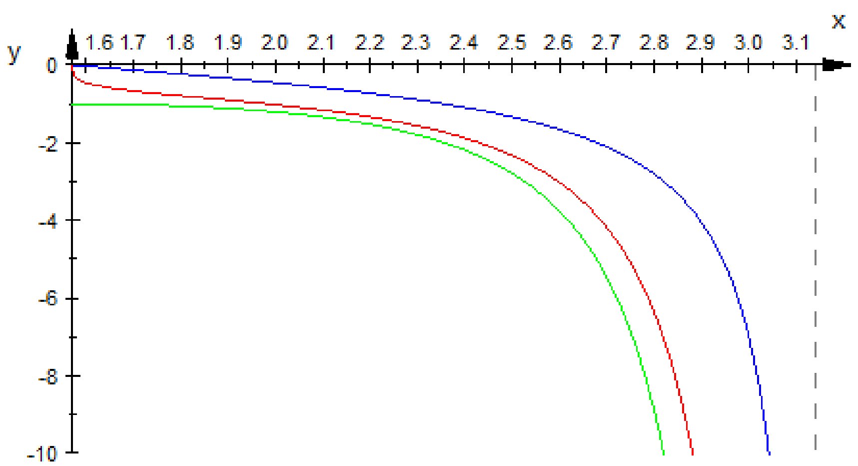
Riemann-Liouville differintegral

```
p:=3/4: h := 10^-11:
f := x-> 1/gamma(1-p)*int((x-t)^(1-p-1)*cot(t), t=PI/2..x)
```

$$x \rightarrow \frac{1}{\Gamma(1-p)}\cdot\int_{\frac{\pi}{2}}^{x}(x-t)^{1-p-1}\cdot\cot(t)\,\mathrm{d}t$$

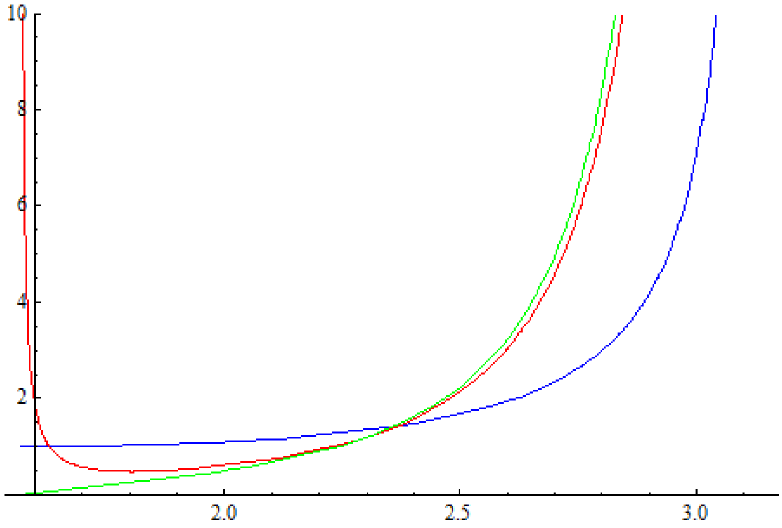
```
float(cotp(3/4,1.7)), float((f(1.7+h)-f(1.7))/h)
```

$-0.6693795932,$ -0.6693795901



Example 2: 14/15 th order derivative of csc x

Only the figure is shown. Blue shows csc x, red shows the 14/15 th order derivative and green shows the 1 st order derivative.



The following *lineal* termwise super derivatives exist for csch x and sech x.

Formula 13.1.3

The following expressions hold for $p \geqq 0$, $x > 0$.

$$(\operatorname{csch} x)^{(p)} = (-1)^{-p} 2 \sum_{k=0}^{\infty} \frac{(2k+1)^p}{e^{(2k+1)x}}$$

$$(\operatorname{sech} x)^{(p)} = (-1)^{-p} 2 \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^p}{e^{(2k+1)x}}$$

Proof

Formula 8.1.3 (**8.1**) was as follows.

$$\int_{\infty}^{x} \sim \int_{\infty}^{x} \operatorname{csch} x \, dx^p = (-1)^p 2 \sum_{k=0}^{\infty} \frac{e^{-(2k+1)x}}{(2k+1)^p}$$

$$\int_{\infty}^{x} \sim \int_{\infty}^{x} \operatorname{sech} x \, dx^p = (-1)^p 2 \sum_{k=0}^{\infty} (-1)^k \frac{e^{-(2k+1)x}}{(2k+1)^p}$$

Since differentiation is the reverse operation of integration, replacing the index p of the integration operator with $-p$, we obtain the desired expressions.

Example: 7/9 th order derivative of csch x .

We calculated the the super differential coefficients on arbitrary one point $x=3.8$ according to the formula and Riemann-Liouville differintegral. As the result, two values were almost corresponding.

Termwize derivative of csch x

```
m = 100;
```

$$\texttt{cschp[p\_, x\_] := } 2\,(-1)^{-p} \sum_{k=0}^{m} \frac{(2\,k+1)^p}{e^{(2\,k+1)\,x}}$$

Riemann-Liouville differintegral

```
p = 7 / 9;  h = 10^-6;
```

$$f[x\_] := \frac{1}{\text{Gamma}[1-p]} \int_{\infty}^{x} (x-t)^{1-p-1}\,\text{Csch}[t]\,dt \quad \text{Rld} = \frac{f[3.8+h]-f[3.8]}{h};$$

```
N[cschp[7 / 9, 3.8]]          N[Rld]
-0.0343144 - 0.0287932 i      -0.0343029  - 0.0287835  i
```

13.2 Termwise Super Derivative of Inverse Trigonometric Functions

Formula 13.2.1

When $\Gamma(x)$ is gamma function and $\uparrow$ is ceilling function, the following expressons hold for $p \geqq 0$ and $0 < x < 1$.

$$\left(\tan^{-1}x\right)^{(p)} = \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{\Gamma(2k+2-p)} x^{2k+1-p}$$

$$\left(\cot^{-1}x\right)^{(p)} = \frac{\pi}{2}\frac{x^{-p}}{\Gamma(1-p)} - \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{\Gamma(2k+2-p)} x^{2k+1-p}$$

$$\left(\sin^{-1}x\right)^{(p)} = \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{\Gamma(2k+2-p)} x^{2k+1-p}$$ **: *collateral***

$$\left(\cos^{-1}x\right)^{(p)} = \frac{\pi}{2}\frac{x^{-p}}{\Gamma(1-p)} - \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{\Gamma(2k+2-p)} x^{2k+1-p}$$ **: *collateral***

Proof

There were the following formulas in " **11 Termwise Higher Derivative "** .

Formula 11.1.1 $$\left(\tan^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{(2k+1-n)!} x^{2k+1-n}$$

Formula 11.1.2 $$\left(\sin^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+1-n)!} x^{2k+1-n}$$

In these formulas, replacing $m!$ with gamma function $\Gamma(1+m)$ and analytically continuing the index of the differentiation operator to $[0,p]$ from $[1,n]$, we obtain $\left(\tan^{-1}x\right)^{(p)}$, $\left(\sin^{-1}x\right)^{(p)}$. Next,

$$\cot^{-1}x = \frac{\pi}{2}x^0 - \tan^{-1}x$$

From this,

$$\left(\cot^{-1}x\right)^{(p)} = \frac{\pi}{2}\left(x^0\right)^{(p)} - \left(\tan^{-1}x\right)^{(p)}$$

Substituting

$$\left(x^0\right)^{(p)} = \frac{x^{-p}}{\Gamma(1-p)} \quad , \quad \left(\tan^{-1}x\right)^{(p)} = \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{\Gamma(2k+2-p)} x^{2k+1-p}$$

for this, we obtain $\left(\cot^{-1}x\right)^{(p)}$. $\left(\cos^{-1}x\right)^{(p)}$ is also obtained in a similar way.

Note

When $p = 1,2,3,\cdots$, $\Gamma(1-p) = \Gamma(0), \Gamma(-1), \Gamma(-2), \cdots$ $i.e.$ $\Gamma(1-p) = \pm\infty$. Then,

$$\frac{x^{-p}}{\Gamma(1-p)} = 0 \quad \text{for } p = 1,2,3,\cdots$$

Therefore, If we replace p with n , $\left(\cot^{-1}x\right)^{(p)}$, $\left(\cos^{-1}x\right)^{(p)}$ results in the following formulas in **11.1** .

Formula 11.1.1 $$\left(\cot^{-1}x\right)^{(n)} = -\sum_{k=\lceil\frac{n-1}{2}\rceil}^{\infty}(-1)^k\frac{(2k)!}{(2k+1-n)!}x^{2k+1-n}$$

Formula 11.1.4 $$\left(\cos^{-1}x\right)^{(n)} = -\sum_{k=\lceil\frac{n-1}{2}\rceil}^{\infty}\frac{\{(2k-1)!!\}^2}{(2k+1-n)!}x^{2k+1-n}$$

Example 1: 9/10 th order derivative of arctan x

We calculated the the super differential coefficients on arbitrary one point $x=0.1$ according to the formula and Riemann-Liouville differintegral. As the result, two values were almost corresponding. Moreover, in the figure, blue shows arctan x, red shows the 9/10 th order derivative and green shows the 1 st order derivative.

Termwize derivative of arctan x

```
m = 200;
```

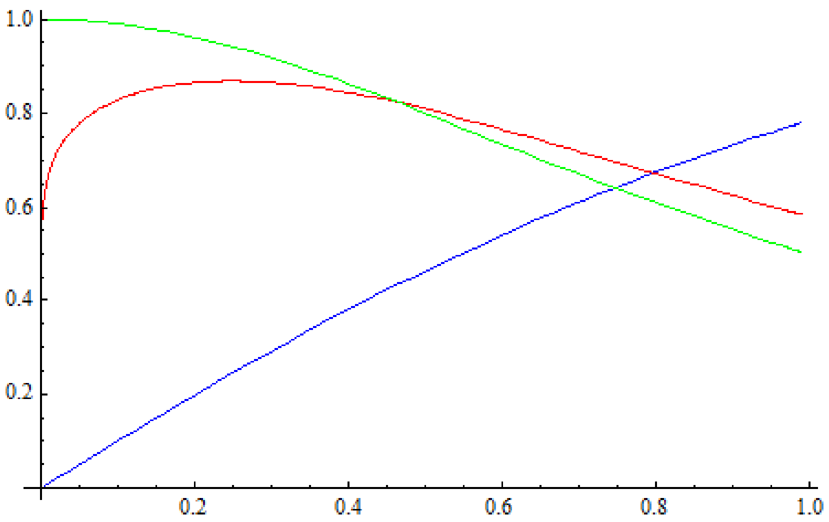
$$\text{arctanp[p\_, x\_] :=} \sum_{k=\text{Ceiling}[\frac{p-1}{2}]}^{m} (-1)^k \frac{(2k)!}{\text{Gamma}[2k+2-p]} x^{2k+1-p}$$

Riemann-Liouville differintegral

```
p = 9/10;  h = 10^-6;
```

$$\text{f[x\_] :=} \frac{1}{\text{Gamma}[1-p]}\int_0^x (x-t)^{1-p-1}\,\text{ArcTan}[t]\,dt \qquad \text{Rld} = \frac{\text{f}[0.1+h]-\text{f}[0.1]}{h};$$

```
N[arctanp [9/10, 0.1]]        N[Rld]
0.827786                      0.827787
```



Example 2: 1/2 th order derivative of arccot x

We calculated the the super differential coefficients on arbitrary one point $x=0.05$ according to the formula and Riemann-Liouville differintegral. As the result, two values were almost corresponding.

Termwize derivative of arccot x

```
m = 200;
```

$$\text{arccotp[p\_, x\_] :=} \frac{\pi}{2}\frac{x^{-p}}{\text{Gamma}[1-p]} - \sum_{k=\text{Ceiling}[\frac{p-1}{2}]}^{m} (-1)^k \frac{(2k)!}{\text{Gamma}[2k+2-p]} x^{2k+1-p}$$

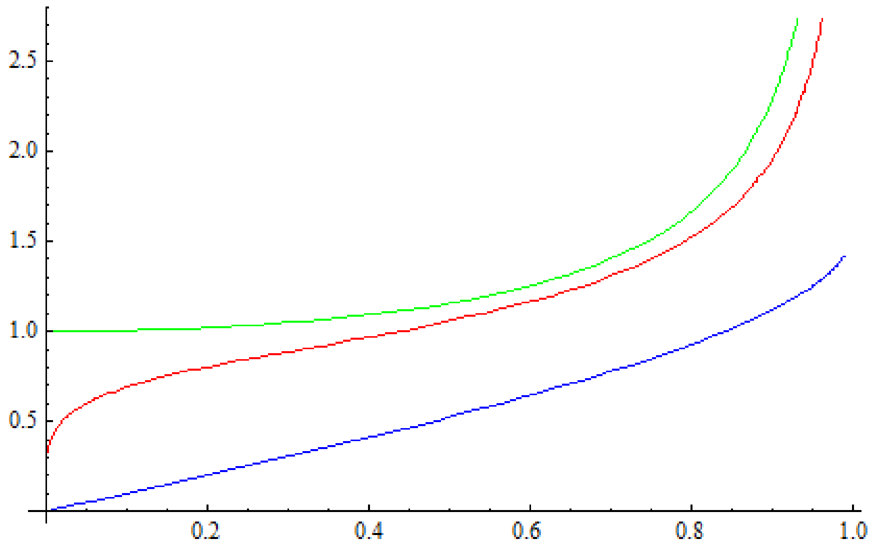
Riemann-Liouville differintegral

```
p = 1 / 2;  h = 10^-6;
f[x_] := 1/Gamma[1 - p] ∫_0^x (x - t)^(1-p-1) ArcCot[t] dt    Rld = (f[0.05 + h] - f[0.05])/h ;

N[arccotp [1 / 2, 0.05 ]]            N[Rld ]
3.71135                              3.71133
```

Example 3: 4/5th order derivative of arcsin x

Only the figure is shown. Blue shows arcsin x, red shows the 4/5th order derivative and green shows the 1st order derivative.



Example 4: 1st order derivative of arccos x

When $p=1$, $\Gamma(1-p)=\Gamma(0)=\infty$. Then $\dfrac{x^{-p}}{\Gamma(1-p)}=0$. Therefore, from the formula,

$$\left(\cos^{-1}x\right)^{(1)}=-\sum_{k=0}^{\infty}\frac{\{(2k-1)!!\}^2}{\Gamma(2k+1)}x^{2k}=-\frac{1}{\sqrt{1-x^2}}$$

In fact, this series converges to the right-hand side on $|x|<1$.

13.3 Termwise Super Derivative of Inverse Hyperbolic Functions

Formula 13.3.1

When $\Gamma(x)$, $\psi(x)$, $\uparrow$, γ are gamma function, digamma function, ceilling function and Euler-Mascheroni constant (= 0.57721566...) , the following expressons hold for $p \geqq 0$ and $0 < x < 1$.

$$\left(tanh^{-1}x\right)^{(p)} = \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} \frac{(2k)!}{\Gamma(2k+2-p)} x^{2k+1-p}$$

$$\left(sinh^{-1}x\right)^{(p)} = \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{\Gamma(2k+2-p)} x^{2k+1-p}$$

: *collateral*

$$\left(sech^{-1}x\right)^{(p)} = \frac{x^{-p}}{\Gamma(1-p)}\left\{log\frac{2}{x} + \psi(1-p) + \gamma\right\} - \sum_{k=\frac{p}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{2k\Gamma(2k-p+1)} x^{2k-p}$$

: *collateral*

$$\left(csch^{-1}x\right)^{(p)} = \frac{x^{-p}}{\Gamma(1-p)}\left\{log\frac{2}{x} + \psi(1-p) + \gamma\right\} - \sum_{k=\frac{p}{2}\uparrow}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{2k\Gamma(2k-p+1)} x^{2k-p}$$

: *collateral*

Proof

There were the following formulas in " **11 Termwise Higher Derivative** "

Formula 11.2.1t
$$\left(tanh^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty} \frac{(2k)!}{(2k+1-n)!} x^{2k+1-n}$$

Formula 11.2.2s
$$\left(sinh^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{(2k+1-n)!} x^{2k+1-n}$$

In these formulas, replacing $m!$ with gamma function $\Gamma(1+m)$ and analytically continuing the index of the differentiation operator to $[0,p]$ from $[1,n]$, we obtain $\left(tanh^{-1}x\right)^{(p)}$, $\left(sinh^{-1}x\right)^{(p)}$.

Next, arcsech x is expanded to Tylor series for $0 < x < 1$ as follows.

$$sech^{-1}x = log\,2\cdot x^0 - log\,x - \sum_{k=1}^{\infty} \frac{\{(2k-1)!!\}^2}{2k\,(2k)!} x^{2k}$$

Differentiating both sides of this with respect to x n times,

$$\left(sech^{-1}x\right)^{(n)} = log\,2\cdot\left(x^0\right)^{(n)} - (log\,x)^{(n)} - \sum_{k=\frac{n}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{2k\,(2k-n)!} x^{2k-n}$$

Substituting

$$\left(x^0\right)^{(n)} = \frac{x^{-n}}{\Gamma(1-n)} \quad , \quad (log\,x)^{(n)} = \frac{log\,x - \psi(1-n) - \gamma}{\Gamma(1-n)} x^{-n}$$

for this,

$$\left(sech^{-1}x\right)^{(n)} = \frac{x^{-n}}{\Gamma(1-n)}\left\{log\frac{2}{x} + \psi(1-n) + \gamma\right\} - \sum_{k=\frac{n}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{2k\,(2k-n)!} x^{2k-n}$$

Replacing n with p, we obtain $\left(sech^{-1}x\right)^{(p)}$. $\left(csch^{-1}x\right)^{(p)}$ is also obtained in a similar way.

Note

According to Formula 1.3.1 (**1.3**) ,

$$\frac{\psi(1-n)}{\Gamma(1-n)} = (-1)^n (n-1)! \qquad n=1,2,3,\cdots$$

If we substitute this for $\left(sech^{-1}x\right)^{(n)}$ in the proof, it results in Formula 11.2.3 (**11.2**) as follows.

$$\left(sech^{-1}x\right)^{(n)} = (-1)^n \frac{(n-1)!}{x^n} - \sum_{k=\lceil \frac{n}{2} \rceil}^{\infty} \frac{\{(2k-1)!!\}^2}{2k\,(2k-n)!} x^{2k-n}$$

Example 1: 3/4th order derivative of arctanh x

We calculated the the super differential coefficients on arbitrary one point $x=0.2$ according to the formula and Riemann-Liouville differintegral. As the result, two values were almost corresponding. Moreover, in the figure, blue shows arctanh x, red shows the 3/4th order derivative and green shows the 1st order derivative.

Termwize derivative of arctanh x

$$m = 200;$$

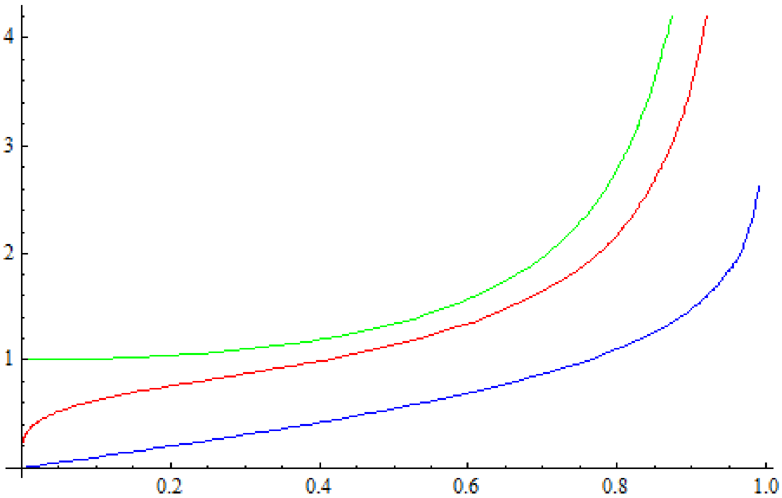
$$\text{atanhp}[p\_, x\_] := \sum_{k=\text{Ceiling}\left[\frac{p-1}{2}\right]}^{m} \frac{(2k)!}{\text{Gamma}[2k+2-p]} x^{2k+1-p}$$

Riemann-Liouville differintegral

$$p = 3/4;\ h = 10^{-6};$$

$$f[x\_] := \frac{1}{\text{Gamma}[1-p]} \int_0^x (x-t)^{1-p-1} \text{ArcTanh}[t]\, dt \qquad \text{Rld} = \frac{f[0.2+h] - f[0.2]}{h};$$

N[atanhp [3 / 4, 0.2]]	N[Rld]
0.759539	0.75954 + 0. i



Example 2: 3/2th order derivative of arcsinh x

Replacing the calculation order of differentiation and integration in Riemann-Liouville differintegral, we obtain

$$f^{(p)}(x) = \frac{1}{\Gamma(n-p)} \int_a^x (x-t)^{n-p-1} \left\{ \frac{d^n}{dt^n} f(t) \right\} dt \qquad n = \lceil p \rceil$$

We calculated the the super differential coefficients on arbitrary one point $x=0.3$ according to the formula and this expression. As the result, two values were corresponding.

Termwize derivative of arcsinh x

```
m = 200;
```

$$asinhp[p\_, x\_] := \sum_{k=Ceiling[\frac{p-1}{2}]}^{m} (-1)^k \frac{((2k-1)!!)^2}{Gamma[2k+2-p]} x^{2k+1-p}$$

Riemann-Liouville differintegral

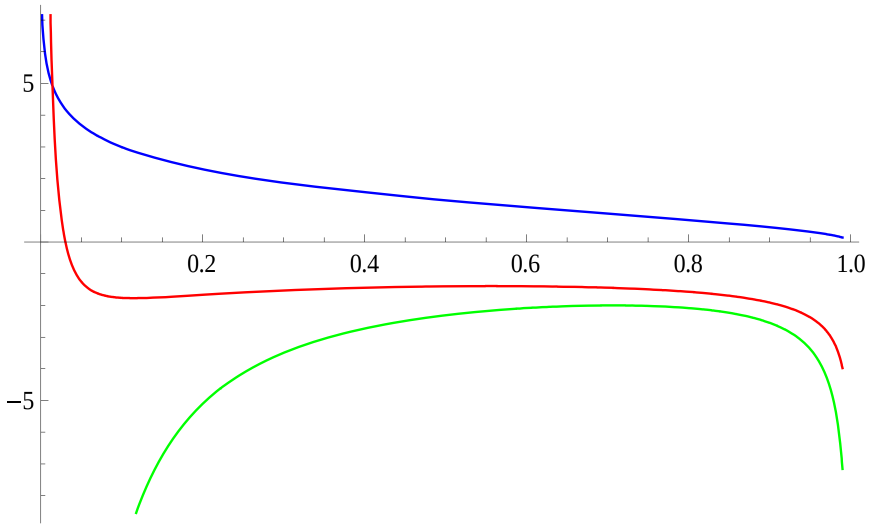
$p = 3/2;\ d2 = \partial_t (\partial_t ArcSinh[t]);$

$$Rld[x\_] := \frac{1}{Gamma[2-p]} \int_0^x (x-t)^{2-p-1} d2\, dt$$

```
N[asinhp[3/2, 0.3]]        N[Rld[0.3]]
-0.113119                  -0.113119
```

Example 3: 6/7 th order derivative of arcsech x

Only the figure is shown. Blue shows arcsech x, red shows the 6/7 th order derivative and green shows the 1 st order derivative.



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Alien's Mathematics