08 Termwise Super Integral

In this chapter, for the function which super (non-integer order) integral cannot be expressed with elementary functions, we integrate with the series expansion of these function non-integer times termwise. Therefore, $e^x, \log x, \sin x, \cos x, \sinh x, \cosh x$ mentioned in "07 Super Integral" are not treated here.

8.1 Termwise Super Integral of Trigonometric Functions & Hyperbolic Functions

In this section, for the formulas in "**05 Termwise Higher Integral (Trigonometric, Hyperbolic)"**, analytically continuing the index of the integration operator to $[0,p]$ from $[1,n]$, we obtain the Termwise Super Integrals.

Formula 8.1.1

When $\Gamma(x)$ is Gamma Function, Bernoulli Number B_{2k} and Euler Number E_{2k} are

$$B_0=1,\ B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ B_6=\frac{1}{42},\ B_8=-\frac{1}{30},\ B_{10}=\frac{5}{66},\ \cdots$$

$$E_0=1,\ E_2=-1,\ E_4=5,\quad E_6=-61,\ E_8=1385,\ E_{10}=-50521,\ \cdots$$

respectively, the following expressions hold for any $0<x<\frac{\pi}{2}$ and any $p \geqq 0$.

$$\int_0^x \sim \int_0^x \tan x\, dx^p = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,\Gamma(2k+p)} x^{2k+p-1}$$

$$\int_0^x \sim \int_0^x \tanh x\, dx^p = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,\Gamma(2k+p)} x^{2k+p-1}$$

$$\int_0^x \sim \int_0^x \sec x\, dx^p = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{\Gamma(2k+p+1)} x^{2k+p}$$ ***: collateral***

$$\int_0^x \sim \int_0^x \operatorname{sech} x\, dx^p = \sum_{k=0}^{\infty} \frac{E_{2k}}{\Gamma(2k+p+1)} x^{2k+p}$$ ***: collateral***

Proof

There were the following formulas in "**05 Termwise Higher Integral**"

Formula 5.1.1 $$\int_0^x \cdots \int_0^x \tan x\, dx^n = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+n-1)!} x^{2k+n-1}$$

Formula 5.2.1 $$\int_0^x \cdots \int_0^x \tanh x\, dx^n = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+n-1)!} x^{2k+n-1}$$

Formula 5.7.1 $$\int_0^x \cdots \int_0^x \sec x\, dx^n = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k+n)!} x^{2k+n}$$ ***: collateral***

In these formulas, replacing $m!$ with gamma function $\Gamma(1+m)$ and analytically continuing the index of the integration operator to $[0,p]$ from $[1,n]$, we obtain the first three expressions. The last expression is obtained by integrating continuously with the following expression from 0 to x.

$$\operatorname{sech} x = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k} \qquad |x|<\frac{\pi}{2}$$

Examples

Below, the examples of the termwise super integral in Formula 8.1.1 are shown. One arbitrary point is chosen suitably. fl is the the function value on the point by the formula and fr is the function value on the same point by Riemann-Liouville Integral. All digits are corresponding to both, and this shows the justification of the above termwise super integrals numerically. In the figure, blue shows the function to be integrated, redshows the termwise super integral, green shows the 1st order integral.

Termwise super integral of tan x (1/2th order)

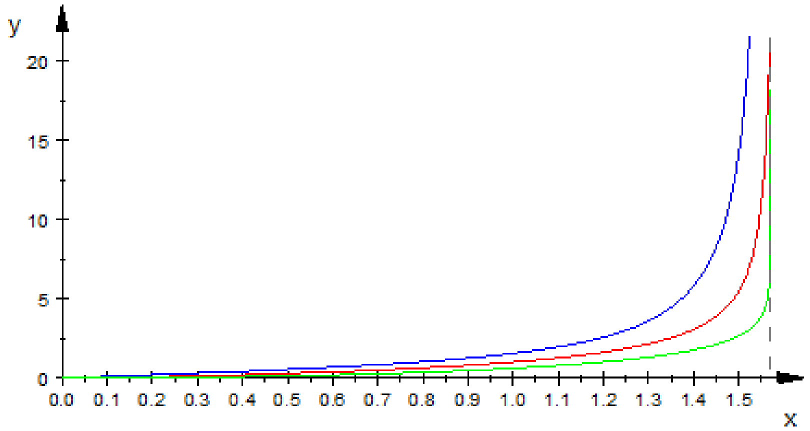
p = 1 / 2; m = 100;

$$fl[x\_] := \sum_{k=1}^{m} \frac{2^{2k}(2^{2k}-1)\,Abs[BernoulliB[2k]]}{2k\,Gamma[2k+p]} x^{2k+p-1}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1}\,Tan[t]\,dt$$

N[fl[1.3]] N[fr[1.3]]

2.15197 2.15197



Termwise super integral of tanh x (1/2th order)

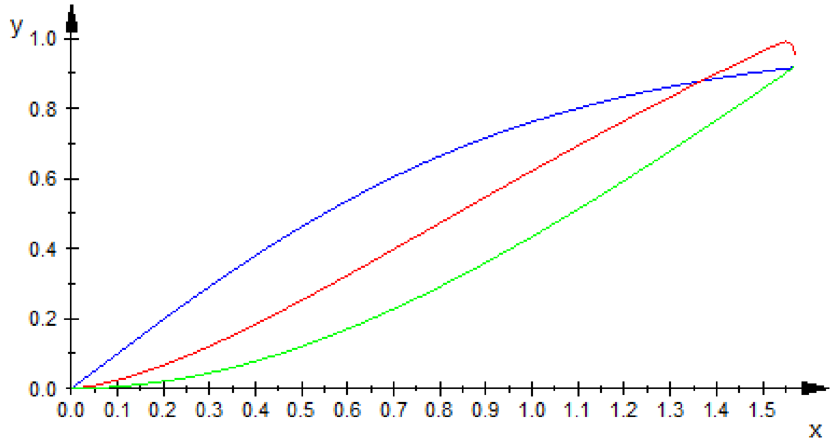
p = 1 / 2; m = 100;

$$fl[x\_] := \sum_{k=1}^{m} \frac{2^{2k}(2^{2k}-1)\,BernoulliB[2k]}{2k\,Gamma[2k+p]} x^{2k+p-1}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1}\,Tanh[t]\,dt$$

N[fl[0.4]] N[fr[0.4]]

0.183688 0.183688



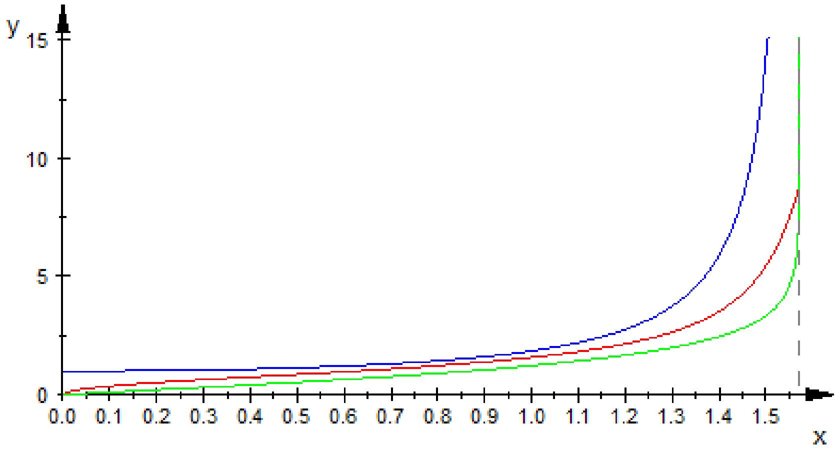
Termwise super integral of sec x (1/2th order)

```
p = 1/2;  m = 100;
```

$$fl[x\_] := \sum_{k=0}^{m} \frac{Abs[EulerE[2\,k]]}{Gamma[2\,k+p+1]}\, x^{2\,k+p}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1}\, Sec[t]\, dt$$

```
N[fl[0.9]]            N[fr[0.9]]
1.38435               1.38435
```



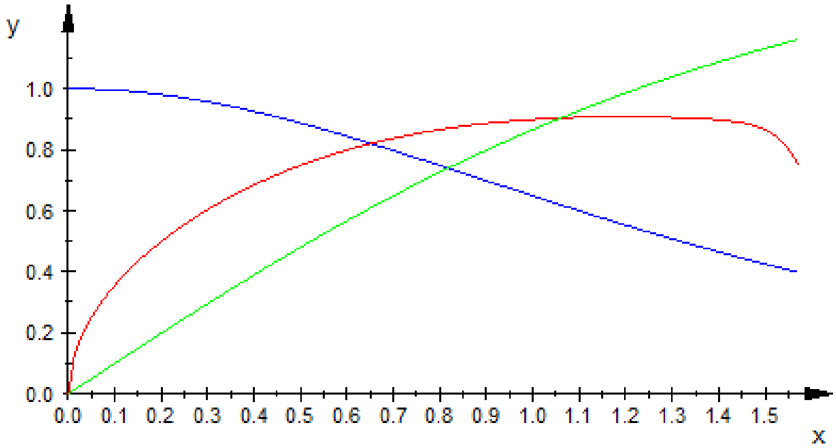
Termwise super integral of sech x (1/2th order)

```
p = 1/2;  m = 100;
```

$$fl[x\_] := \sum_{k=0}^{m} \frac{EulerE[2\,k]}{Gamma[2\,k+p+1]}\, x^{2\,k+p}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1}\, Sech[t]\, dt$$

```
N[fl[0.1]]            N[fr[0.1]]
0.355876              0.355876
```



Formula 8.1.2

When B_{2k} , E_{2k} , $\Gamma(x)$ denote Bernoulli Numbers, Euler Numbers, Gamma Function respectively, the following expressions hold for any $p \geqq 0$ and any $\pi/2 < x < \pi$.

$$\int_{\frac{\pi}{2}}^{x}\sim\int_{\frac{\pi}{2}}^{x} \cot x\, dx^{p} = -\sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k\,\Gamma(2k+p)}\left(x-\frac{\pi}{2}\right)^{2k+p-1}$$

$$\int_{\frac{\pi}{2}}^{x}\sim\int_{\frac{\pi}{2}}^{x} \csc x\, dx^{p} = \sum_{k=0}^{\infty}\frac{\left|E_{2k}\right|}{\Gamma(2k+p+1)}\left(x-\frac{\pi}{2}\right)^{2k+p}$$

: *collateral*

Proof

There were the following formulas in "**05 Termwise Higher Integral**"

Formula 5.3.1'
$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} \cot x\, dx^{n} = -\sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k(2k+n-1)!}\left(x-\frac{\pi}{2}\right)^{2k+n-1}$$

Formula 5.5.1'
$$\int_{\frac{\pi}{2}}^{x}\cdots\int_{\frac{\pi}{2}}^{x} \csc x\, dx^{n} = \sum_{k=0}^{\infty}\frac{\left|E_{2k}\right|}{(2k+n)!}\left(x-\frac{\pi}{2}\right)^{2k+n}$$

: *collateral*

In these formulas, replacing $m!$ with gamma function $\Gamma(1+m)$ and analytically continuing the index of the integration operator to $[0,p]$ from $[1,n]$, we obtain the desired expressions

Examples

Below, the examples of the termwise super integral in Formula 8.1.2 are shown. One arbitrary point is chosen suitably. fl is the the function value on the point by the formula and fr is the function value on the same point by Riemann-Liouville Integral. All digits are corresponding to both, and this shows the justification of the above termwise super integrals numerically. In the figure, blue shows the function to be integrated, redshows the termwise super integral, green shows the 1st order integral.

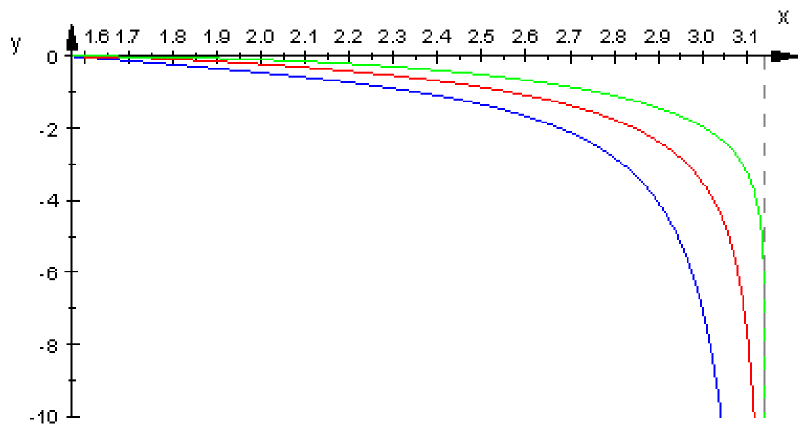
Termwise super integral of cot x (1/2th order)

```
p = 1 / 2;  m = 100;
```

$$\mathrm{fl}[x\_] := -\sum_{k=1}^{m}\frac{2^{2k}\left(2^{2k}-1\right)\mathrm{Abs}[\mathrm{BernoulliB}[2k]]}{2k\,\mathrm{Gamma}[2k+p]}\left(x-\frac{\pi}{2}\right)^{2k+p-1}$$

$$\mathrm{fr}[x\_] := \frac{1}{\mathrm{Gamma}[p]}\int_{\frac{\pi}{2}}^{x}(x-t)^{p-1}\,\mathrm{Cot}[t]\,dt$$

```
N[fl[2.8]]          N[fr[2.8]]
-1.75789            -1.75789
```



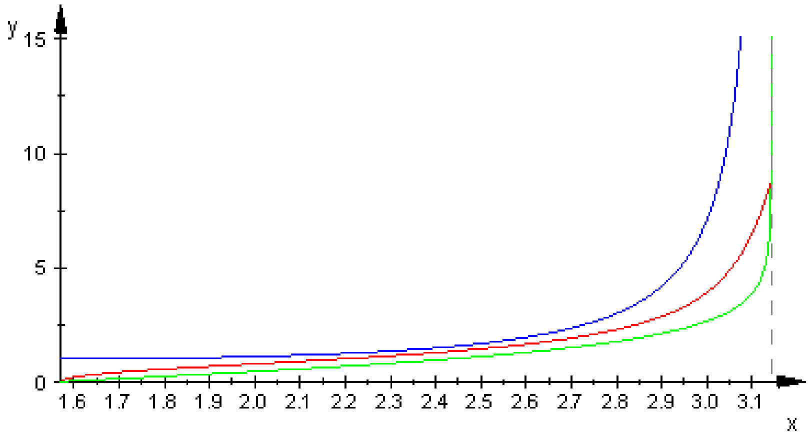
Termwise super integral of csc x (1/2th order)

```
p = 1 / 2;  m = 100;
```

$$fl[x\_] := \sum_{k=0}^{m} \frac{Abs[EulerE[2k]]}{Gamma[2k+p+1]} \left(x - \frac{\pi}{2}\right)^{2k+p}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_{\frac{\pi}{2}}^{x} (x-t)^{p-1} Csc[t]\, dt$$

```
N[fl[1.9]]          N[fr[1.9]]
0.666801            0.666801
```



The following *lineal* termwise super integrals exist for csch x and sech x .

Formula 8.1.3

The following expressions hold for $p \geqq 0$, $x > 0$.

$$\int_{\infty}^{x} \sim \int_{\infty}^{x} \operatorname{csch} x\, dx^p = (-1)^p 2 \sum_{k=0}^{\infty} \frac{e^{-(2k+1)x}}{(2k+1)^p}$$

$$\int_{\infty}^{x} \sim \int_{\infty}^{x} \operatorname{sech} x\, dx^p = (-1)^p 2 \sum_{k=0}^{\infty} (-1)^k \frac{e^{-(2k+1)x}}{(2k+1)^p}$$

Proof

There were the following formulas in " **05 Termwise Higher Integral** "

Formula 5.6.2 $$\int_{\infty}^{x} \cdots \int_{\infty}^{x} \operatorname{csch} x\, dx^n = (-1)^n 2 \sum_{k=0}^{\infty} \frac{e^{-(2k+1)x}}{(2k+1)^n}$$

Formula 5.8.2 $$\int_{\infty}^{x} \cdots \int_{\infty}^{x} \operatorname{sech} x\, dx^n = (-1)^n 2 \sum_{k=0}^{\infty} (-1)^k \frac{e^{-(2k+1)x}}{(2k+1)^n}$$

In these formulas, analytically continuing the index of the integration operator to $[0,p]$ from $[1,n]$, we obtain the desired expressions

Examples

Below, the examples of the termwise super integral in Formula 8.1.3 are shown. One arbitrary point is chosen suitably. fl is the the function value on the point by the formula and fr is the function value on the same point by Riemann-Liouville Integral. All digits are corresponding to both, and this shows the justification of the above termwise super integrals numerically.

Termwise super integral of csch x (5/8th order)

```
p = 5 / 8;  m = 100;
```

$$fl[x\_] := (-1)^p\, 2 \sum_{k=0}^{m} \frac{e^{-(2k+1)x}}{(2k+1)^p}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_{\infty}^{x} (x-t)^{p-1}\, Csch[t]\, dt$$

```
N[fl[1.7]]                      N[fr[1.7]]
-0.142227 + 0.343366 i          -0.142227 + 0.343366 i
```

Termwise super integral of sech x (3/7th order)

```
p = 3 / 7;  m = 100;
```

$$fl[x\_] := (-1)^p\, 2 \sum_{k=0}^{m} (-1)^k \frac{e^{-(2k+1)x}}{(2k+1)^p}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_{\infty}^{x} (x-t)^{p-1}\, Sech[t]\, dt$$

```
N[fl[0.3]]                      N[fr[0.3]]
0.250622 + 1.09805 i            0.250622 + 1.09805 i
```

8.2 Termwise Super Integral of Inverse Trigonometric Functions

In Formula 6.1.1 ~ Formula 6.4.1 in "**06 Termwise Higher Integral**", replacing a factorial with a gamma function and analytically continuing the index of the integration operator to $[0,p]$ from $[1,n]$, we obtain the following formula. When the formula of a foundation is Collateral Higher Integrals, naturally this becomes *Collateral Super Integral*.

Formula 8.2.1

When $\Gamma(x)$ denotes Gamma Function, the following expressons hold for $0<x<1$ and $p \geqq 0$.

$$\int_0^x \sim \int_0^x tan^{-1} x\, dx^p = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{\Gamma((2k+p+2))} x^{2k+p+1}$$

$$\int_0^x \sim \int_0^x cot^{-1} x\, dx^p = \frac{\pi}{2} \frac{x^p}{\Gamma(1+p)} - \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{\Gamma(2k+p+2)} x^{2k+p+1}$$

$$\int_0^x \sim \int_0^x sin^{-1} x\, dx^p = \sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{\Gamma(2k+p+2)} x^{2k+p+1} \quad : \textit{collateral}$$

$$\int_0^x \sim \int_0^x cos^{-1} x\, dx^p = \frac{\pi}{2} \frac{x^p}{\Gamma(1+p)} - \sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{\Gamma(2k+p+2)} x^{2k+p+1} \quad : \textit{collateral}$$

Examples

Below, the examples of the termwise super integral in Formula 8.2.1 are shown. One arbitrary point is chosen suitably. fl is the the function value on the point by the formula and fr is the function value on the same point by Riemann-Liouville Integral. All digits are corresponding to both, and this shows the justification of the above termwise super integrals numerically. In the figure, blue shows the function to be integrated, redshows the termwise super integral, green shows the 1st order integral.

Termwise super integral of arctan x (1/2th order)

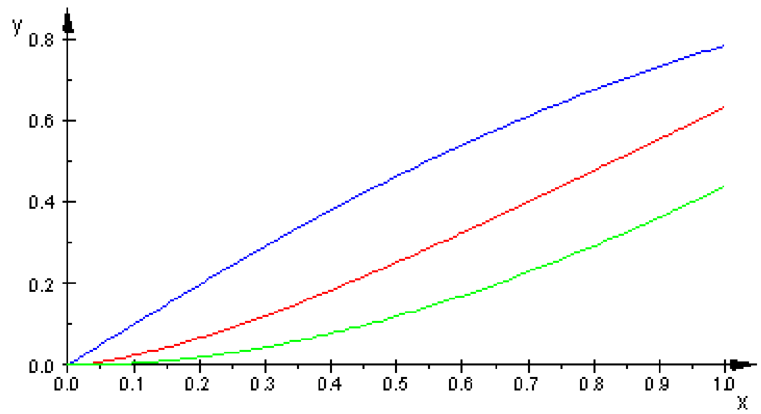
p = 1 / 2 ; m = 100 ;

$$fl[x\_] := \sum_{k=0}^{m} (-1)^k \frac{(2\,k)!}{Gamma[2\,k+p+2]} x^{2\,k+p+1}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1} ArcTan[t]\, dt$$

N[fl[0.8]] N[fr[0.8]]

0.477246 0.477246



Termwise super integral of arccot x (1/2th order)

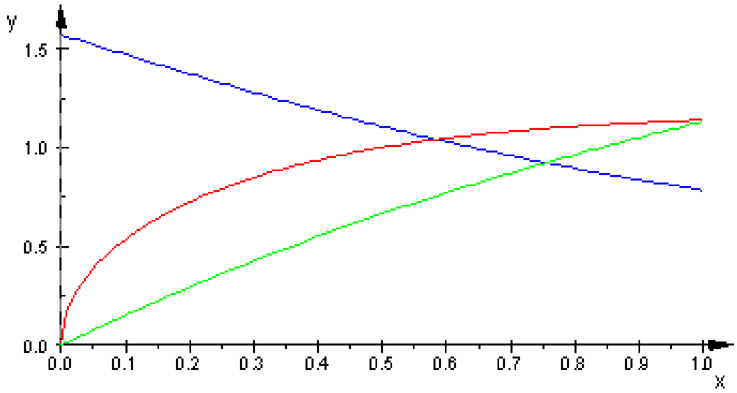
p = 1 / 2 ; m = 100 ;

$$fl[x\_] := \frac{\pi}{2}\frac{x^p}{Gamma[1+p]} - \sum_{k=0}^{m} (-1)^k \frac{(2k)!}{Gamma[2k+p+2]} x^{2k+p+1}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1} ArcCot[t] \, dt$$

N[fl[0.73]] 1.09114

N[fr[0.73]] 1.09114



Termwise super integral of arcsin x (1/2th order)

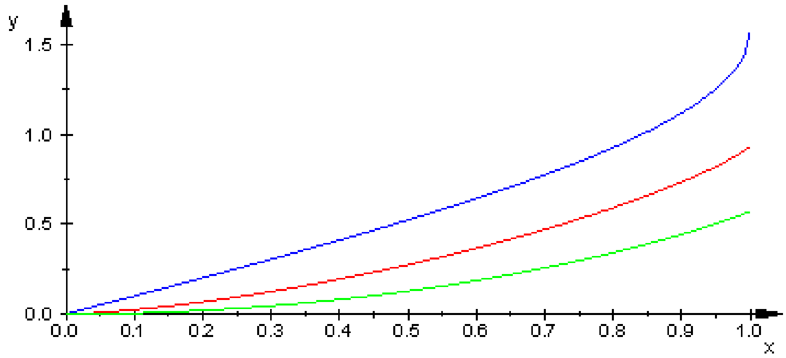
p = 1 / 2 ; m = 100 ;

$$fl[x\_] := \sum_{k=0}^{m} \frac{((2k-1)!!)^2}{Gamma[2k+p+2]} x^{2k+p+1}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1} ArcSin[t] \, dt$$

N[fl[0.7]] 0.471235

N[fr[0.7]] 0.471235 + 0. i



Termwise super integral of arccos x (1/2th order)

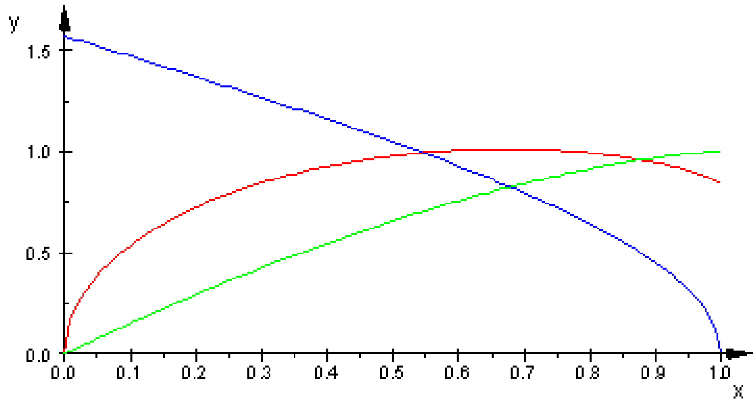
p = 1 / 2 ; m = 100 ;

$$fl[x\_] := \frac{\pi}{2}\frac{x^p}{Gamma[1+p]} - \sum_{k=0}^{m} \frac{((2k-1)!!)^2}{Gamma[2k+p+2]} x^{2k+p+1}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1} ArcCos[t] \, dt$$

N[fl[0.3]] 0.84589

N[fr[0.3]] 0.84589 + 0. i


y
1.5
1.0
0.5
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x

8.3 Termwise Super Integral of Inverse Hyperbolic Functions

In Formula 6.5.1 ~ Formula 6.8.1 in "**06 Termwise Higher Integral**" , replacing a factorial with a gamma function and analytically continuing the index of the integration operator to $[0,p]$ from $[1,n]$, we obtain the following formula. When the formula of a foundation is Collateral Higher Integrals, naturally this becomes *Collateral Super Integral* .

Formula 8.3.1

When $\Gamma(x)$, $\psi(x)$, γ denote Gamma Function, Digamma Function, Euler-Mascheroni Constant (= 0.57721566...) respectively, the following expressons hold for $0<x<1$ and $p \geqq 0$.

$$\int_0^x \sim \int_0^x \tanh^{-1} x\, dx^p = \sum_{k=0}^{\infty} \frac{(2k)!}{\Gamma(2k+p+2)} x^{2k+p+1}$$

$$\int_0^x \sim \int_0^x \sinh^{-1} x\, dx^p = \sum_{k=0}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{\Gamma(2k+p+2)} x^{2k+p+1}$$ ***collateral***

$$\int_0^x \sim \int_0^x \mathrm{sech}^{-1} x\, dx^p = \frac{x^p}{\Gamma(1+p)} \left\{ \log\frac{2}{x} + \psi(1+p) + \gamma \right\} - \sum_{k=1}^{\infty} \frac{\{(2k-1)!!\}^2}{2k\,\Gamma(2k+p+1)} x^{2k+p}$$ ***collateral***

$$\int_0^x \sim \int_0^x \mathrm{csch}^{-1} x\, dx^p = \frac{x^p}{\Gamma(1+p)} \left\{ \log\frac{2}{x} + \psi(1+p) + \gamma \right\} - \sum_{k=1}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{2k\,\Gamma(2k+p+1)} x^{2k+p}$$ ***collateral***

Examples

Below, the examples of the termwise super integral in Formula 8.3.1 are shown. One arbitrary point is chosen suitably. fl is the the function value on the point by the formula and fr is the function value on the same point by Riemann-Liouville Integral. All digits are corresponding to both, and this shows the justification of the above termwise super integrals numerically. In the figure, blue shows the function to be integrated, redshows the termwise super integral, green shows the 1st order integral.

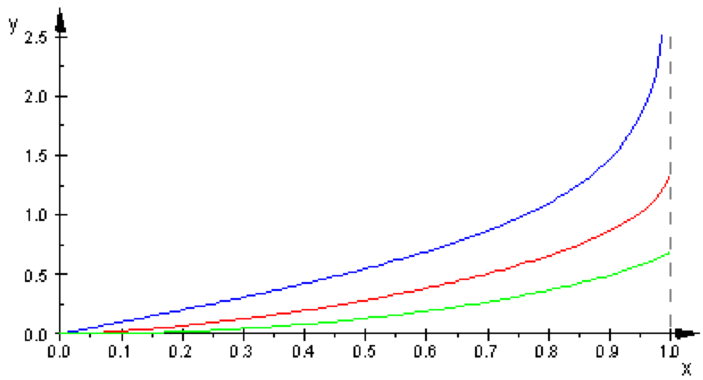
Termwise super integral of arctanh x (1/2th order)

```
p = 1 / 2 ;  m = 100 ;
```

$$fl[x\_] := \sum_{k=0}^{m} \frac{(2k)!}{Gamma[2k+p+2]} x^{2k+p+1}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1} ArcTanh[t]\, dt$$

```
N[fl[0.7]]          N[fr[0.7]]
0.507089            0.507089  + 0. i
```



Termwise super integral of arcsinh x (1/2th order)

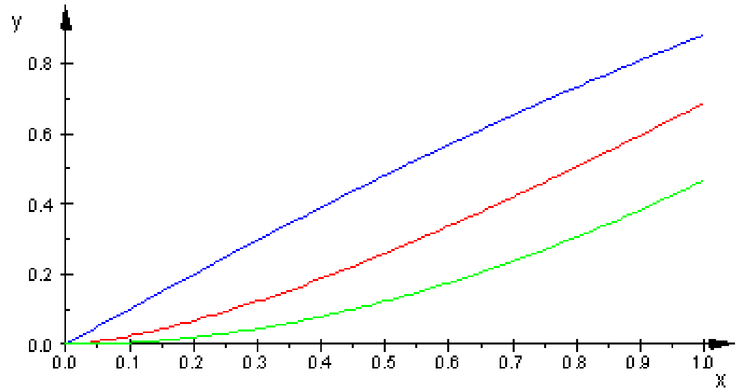
p = 1 / 2; m = 100;

$$fl[x\_] := \sum_{k=0}^{m} (-1)^k \frac{((2k-1)!!)^2}{Gamma[2k+p+2]} x^{2k+p+1}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1} ArcSinh[t] \, dt$$

N[fl[0.88]] N[fr[0.88]]

0.577282 0.577282



Termwise super integral of arcsech x (1/2th order)

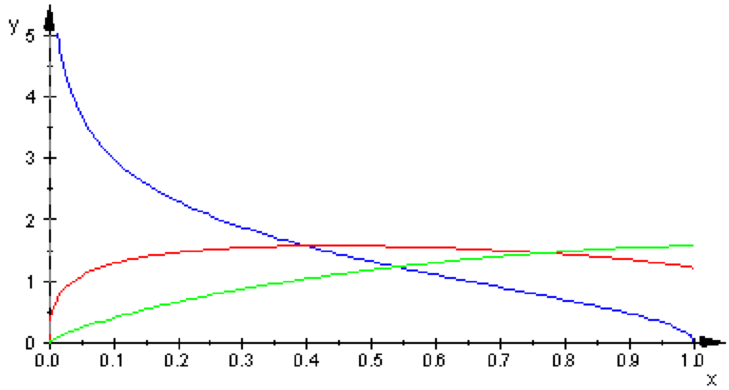
p = 1 / 2; m = 100;

$$fl[x\_] := \frac{x^p}{Gamma[1+p]} \left(Log\left[\frac{2}{x}\right] + PolyGamma[1+p] + EulerGamma \right) - \sum_{k=1}^{m} \frac{((2k-1)!!)^2}{2k\, Gamma[2k+p+1]} x^{2k+p}$$

$$fr[x\_] := \frac{1}{Gamma[p]} \int_0^x (x-t)^{p-1} ArcSech[t] \, dt$$

N[fl[0.03]] N[fr[0.03]]

0.940714 0.940714



Termwise super integral of arccsch x (1/2th order)

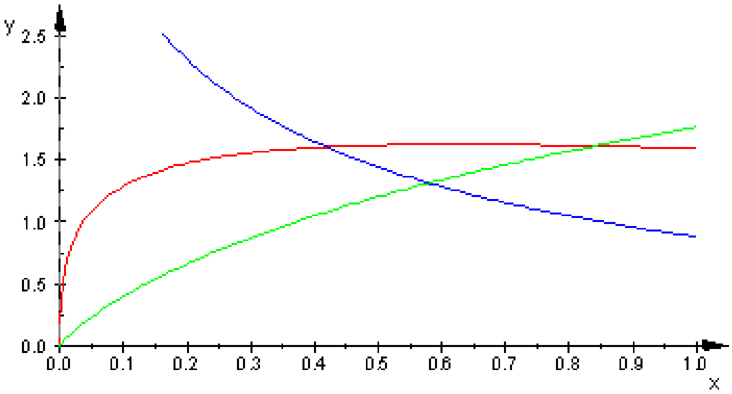
p = 1 / 2; m = 100;

$$fl[x\_] := \frac{x^p}{Gamma[1+p]} \left(Log\left[\frac{2}{x}\right] + PolyGamma[1+p] + EulerGamma \right) - \sum_{k=1}^{m} (-1)^k \frac{((2k-1)!!)^2}{2k\, Gamma[2k+p+1]} x^{2k+p}$$

$$fr[x\_] := \frac{1}{\text{Gamma}[p]} \int_0^x (x - t)^{p-1} \text{ArcCsch}[t] \, dt$$

N[f1[0.6]]
1.62696

N[fr[0.6]]
1.62696



2006.10.15
2012.06.10 Renewal

K. Kono

Alien's Mathematics