10 Vieta's Formulas on Completed Riemann Zeta

In the previous chapter, Maclaurin series of the completed Riemann zeta was obtained. If we put the series as 0, it is an infinite-degree equation. As seen in Chap.8, completed Riemann zeta is completely factored by the roots (zeros) . So, the relationship between zeros (roots) and coefficients is obtained by Vieta's formula.

10.1 Zeros and Coefficients on $\xi(z)$

10.1.1 Coefficients of Maclaurin series of $\xi(z)$

Maclaurin series of the completed Riemann zeta $\xi(z)$ was obtained by Theorem 9.1.3 in previous chapter. When this is slightly modified and reprinted, it is as follows.

Theorem 10.1.1

Let completed Riemann zeta $\xi(z)$ and the Maclaurin series are as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \sum_{r=0}^{\infty} A_r z^r \tag{1.0}$$

Then, these coefficients A_r $r=0,1,2,3,\cdots$ are given by

$$A_r = \sum_{s=0}^{r}\sum_{t=0}^{s} \frac{\log^{r-s}\pi}{2^{r-s}(r-s)!} \frac{(-1)^{s-t} g_{s-t}(3/2)}{2^{s-t}(s-t)!} c_t \tag{1.a}$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r=0 \\ \displaystyle\sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \cdots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ -\dfrac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

The first 4 are as follows.

$$A_0 = \frac{\log^0\pi}{2^0 0!} \frac{(-1)^0 g_0(3/2)}{2^0 0!} c_0 = 1$$

$$A_1 = \frac{\log^1\pi}{2^1 1!} - \frac{g_1(3/2)}{2^1 1!} - \frac{\gamma_0}{0!}$$

$$A_2 = \frac{\log^2\pi}{2^2 2!} + \frac{g_2(3/2)}{2^2 2!} - \frac{\gamma_1}{1!} - \frac{\log^1\pi}{2^1 1!}\frac{g_1(3/2)}{2^1 1!} + \frac{g_1(3/2)}{2^1 1!}\frac{\gamma_0}{0!} - \frac{\log^1\pi}{2^1 1!}\frac{\gamma_0}{0!}$$

$$A_3 = \frac{\log^3\pi}{2^3 3!} - \frac{g_3(3/2)}{2^3 3!} - \frac{\gamma_2}{2!} - \frac{\log^2\pi}{2^2 2!}\frac{g_1(3/2)}{2^1 1!} - \frac{\log^2\pi}{2^2 2!}\frac{\gamma_0}{0!} - \frac{g_2(3/2)}{2^2 2!}\frac{\gamma_0}{0!}$$
$$+ \frac{\log^1\pi}{2^1 1!}\frac{g_2(3/2)}{2^2 2!} - \frac{\log^1\pi}{2^1 1!}\frac{\gamma_1}{1!} + \frac{g_1(3/2)}{2^1 1!}\frac{\gamma_1}{1!} + \frac{\log^1\pi}{2^1 1!}\frac{g_1(3/2)}{2^1 1!}\frac{\gamma_0}{0!}$$

When $A_1 \sim A_4$ are calculated with the formula manipulation software ***Mathematica***, it is as follows.

$$\text{Tbl}\psi[r\_, z\_] := \text{Table}[\text{PolyGamma}[k, z], \{k, 0, r-1\}]$$

$$g_{r\_}\left[\frac{3}{2}\right] := \text{If}\left[r == 0, 1, \sum_{k=1}^{r} \text{BellY}\left[r, k, \text{Tbl}\psi\left[r, \frac{3}{2}\right]\right]\right]$$

$$\gamma_{s\_} := \text{StieltjesGamma}[s] \qquad c_{r\_} := \text{If}\left[r == 0, 1, -\frac{\gamma_{r-1}}{(r-1)!}\right]$$

$$A_{r\_} := \sum_{s=0}^{r}\sum_{t=0}^{s} \frac{\text{Log}[\pi]^{r-s}}{2^{r-s}\,(r-s)!}\,\frac{(-1)^{s-t}\,g_{s-t}[3/2]}{2^{s-t}\,(s-t)!}\,c_t$$

```
SetPrecision[{A1, A2, A3, A4}, 12]
{-0.02309570897, 0.02334386453, -0.00049798385, 0.00025318173}
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10.1.2 Vieta's Formulas on $\xi(z)$

Theorem 10.1.2

Let completed Riemann zeta $\xi(z)$ and the Maclaurin series are as follows.

$$\xi(z) = -z(1-z)\,\pi^{-\frac{z}{2}}\,\Gamma\!\left(\frac{z}{2}\right)\zeta(z) = \sum_{r=0}^{\infty} B_r z^r \tag{1.0}$$

Then,

(1) The following expressions hold for non-trivial zeros $z_k = x_k \pm i y_k$, $y_k \neq 0$ $k=1,2,3,\cdots$ of $\zeta(z)$.

$$B_1 = -\sum_{r_1=1}^{\infty} \frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}$$

$$B_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^2\,x_{r_1}x_{r_2}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{2^0}{x_{r_1}^2+y_{r_1}^2}$$

$$B_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{2^3\,x_{r_1}x_{r_2}x_{r_3}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^1\left(x_{r_1}+x_{r_2}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$$B_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} \frac{2^4 x_{r_1}x_{r_2}x_{r_3}x_{r_4}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{2^2\left(x_{r_1}x_{r_2}+x_{r_1}x_{r_3}+x_{r_2}x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$\vdots$

$$B_{2n-1} = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-1}x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-2}=r_{2n-3}+1}^{\infty}\frac{2^{2n-3}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-2}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-2}}^2+y_{r_{2n-2}}^2\right)}$$

$$\vdots$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}+\cdots+x_{r_n}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

$$B_{2n} = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty}\frac{2^{2n}x_{r_1}x_{r_2}\cdots x_{r_{2n}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-2}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$$\vdots$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

(2) When A_n is a coefficient in **Theorem 10.1.1** , $B_n = A_n \quad n=1,2,3,\cdots$.

Proof

(1) According to Theorem 8.3.1 in **" 08 Factorization of Completed Riemann Zeta "**, when the non-trivial zeros of Riemann zeta $\zeta(z)$ are $z_k = x_k \pm i y_k$, $y_k \neq 0$ ($k=1,2,3,\cdots$) , completed zeta (1.0) is factored as follows.

$$\xi(z) = \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)$$

On the other hand, according to Formula 3.5.1 in **" 03 Vieta's Formulas in Infinite-degree Equation " (Infinite-degree Equation)** , such an infinite product $\xi(z)$ is expanded in the Maclaurin series as follows

$$\xi(z) = 1+B_1 z^1+B_2 z^2+B_3 z^3+B_4 z^4+\cdots$$

Where,

$$B_1 = -\sum_{r_1=1}^{\infty}\frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}$$

$$B_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^2 x_{r_1}x_{r_2}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}+\sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2}$$

$$B_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^3 x_{r_1}x_{r_2}x_{r_3}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)}-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$$B_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{2^4 x_{r_1}x_{r_2}x_{r_3}x_{r_4}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\left(x_{r_1}x_{r_2}+x_{r_1}x_{r_3}+x_{r_2}x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$\vdots$

(2) Due to the uniqueness of the power series, it has to be $B_n = A_n \quad n=1,2,3,\cdots$.

10.1.3 Proposition1 equivalent to the Riemann Hypothesis

If Riemann Hypothesis is true, the following proposition has to hold.

Proposition 10.1.3

When the non-trivial zeros of Riemann zeta $\zeta(z)$ are $z_k = 1/2 \pm i y_k$, $y_k \neq 0$ ($k=1,2,3,\cdots$) , the following expressions hold.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = -A_1 = 0.0230957089\cdots \qquad (1.3_1)$$

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)} = A_2 + A_1 = 0.0002481555\cdots \qquad (1.3_2)$$

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)} = -A_3 - 2\left(A_2 + A_1\right)$$
$$= 0.0000016727\cdots \qquad (1.3_3)$$

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)\left(1/4+y_u^2\right)}$$
$$= A_4 + 3A_3 + 5\left(A_2 + A_1\right) = 8.021073428\times 10^{-9} \qquad (1.3_4)$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$A_r = \sum_{s=0}^{r}\sum_{t=0}^{s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\,\frac{(-1)^{s-t}g_{s-t}(3/2)}{2^{s-t}(s-t)!}c_t$$

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r}B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ -\dfrac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

Proof of equivalence

If the Riemann hypothesis holds, The real part of the non-trivial zeros $z_k = x_k \pm i y_k$ of $\zeta(z)$ is $x_k = 1/2$ $(k=1,2,3,\cdots)$. Substituting this for $B_1 \sim B_4$ in Theorem 10.1.2 ,

$$B_1 = -\sum_{r=1}^{\infty} \frac{1}{1/4 + y_r^2}$$

$$B_2 = \sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \frac{1}{(1/4+y_r^2)(1/4+y_s^2)} + \sum_{r=1}^{\infty} \frac{1}{1/4+y_r^2}$$

$$B_3 = -\sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \sum_{t=s+1}^{\infty} \frac{1}{(1/4+y_r^2)(1/4+y_s^2)(1/4+y_t^2)} - \sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \frac{2}{(1/4+y_r^2)(1/4+y_s^2)}$$

$$B_4 = \sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \sum_{t=s+1}^{\infty} \sum_{u=t+1}^{\infty} \frac{1}{(1/4+y_r^2)(1/4+y_s^2)(1/4+y_t^2)(1/4+y_u^2)}$$
$$+ \sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \sum_{t=s+1}^{\infty} \frac{3}{(1/4+y_r^2)(1/4+y_s^2)(1/4+y_t^2)}$$
$$+ \sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \frac{1}{(1/4+y_r^2)(1/4+y_s^2)}$$

Since $B_n = A_n \quad n=1,2,3,\cdots$ from Theorem 10.1.2 (2) , replacing $B_1 \sim B_4$ with $A_1 \sim A_4$ and substituting a top one by one downward,

$$A_1 = -\sum_{r=1}^{\infty} \frac{1}{1/4 + y_r^2}$$

$$A_2 = \sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \frac{1}{(1/4+y_r^2)(1/4+y_s^2)} - A_1$$

$$A_3 = -\sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \sum_{t=s+1}^{\infty} \frac{1}{(1/4+y_r^2)(1/4+y_s^2)(1/4+y_t^2)} - 2(A_2 + A_1)$$

$$A_4 = \sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \sum_{t=s+1}^{\infty} \sum_{u=t+1}^{\infty} \frac{1}{(1/4+y_r^2)(1/4+y_s^2)(1/4+y_t^2)(1/4+y_u^2)}$$
$$-3A_3 - 5(A_2 + A_1)$$

From these, we obtain $(1.3_2) \sim (1.3_4)$. By **Theorem 8.2.4 (" 08 Factorization of Completed Riemann Zeta ")**, (1.3_1) is equivalent to Riemann Hypothesis. Since $(1.3_2) \sim (1.3_3)$ contain (1.3_1) in some way, they must also be equivalent to Riemann Hypothesis respectively.

Q.E.D.

Direct Calculation

Since (1.3_1) has already been calculated in the previous chapter **8.2** , it is not calculated here. Both sides of (1.3_2) and (1.3_3) are calculated with the formula manipulation software ***Mathematica*** , it is as follows, f_2 is calculated in each 3000 terms and is matched up to the 5th decimal place, f_3 is calculated in each 300 terms and is matched up to the 6th decimal place

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y_r_ := Im[ZetaZero[r]]
```

$$f2[m\_] := \sum_{r=1}^{m}\sum_{s=r+1}^{m}\frac{1}{\left(1/4+y_r{}^2\right)\left(1/4+y_s{}^2\right)}$$

N[f2[3000]]	N[A2 + A1]
0.000240583	0.000248156

$$f3[m\_] := \sum_{r=1}^{m}\sum_{s=r+1}^{m}\sum_{t=s+1}^{m}\frac{1}{\left(1/4+y_r{}^2\right)\left(1/4+y_s{}^2\right)\left(1/4+y_t{}^2\right)}$$

N[f3[300]]	N[-A3 - 2 (A2 + A1)]
1.304663×10^{-6}	1.67271×10^{-6}

Indirect Calculation

The direct calculations of $(1.3_2)\sim(1.3_4)$ are so slow in convergence. So, we calculate these indirectly using the following formula. (See Formula 1.3.1 in **" 01 Power of Infinite Series " (Infinite-degree Equation)**)

$$\left(\sum_{r=0}^{\infty}a_r\right)^2=\sum_{r=0}^{\infty}a_r^2+2\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}a_r a_s \tag{3.2}$$

$$\left(\sum_{r=0}^{\infty}a_r\right)^3=\sum_{r=0}^{\infty}a_r^3+3\sum_{r=0}^{\infty}a_r\cdot\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}a_r a_s-3\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}a_r a_s a_t \tag{3.3}$$

$$\begin{aligned}\left(\sum_{r=0}^{\infty}a_r\right)^4&=2\sum_{r=0}^{\infty}a_r^4-\left(\sum_{r=0}^{\infty}a_r^2\right)^2+4\left(\sum_{r=0}^{\infty}a_r\right)^2\cdot\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}a_r a_s\\&-8\sum_{r=0}^{\infty}a_r\cdot\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}a_r a_s a_t+8\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty}a_r a_s a_t a_u\end{aligned} \tag{3.4}$$

Applying (3.2) to (1.3_2) ,

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)}=\frac{1}{2}\left\{\left(\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}\right)^2-\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2\right\}$$

Substituting (1.3_1) for the right hand,

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)}=-\frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2+\frac{A_1^2}{2} \tag{1.3$_2$'}$$

Applying (3.3) to (1.3_3) ,

$$\begin{aligned}\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)}&=\frac{1}{3}\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^3\\&+\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}\cdot\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)}-\frac{1}{3}\left(\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}\right)^3\end{aligned}$$

Substituting (1.3_1) and (1.3_2) for the right hand,

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)}=\frac{1}{3}\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^3+\frac{A_1^3}{3}-A_1\left(A_2+A_1\right) \tag{1.3$_3$'}$$

Applying (3.4) to (1.3_4) ,

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)\left(1/4+y_u^2\right)}$$

$$=\frac{1}{8}\left(\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}\right)^4-\frac{1}{4}\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^4+\frac{1}{8}\left(\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2\right)^2$$

$$-\frac{1}{2}\left(\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}\right)^2\left(\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)}\right)$$

$$+\left(\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}\right)\left(\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)}\right)$$

Substituting (1.3_1) ~ (1.3_3) for the right side,

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)\left(1/4+y_u^2\right)}$$

$$=-\frac{1}{4}\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^4+\frac{1}{8}\left(\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2\right)^2$$

$$+\frac{A_1^4}{8}-\left(\frac{1}{2}A_1-2\right)A_1\left(A_1+A_2\right)+A_1A_3$$

Here,

$$\sum_{r=0}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2=\left(\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}\right)^2-2\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)}$$

$$=A_1^2-2\left(A_2+A_1\right)$$

Using this,

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)\left(1/4+y_u^2\right)}$$

$$=-\frac{1}{4}\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^4+\frac{A_1^4}{4}-A_1^3+A_1^2\left(\frac{5}{2}-A_2\right)$$

$$+\frac{A_2^2}{2}+A_1\left(3A_2+A_3\right) \qquad (1.3_4')$$

If $(1.3_2')$ ~ $(1.3_4')$ are used to calculate the left sides of (1.3_2) ~ (1.3_4) , it is as follows.

f_2 is calculated in 500 terms and is matched up to the 9th decimal place, f_3 is calculated in 30 terms and is matched up to the 11th decimal place, and f_4 is calculated in 20 terms and is matched up to the 14th decimal place. The speed of these convergence are much faster than the direct calculations above.

```
y_r_ := Im[ZetaZero[r]]
```

$$f_2[m\_] := -\frac{1}{2}\sum_{r=1}^{m}\left(\frac{1}{1/4+y_r^2}\right)^2+\frac{A_1^2}{2}$$

```
N[f2[500], 7]
0.0002481558
```

```
N[A2 + A1, 7]
0.0002481556
```

$$f_3[m\_] := \frac{1}{3}\sum_{r=1}^{m}\left(\frac{1}{1/4+y_r^2}\right)^3 + \frac{A_1{}^3}{3} - A_1\,(A_1 + A_2)$$

```
N[f3[30], 7]
1.672711 x 10^-6
```

```
N[-A3 - 2 (A2 + A1), 7]
1.672714 x 10^-6
```

$$f_4[m\_] := -\frac{1}{4}\sum_{r=1}^{m}\left(\frac{1}{1/4+y_r^2}\right)^4 + \frac{A_1^4}{4} - A_1^3 + A_1^2\left(\frac{5}{2} - A_2\right) + \frac{A_2^2}{2} + A_1\,(3\,A_2 + A_3)$$

```
N[f4[20], 7]
8.021074 x 10^-9
```

```
N[A4 + 3 A3 + 5 (A2 + A1), 7]
8.021073 x 10^-9
```

The above indirect calculation also can be represented as follows.

Proposition 10.1.3'

When $z_k = 1/2 \pm i y_k$, $y_k \neq 0$ $(\ k = 1, 2, 3, \cdots\ )$ are the non-trivial zeros of Riemann zeta $\zeta(z)$ and A_r $r = 1, 2, 3, \cdots$ are constants given by Theorem 10.1.1 , the following expressions hold.

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2 = A_1^2 - 2(A_1 + A_2) = 0.00003710063\cdots$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^3 = -A_1^3 + 3(A_1 - 2)(A_1 + A_2) - 3A_3 = 0.00000014367786\cdots$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^4 = A_1^4 - 4A_1^3 + 4A_1^2\left(\frac{5}{2} - A_2\right) + 4A_1(3A_2 + A_3 - 5)$$
$$+ 2A_1^2 - 20A_2 - 12A_3 - 4A_4 = 6.59827915\times 10^{-10}$$

cf.

Proposition 4.4.1 in **" 04 Sum of series equivalent to the Riemann hypothesis " (Infinite-degree Equation)** was

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2 = \gamma_0^2 + 2\gamma_0 + 2\gamma_1 - \log\pi + \psi_0\left(\frac{3}{2}\right) - \frac{1}{4}\psi_1\left(\frac{3}{2}\right)$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^3 = \gamma_0^3 + 3\gamma_0^2 + 6\gamma_0 + 6\gamma_1 + 3\gamma_0\gamma_1 + \frac{3}{2}\gamma_2 - 3\log\pi$$
$$+ 3\psi_0\left(\frac{3}{2}\right) - \frac{3}{4}\psi_1\left(\frac{3}{2}\right) + \frac{1}{16}\psi_2\left(\frac{3}{2}\right)$$

These are what $A_1 \sim A_3$ are expanded.

The calculation results with the formula manipulation software ***Mathematica*** are as follows.

$y_{r\_} := \mathrm{Im}[\mathrm{ZetaZero}[r]]$

$$N\left[\left\{\sum_{r=1}^{2000}\left(\frac{1}{1/4+y_r{}^2}\right)^2,\ A_1{}^2-2\,(A_1+A_2)\right\},\ 7\right]$$

$\{0.00003710062,\ 0.00003710064\}$

$$N\left[\left\{\sum_{r=1}^{100}\left(\frac{1}{1/4+y_r{}^2}\right)^3,\ -A_1{}^3+3\,(A_1-2)\,(A_1+A_2)-3\,A_3\right\},\ 7\right]$$

$\{1.436777\times 10^{-7},\ 1.436779\times 10^{-7}\}$

$$f4 := A_1^4-4\,A_1^3+4\,A_1^2\left(\frac{5}{2}-A_2\right)+4\,A_1\,(3\,A_2+A_3-5)+2\,A_2^2-20\,A_2-12\,A_3-4\,A_4$$

$$N\left[\left\{\sum_{r=1}^{25}\left(\frac{1}{1/4+y_r{}^2}\right)^4,\ f4\right\},\ 7\right]$$

$\{6.598267\times 10^{-10},\ 6.598279\times 10^{-10}\}$

As far as the above calculation results are concerned, the Riemann hypothesis seems to be true.

10.2 Zeros and Coefficients on $\Xi(z)$

10.2.1 Coefficients of Maclaurin series of $\Xi(z)$

Maclaurin series of the completed Riemann zeta $\Xi(z)$ was obtained by Theorem 9.2.3 in previous chapter. When this is slightly modified and reprinted, it is as follows.

Theorem 10.2.1

Let completed Riemann zeta $\Xi(z)$ and the Maclaurin series are as follows.

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) \tag{2.0}$$

$$= \Xi(0)\left(1+A_1 z^1+A_2 z^2+A_3 z^3+A_4 z^4+\cdots\right)$$

Then, these coefficients A_r $r=0,1,2,3,\cdots$ are given by

$$A_r = \sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{g_{s-t}(5/4)}{2^{s-t}(s-t)!}c_t \tag{2.a}$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \displaystyle\sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \ldots, \psi_{r-1}\left(\frac{5}{4}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ \displaystyle\frac{2}{\zeta(1/2)}\sum_{s=r}^{\infty}(-1)^r\frac{\gamma_{s-1}}{(s-1)!}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

The first 4 are as follows.

$$A_0 = \frac{\log^0\pi}{2^0 0!}\frac{g_0(5/4)}{2^0 0!}c_0 = 1$$

$$A_1 = -\frac{\log^1\pi}{2^1 1!}+\frac{g_1(5/4)}{2^1 1!}+c_1$$

$$A_2 = \frac{\log^2\pi}{2^2 2!}+\frac{g_2(5/4)}{2^2 2!}+c_2-\frac{\log^1\pi}{2^1 1!}\frac{g_1(5/4)}{2^1 1!}+\frac{g_1(5/4)}{2^1 1!}c_1-\frac{\log^1\pi}{2^1 1!}c_1$$

$$A_3 = -\frac{\log^3\pi}{2^3 3!}+\frac{g_3(5/4)}{2^3 3!}+c_3+\frac{\log^2\pi}{2^2 2!}\frac{g_1(5/4)}{2^1 1!}+\frac{\log^2\pi}{2^2 2!}c_1+\frac{g_2(5/4)}{2^2 2!}c_1$$
$$-\frac{\log^1\pi}{2^1 1!}\frac{g_2(5/4)}{2^2 2!}-\frac{\log^1\pi}{2^1 1!}c_2+\frac{g_1(5/4)}{2^1 1!}c_2-\frac{\log^1\pi}{2^1 1!}\frac{g_1(5/4)}{2^1 1!}c_1$$

When $A_1 \sim A_8$ are calculated with the formula manipulation software ***Mathematica***, it is as follows.

```
Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]
γs_ := StieltjesGamma[s]
```

```
g_r[5/4] := If[r == 0, 1, Sum[BellY[r, k, Tbl𝜓[r, 5/4]], {k, 1, r}]]

c_r := If[r == 0, 1, 2/Zeta[1/2] Sum[(-1)^r γ_(s-1)/(s-1)! Binomial[s, r] (1/2)^(s-r), {s, r, 1000}]]

A_r := Sum[Sum[(-1)^(r-s) Log[π]^(r-s)/(2^(r-s) (r-s)!) g_(s-t)[5/4]/(2^(s-t) (s-t)!) c_t, {t, 0, s}], {s, 0, r}]

SetPrecision[{A_2, A_4, A_6, A_8}, 13]
{0.02310499312, 0.000248334054, 1.674353×10^-6, 8.0307×10^-9}

SetPrecision[{A_1, A_3, A_5, A_7}, 13]
{0.×10^-12, 0.×10^-12, 0.×10^-13, 0.×10^-14}
```

10.2.2 Vieta's Formulas on $\Xi(z)$

Theorem 10.2.2

Let completed Riemann zeta $\Xi(z)$ and the Maclaurin series are as follows.

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) \tag{2.0}$$

$$= \Xi(0)\left(1+B_1z^1+B_2z^2+B_3z^3+B_4z^4+\cdots\right)$$

Then,

(1) The following expressions hold for non-trivial zeros $z_k = x_k \pm iy_k$, $y_k \neq 0$ $k=1,2,3,\cdots$ of $\zeta(z)$.

$$\Xi(0) = \prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2} = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.9942415563\cdots$$

$$B_1 = -\sum_{r_1=1}^{\infty}\frac{2(x_{r_1}-1/2)}{(x_{r_1}-1/2)^2+y_{r_1}^2}$$

$$B_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^2(x_{r_1}-1/2)(x_{r_2}-1/2)}{\left\{(x_{r_1}-1/2)^2+y_{r_1}^2\right\}\left\{(x_{r_2}-1/2)^2+y_{r_2}^2\right\}} + \sum_{r_1=1}^{\infty}\frac{2^0}{\left\{(x_{r_1}-1/2)^2+y_{r_1}^2\right\}}$$

$$B_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^3(x_{r_1}-1/2)(x_{r_2}-1/2)(x_{r_3}-1/2)}{\left\{(x_{r_1}-1/2)^2+y_{r_1}^2\right\}\left\{(x_{r_2}-1/2)^2+y_{r_2}^2\right\}\left\{(x_{r_3}-1/2)^2+y_{r_3}^2\right\}}$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left\{(x_{r_1}-1/2)+(x_{r_2}-1/2)\right\}}{\left\{(x_{r_1}-1/2)^2+y_{r_1}^2\right\}\left\{(x_{r_2}-1/2)^2+y_{r_2}^2\right\}}$$

$$B_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_4=r_3+1}^{\infty}\frac{2^4(x_{r1}-1/2)(x_{r2}-1/2)\cdots(x_{r4}-1/2)}{\left\{(x_{r1}-1/2)^2+y_{r1}^2\right\}\left\{(x_{r2}-1/2)^2+y_{r2}^2\right\}\cdots\left\{(x_{r4}-1/2)^2+y_{r4}^2\right\}}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\{(x_{r1}-1/2)(x_{r2}-1/2)+\cdots+(x_{r2}-1/2)(x_{r3}-1/2)\}}{\{(x_{r_1}-1/2)^2+y_{r_1}^2\}\{(x_{r_2}-1/2)^2+y_{r_2}^2\}\{(x_{r_3}-1/2)^2+y_{r_3}^2\}}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\{(x_{r_1}-1/2)^2+y_{r_1}^2\}\{(x_{r_2}-1/2)^2+y_{r_2}^2\}}$$

⋮

(2) When A_n is a coefficient in **Theorem 10.2.1** , $B_n = A_n \quad n=1,2,3,\cdots$.

Proof

(1) According to Theorem 8.4.1 in **" 08 Factorization of Completed Riemann Zeta "**, when the non-trivial zeros of Riemann zeta $\zeta(z)$ are $z_k = x_k \pm i y_k$, $y_k \neq 0$ ($k=1,2,3,\cdots$) , completed zeta (2.0) is factored as follows.

$$\Xi(z) = \Xi(0)\prod_{n=1}^{\infty}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\}$$

$$\text{Where, } \Xi(0)=\prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2} = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.9942415563\cdots$$

On the other hand, according to Formula 3.5.1 in **" 03 Vieta's Formulas in Infinite-degree Equation " (Infinite-degree Equation)** , an infinite product

$$f(z) = \prod_{n=1}^{\infty}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\}$$

is expanded in the Maclaurin series as follows

$$f(z) = 1+B_1 z^1+B_2 z^2+B_3 z^3+B_4 z^4+\cdots$$

Where,

$$B_1 = -\sum_{r_1=1}^{\infty}\frac{2(x_{r_1}-1/2)}{(x_{r_1}-1/2)^2+y_{r_1}^2}$$

$$B_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^2(x_{r_1}-1/2)(x_{r_2}-1/2)}{\{(x_{r_1}-1/2)^2+y_{r_1}^2\}\{(x_{r_2}-1/2)^2+y_{r_2}^2\}} + \sum_{r_1=1}^{\infty}\frac{2^0}{\{(x_{r_1}-1/2)^2+y_{r_1}^2\}}$$

$$B_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^3(x_{r_1}-1/2)(x_{r_2}-1/2)(x_{r_3}-1/2)}{\{(x_{r_1}-1/2)^2+y_{r_1}^2\}\{(x_{r_2}-1/2)^2+y_{r_2}^2\}\{(x_{r_3}-1/2)^2+y_{r_3}^2\}}$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\{(x_{r_1}-1/2)+(x_{r_2}-1/2)\}}{\{(x_{r_1}-1/2)^2+y_{r_1}^2\}\{(x_{r_2}-1/2)^2+y_{r_2}^2\}}$$

$$B_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_4=r_3+1}^{\infty}\frac{2^4(x_{r1}-1/2)(x_{r2}-1/2)\cdots(x_{r4}-1/2)}{\{(x_{r1}-1/2)^2+y_{r1}^2\}\{(x_{r2}-1/2)^2+y_{r2}^2\}\cdots\{(x_{r4}-1/2)^2+y_{r4}^2\}}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\{(x_{r1}-1/2)(x_{r2}-1/2)+\cdots+(x_{r2}-1/2)(x_{r3}-1/2)\}}{\{(x_{r_1}-1/2)^2+y_{r_1}^2\}\{(x_{r_2}-1/2)^2+y_{r_2}^2\}\{(x_{r_3}-1/2)^2+y_{r_3}^2\}}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\{(x_{r_1}-1/2)^2+y_{r_1}^2\}\{(x_{r_2}-1/2)^2+y_{r_2}^2\}}$$

$\vdots$

Since $\Xi(z)=\Xi(0)f(z)$, **(1)** holds.

(2) Due to the uniqueness of the power series, it has to be $B_n=A_n \quad n=1,2,3,\cdots$.

Q.E.D.

10.2.3 Proposition2 equivalent to the Riemann Hypothesis

If Riemann Hypothesis is true, the following proposition has to hold.

Proposition 10.2.3

When the non-trivial zeros of Riemann zeta $\zeta(z)$ are $z_k=1/2\pm iy_k$, $y_k\neq 0$ ($k=1,2,3,\cdots$), the following expressions hold.

$$\sum_{r1=1}^{\infty}\frac{1}{y_{r1}^2}=A_2=0.0231049931\cdots \tag{2.3$_2$}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{y_{r_1}^2 y_{r_2}^2}=A_4=0.0002483340\cdots \tag{2.3$_4$}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{y_{r_1}^2 y_{r_2}^2 y_{r_3}^2}=A_6=0.00000167435\cdots \tag{2.3$_6$}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{1}{y_{r_1}^2 y_{r_2}^2 y_{r_3}^2 y_{r_4}^2}=A_8=8.030697\times 10^{-9} \tag{2.3$_8$}$$

$\vdots$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty}\frac{1}{y_{r_1}^2 y_{r_2}^2\cdots y_{r_{2n}}^2}=A_{2n}$$

$$=\sum_{s=0}^{2n}\sum_{t=0}^{s}(-1)^{2n-s}\frac{log^{2n-s}\pi}{2^{2n-s}(r-s)!}\frac{g_{s-t}(5/4)}{2^{s-t}(s-t)!}c_t \tag{2.3$_{2n}$}$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1,f_2,\ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{5}{4}\right)=\begin{cases}1 & r=0\\ \displaystyle\sum_{k=1}^{r}B_{r,k}\left(\psi_0\left(\frac{5}{4}\right),\psi_1\left(\frac{5}{4}\right),\ldots,\psi_{r-1}\left(\frac{5}{4}\right)\right) & r=1,2,3,\cdots\end{cases}$$

$$c_r=\begin{cases}1 & r=0\\ \displaystyle\frac{2}{\zeta(1/2)}\sum_{s=r}^{\infty}(-1)^r\frac{\gamma_{s-1}}{(s-1)!}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots\end{cases}$$

Proof of equivalence

If the Riemann hypothesis holds, The real part of the non-trivial zeros $z_k = x_k \pm i y_k$ of $\zeta(z)$ is $x_k = 1/2$ $(k=1,2,3,\cdots)$. Substituting this for each expressions in Theorem 10.2.2 and replacing B_r with A_r, we obtain the desired expressions. According to Theorem 8.2.4 (**" 08 Factorization of Completed Riemann Zeta "**), (2.3_2) is equivalent to Riemann Hypothesis. Since (2.3_4) ~ (2.3_{2n}) contain (2.3_2) in some way, they must also be equivalent to Riemann Hypothesis respectively.

Direct Calculation

Both sides of (2.3_2), (2.3_4) and (2.3_6) are calculated with formula manipulation software ***Mathematica***, it is as follows, B_2 is calculated in 100,000 terms and is matched up to the 3th decimal place, B_4 is calculated in each 3,000 terms and is matched up to the 5th decimal place, B_6 is calculated in each 300 terms and is matched up to the 6th decimal place

```
y_r := Im[ZetaZero[r]]
```

$$B_2[m\_] := \sum_{r=1}^{m} \frac{1}{y_r^2}$$

N[B2[100000]] → 0.0230829 N[A2] → 0.023105

$$B_4[m\_] := \sum_{r=1}^{m} \sum_{s=r+1}^{m} \frac{1}{y_r^2 y_s^2}$$

N[B4[3000]] → 0.000240759 N[A4] → 0.000248334

$$B_6[m\_] := \sum_{r=1}^{m} \sum_{s=r+1}^{m} \sum_{t=s+1}^{m} \frac{1}{y_r^2 y_s^2 y_t^2}$$

N[B6[300]] → 1.30603×10^{-6} N[A6] → 1.67435×10^{-6}

Indirect Calculation

Here, (2.3_2) is not calculated and $B_2 = A_2 = 0.02310499311\cdots$ is assumed. And using this, the left side of (2.3_4) ~ (2.3_8) are calculated indirectly.

According to Formula 1.3.1 in **" 01 Power of Infinite Series " (Infinite-degree Equation)** ,

$$\left(\sum_{r=0}^{\infty} a_r\right)^2 = \sum_{r=0}^{\infty} a_r^2 + 2\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s \tag{3.2}$$

$$\left(\sum_{r=0}^{\infty} a_r\right)^3 = \sum_{r=0}^{\infty} a_r^3 + 3\sum_{r=0}^{\infty} a_r \cdot \sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s - 3\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t \tag{3.3}$$

$$\begin{aligned}\left(\sum_{r=0}^{\infty} a_r\right)^4 &= 2\sum_{r=0}^{\infty} a_r^4 - \left(\sum_{r=0}^{\infty} a_r^2\right)^2 + 4\left(\sum_{r=0}^{\infty} a_r\right)^2 \cdot \sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} a_r a_s \\ &\quad - 8\sum_{r=0}^{\infty} a_r \cdot \sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty} a_r a_s a_t + 8\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty} a_r a_s a_t a_u\end{aligned} \tag{3.4}$$

Replacing $r=0$ with $r=1$ and a_r with $1/y_r^2$,

$$\left(\sum_{r=0}^{\infty} \frac{1}{y_r^2}\right)^2 = \sum_{r=0}^{\infty}\left(\frac{1}{y_r^2}\right)^2 + 2\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty} \frac{1}{y_r^2 y_s^2}$$

$$\left(\sum_{r=0}^{\infty}\frac{1}{y_r^2}\right)^3=\sum_{r=0}^{\infty}\left(\frac{1}{y_r^2}\right)^3+3\sum_{r=0}^{\infty}\frac{1}{y_r^2}\cdot\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{y_r^2y_s^2}-3\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{y_r^2y_s^2y_t^2}$$

$$\left(\sum_{r=1}^{\infty}\frac{1}{y_r^2}\right)^4=2\sum_{r=1}^{\infty}\left(\frac{1}{y_r^2}\right)^4-\left\{\sum_{r=1}^{\infty}\left(\frac{1}{y_r^2}\right)^2\right\}^2+4\left(\sum_{r=1}^{\infty}\frac{1}{y_r^2}\right)^2\cdot\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{y_r^2y_s^2}$$
$$-8\sum_{r=1}^{\infty}\frac{1}{y_r^2}\cdot\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{y_r^2y_s^2y_t^2}+8\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty}\frac{1}{y_r^2y_s^2y_t^2y_u^2}$$

Substituting $(2.3_2)\sim(2.3_8)$ for these,

$$A_2^2=\sum_{r=0}^{\infty}\frac{1}{y_r^4}+2A_4 \tag{w4}$$

$$A_2^3=\sum_{r=0}^{\infty}\frac{1}{y_r^6}+3A_2A_4-3A_6 \tag{w6}$$

$$A_2^4=2\sum_{r=1}^{\infty}\frac{1}{y_r^8}-\left(\sum_{r=1}^{\infty}\frac{1}{y_r^4}\right)^2+4A_2^2A_4-8A_2A_6+8A_8$$
$$=2\sum_{r=1}^{\infty}\frac{1}{y_r^8}-\left(A_2^2-2A_4\right)^2+4A_2^2A_4-8A_2A_6+8A_8$$

i.e.

$$A_2^4=\sum_{r=1}^{\infty}\frac{1}{y_r^8}-2A_4^2+4A_2^2A_4-4A_2A_6+4A_8 \tag{w8}$$

From (w4), (w6), (w8) ,

$$A_4=-\frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{y_r^4}+\frac{1}{2}A_2^2\quad,\quad A_6=\frac{1}{3}\sum_{r=0}^{\infty}\frac{1}{y_r^6}-\frac{1}{3}A_2^3+A_2A_4$$

$$A_8=-\frac{1}{4}\sum_{r=1}^{\infty}\frac{1}{y_r^8}+\frac{1}{4}A_2^4+\frac{1}{2}A_4^2-A_2^2A_4+A_2A_6$$

Since $A_n=B_n\quad n=1,2,3,\cdots$,

$$B_4=-\frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{y_r^4}+\frac{1}{2}A_2^2 \tag{2.3$_4$'}$$

$$B_6=\frac{1}{3}\sum_{r=0}^{\infty}\frac{1}{y_r^6}-\frac{1}{3}A_2^3+A_2A_4 \tag{2.3$_6$'}$$

$$B_8=-\frac{1}{4}\sum_{r=1}^{\infty}\frac{1}{y_r^8}+\frac{1}{4}A_2^4+\frac{1}{2}A_4^2-A_2^2A_4+A_2A_6 \tag{2.3$_8$'}$$

We should just calculate $(2.3_4')$, $(2.3_6')$ and $(2.3_8')$ instead of

$$B_4=\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{y_r^2y_s^2}\ ,\ B_6=\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{y_r^2y_s^2y_t^2}\ ,\ B_8=\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty}\frac{1}{y_r^2y_s^2y_t^2y_u^2}$$

The calculation results with the formula manipulation software ***Mathematica*** are as follows. B_4 is calculated in 500 terms and is matched up to the 9th decimal place, B_6 is calculated in 30 terms and is matched up to the 11th decimal place, and B_8 is calculated in 20 terms and is matched up to the 13th decimal place. The speed of these convergence are much faster than the direct calculations above.

$$B_4[m\_] := -\frac{1}{2}\sum_{r=1}^{m}\frac{1}{y_r^{\,4}} + \frac{A_2^{\,2}}{2}$$

N[B_4[500]]
0.000248334

N[A_4]
0.000248334

$$B_6[m\_] := \frac{1}{3}\sum_{r=1}^{m}\frac{1}{y_r^{\,6}} + A_2 A_4 - \frac{A_2^{\,3}}{3}$$

N[B_6[30]]
1.67435×10^{-6}

N[A_6]
1.67435×10^{-6}

$$B_8[m\_] := -\frac{1}{4}\sum_{r=1}^{m}\frac{1}{y_r^{\,8}} + \frac{A_2^{\,4}}{4} + \frac{A_4^{\,2}}{2} - A_2^{\,2} A_4 + A_2 A_6$$

N[B_8[20]]
8.0307×10^{-9}

N[A_8]
8.0307×10^{-9}

(w4), (w6) and (w8) also can be represented as follows.

Proposition 10.2.3'

When $z_k = 1/2 \pm i y_k$, $y_k \neq 0$ ($k=1,2,3,\cdots$) are the non-trivial zeros of Riemann zeta $\zeta(z)$ and A_r $r=1,2,3,\cdots$ are constants given by Theorem 10.2.1 , the following expressions hold.

$$\sum_{r=1}^{\infty}\frac{1}{y_r^4} = A_2^2 - 2A_4 = 0.00003717259\cdots$$

$$\sum_{r=1}^{\infty}\frac{1}{y_r^6} = A_2^3 - 3A_2A_4 + 3A_6 = 0.00000014417393\cdots$$

$$\sum_{r=1}^{\infty}\frac{1}{y_r^8} = A_2^4 + 2A_4^2 - 4A_2^2A_4 + 4A_2A_6 - 4A_8 = 6.6303\times10^{-10}$$

The calculation results with the formula manipulation software ***Mathematica*** are as follows. Higher precision is required for calculation of the right side.

$y_{r\_}$:= Im[ZetaZero[r]]

$$N\left[\left\{\sum_{r=1}^{2000}\frac{1}{y_r^{\,4}},\ A_2^{\,2} - 2A_4\right\},\ 7\right]$$

{0.00003717258, 0.0000371726}

$$N\left[\left\{\sum_{r=1}^{100}\frac{1}{y_r^{\,6}},\ A_2^{\,3} - 3A_2A_4 + 3A_6\right\},\ 7\right]$$

$\{1.441738\times10^{-7},\ 1.44174\times10^{-7}\}$

$$N\left[\left\{\sum_{r=1}^{25}\frac{1}{y_r{}^8},\ A_2{}^4+2\,A_4{}^2-4\,A_2{}^2\,A_4+4\,A_2\,A_6-4\,A_8\right\},\ 10\right]$$

$$\{6.63030429\times 10^{-10},\ 6.63032\times 10^{-10}\}$$

As far as the above calculation results are concerned, the Riemann hypothesis seems to be true.

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2018.09.19 Added the 4th power and the 8th power.

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