03 Vieta's Formulas in Infinite-degree Equation

3.1 Properties of Infinite-degree Equation

Unlike a finite-degree equations, there are inherent properties in an infinite-degree equation. In this section, it is clarified first.

(1) Fundamental theorem of algebra does not hold generally.

Fundamental theorem of algebra is that "every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots".

However, in the case of an infinite-degree equation, this does not hold generally. For example, infinite-degree equation

$$1+\frac{1}{1!}z^1+\frac{1}{2!}z^2+\frac{1}{3!}z^3+\cdots=0$$

has no roots on the complex plane. Because, this series is equal to e^z and it is known that this function has no zero on the complex plane.

If so, when an infinite-degree equation has a root, does the root exist infinitely? It is not necessarily so. For example, infinite-degree equation

$$1+\frac{2}{1!}z^1+\frac{3}{2!}z^2+\frac{4}{3!}z^3+\cdots=0$$

has only $z=-1$ as the root. Because, this series is equal to $(z+1)e^z$.

(2) Vieta's Formulas does not hold generally.

Let quadratic equation be

$$1+a_1z^1+a_2z^2=0$$

When α_1,α_2 denote the roots , the following expressions hold.

$$a_1=-\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right) \qquad , \qquad a_2=\frac{1}{\alpha_1\alpha_2}$$

This is Vieta's formulas in the quadratic equation.

However, in the case of an infinite-degree equation, this does not hold generally. For example, when $\psi_r(z)$ is the polygamma function, $B_{r,k}(f_1,f_2,\ldots)$ are Bell polynomials, γ is Euler-Mascheroni constant and

$$a_r=\sum_{k=1}^{r}(-1)^kB_{r,k}(\psi_0(1),\psi_1(1),\ldots,\psi_{r-1}(1)) \qquad r=1,2,3,\cdots$$

Infinite-degree equation

$$1+\left(\frac{\gamma^1}{1!}-\frac{a_1\gamma^0}{1!0!}\right)z^1+\left(\frac{\gamma^2}{2!}-\frac{a_1\gamma^1}{1!1!}+\frac{a_2\gamma^0}{2!0!}\right)z^2$$
$$+\left(\frac{\gamma^3}{3!}-\frac{a_1\gamma^2}{1!2!}+\frac{a_2\gamma^1}{2!1!}-\frac{a_3\gamma^0}{3!0!}\right)z^3+\cdots=0$$

has $z=1,2,3,\cdots$ as the roots. (See " **02 Infinite-degree Equation with Integers as Roots** ")

In this equation, Vieta's formulas does not hold as follows.

$$a_1 = \frac{\gamma^1}{1!} - \frac{a_1\gamma^0}{1!0!} = 0 \neq -\infty = -\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\right) = -\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\cdots\right)$$

(3) Roots of an infinite-degree equation with rational coefficients are not algebraic numbers generally

Roots of a nonzero polynomial in one variable with rational coefficients are always algebraic numbers. Because, this is the definition of an algebraic number.

However, roots of an infinite-degree equation with rational coefficients are not algebraic numbers in many cases. For example, let infinite-degree equation be

$$1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} +-\cdots = 0$$

Although all the coefficients of this are rational numbers, the roots are $\pm 1\pi/2, \pm 3\pi/2, \pm 5\pi/2 \cdots$ and they are not algebraic numbers.

Of course, there is an infinite-degree equation with rational coefficients which has an algebraic number as a root. For exmple, infinite-degree equation mentoned in **(1)**

$$1+\frac{2}{1!}z^1+\frac{3}{2!}z^2+\frac{4}{3!}z^3+\cdots = 0$$

has an algebraic number $z=-1$ as the root.

3.2 Vieta's Formulas (Part1)

As mentioned in the previous section, in the infinite-degree equation, the fundamental theorem of algebra and Vieta's formulas are not guaranteed. However, if a function $f(z)$ has zeros and it is fully factored at the zeros, the Vieta's formulas holds for the infinite product. In this section, we consider only such cases.

Formula 3.2.1 (Vieta's Formulas)

Assume that the function $f(z)$ on the complex plane has zeros $z_1, z_2, z_3, z_4, \cdots$ and is completely factored as follows.

$$f(z) = \left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\left(1-\frac{z}{z_3}\right)\left(1-\frac{z}{z_4}\right)\cdots$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+a_1z^1+a_2z^2+a_3z^3+a_4z^4+\cdots \tag{2.s}$$

Where,

$$a_1 = -\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}$$

$$a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}}$$

$$a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}}$$

$$a_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}z_{r_3}}$$

$$\vdots$$

$$a_n = (-1)^n\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}\cdots z_{r_n}}$$

Proof

When $z_1, z_2, z_3, \cdots$ denote the roots of (2.s) ,

$$\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\left(1-\frac{z}{z_3}\right)\left(1-\frac{z}{z_4}\right)\cdots = 0 \tag{2.p}$$

Expanding (2.p) to a power series,

$$1-\sum_{r=1}^{\infty}\frac{1}{z_r}z^1+\sum_{r<s}^{\infty}\frac{1}{z_rz_s}z^2-\sum_{r<s<t}^{\infty}\frac{1}{z_rz_sz_t}z^3+\sum_{r<s<t<u}^{\infty}\frac{1}{z_rz_sz_tz_u}z^4-+\cdots = 0 \tag{2.c}$$

Since (2.s) and (2.p) express the same function, both coefficient have to be equal by the uniqueness of a series. So, the following expressions hold.

$$a_1 = -\sum_{r=1}^{\infty}\frac{1}{z_r}\ ,\ a_2 = \sum_{r<s}\frac{1}{z_rz_s}\ ,\ a_3 = -\sum_{r<s<t}\frac{1}{z_rz_sz_t}\ ,\ a_4 = \sum_{r<s<t<u}\frac{1}{z_rz_sz_tz_u}\ ,\ -+\cdots$$

The right sides of $a_2, a_3, a_4, \cdots$ are calculated further as follows.

$$\sum_{r<s}^{\infty}\frac{1}{z_rz_s} = \frac{1}{z_1}\left(\frac{1}{z_2}+\frac{1}{z_3}+\frac{1}{z_4}+\cdots\right)+\frac{1}{z_2}\left(\frac{1}{z_3}+\frac{1}{z_4}+\frac{1}{z_5}+\cdots\right)+\frac{1}{z_3}\left(\frac{1}{z_4}+\frac{1}{z_5}+\frac{1}{z_6}+\cdots\right)+\cdots$$

$$= \sum_{r=1}^{\infty}\frac{1}{z_r}\sum_{s=r+1}^{\infty}\frac{1}{z_s} = \sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{z_rz_s}$$

$$\sum_{r<s<t}^{\infty} \frac{1}{z_r z_s z_t}$$

$$= \frac{1}{z_1}\frac{1}{z_2}\left(\frac{1}{z_3}+\frac{1}{z_4}+\frac{1}{z_5}+\cdots\right)+\frac{1}{z_1}\frac{1}{z_3}\left(\frac{1}{z_4}+\frac{1}{z_5}+\frac{1}{z_6}+\cdots\right)+\frac{1}{z_1}\frac{1}{z_4}\left(\frac{1}{z_5}+\frac{1}{z_6}+\frac{1}{z_7}+\cdots\right)+\cdots$$

$$+\frac{1}{z_2}\frac{1}{z_3}\left(\frac{1}{z_4}+\frac{1}{z_5}+\frac{1}{z_3}+\cdots\right)+\frac{1}{z_2}\frac{1}{z_4}\left(\frac{1}{z_5}+\frac{1}{z_6}+\frac{1}{z_7}+\cdots\right)+\frac{1}{z_2}\frac{1}{z_5}\left(\frac{1}{z_6}+\frac{1}{z_7}+\frac{1}{z_8}+\cdots\right)+\cdots$$

$$+\frac{1}{z_3}\frac{1}{z_4}\left(\frac{1}{z_5}+\frac{1}{z_6}+\frac{1}{z_7}+\cdots\right)+\frac{1}{z_3}\frac{1}{z_5}\left(\frac{1}{z_5}+\frac{1}{z_6}+\frac{1}{z_7}+\cdots\right)+\frac{1}{z_3}\frac{1}{z_6}\left(\frac{1}{z_7}+\frac{1}{z_8}+\frac{1}{z_9}+\cdots\right)+\cdots$$

$$+$$

$$\vdots$$

$$= \sum_{r=1}^{\infty}\frac{1}{z_r}\sum_{s=r+1}^{\infty}\frac{1}{z_s}\sum_{t=s+1}^{\infty}\frac{1}{z_t} = \sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{z_r z_s z_t}$$

Hereafter, applying induction and replacing $r, s, t \cdots$ with $r_1, r_2, r_3, \cdots$, we obtain the desired expressions.

Example Infinite-degree Equation with the 1.5th power of a natural number as roots

Such a function $f_p(z)$ is expressed with the following infinite products.

$$f_p(z) = \prod_{r=1}^{\infty}\left(1-\frac{z}{r^{1.5}}\right)$$

Assume that this is expanded to a power series as follows.

$$f_s(z) = 1+a_1 z^1+a_2 z^2+a_3 z^3+a_4 z^4+\cdots$$

If the above formula is applied, since $z_r = r^{1.5}$ $(r=1,2,3,\cdots)$, these coefficients are expressed as follows.

$$a_1 = -\sum_{r_1=1}^{\infty}\frac{1}{r_1^{1.5}}$$

$$a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{(r_1 r_2)^{1.5}}$$

$$a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{(r_1 r_2 r_3)^{1.5}}$$

$$\vdots$$

$$a_n = (-1)^n \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{1}{(r_1 r_2 \cdots r_n)^{1.5}}$$

When $f_p(z)$ is expanded up to z^3 and a_1, a_2, a_3 are calculated by formula manipulation software ***Mathematica***, it is as follows. Although both are calculated by 1000 terms, the coefficients of both are exactly the same.

$$\mathrm{fp}[z\_, m\_] := \prod_{r=1}^{m}\left(1-\frac{z}{r^{1.5}}\right)$$

$$a_1[m\_] := -\sum_{r_1=1}^{m}\frac{1}{r_1^{1.5}} \qquad a_2[m\_] := \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{1}{(r_1\ r_2)^{1.5}} \qquad a_3[m\_] := -\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m}\frac{1}{(r_1\ r_2\ r_3)^{1.5}}$$

```
Series[fp[z, 1000], {z, 0, 3}]
1 - 2.54915 z + 2.64804 z^2 - 1.58025 z^3 + O[z]^4
{a1[1000], a2[1000], a3[1000]}
{-2.54915, 2.64804, -1.58025}
```

Compare this formula with Formula 2.2.1 (**02 Infinite-degree Equation with Integers as Roots**), we obtain the following formula.

Formula 3.2.2

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{1}{\left(r_1 r_2\cdots r_n\right)^2}=\frac{\pi^{2n}}{(2n+1)\,!} \tag{2.1}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{1}{\left\{(2r_1-1)(2r_2-1)\cdots(2r_n-1)\right\}^2}=\frac{\pi^{2n}}{(4n)!!} \tag{2.2}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{(-1)^{r_1+r_2+\cdots+r_n}}{r_1 r_2\cdots r_n}=\frac{\alpha_n+(-1)^n\beta_n}{(2n)!!}+\sum_{s=1}^{n-1}\frac{(-1)^{n-s}\alpha_s\beta_{n-s}}{(2s)!!\{2(n-s)\}!!} \tag{2.3}$$

Where, $\psi_r(z)$ is the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials and

$$\alpha_r=\sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

$$\beta_r=\sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right), \psi_1\left(\frac{1}{2}\right), \ldots, \psi_{r-1}\left(\frac{1}{2}\right)\right) \qquad r=1,2,3,\cdots$$

Proof

From Formula 2.2.1 (**02 Infinite-degree Equation with Integers as Roots**),

$$\prod_{r=1}^{\infty}\left(1-\frac{z^2}{r^2}\right)=\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(2r+1)\,!}z^{2r} \tag{evn}$$

$$\prod_{r=1}^{\infty}\left\{1-\left(\frac{z}{2r-1}\right)^2\right\}=\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(4r)!!}z^{2r} \tag{odd}$$

The left side of (evn) is merely a replacement of z/z_r with z^2/r^2 in Formula 3.2.1 . So,

$$a_1=-\sum_{r=1}^{\infty}\frac{1}{r^2}=-\frac{\pi^2}{3\,!}$$

$$a_2=\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{r^2 s^2}=\frac{\pi^4}{5\,!}$$

$$a_3=-\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{r^2 s^2 t^2}=-\frac{\pi^6}{7\,!}$$

⋮

$$a_n = (-1)^n \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \cdots \sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{(r_1 r_2 \cdots r_n)^2} = (-1)^n \frac{\pi^{2n}}{(2n+1)!}$$

Thus, (2.1) is obtained.

The left side of (odd) is merely a replacement of z/z_r with $z^2/(2r-1)^2$ in Formula 3.2.1 . Therefore, in a similar way as the above, we obtain (2.2) .

From Formula 2.2.1 (**02 Infinite-degree Equation with Integers as Roots**) ,

$$\prod_{r=1}^{\infty} \left(1 - \frac{z}{2r-1}\right)\left(1 + \frac{z}{2r}\right) = 1 + \frac{\alpha_1 - \beta_1}{2!!} z^1$$
$$+ \sum_{r=2}^{\infty} \left\{ \frac{\alpha_r + (-1)^r \beta_r}{(2r)!!} + \sum_{s=1}^{r-1} \frac{(-1)^{r-s} \alpha_s \beta_{r-s}}{(2s)!!\{2(r-s)\}!!} \right\} z^r$$

If Formula 3.2.1 is applied to this, since $z_r = -(-1)^r r \quad (r=1,2,3,\cdots)$, these coefficients are expressed as follows.

$$a_1 = \sum_{r=1}^{\infty} \frac{(-1)^r}{r} = \frac{\alpha_1 - \beta_1}{2!!}$$

$$a_2 = \sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \frac{(-1)^{r+s}}{r\,s} = \frac{\alpha_2 + \beta_2}{4!!} - \frac{\alpha_1}{2!!} \frac{\beta_1}{2!!}$$

$$a_3 = \sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \sum_{t=s+1}^{\infty} \frac{(-1)^{r+s+t}}{r\,s\,t} = \frac{\alpha_3 - \beta_3}{6!!} + \frac{\alpha_1}{2!!} \frac{\beta_2}{4!!} - \frac{\alpha_2}{4!!} \frac{\beta_1}{2!!}$$

$$\vdots$$

$$a_n = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \cdots \sum_{r_n=r_{n-1}+1}^{\infty} \frac{(-1)^{r_1+r_2+\cdots+r_n}}{r_1 r_2 \cdots r_n}$$
$$= \frac{\alpha_n + (-1)^n \beta_n}{(2n)!!} + \sum_{s=1}^{n-1} (-1)^{n-s} \frac{\alpha_s \beta_{n-s}}{(2s)!!\{2(n-s)\}!!}$$

That is, we obtain (2.3) .

Example1

When both sides of (2.1) are calculated for $n=3$, it is as follows. Althogh the left side is calculated up to $m=1,100$, both sides is equal to up to 3 decimal places

$$a_3[m\_] := \sum_{r=1}^{m} \sum_{s=r+1}^{m} \sum_{t=s+1}^{m} \frac{1}{(r\,s\,t)^2} \qquad f[n\_] := \frac{\pi^{2n}}{(2n+1)!}$$

```
N[{a3[1100] , f[3]}]
{0.190015 , 0.190752}
```

Example2

When both sides of (2.2) are calculated for $n=3$, it is as follows. Althogh the left side is calculated up to $m=1,500$, both sides is equal to up to 4 decimal places

$$a_3[m\_] := \sum_{r=1}^{m}\sum_{s=r+1}^{m}\sum_{t=s+1}^{m}\frac{1}{((2r-1)(2s-1)(2t-1))^2} \qquad f[n\_] := \frac{\pi^{2n}}{(4n)!!}$$

N[{a3[1500] , f[3]}]

{0.0208212 , 0.0208635}

Example3

When both sides of (2.3) are calculated for $n=2$, it is as follows. Here, the polynomial $B_{r,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica.*** Althogh the left side is calculated up to $m=5000$, both sides is equal to up to 4 decimal places

$$a_2[m\_] := \sum_{r=1}^{m}\sum_{s=r+1}^{m}\frac{(-1)^{r+s}}{r\,s}$$

Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]

$$\alpha_{r\_} := \sum_{k=1}^{r}(-1)^k \text{BellY}[r,k,\text{Tbl}\psi[r,1]] \qquad \beta_{r\_} := \sum_{k=1}^{r}(-1)^k \text{BellY}\left[r,k,\text{Tbl}\psi\left[r,\frac{1}{2}\right]\right]$$

$$N\left[\left\{a_2[5000],\ \frac{\alpha_2+\beta_2}{4!!}-\frac{\alpha_1\,\beta_1}{2!!\quad 2!!}\right\}\right]$$

{-0.58221 , -0.58224}

3.3 Vieta's Formulas (Part2)

In this section, we consider the relationship between roots and coefficients when a function $f(z)$ is factored incompletely at the zeros.

Formula 3.3.1

Assume that the function $f(z)$ on the complex plane has zeros $z_1, z_2, z_3, z_4, \cdots$ and is incompletely factored as follows.

$$f(z) = \prod_{r=1}^{\infty}\left(1-\frac{z}{z_r}\right)e^{\frac{z}{z_r}}$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+c_1 z^1 + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots \qquad (3.s)$$

$$c_1 = \frac{a_1 a_1^0}{0!} - \frac{a_0 a_1^1}{1!}$$

$$c_2 = \frac{a_2 a_1^0}{0!} - \frac{a_1 a_1^1}{1!} + \frac{a_0 a_1^2}{2!}$$

$$c_3 = \frac{a_3 a_1^0}{0!} - \frac{a_2 a_1^1}{1!} + \frac{a_1 a_1^2}{2!} - \frac{a_0 a_1^3}{3!}$$

$$\vdots$$

$$c_r = \sum_{s=0}^{r} (-1)^s \frac{a_{r-s} a_1^s}{s!}$$

Where,

$$a_0 = 1\ ,\ a_1 = -\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\ ,\ a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1} z_{r_2}}\ ,\ a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1} z_{r_2} z_{r_3}}\ ,\ \cdots$$

Especially, $c_1 \sim c_4$ are expressed briefly as follows.

$$c_1 = 0$$

$$c_2 = -\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{z_r^2}$$

$$c_3 = -\frac{1}{3}\sum_{r=1}^{\infty}\frac{1}{z_r^3}$$

$$c_4 = -\frac{1}{8}\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}^4} + \frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}^2 z_{r_2}^2}$$

Proof

Divide the function $f(z)$ as follows.

$$f(z) = \prod_{r=1}^{\infty}\left(1-\frac{z}{z_r}\right)e^{\frac{z}{z_r}} = \prod_{r=1}^{\infty}\left(1-\frac{z}{z_r}\right)\cdot e^{\sum_{r=1}^{\infty}\frac{z}{z_r}} \equiv f_1(z) f_2(z)$$

Then, $f_1(z)$ and $f_2(z)$ are expanded to the Maclaurin series respectively as follows.

$$f_1(z) = \prod_{r=1}^{\infty}\left(1-\frac{z}{z_r}\right) = 1+a_1z^1+a_2z^2+a_3z^3+a_4z^4+\cdots$$

$$f_2(z) = e^{z\sum_{r=1}^{\infty}\frac{1}{z_r}} = 1+b_1z^1+b_2z^2+b_3z^3+b_4z^4+\cdots$$

Where,

$$a_0 = 1\ ,\ a_1 = -\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\ ,\ a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}}\ ,\ a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}}\ ,\cdots$$

$$b_0 = 1\ ,\ b_1 = \frac{1}{1!}\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\ ,\ b_2 = \frac{1}{2!}\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^2\ ,\ b_3 = \frac{1}{3!}\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^3\ ,\cdots$$

Taking the Cauchy Product of $f_1(z)$ and $f_2(z)$,

$$f_1(z)f_2(z) = \left(\sum_{r=0}^{\infty}a_rz^r\right)\left(\sum_{r=0}^{\infty}b_rz^r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}a_{r-s}b_sz^r$$

Here,

$$b_s = (-1)^s\frac{a_1^s}{s!} \qquad s=0,1,2,\cdots$$

Using this,

$$\sum_{r=0}^{\infty}\sum_{s=0}^{r}a_{r-s}b_sz^r = \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^s\frac{a_{r-s}a_1^s}{s!}z^r$$

Therefore,

$$f(z) = \prod_{r=1}^{\infty}\left(1-\frac{z}{z_r}\right)e^{\frac{z}{z_r}} = \sum_{r=0}^{\infty}\left\{\sum_{s=0}^{r}(-1)^s\frac{a_{r-s}a_1^s}{s!}\right\}z^r$$

If the first few terms are written down, it is as follows.

$$f(z) = 1+\left(\frac{a_1a_1^0}{0!}-\frac{a_0a_1^1}{1!}\right)z^1+\left(\frac{a_2a_1^0}{0!}-\frac{a_1a_1^1}{1!}+\frac{a_0a_1^2}{2!}\right)z^2$$
$$+\left(\frac{a_3a_1^0}{0!}-\frac{a_2a_1^1}{1!}+\frac{a_1a_1^2}{2!}-\frac{a_0a_1^3}{3!}\right)z^3+\cdots$$

Thus, we obtain $c_1, c_2, c_3, \cdots$.

Especially,

$$c_1 = \frac{a_1a_1^0}{0!}-\frac{a_0a_1^1}{1!} = a_1 - a_1 = 0$$

$$c_2 = \frac{a_2a_1^0}{0!}-\frac{a_1a_1^1}{1!}+\frac{a_0a_1^2}{2!} = a_2-\frac{a_1^2}{2} = -\frac{1}{2}\left(a_1^2-2a_2\right)$$
$$= -\frac{1}{2}\left\{\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^2-2\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}}\right\} = -\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{z_r^2}$$

Next,

$$c_3 = \frac{a_3a_1^0}{0!}-\frac{a_2a_1^1}{1!}+\frac{a_1a_1^2}{2!}-\frac{a_0a_1^3}{3!} = a_3-a_2a_1+\frac{a_1^3}{3} = -\frac{1}{3}\left(-a_1^3+3a_1a_2-3a_3\right)$$

$$= -\frac{1}{3}\left\{\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^3 - 3\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\cdot\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}} + 3\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}}\right\}$$

Here, according to Formula 1.3.1 (**01 Power of Infinite Series**),

$$\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^3 = \sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}^3} + 3\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\cdot\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}} - 3\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}}$$

Substituting this for the above,

$$c_3 = -\frac{1}{3}\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}^3}$$

Next,

$$c_4 = \frac{a_4 a_1^0}{0!} - \frac{a_3 a_1^1}{1!} + \frac{a_2 a_1^2}{2!} - \frac{a_1 a_1^3}{3!} + \frac{a_0 a_1^4}{4!} = a_4 - a_3 a_1 + \frac{a_2 a_1^2}{2} - \frac{a_1^4}{8}$$

$$= \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}z_{r_4}} - \left(\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}}\right)\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}$$

$$+ \frac{1}{2}\left(\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}}\right)\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^2 - \frac{1}{8}\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^4$$

Here, according to Formula 1.3.1 (**01 Power of Infinite Series**),

$$\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^4 = \sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}^4} - 2\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}^2 z_{r_2}^2} + 4\left(\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}}\right)\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^2$$

$$- 8\left(\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}}\right)\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}$$

$$+ 8\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}z_{r_4}}$$

Substituting this for the above,

$$c_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}z_{r_4}} - \left(\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}}\right)\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}$$

$$+ \frac{1}{2}\left(\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}}\right)\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^2 - \frac{1}{8}\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}^4}$$

$$+ \frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}^2 z_{r_2}^2} - \frac{1}{2}\left(\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}}\right)\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^2$$

$$+ \left(\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}}\right)\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}$$

$$- \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}z_{r_4}}$$

i.e.

$$c_4 = -\frac{1}{8}\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}^4} + \frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}^2 z_{r_2}^2}$$

Example Infinite-degree Equation with Natural Numbers as Roots

Such a function $f_p(z)$ is expressed with the following infinite products.

$$f_p(z) = \prod_{r=1}^{\infty}\left(1-\frac{z}{r}\right)e^{\frac{z}{r}}$$

Assume that this is expanded to a power series as follows.

$$f_s(z) = 1+c_1z^1+c_2z^2+c_3z^3+c_4z^4+\cdots$$

If the above formula is applied, since $z_r = r \ \ (r=1,2,3,\cdots)$, these coefficients are expressed as follows.

$$c_r = \sum_{s=0}^{r}(-1)^s\frac{a_{r-s}\,a_1^s}{s\,!} \qquad r=1,2,3,\cdots$$

Where,

$$a_0 = 1 \ ,\ a_1 = -\sum_{r_1=1}^{\infty}\frac{1}{r_1} \ ,\ a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{r_1 r_2} \ ,\ a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{r_1 r_2 r_3} \ ,\ \cdots$$

When $f_p(z)$ is expanded up to z^3 and c_1, c_2, c_3 are calculated by formula manipulation software ***Mathematica***, it is as follows. Although both are calculated by 1000 terms, the coefficients of both are exactly the same.

$$a_0[m\_] := 1 \quad a_1[m\_] := -\sum_{r_1=1}^{m}\frac{1}{r_1} \quad a_2[m\_] := \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{1}{r_1 r_2} \quad a_3[m\_] := -\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m}\frac{1}{r_1 r_2 r_3}$$

$$\mathrm{fp}[z\_,\ m\_] := \prod_{r=1}^{m}\left(1-\frac{z}{r}\right)\equiv^{\frac{z}{r}} \qquad c_{r\_}[m\_] := \sum_{s=0}^{r}(-1)^s\frac{a_{r-s}[m]\ a_1[m]^s}{s\,!}$$

```
N[Series[fp[z, 1000], {z, 0, 3}]]
```

$$1. - 0.821967\,(z+0.)^2 - 0.400685\,(z+0.)^3 + O[z+0.]^4$$

```
N[{c1[1000], c2[1000], c3[1000]}]
{0., -0.821967, -0.400685}
```

Especially, $c_1 \sim c_4$ are expressed briefly as follows.

$$c_1 = 0 \quad ,\quad c_2 = -\frac{1}{2}\sum_{r_1=1}^{\infty}\frac{1}{r_1^2} = -\frac{\zeta(2)}{2} \quad ,\quad c_3 = -\frac{1}{3}\sum_{r_1=1}^{\infty}\frac{1}{r_1^3} = -\frac{\zeta(3)}{3}$$

$$c_4 = -\frac{1}{8}\sum_{r_1=1}^{\infty}\frac{1}{r_1^4} + \frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{r_1^2 r_2^2} = -\frac{\zeta(4)}{8} + \frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{r_1^2 r_2^2}$$

Now, if the accuracy is increased, and if the Maclaurin series of $f_p(z)$ and the coefficients c_2, c_3, c_4 are calculated, it is as follows. Although $f_p(z)$ is calculated up to $m=10{,}000$ and c_4 is calculated up to $m=5{,}000$, both are almost equal.

$$c_2 := -\frac{\mathrm{Zeta}[2]}{2} \qquad c_3 := -\frac{\mathrm{Zeta}[3]}{3} \qquad c_4[m\_] := -\frac{\mathrm{Zeta}[4]}{8} + \frac{1}{4}\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{1}{(r_1 r_2)^2}$$

```
N[Series[fp[z, 10000], {z, 0, 4}]]
```

$$1. - 0.822417\,(z+0.)^2 - 0.4006856\,(z+0.)^3 + 0.0676041\,(z+0.)^4 + O[z+0.]^5$$

```
N[{c2, c3, c4[5000]}]
{-0.822467, -0.400686, 0.067563}
```

Compare this formula with Formula 2.1.1 (**02 Infinite-degree Equation with Integers as Roots**) , we obtain the following formula.

Formula 3.3.2

Let $\zeta(z)$ be the Riemann zeta function, $\psi_r(z)$ be the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials, γ be Euler-Mascheroni constant and α_r , a_r are the following constants.

$$\alpha_r = \sum_{k=1}^{r} (-1)^k B_{r,k}(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)) \qquad r=1,2,3,\cdots$$

$$a_0 = 1 , \; a_1 = -\sum_{r_1=1}^{\infty} \frac{1}{r_1} , \; a_2 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{r_1 r_2} , \; a_3 = -\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{r_1 r_2 r_3} , \cdots$$

Then, the following expression holds.

$$\sum_{s=0}^{r} (-1)^s \frac{a_{r-s} a_1^s}{s!} = \frac{\gamma^r}{r!} + \sum_{s=1}^{r} \frac{(-1)^s \alpha_s \gamma^{r-s}}{s!(r-s)!} \qquad r=1,2,3,\cdots$$

Especially,

$$-\frac{\zeta(2)}{2} = \frac{\gamma^2}{2!} - \frac{\alpha_1 \gamma^1}{1!1!} + \frac{\alpha_2 \gamma^0}{2!0!}$$

$$-\frac{\zeta(3)}{3} = \frac{\gamma^3}{3!} - \frac{\alpha_1 \gamma^2}{1!2!} + \frac{\alpha_2 \gamma^1}{2!1!} - \frac{\alpha_3 \gamma^0}{3!0!}$$

$$-\frac{\zeta(4)}{8} + \frac{1}{4} \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{r_1^2 r_2^2} = \frac{\gamma^4}{4!} - \frac{\alpha_1 \gamma^3}{1!3!} + \frac{\alpha_2 \gamma^2}{2!2!} - \frac{\alpha_3 \gamma^1}{3!1!} + \frac{\alpha_4 \gamma^0}{4!0!}$$

Proof

Substituting $z_r = r \quad r=1,2,3,\cdots$ for Formula 3.3.1 ,

$$\prod_{r=1}^{\infty} \left(1 - \frac{z}{r}\right) e^{\frac{z}{r}} = 1 + c_1 z^1 + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots$$

$$c_r = \sum_{s=0}^{r} (-1)^s \frac{a_{r-s} a_1^s}{s!} \tag{3.c}$$

Where,

$$a_0 = 1 , \; a_1 = -\sum_{r_1=1}^{\infty} \frac{1}{r_1} , \; a_2 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{r_1 r_2} , \; a_3 = -\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{r_1 r_2 r_3} , \cdots$$

Especially, c_2, c_3, c_4 are expressed briefly as follows.

$$c_2 = -\frac{1}{2} \sum_{r=1}^{\infty} \frac{1}{r^2} = -\frac{\zeta(2)}{2}$$

$$c_3 = -\frac{1}{3} \sum_{r=1}^{\infty} \frac{1}{r^3} = -\frac{\zeta(3)}{3}$$

$$c_4 = -\frac{1}{8}\sum_{r=1}^{\infty}\frac{1}{r^4} + \frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{r_1^2 r_2^2} = -\frac{\zeta(4)}{8} + \frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{r_1^2 r_2^2}$$

On the other hand, according to Formula 2.1.1 (**02 Infinite-degree Equation with Integers as Roots**),

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{r}\right)e^{\frac{z}{r}} = 1 + \sum_{r=1}^{\infty}\left\{\frac{\gamma^r}{r!} + \sum_{s=1}^{r}\frac{(-1)^s\alpha_s\gamma^{r-s}}{s!(r-s)!}\right\}z^r \tag{1.1$_+$}$$

From (3.c) and (1.1$_+$),

$$c_r = \sum_{s=0}^{r}(-1)^s\frac{a_{r-s}a_1^s}{s!} = \frac{\gamma^r}{r!} + \sum_{s=1}^{r}\frac{(-1)^s\alpha_s\gamma^{r-s}}{s!(r-s)!} \qquad r=1,2,3,\cdots$$

Especially,

$$c_2 = -\frac{\zeta(2)}{2} = \frac{\gamma^2}{2!} - \frac{\alpha_1\gamma^1}{1!1!} + \frac{\alpha_2\gamma^0}{2!0!}$$

$$c_3 = -\frac{\zeta(3)}{3} = \frac{\gamma^3}{3!} - \frac{\alpha_1\gamma^2}{1!2!} + \frac{\alpha_2\gamma^1}{2!1!} - \frac{\alpha_3\gamma^0}{3!0!}$$

$$c_4 = -\frac{\zeta(4)}{8} + \frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{r_1^2 r_2^2} = \frac{\gamma^4}{4!} - \frac{\alpha_1\gamma^3}{1!3!} + \frac{\alpha_2\gamma^2}{2!2!} - \frac{\alpha_3\gamma^1}{3!1!} + \frac{\alpha_4\gamma^0}{4!0!}$$

Example c_3

We verify the following equations.

$$-\frac{\zeta(3)}{3} = \frac{\gamma^3}{3!} - \frac{\alpha_1\gamma^2}{1!2!} + \frac{\alpha_2\gamma^1}{2!1!} - \frac{\alpha_3\gamma^0}{3!0!}$$

Substituting $\alpha_1 = -\psi_0(1)$, $\alpha_2 = \psi_0^2(1) - \psi_1(1)$, $\alpha_3 = -\psi_0^3(1) + 3\psi_0(1)\psi_1(1) - \psi_2(1)$ for the right side,

$$\frac{\gamma^3}{3!} - \frac{\alpha_1\gamma^2}{1!2!} + \frac{\alpha_2\gamma^1}{2!1!} - \frac{\alpha_3\gamma^0}{3!0!} = \frac{\gamma^3}{3!} + \frac{\psi_0(1)\gamma^2}{1!2!} + \frac{\{\psi_0^2(1) - \psi_1(1)\}\gamma^1}{2!1!} + \frac{\{\psi_0^3(1) - 3\psi_0(1)\psi_1(1) + \psi_2(1)\}\gamma^0}{3!0!}$$

Further, substituting $\psi_0[1] = -\gamma$ for the right side,

$$\frac{\gamma^3}{3!} - \frac{\alpha_1\gamma^2}{1!2!} + \frac{\alpha_2\gamma^1}{2!1!} - \frac{\alpha_3\gamma^0}{3!0!} = \frac{\gamma^3}{3!} - \frac{\gamma^3}{1!2!} + \frac{\{\gamma^2 - \psi_1(1)\}\gamma^1}{2!1!} + \frac{-\gamma^3 + 3\gamma\psi_1(1) + \psi_2(1)}{3!0!}$$

$$= -\frac{\psi_1(1)\gamma}{2!} + \frac{3\psi_1(1)\gamma}{3!} + \frac{\psi_2(1)}{3!}$$

i.e.

$$\frac{\gamma^3}{3!} - \frac{\alpha_1\gamma^2}{1!2!} + \frac{\alpha_2\gamma^1}{2!1!} - \frac{\alpha_3\gamma^0}{3!0!} = \frac{\psi_2(1)}{3!}$$

So,

$$-\frac{\zeta(3)}{3}=\frac{\psi_2(1)}{3!}$$

i.e.

$$\zeta(3)=-\frac{\psi_2(1)}{2}$$

This relational expression is well known. Therefore, the equality of this example holds.

In a similar way as the proof of Formula 3.3.2 , we obtain the following formula.

Formula 3.3.3

Let $\lambda(z)$ be the Dirichlet lamda function, $\psi_r(z)$ be the polygamma function, $B_{r,k}(f_1, f_2, \ldots)$ are Bell polynomials, γ be Euler-Mascheroni constant and β_r , a_r are the following constants.

$$\beta_r=\sum_{k=1}^{r}(-1)^k B_{r,k}\left(\psi_0\left(\frac{1}{2}\right),\psi_1\left(\frac{1}{2}\right),\ldots,\psi_{r-1}\left(\frac{1}{2}\right)\right)\qquad r=1,2,3,\cdots$$

$$a_0=1\quad,\quad a_1=-\sum_{r_1=1}^{\infty}\frac{1}{2r_1-1}\quad,\quad a_2=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{(2r_1-1)(2r_2-1)}$$

$$a_3=-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{(2r_1-1)(2r_2-1)(2r_3-1)}\quad,\ \cdots$$

Then, the following expression holds.

$$\sum_{s=0}^{r}(-1)^s\frac{a_{r-s}\,a_1^s}{s\,!}=\frac{\left(\frac{\gamma}{2}+\log 2\right)^r}{r\,!}+\sum_{s=1}^{r}\frac{(-1)^s\beta_s\left(\frac{\gamma}{2}+\log 2\right)^{r-s}}{(2s)!!\,(r-s)\,!}$$

Especially,

$$-\frac{\lambda(2)}{2}=\frac{\left(\frac{\gamma}{2}+\log 2\right)^2}{2\,!}+\sum_{s=1}^{2}\frac{(-1)^s\beta_s\left(\frac{\gamma}{2}+\log 2\right)^{2-s}}{(2s)!!\,(2-s)\,!}$$

$$-\frac{\lambda(3)}{3}=\frac{\left(\frac{\gamma}{2}+\log 2\right)^3}{3\,!}+\sum_{s=1}^{3}\frac{(-1)^s\beta_s\left(\frac{\gamma}{2}+\log 2\right)^{3-s}}{(2s)!!\,(3-s)\,!}$$

$$-\frac{\lambda(4)}{8}+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{(2r_1-1)^2(2r_2-1)^2}=\frac{\left(\frac{\gamma}{2}+\log 2\right)^4}{4\,!}+\sum_{s=1}^{4}\frac{(-1)^s\beta_s\left(\frac{\gamma}{2}+\log 2\right)^{4-s}}{(2s)!!\,(4-s)\,!}$$

If three lines of "Especially" are calculated, it is as follows. The left side is f and the right side is g . Here, the polynomial $B_{r,k}(f_1, f_2, \ldots)$ is generated using the function $BellY[\,]$ of formula manipulation software ***Mathematica***. Both sides are exactly matched for the second and third lines. Both sides of the fourth line are almost equal.

```
λ[z_] := DirichletLambda[z]        γ := EulerGamma
```

$$f[r\_] := -\frac{\lambda[r]}{r}\qquad f4[m\_] := -\frac{\lambda[4]}{8}+\frac{1}{4}\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{1}{(2\,r_1-1)^2(2\,r_2-1)^2}$$

```
Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]
```

$$\beta_{s\_} := \sum_{k=1}^{s} (-1)^k \text{BellY}\left[s, k, \text{Tbl}\psi\left[s, \frac{1}{2}\right]\right]$$

$$g[r\_] := \frac{\left(\frac{\gamma}{2} + \text{Log}[2]\right)^r}{r!} + \sum_{s=1}^{r} \frac{(-1)^s \beta_s \left(\frac{\gamma}{2} + \text{Log}[2]\right)^{r-s}}{(2 s)!! (r - s)!}$$

```
N[{f[2], g[2]}]            N[{f[3], g[3]}]          N[{f4[5000], g[4]}]
{-0.61685, -0.61685}       {-0.3506, -0.3506}       {-0.0634328, -0.0634174}
```

3.4 Infinite-degree Equation with Real Coefficients

In this section, we describe the properties of an infinite-degree equation with real coefficients.

Definition

When the function $f(z)$ defined in the domain D satisfies

$$f\left(\overline{z}\right) = \overline{f(z)} \qquad z \in D$$

we say that the $f(z)$ has **complex conjugate property**. ($\overline{z}$ denotes the conjugate complex number of z)

Theorem 3.4.1

If $a_k \ (k=0,1,2,\cdots)$ are real numbers in function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then $f\left(\overline{z}\right) = \overline{f(z)}$.

Proof

Convert cartesian coordinates $z = x+iy$ to polar coordinates as follows.

$$z = r(\cos\theta + i\sin\theta) \qquad (r=|z| \ , \ \theta = \arg z \ , \ -\pi < \theta \le \pi)$$

Then,

$$z^k = r^k(\cos k\theta + i\sin k\theta) \qquad k=0,1,2,\cdots$$

Using this,

$$f(z) = \sum_{k=0}^{\infty} a_k r^k(\cos k\theta + i\sin k\theta) = \sum_{k=0}^{\infty} a_k r^k \cos k\theta + i\sum_{k=0}^{\infty} a_k r^k \sin k\theta$$

$$\because \ a_k \ (k=0,1,2,\cdots) \ \text{are real numbers}$$

On the other hand, from $\overline{z} = r(\cos\theta - i\sin\theta)$

$$\left(\overline{z}\right)^k = r^k(\cos k\theta - i\sin k\theta) \qquad k=0,1,2,\cdots$$

Using this,

$$f\left(\overline{z}\right) = \sum_{k=0}^{\infty} a_k r^k(\cos k\theta - i\sin k\theta) = \sum_{k=0}^{\infty} a_k r^k \cos k\theta - i\sum_{k=0}^{\infty} a_k r^k \sin k\theta$$

$$\because \ a_k \ (k=0,1,2,\cdots) \ \text{are real numbers}$$

$$\therefore \ f\left(\overline{z}\right) = \overline{f(z)}$$

From this theorem, the following important corollary on the infinite-degree equation with real coefficients is derived.

Corollary 3.4.1

When $a_k \ (k=0,1,2,\cdots)$ are real numbers in the infinite-degree equation $\sum_{k=0}^{\infty} a_k z^k = 0$,

if z_0 is the root, $\overline{z_0}$ is also the root.

Proof

Let

$$\sum_{k=0}^{\infty} a_k z^k = f(z) = u(x) + i v(z)$$

When z_0 is the root of this,

$$f(z_0) = u(x_0) + i v(z_0) = 0$$

From this, since $u(z_0) = 0$, $v(z_0) = 0$,

$$\overline{f(z_0)} = u(x_0) - i v(z_0) = 0$$

From Theorem 3.4.1 ,

$$\overline{f(z_0)} = f(\overline{z_0})$$

Therefore, $\overline{z_0}$ is the root of $f(z) = \sum_{k=0}^{\infty} a_k z^k = 0$.

3.5 Vieta's Formulas (Part3)

In this section, we consider the relationship between roots and coefficients in the infinite-degree equation which has real coefficients and is factored completely.

Formula 3.5.1 (Infinite-degree Equation with Conjugate Complex Roots)

Assume that the function $f(z)$ on the complex plane has zeros $z_k = x_k \pm i y_k \ , y_k \neq 0 \quad (k=1,2,3,\cdots)$ and is completely factored as follows.

$$f(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right) = \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+a_1z^1+a_2z^2+a_3z^3+a_4z^4+\cdots \tag{5.s}$$

Where,

$$a_1 = -\sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}$$

$$a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^2 x_{r_1}x_{r_2}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}$$

$$a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^3 x_{r_1}x_{r_2}x_{r_3}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$$\begin{aligned} a_4 = &\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{2^4 x_{r_1}x_{r_2}x_{r_3}x_{r_4}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)} \\ &+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\left(x_{r_1}x_{r_2}+x_{r_1}x_{r_3}+x_{r_2}x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} \\ &+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} \end{aligned}$$

$$\vdots$$

$$\begin{aligned} a_{2n-1} = &-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-1} x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)} \\ &-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-2}=r_{2n-3}+1}^{\infty}\frac{2^{2n-3}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-2}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-2}}^2+y_{r_{2n-2}}^2\right)} \\ &\ \vdots \\ &-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}+\cdots+x_{r_n}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)} \end{aligned}$$

$$a_{2n} = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty}\frac{2^{2n} x_{r_1}x_{r_2}\cdots x_{r_{2n}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-2}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$\vdots$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

Proof

When the roots of (5.s) are $z_k=x_k\pm iy_k$, $y_k\neq 0$ $(k=1,2,3,\cdots)$,

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)=\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right)=0$$

i.e.

$$\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)=0 \tag{5.p}$$

For simplicity, we make the following substitution.

$$\frac{2x_r}{x_r^2+y_r^2}=X_r \quad , \quad \frac{1}{x_r^2+y_r^2}=I_r$$

Then, (5.p) becomes

$$\prod_{r=1}^{\infty}\left(1-X_r z+I_r z^2\right)=\left(1-X_1 z+I_1 z^2\right)\left(1-X_2 z+I_2 z^2\right)\left(1-X_3 z+I_3 z^2\right)\cdots \tag{5.p'}$$

If (5.s) and (5.p') are compared and the coefficient of (5.s) is calculated, it is as follows.

$$a_1=-X_1-X_2-X_3-\cdots=-\sum_{r_1=1}^{\infty}X_{r_1}$$

$$a_2=X_1\left(X_2+X_3+X_4+\cdots\right)+X_2\left(X_3+X_4+X_5+\cdots\right)+X_3\left(X_4+X_5+X_6+\cdots\right)+\cdots+I_1+I_2+I_3+\cdots$$

$$=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}X_{r_1}X_{r_2}+\sum_{r_1=1}^{\infty}I_{r_1}$$

$$a_3=-X_1X_2\left(X_3+X_4+X_5+\cdots\right)-X_1X_3\left(X_4+X_5+X_6+\cdots\right)-X_1X_4\left(X_5+X_6+X_4+\cdots\right)-\cdots$$
$$-X_2X_3\left(X_4+X_5+X_6+\cdots\right)-X_2X_4\left(X_5+X_6+X_4+\cdots\right)-X_2X_5\left(X_6+X_7+X_8+\cdots\right)-\cdots$$
$$\vdots$$
$$-X_1\left(I_2+I_3+I_4+\cdots\right)-X_2\left(I_3+I_4+I_5+\cdots\right)-X_3\left(I_4+I_5+I_6+\cdots\right)-\cdots$$
$$-I_1\left(X_2+X_3+X_4+\cdots\right)-I_2\left(X_3+X_4+X_5+\cdots\right)-I_3\left(X_4+X_5+X_6+\cdots\right)-\cdots$$

$$=-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}X_{r_1}X_{r_2}X_{r_3}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\left(X_{r_1}I_{r_2}+I_{r_1}X_{r_2}\right)$$

$$a_4=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}X_{r_1}X_{r_2}X_{r_3}X_{r_4}$$
$$+X_1X_2\left(I_3+I_4+I_5+\cdots\right)+X_1X_3\left(I_4+I_5+I_6+\cdots\right)+X_1X_4\left(I_5+I_6+I_4+\cdots\right)+\cdots$$
$$+X_2X_3\left(I_4+I_5+I_6+\cdots\right)+X_2X_4\left(I_5+I_6+I_4+\cdots\right)+X_2X_5\left(I_6+I_7+I_8+\cdots\right)+\cdots$$

$$\vdots$$

$$+X_1I_2(X_3+X_4+X_5+\cdots)+X_1I_3(X_4+X_5+X_6+\cdots)+X_1I_4(X_5+X_6+X_4+\cdots)+\cdots$$

$$+X_2I_3(X_4+X_5+X_6+\cdots)+X_2I_4(X_5+X_6+X_4+\cdots)+X_2I_5(X_6+X_7+X_8+\cdots)+\cdots$$

$$\vdots$$

$$+I_1X_2(X_3+X_4+X_5+\cdots)+I_1X_3(X_4+X_5+X_6+\cdots)+I_1X_4(X_5+X_6+X_4+\cdots)+\cdots$$

$$+I_2X_3(X_4+X_5+X_6+\cdots)+I_2X_4(X_5+X_6+X_4+\cdots)+I_2X_5(X_6+X_7+X_8+\cdots)+\cdots$$

$$\vdots$$

$$+I_1(I_2+I_3+I_4+\cdots)+I_2(I_3+I_4+I_5+\cdots)+I_3(I_4+I_5+I_6+\cdots)+\cdots$$

$$=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}X_{r_1}X_{r_2}X_{r_3}X_{r_4}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\left(X_{r_1}X_{r_2}I_{r_3}+X_{r_1}I_{r_2}X_{r_3}+I_{r_1}X_{r_2}X_{r_3}\right)+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}I_{r_1}I_{r_2}$$

Returning to the original symbol, we obtain $a_1 \sim a_4$. And we obtain a_{2n-1}, a_{2n} by induction.

Example When $z_r = r^2 \pm ir \quad (r=1,2,3,\cdots)$

$$f_p(z) = \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right) = \prod_{r=1}^{\infty}\left(1-\frac{2z}{r^2+1}+\frac{z^2}{r^4+r^2}\right)$$

If $f_p(z)$ is expanded up to z^3 by formula manipulation software ***Mathematica***, it is as follows.

$$\text{fp[z\_, m\_]} := \prod_{r=1}^{m}\left(1-\frac{2z}{r^2+1}+\frac{z^2}{r^4+r^2}\right)$$

```
N[Series[fp[z, 10000], {z, 0, 3}]]
```

$$1. - 2.1535\,(z+0.) + 2.27261\,(z+0.)^2 - 1.40118\,(z+0.)^3 + O[z+0.]^4$$

On the other hand, if a_1, a_2, a_3 are calculated, it is as follows. Both coefficients are approximately equal.

$$a_1[m\_] := -\sum_{r_1=1}^{m}\frac{2\,r_1^2}{r_1^4+r_1^2} \qquad a_2[m\_] := \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{2^2\,r_1^2\,r_2^2}{(r_1^4+r_1^2)\,(r_2^4+r_2^2)}+\sum_{r_1=1}^{m}\frac{2^0}{r_1^4+r_1^2}$$

$$a_3[m\_] := -\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m}\frac{2^3\,r_1^2\,r_2^2\,r_3^2}{(r_1^4+r_1^2)\,(r_2^4+r_2^2)\,(r_3^4+r_3^2)}-\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{2^1\,(r_1^2+r_2^2)}{(r_1^4+r_1^2)\,(r_2^4+r_2^2)}$$

N[a_1[1000]]	N[a_2[1500]]	N[a_3[2000]]
-2.15135	2.27017	-1.39936

Corollary 3.5.1 (Infinite-degree Equation with Conjugate Imaginary Roots)

When an infinite-degree equation

$$1+a_1z^1+a_2z^2+a_3z^3+a_4z^4+\cdots = 0$$

has the roots $z_k = \pm iy_k$, $y_k \neq 0 \;\; (k=1,2,3,\cdots)$ and is completely factored by these roots,

$$a_{2r-1} = 0 \qquad (r=1,2,3,\cdots)$$

$$a_2 = \sum_{r_1=1}^{\infty}\frac{1}{y_{r_1}^2}$$

$$a_4 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2}$$

$$a_6 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2 y_{r_3}^2}$$

$$a_8 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2 y_{r_3}^2 y_{r_4}^2}$$

$$\vdots$$

Example Infinite-degree Equation with All Imaginary Integers as Roots

$$1 + \frac{\pi^2}{3!} z^2 + \frac{\pi^4}{5!} z^4 + \frac{\pi^6}{7!} z^6 + \cdots = 0$$

The left side is factored as follows.

$$\sum_{r=0}^{\infty} \frac{\pi^{2r}}{(2r+1)!} z^{2r} = \frac{\sinh \pi z}{\pi z} = \prod_{r=1}^{\infty} \left(1+\frac{z}{ir}\right)\left(1-\frac{z}{ir}\right)$$

Since the roots are $z_r = \pm ir \quad (r=1,2,3,\cdots)$, according to the corollary ,

$$a_2 = \sum_{r_1=1}^{\infty} \frac{1}{r_1^2} = \frac{\pi^2}{3!}$$

$$a_4 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{r_1^2 r_2^2} = \frac{\pi^4}{5!}$$

$$a_6 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{r_1^2 r_2^2 r_3^2} = \frac{\pi^6}{7!}$$

$$\vdots$$

These are consistent with Formula 3.2.2 (2.1) .

Note

An infinite-degree equation with only conjugate imaginary roots is a positive term series composed of only even powers. This contrary does not hold. That is, when a positive term series composed of only even powers has zeros, the real part is not necessarily 0. For example, although an infinite-degree equation

$$1 + \left(1+\frac{\pi^2}{3!}\right) z^2 + \left(\frac{\pi^0}{1!}+\frac{\pi^2}{3!}+\frac{\pi^4}{5!}\right) z^4 + \left(\frac{\pi^2}{3!}+\frac{\pi^4}{5!}+\frac{\pi^6}{7!}\right) z^6 + \cdots = 0$$

has roots $z_r = \pm ir \quad r=1,2,3,\cdots$, in addition to these, it has also roots $z_s = \pm\frac{1\pm i\sqrt{3}}{2} \quad s=1,2,3,4$.

In fact, this left side is made as follows.

$$\left(1+z^2+z^4\right) \sum_{r=0}^{\infty} \frac{\pi^{2r}}{(2r+1)!} z^{2r} = 1 + \left(1+\frac{\pi^2}{3!}\right) z^2$$

$$+ \sum_{r=0}^{\infty} \left\{ \frac{\pi^{2r}}{(2r+1)!} + \frac{\pi^{2r+2}}{(2r+3)!} + \frac{\pi^{2r+4}}{(2r+5)!} \right\} z^{2r+4}$$

Infinite-degree Equation with Conjugate Real Roots

Although Corollary 3.5.1 is resembled, the completely different following formula holds.

Formula 3.5.2

When an infinite-degree equation

$$1+a_1z^1+a_2z^2+a_3z^3+a_4z^4+\cdots = 0$$

has the roots $z_k = \pm x_k$, $x_k \ne 0$ $(k=1,2,3,\cdots)$ and is completely factored by these roots,

$$a_{2r-1} = 0 \quad (r=1,2,3,\cdots)$$

$$a_2 = -\sum_{r_1=1}^{\infty}\frac{1}{x_{r_1}^2}$$

$$a_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{x_{r_1}^2x_{r_2}^2}$$

$$a_6 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{x_{r_1}^2x_{r_2}^2x_{r_3}^2}$$

$$a_8 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{1}{x_{r_1}^2x_{r_2}^2x_{r_3}^2x_{r_4}^2}$$

$$\vdots$$

Proof

$$1+c_1z^1+c_2z^2+c_3z^3+c_4z^4+\cdots = 0 \qquad (2.s)$$

When the roots of (2.s) are $z_k = \pm i x_k$, $k_k \ne 0$ $(k=1,2,3,\cdots)$,

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right) = \prod_{r=1}^{\infty}\left(1+\frac{z}{x_r}\right)\left(1-\frac{z}{x_r}\right) = 0$$

i.e.

$$\prod_{r=1}^{\infty}\left(1-\frac{z^2}{x_r^2}\right) = 0$$

This is merely a replacement of z/z_r with z^2/r^2 in Formula 3.2.1 (2.s) .

As first, replacing z with z^2 in (2.s) ,

$$1+c_1z^2+c_2z^4+c_3z^6+c_4z^8+\cdots = 0$$

As second, replacing z_r with x_r^2 in $c_1, c_2, c_3, \cdots$,

$$c_1 = -\sum_{r_1=1}^{\infty}\frac{1}{x_{r_1}^2}$$

$$c_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{x_{r_1}^2x_{r_2}^2}$$

$$c_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{x_{r_1}^2x_{r_2}^2y_{r_3}^2}$$

$$c_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{1}{x_{r_1}^2 x_{r_2}^2 x_{r_3}^2 x_{r_4}^2}$$

s

$$\vdots$$

Last, replacing c_r with a_{2r} and adding the odd number terms $a_{2r-1}z^{2r-1}$ $\left(a_{2r-1}=0\right)$ to this, we obtain the desired expression.

Example Infinite-degree Equation with All Odd Number as Roots

$$1-\frac{\pi^2}{4!!}z^2+\frac{\pi^4}{8!!}z^4-\frac{\pi^6}{12!!}z^6+-\cdots=0$$

The left side is factored as follows.

$$\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(4r)!!}z^{2r}=\cos\frac{\pi z}{2}=\prod_{r=1}^{\infty}\left(1+\frac{z}{2r-1}\right)\left(1-\frac{z}{2r-1}\right)$$

Since the roots are $z_r=\pm(2r-1)$ $\left(r=1,2,3,\cdots\right)$, according to the formula ,

$$a_2=-\sum_{r_1=1}^{\infty}\frac{1}{x_{r_1}^2}=-\frac{\pi^2}{4!!}$$

$$a_4=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{x_{r_1}^2x_{r_2}^2}=\frac{\pi^4}{8!!}$$

$$a_6=-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{x_{r_1}^2x_{r_2}^2x_{r_3}^2}=-\frac{\pi^6}{12!!}$$

$$\vdots$$

These are consistent with Formula 3.2.2 (2.2) .

Note

An infinite-degree equation with only the conjugate real numbers as its root is an alternating series composed of only even powers. This contrary does not hold. That is, when an alternating series composed of only even powers has zeros, the imaginary part is not necessarily 0. For example, although an infinite-degree equation

$$1-\left(1+\frac{\pi^2}{4!!}\right)z^2+\left(\frac{\pi^0}{0!!}+\frac{\pi^2}{4!!}+\frac{\pi^4}{8!!}\right)z^4-\left(\frac{\pi^2}{4!!}+\frac{\pi^4}{8!!}+\frac{\pi^6}{12!!}\right)z^6+-\cdots=0$$

has roots $z_r=\pm(2r-1)$ $r=1,2,3,\cdots$, in addition to these, it has also roots $z_r=\pm\dfrac{\sqrt{3}\pm i}{2}$ $s=1\sim4$.
In fact, this left side is made as follows.

$$\left(1-z^2+z^4\right)\sum_{r=0}^{\infty}\frac{(-1)^r\pi^{2r}}{(4r)!!}z^{2r}=1-\left(1+\frac{\pi^2}{4!!}\right)z^2$$

$$+\sum_{r=0}^{\infty}(-1)^r\left\{\frac{\pi^{2r}}{(4r)!!}+\frac{\pi^{2r+2}}{(4r+4)!!}+\frac{\pi^{2r+4}}{(4r+8)!!}\right\}z^{2r+4}$$

3.6 Vieta's Formulas (Part4)

In this section, we consider the relationship between roots and coefficients in the infinite-degree equation which has real coefficients and is factored incompletely.

Formula 3.6.1 (Infinite-degree Equation with Conjugate Complex Roots)

Assume that the function $f(z)$ on the complex plane has zeros $z_k = x_k \pm i y_k \ , y_k \neq 0 \quad (k=1,2,3,\cdots)$ and is incompletely factored as follows.

$$f(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} = \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)e^{\frac{2x_r z}{x_r^2+y_r^2}}$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+c_1 z^1+c_2 z^2+c_3 z^3+c_4 z^4+\cdots \qquad (6.s)$$

$$c_r = \sum_{s=0}^{r}(-1)^s \frac{a_{r-s}\, a_1^s}{s\,!}$$

Where,

$$a_0 = 1$$

$$a_1 = -\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}$$

$$a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^2 x_{r1}x_{r2}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)} + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}$$

$$a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^3 x_{r1}x_{r2}x_{r3}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

$$a_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{2^4 x_{r1}x_{r2}x_{r3}x_{r4}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)\left(x_{r4}^2+y_{r4}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\left(x_{r1}x_{r2}+x_{r_1}x_{r_3}+x_{r2}x_{r3}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

$\vdots$

$$a_{2n-1} = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-1}x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-2}=r_{2n-3}+1}^{\infty}\frac{2^{2n-3}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-2}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-2}}^2+y_{r_{2n-2}}^2\right)}$$

$\vdots$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}+\cdots+x_{r_n}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

$$a_{2n}=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty}\frac{2^{2n}x_{r_1}x_{r_2}\cdots x_{r_{2n}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-2}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$$\vdots$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

Especially, $c_1 \sim c_4$ are represented by a faster formula as follows.

$$c_1 = 0$$

$$c_2 = -\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^2+\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}$$

$$c_3 = -\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^3+\sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\cdot\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$$c_4 = -\frac{1}{8}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^4+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\left(\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^2\left(\frac{2x_{r_2}}{x_{r_2}^2+y_{r_2}^2}\right)^2$$

$$-\left\{\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}\right\}\sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}+\frac{1}{2}\left(\sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}\right)^2\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\left(x_{r_1}x_{r_2}+x_{r_1}x_{r_3}+x_{r_2}x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)}+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

Proof

Divide the function $f(z)$ as follows.

$$f(z) = \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)e^{\frac{2x_r z}{x_r^2+y_r^2}}$$

$$= \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)\cdot e^{\sum_{r=1}^{\infty}\frac{2x_r z}{x_r^2+y_r^2}} \equiv f_1(z)f_2(z)$$

Then, $f_1(z)$ and $f_2(z)$ are expanded to the Maclaurin series respectively as follows.

$$f_1(z) = \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right) = 1+a_1z^1+a_2z^2+a_3z^3+a_4z^4+\cdots$$

$$f_2(z) = e^{z\sum_{r=1}^{\infty}\frac{2x_r z}{x_r^2+y_r^2}} = 1+b_1z^1+b_2z^2+b_3z^3+b_4z^4+\cdots$$

Here, according to Formula 3.5.1 , coefficients $a_1, a_2, a_3, \cdots$ of $f_1(z)$ becomes like a proviso. Further, taking the Cauchy Product of $f_1(z)$ and $f_2(z)$, we obtain c_r. (See the proof of Formula 3.3.1)

Especially,

$$c_1 = \frac{a_1 a_1^0}{0!} - \frac{a_0 a_1^1}{1!} = a_1 - a_1 = 0$$

$$c_2 = \frac{a_2 a_1^0}{0!} - \frac{a_1 a_1^1}{1!} + \frac{a_0 a_1^2}{2!} = a_2 - \frac{a_1^2}{2}$$

$$= \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^2 x_{r1} x_{r2}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{2^0}{x_{r1}^2+y_{r1}^2} - \frac{1}{2}\left(\sum_{r_1=1}^{\infty} \frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2$$

$$= -\frac{1}{2}\left\{\left(\sum_{r_1=1}^{\infty} \frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2 - 2\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^2 x_{r1} x_{r2}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}\right\} + \sum_{r_1=1}^{\infty} \frac{2^0}{x_{r1}^2+y_{r1}^2}$$

$$= -\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2 + \sum_{r_1=1}^{\infty} \frac{2^0}{x_{r1}^2+y_{r1}^2}$$

Next,

$$c_3 = \frac{a_3 a_1^0}{0!} - \frac{a_2 a_1^1}{1!} + \frac{a_1 a_1^2}{2!} - \frac{a_0 a_1^3}{3!} = a_3 - a_2 a_1 + \frac{a_1^3}{3}$$

$$= -\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{2^3 x_{r1} x_{r2} x_{r3}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)} - \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

$$+ \sum_{r_1=1}^{\infty} \frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\left\{\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^2 x_{r1} x_{r2}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{2^0}{x_{r1}^2+y_{r1}^2}\right\}$$

$$-\frac{1}{3}\left(\sum_{r_1=1}^{\infty} \frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3$$

$$= -\frac{1}{3}\left\{\left(\sum_{r_1=1}^{\infty} \frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3 - 3\sum_{r_1=1}^{\infty} \frac{2x_{r1}}{x_{r1}^2+y_{r1}^2} \cdot \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^2 x_{r1} x_{r2}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}\right.$$

$$\left. + 3\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{2^3 x_{r1} x_{r2} x_{r3}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)}\right\}$$

$$- \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{2x_{r1}}{x_{r1}^2+y_{r1}^2} \cdot \sum_{r_1=1}^{\infty} \frac{2^0}{x_{r1}^2+y_{r1}^2}$$

Here, from Formula 1.3.1 (**01 Power of Infinite Series**),

$$\left(\sum_{r_1=1}^{\infty} \frac{1}{z_{r1}}\right)^3 = \sum_{r_1=1}^{\infty} \frac{1}{z_{r1}^3} + 3\sum_{r_1=1}^{\infty} \frac{1}{z_{r1}} \cdot \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{z_{r1} z_{r2}} - 3\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{z_{r1} z_{r2} z_{r3}}$$

Substituting $\frac{1}{z_r} = \frac{2x_r}{x_r^2+y_r^2}$ for this and applying it to the blue part above, we obtain

$$c_3 = -\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^3 + \sum_{r_1=1}^{\infty} \frac{2x_{r1}}{x_{r1}^2+y_{r1}^2} \cdot \sum_{r_1=1}^{\infty} \frac{2^0}{x_{r1}^2+y_{r1}^2} - \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

Last,

$$c_4=\frac{a_4 a_1^0}{0!}-\frac{a_3 a_1^1}{1!}+\frac{a_2 a_1^2}{2!}-\frac{a_1 a_1^3}{3!}+\frac{a_0 a_1^4}{4!}=a_4-a_3 a_1+\frac{a_2 a_1^2}{2}-\frac{a_1^4}{8}$$

$$=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{2^4 x_{r1}x_{r2}x_{r3}x_{r4}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)\left(x_{r4}^2+y_{r4}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\left(x_{r1}x_{r2}+x_{r1}x_{r3}+x_{r2}x_{r3}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}$$

$$-\left\{\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^3 x_{r1}x_{r2}x_{r3}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)}\right\}\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}$$

$$-\left\{\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}\right\}\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}$$

$$+\frac{1}{2}\left\{\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^2 x_{r1}x_{r2}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}+\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}\right\}\left(\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2$$

$$-\frac{1}{8}\left(\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^4$$

$$=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{2^4 x_{r1}x_{r2}x_{r3}x_{r4}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)\left(x_{r4}^2+y_{r4}^2\right)}$$

$$-\left\{\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^3 x_{r1}x_{r2}x_{r3}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)}\right\}\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}$$

$$+\frac{1}{2}\left\{\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^2 x_{r1}x_{r2}}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}\right\}\left(\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2$$

$$-\frac{1}{8}\left(\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^4$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\left(x_{r1}x_{r2}+x_{r1}x_{r3}+x_{r2}x_{r3}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)}$$

$$-\left\{\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}\right\}\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}+\frac{1}{2}\left(\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}$$

Here, from Formula 1.3.1 (**01 Power of Infinite Series**),

$$\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^4=\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}^4}-2\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}^2 z_{r_2}^2}+4\left(\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}}\right)\left(\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}\right)^2$$

$$-8\left(\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}}\right)\sum_{r_1=1}^{\infty}\frac{1}{z_{r_1}}$$

$$+8\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{1}{z_{r_1}z_{r_2}z_{r_3}z_{r_4}}$$

Substituting $\frac{1}{z_r}=\frac{2x_r}{x_r^2+y_r^2}$ for this and applying it to the blue part above , we obtain

$$c_4=-\frac{1}{8}\sum_{r_1=1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^4+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\left(\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2\left(\frac{2x_{r2}}{x_{r2}^2+y_{r2}^2}\right)^2$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\left(x_{r1}x_{r2}+x_{r1}x_{r3}+x_{r2}x_{r3}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)\left(x_{r3}^2+y_{r3}^2\right)}$$

$$-\left\{\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r1}+x_{r2}\right)}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}\right\}\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r1}^2+y_{r1}^2\right)\left(x_{r2}^2+y_{r2}^2\right)}+\frac{1}{2}\left(\sum_{r_1=1}^{\infty}\frac{2x_{r1}}{x_{r1}^2+y_{r1}^2}\right)^2\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r1}^2+y_{r1}^2}$$

Example When $z_r=r\pm ir$ $(r=1,2,3,\cdots)$

$$f_p(z)=\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)e^{\frac{2x_r z}{x_r^2+y_r^2}}=\prod_{r=1}^{\infty}\left(1-\frac{z}{r}+\frac{z^2}{2r^2}\right)e^{\frac{z}{r}}$$

If $f_p(z)$ is expanded up to z^4 by formula manipulation software ***Mathematica***, it is as follows.

$$\text{fp}[z\_,m\_]:=\prod_{r=1}^{m}\left(1-\frac{z}{r}+\frac{z^2}{2r^2}\right)e^{\frac{z}{r}}$$

```
N[Series[fp[z, 1000], {z, 0, 4}]]
```

$$1.+0.200343\,(z+0.)^3+0.13529\,(z+0.)^4+O[z+0.]^5$$

On the other hand, if $c_1\sim c_4$ are calculated, it is as follows. These are exactly consistent with the coefficients of the series.

$$a_0[m\_]:=1 \qquad a_1[m\_]:=-\sum_{r_1=1}^{m}\frac{1}{r_1} \qquad a_2[m\_]:=\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{1}{r_1 r_2}+\frac{1}{2}\sum_{r_1=1}^{m}\frac{1}{r_1^2}$$

$$a_3[m\_]:=-\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m}\frac{1}{r_1 r_2 r_3}-\frac{1}{2}\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{r_1+r_2}{r_1^2 r_2^2}$$

$$a_4[m\_]:=\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m}\sum_{r_4=r_3+1}^{m}\frac{1}{r_1 r_2 r_3 r_4}+\frac{1}{2}\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m}\frac{r_1 r_2+r_1 r_3+r_2 r_3}{r_1^2 r_2^2 r_3^2}+\frac{1}{4}\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{1}{r_1^2 r_2^2}$$

$$c_{r\_}[m\_] := \sum_{s=0}^{r} (-1)^s \frac{a_{r-s}[m]\, a_1[m]^s}{s!}$$

```
N[{c1[100], c2[100], c3[600], c4[100]}]
 {0.   ,   0.   ,   0.200343, 0.13529}
```

Especially, $c_1 \sim c_4$ are represented by a faster formula as follows.

$$c_1 = 0$$

$$c_2 = -\frac{1}{2}\sum_{r_1=1}^{\infty}\frac{1}{r_1^2} + \sum_{r_1=1}^{\infty}\frac{2^0}{2r_1^2} = 0$$

$$c3[m\_] := -\frac{1}{3}\sum_{r_1=1}^{m}\frac{1}{r_1^3} - \frac{1}{2}\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{r_1+r_2}{r_1^2\, r_2^2} + \frac{1}{2}\sum_{r_1=1}^{m}\frac{1}{r_1} \times \sum_{r_1=1}^{m}\frac{1}{r_1^2}$$

$$c4[m\_] := -\frac{1}{8}\sum_{r_1=1}^{m}\frac{1}{r_1^4} + \frac{1}{2}\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{1}{r_1^2\, r_2^2} + \frac{1}{4}\sum_{r_1=1}^{m}\frac{1}{r_1^2}\left(\sum_{r_1=1}^{m}\frac{1}{r_1}\right)^2$$
$$+\frac{1}{2}\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m}\frac{r_1 r_2 + r_1 r_3 + r_2 r_3}{r_1^2\, r_2^2\, r_3^2} - \frac{1}{2}\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{r_1+r_2}{r_1^2\, r_2^2} \times \sum_{r_1=1}^{m}\frac{1}{r_1}$$

If c_3, c_4 are calculated by these, it is as follows. The coefficients are exactly consistent with the above series .

```
N[{c3[600], c4[50]}]
 {0.200343, 0.13529}
```

.

As a special case of this formula, the following corollary is obtained.

Corollary 3.6.1 (Infinite-degree Equation with roots whose real part is 1/2)

Assume that the function $f(z)$ on the complex plane has zeros $z_k = 1/2 \pm i y_k$, $y_k \neq 0$ $(k=1,2,3,\cdots)$ and is incompletely factored as follows.

$$f(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}} = \prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)e^{\frac{z}{1/4+y_r^2}}$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1 + c_1 z^1 + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots \qquad (6.s')$$

$$c_r = \sum_{s=0}^{r}(-1)^s \frac{a_{r-s}\, a_1^s}{s!}$$

Where,

$$a_0 = 1$$

$$a_1 = -\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r_1}^2}$$

$$a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r_1}^2}$$

$$a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)\left(1/4+y_{r3}^2\right)}$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}$$

$$a_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)\left(1/4+y_{r3}^2\right)\left(1/4+y_{r4}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{3}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)\left(1/4+y_{r3}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}$$

$$\vdots$$

Especially, $c_1 \sim c_4$ are represented by a faster formula as follows.

$$c_1 = 0$$

$$c_2 = -\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2 + \sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}$$

$$c_3 = -\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3 + \sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2$$

$$c_4 = -\frac{1}{8}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^4 + \sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3 - \frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}$$

$$+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_{r=1}+1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2\left(\frac{1}{1/4+y_{r2}^2}\right)^2 + \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}$$

Proof

Substituting $x_r = 1/2 \quad r=1,2,3,\cdots$ for Formula 3.6.1 , we obtain $a_1, a_2, a_3 \cdots$ and c_2.

c_3 is obtained as follows.

$$c_3 = -\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3 - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)} + \left(\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}\right)^2$$

$$= -\frac{1}{3}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3 + \sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2$$

c_4 is obtained as follows.

$$c_4 = -\frac{1}{8}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^4 + \frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2\left(\frac{1}{1/4+y_{r2}^2}\right)^2$$

$$-\left\{\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}\right\}\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2} + \frac{1}{2}\left(\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}\right)^3$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{3}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)\left(1/4+y_{r3}^2\right)}+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}$$

$$=-\frac{1}{8}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^4+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2\left(\frac{1}{1/4+y_{r2}^2}\right)^2$$

$$+3\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)\left(1/4+y_{r3}^2\right)}$$

$$-3\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}+\left(\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}\right)^3$$

$$+\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}\cdot\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}-\frac{1}{2}\left(\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}\right)^3$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}$$

Here, from Formula 1.3.1 (**01 Power of Infinite Series**),

$$\left(\sum_{r1=1}^{\infty}\frac{1}{z_{r1}}\right)^3=\sum_{r1=1}^{\infty}\frac{1}{z_{r1}^3}+3\sum_{r_1=1}^{\infty}\frac{1}{z_{r1}}\cdot\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{z_{r1}z_{r2}}-3\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{z_{r1}z_{r2}z_{r3}}$$

Substituting $\dfrac{1}{z_r}=\dfrac{1}{1/4+y_r^2}$ for this and applying it to the blue part above, we obtain

$$c_4=-\frac{1}{8}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^4+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2\left(\frac{1}{1/4+y_{r2}^2}\right)^2$$

$$+\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}$$

$$-\frac{1}{2}\left\{\left(\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}\right)^2-2\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}\right\}\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}$$

$$=-\frac{1}{8}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^4+\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^3-\frac{1}{2}\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r1}^2}$$

$$+\frac{1}{4}\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\left(\frac{1}{1/4+y_{r1}^2}\right)^2\left(\frac{1}{1/4+y_{r2}^2}\right)^2+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r1}^2\right)\left(1/4+y_{r2}^2\right)}$$

Example When $z_r=1/2\pm ir$ ($r=1,2,3,\cdots$)

$$f_p(z)=\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+r^2}+\frac{z^2}{1/4+r^2}\right)e^{\frac{z}{1/4+r^2}}$$

If $f_p(z)$ is expanded up to z^4 by formula manipulation software ***Mathematica***, it is as follows.

```
fp[z_, m_] := Product[(1 - z/(1/4 + r^2) + z^2/(1/4 + r^2)) E^(z/(1/4 + r^2)), {r, 1, m}]
```

$$\mathrm{N[Series[fp[z,\ 1000],\ \{z,\ 0,\ 4\}]]}$$

$$1. + 1.0672\,(z+0.)^2 + 0.538818\,(z+0.)^3 + 0.635695\,(z+0.)^4 + O[z+0.]^5$$

On the other hand, if $c_1 \sim c_4$ are calculated, it is as follows. Both coefficients are exactly equal for c_1, c_2, c_3, and both coefficients are roughly equal for c_4.

$$a_0[m\_] := 1 \qquad a_1[m\_] := -\sum_{r_1=1}^{m} \frac{1}{1/4+r_1^2} \qquad a_2[m\_] := \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m} \frac{1}{(1/4+r_1^2)(1/4+r_2^2)} + \sum_{r_1=1}^{m} \frac{1}{1/4+r_1^2}$$

$$a_3[m\_] := -\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m} \frac{1}{(1/4+r_1^2)(1/4+r_2^2)(1/4+r_3^2)} - \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m} \frac{2}{(1/4+r_1^2)(1/4+r_2^2)}$$

$$a_4[m\_] := \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m}\sum_{r_4=r_3+1}^{m} \frac{1}{(1/4+r_1^2)(1/4+r_2^2)(1/4+r_3^2)(1/4+r_4^2)}$$
$$+ \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m} \frac{3}{(1/4+r_1^2)(1/4+r_2^2)(1/4+r_3^2)} + \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m} \frac{1}{(1/4+r_1^2)(1/4+r_2^2)}$$

$$c_{r\_}[m\_] := \sum_{s=0}^{r} (-1)^s \frac{a_{r-s}[m]\, a_1[m]^s}{s!}$$

$$\mathrm{N[\{c_1[200],\ c_2[1000],\ c_3[200],\ c_4[200]\}]}$$

$$\{0.\ ,\ 1.0672,\ 0.538818,\ 0.631447\}$$

Especially, $c_1 \sim c_4$ are represented by a faster formula as follows.

$$c_1 = 0$$

$$c2[m\_] := -\frac{1}{2}\sum_{r_1=1}^{m}\left(\frac{1}{1/4+r_1^2}\right)^2 + \sum_{r_1=1}^{m}\frac{1}{1/4+r_1^2}$$

$$c3[m\_] := -\frac{1}{3}\sum_{r_1=1}^{m}\left(\frac{1}{1/4+r_1^2}\right)^3 + \sum_{r_1=1}^{m}\left(\frac{1}{1/4+r_1^2}\right)^2$$

$$c4[m\_] := -\frac{1}{8}\sum_{r_1=1}^{m}\left(\frac{1}{1/4+r_1^2}\right)^4 + \sum_{r_1=1}^{m}\left(\frac{1}{1/4+r_1^2}\right)^3 - \frac{1}{2}\sum_{r_1=1}^{m}\left(\frac{1}{1/4+r_1^2}\right)^2 \sum_{r_1=1}^{m}\frac{1}{1/4+r_1^2}$$
$$+ \frac{1}{4}\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\left(\frac{1}{1/4+r_1^2}\right)^2\left(\frac{1}{1/4+r_2^2}\right)^2 + \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\frac{1}{(1/4+r_1^2)(1/4+r_2^2)}$$

When c_2, c_3, c_4 are calculated by these, it is as follows. These are exactly consistent with the coefficients of the above series .

$$\mathrm{N[\{c2[1000],\ c3[200],\ c4[1000]\}]}$$

$$\{1.0672,\ 0.538818,\ 0.635695\}$$

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2017.10.29 Added Sec.6

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Alien's Mathematics