06 Zeros on the Critical Line of Dirichlet Beta

6.1 Calculation by Sign Function

6.1.0 Calculation by $\Omega(z)$ Function

Completed Dirichlet Beta Function $\Omega(z)$ is as follows. (See " **04 Completed Dirichlet Beta** ").

$$\Omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+z} \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+z\right)\right\}\beta\left(\frac{1}{2}+z\right) \qquad (1.0z)$$

According to 4.2.1, the zeros of this function all coincide with the nontrivial zeros of Dirichlet Beta $\beta(1/2+z)$.
So, putting $z = 0+iy$,

$$\Omega_h(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+iy} \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right) \qquad (1.0y)$$

The zeros of this function all become nontrivial zeros of Dirichlet Beta $\beta(z)$ on the critical line. Then, when this is drawn, it is as follows .
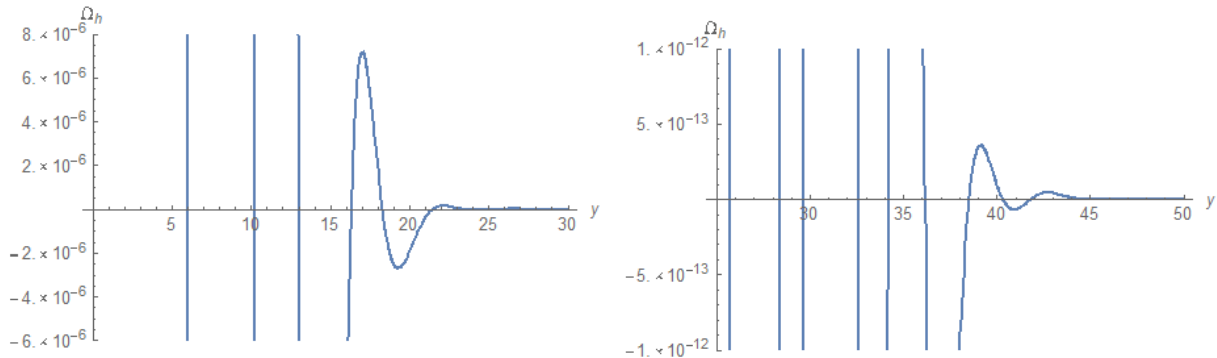
Although drawn on the left figure up to $y=30$, the zeros of this vicinity cannot be distinguished. Although this is expanded in the right figure and from $y=25$ to $y=50$ is drawn, the zeros of this vicinity cannot be distinguished, either. If $y=917$, *Mathematica* can somehow find zeros, but if $y>917$, *Mathematica* can not plot. That is, although $\Omega(z)$ has analytically excellent properties, it is unsuitable for numerical calculation.

6.1.1 Calculation by Sign Function

The reason for this is that the function value of $\Omega_h(y)$ at large y is too small. If so, the idea of increasing the function value is very natural. So, we normalize $\Omega_h(y)$ and call it **sign function** . That is,

$$sgn(y) = -\frac{\Omega_h(y)}{|\Omega_h(y)|}$$

According to " **04 Completed Dirichlet Beta** " **4.2.1 (5)** , $Im\{\Omega_h(y)\} = 0$ for arbitrary real number y .
Then,

$$sgn(y) = -\frac{\Omega_h(y)}{|\Omega_h(y)|}$$

$$= -\frac{\left(\dfrac{2}{\sqrt{\pi}}\right)^{3/2+iy}\Gamma\left\{\dfrac{1}{2}\left(\dfrac{3}{2}+iy\right)\right\}\beta\left(\dfrac{1}{2}+iy\right)}{\left|\left(\dfrac{2}{\sqrt{\pi}}\right)^{3/2+iy}\Gamma\left\{\dfrac{1}{2}\left(\dfrac{3}{2}+iy\right)\right\}\beta\left(\dfrac{1}{2}+iy\right)\right|}$$

Here,

$$\left(\frac{2}{\sqrt{\pi}}\right)^{3/2+iy}\Bigg/\left|\left(\frac{2}{\sqrt{\pi}}\right)^{3/2+iy}\right| = \left(\frac{2}{\sqrt{\pi}}\right)^{iy}\Bigg/\left|\left(\frac{2}{\sqrt{\pi}}\right)^{iy}\right| = \left(\frac{2}{\sqrt{\pi}}\right)^{iy}$$

$$\left(\because \ \left|\left(2/\sqrt{\pi}\right)^{iy}\right| = 1\right)$$

Then,

$$sgn(y) = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy}\frac{\Gamma\left\{\dfrac{1}{2}\left(\dfrac{3}{2}+iy\right)\right\}\beta\left(\dfrac{1}{2}+iy\right)}{\left|\Gamma\left\{\dfrac{1}{2}\left(\dfrac{3}{2}+iy\right)\right\}\beta\left(\dfrac{1}{2}+iy\right)\right|} \qquad (1.1)$$

Note: Attention when drawing

   Although the numerator of (1.1) is a real number theoretically, small imaginary number is included in numerical calculation. In numerical calculation, there is no problem if the imaginary part is ignored. However, at the time of drawing, $sgn(y)$ may be regarded as a complex number and may not be drawn. In that case, we only need to draw $Re[sgn(y)]$.

   Since this function has an absolute value of 1 , it must be easy to find the zero of $\Omega_h(y)$. In fact, when this is drawn, it becomes as follows.
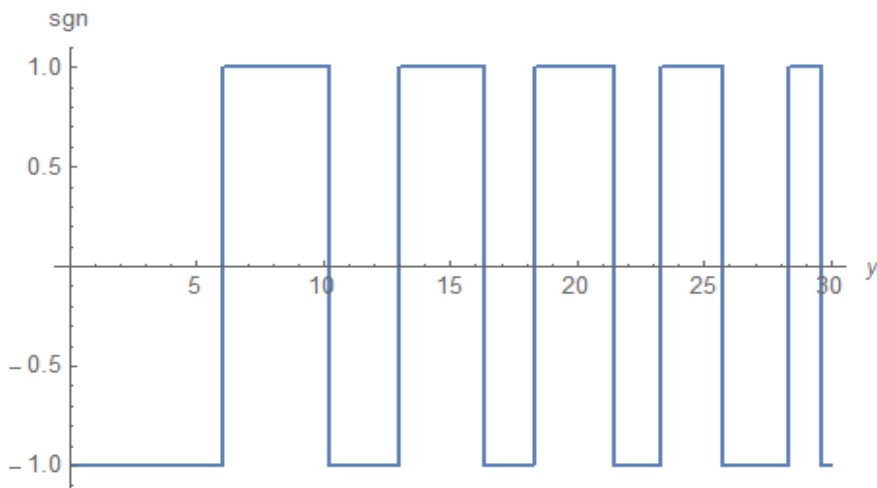

   After finding the approximate position of the zero with this function, we calculate the value of the zero point directly from $\beta(1/2+iy)$. For example, for a zero point near $y=28$, we use the function $FindRoot$ of $Mathematica$ as follows. (Imaginary part ignored.)

$$\text{FindRoot}\left[\beta\left[\frac{1}{2}+i\,y\right],\,\{y,\,28\}\right] \qquad \{y \to 28.3596 + 6.63172 \times 10^{-16}\,i\}$$

6.2 Invert the sign of |β(1/2+iy)|

The absolute value of the Dirichlet beta function $\beta(1/2+iy)$ is as drawn on the left figure.

If the sign function $sgn(y)$ of the previous section is superimposed on this, it becoms a right figure.
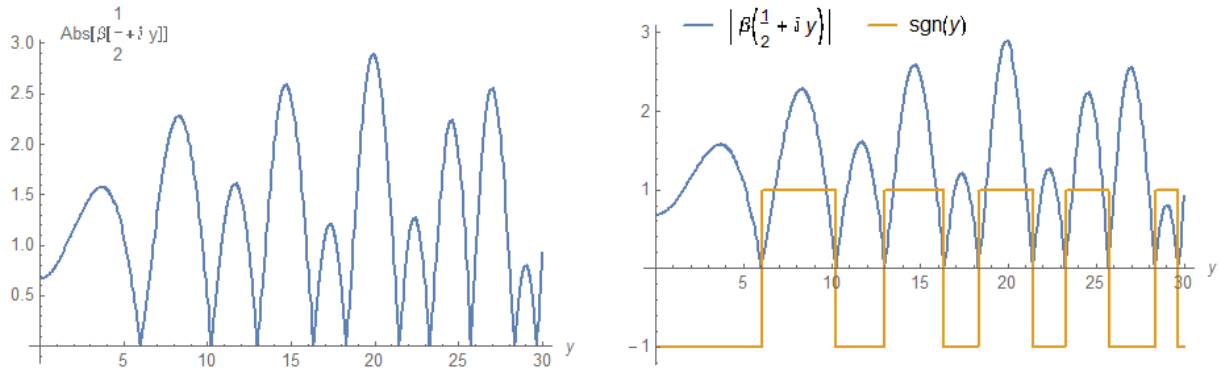
To be noted is the right figure. The values of $sgn(y)$ in the intervals we want to be negative are exactly negative. The product of the two can be expected to be a smooth function. In fact, the product of the two is drawn as follows.
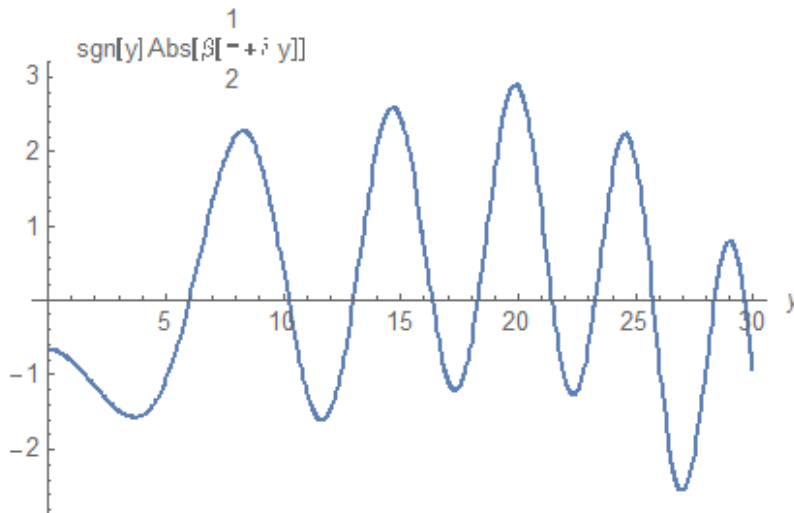
Naturally, all the zeros of this function coincide with the zeros on the critical line of the Dirichlet beta $\beta(z)$. Moreover, the absolute value is much larger than $\Omega_h(y)$.

B function

We call this product the *B* function. Then, it is expressed as follows.

$$B(y) = sgn(y)\left|\beta\left(\frac{1}{2}+iy\right)\right|$$

From the previous section,

$$sgn(y) = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right)}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right)\right|} \tag{1.1}$$

Substituting this for the above,

$$B(y) = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy} \frac{\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right)}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right)\right|}\left|\beta\left(\frac{1}{2}+iy\right)\right|$$

From this, we obtain

$$B(y) = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy} \frac{\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\right|}\beta\left(\frac{1}{2}+iy\right) \tag{2.1}$$

Using this function, we can find the zeros on the critical line of $\beta(z)$ by the intersection of the curve and the y-axis.

Note: Attention when drawing

Although the numerator of (2.1) is a real number theoretically, small imaginary number is included in numerical calculation. In numerical calculation, there is no problem if the imaginary part is ignored. However, at the time of drawing, $B(y)$ may be regarded as a complex number and may not be drawn. In that case, we only need to draw $Re[B(y)]$.

Zeros near y= 1 millin

When this is drawn using (2.1) and *Mathematica*, it is as follows.
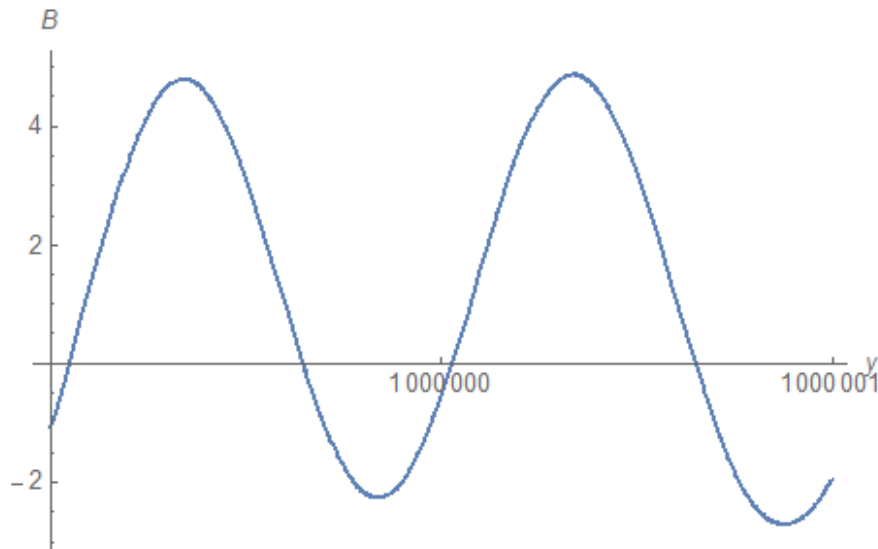
The zero point near the center of this figure is calculated as follows.

```
SetPrecision[FindRoot[B[y], {y, 1 000 000}], 20]
{y → 1.00000002797520882450 × 10⁶ - 1.4374 × 10⁻¹⁰ i}

N[DirichletBeta[1/2 + i 1 000 000.0279752088245], 20]
-3.5 × 10⁻¹⁰ + 1.33 × 10⁻⁹ i
```

It is difficult in time to calculate the zeros at $y = 10$ million or more using (2.1) and *Mathematica*.

6.3 Riemann-Siegel Style B function

We can derive Riemann-Siegel style B function from B function in the previous section. For the purpose, we prepare a Lemma, which is a repeat from " **11 Zeros on the Critical Line of Riemann Zeta** ".

Lemma 6.3.1 (Repeat)

When $f(z)$ is a complex function defined on the domain D, the following expression holds.

$$e^{i\,\mathrm{Im}\,\log f(z)} = \frac{f(z)}{|f(z)|}$$

Riemann-Siegel style B function

From the previous section,

$$B(y) = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy} \frac{\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\right|} \beta\left(\frac{1}{2}+iy\right) \tag{2.1}$$

Applying Lemma 6.3.1 to the gamma function,

$$B(y) = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy} e^{\,i\,\mathrm{Im}\,\log \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}} \beta\left(\frac{1}{2}+iy\right)$$

$$= -e^{\,i\left[\mathrm{Im}\,\log \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\} - \frac{y}{2}\log\frac{\pi}{4}\right]} \beta\left(\frac{1}{2}+iy\right)$$
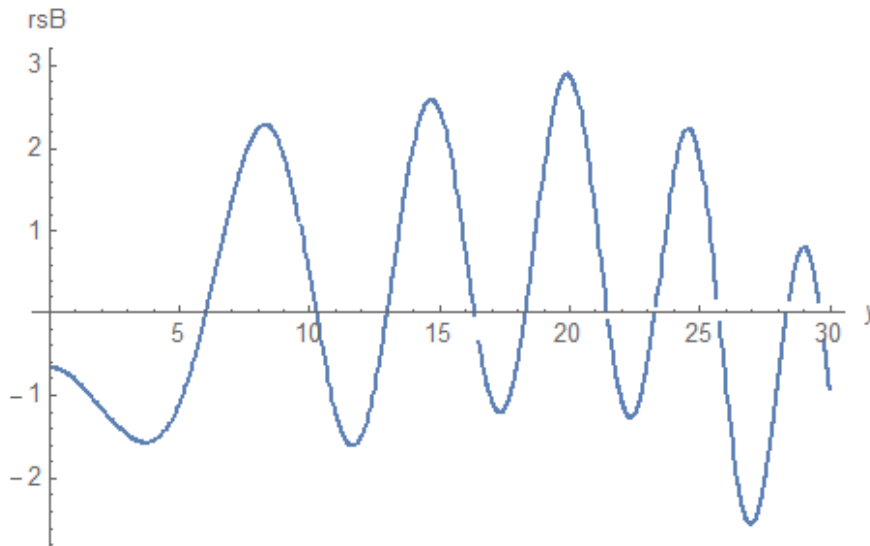
Here, let

$$\theta(y) = \mathrm{Im}\,\log\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\} - \frac{y}{2}\log\frac{\pi}{4}$$

Then,

$$B(y) = -e^{i\theta(y)}\beta\left(\frac{1}{2}+iy\right) \tag{3.1}$$

This is Riemann-Siegel style B function, which is drawn as follows.

We can see that this figure is the same as in the previous section .

Note

　It is difficult in time to calculate the zeros at $y=10$ million or more using (3.1) and *Mathematica*. In order to calculate the zeros at y greater than this, a formula such as the Riemann-Siegel asymptotic formula in the zeta function is required.

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Alien's Mathematics