11 Zeros on the Critical Line of Riemann Zeta

11.1 Calculation by Sign Function

11.1.0 Calculation by $\Xi(z)$ Function

Completed Riemann Zeta Function $\Xi(z)$ is as follows. (See " **07 Completed Riemann Zeta** ").
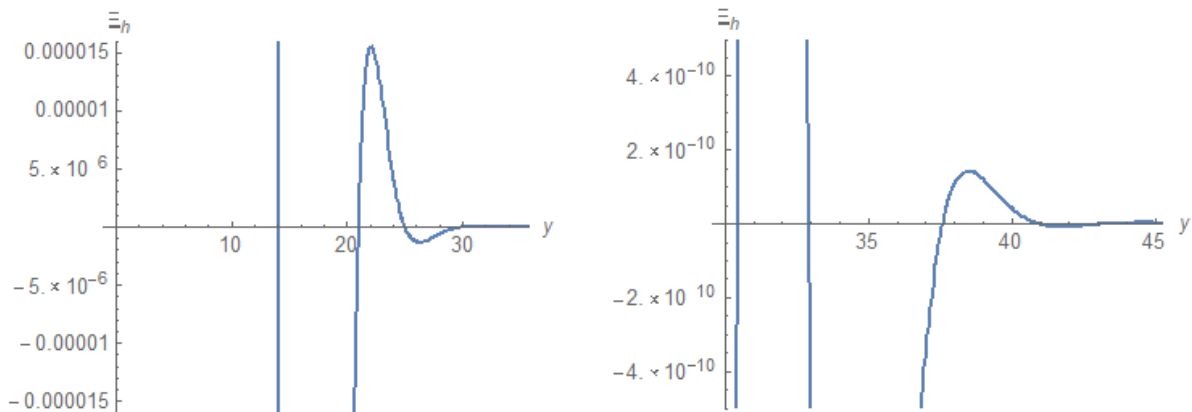
$$\Xi(z) \;=\; -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) \tag{1.0z}$$

According to 7.4.1, the zeros of this function all coincide with the nontrivial zeros of Riemann Zeta $\zeta(1/2+z)$.
So, putting $z=0+iy$,

$$\Xi_h(y) \;=\; -\left(\frac{1}{2}+iy\right)\left(\frac{1}{2}-iy\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+iy\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right) \tag{1.0y}$$

the zeros of this function all become nontrivial zeros of Riemann Zeta $\zeta(z)$ on the critical line. Then, when
this is drawn, it is as follows .

Although drawn on the left figure up to $y=35$, the zeros of this vicinity cannot be distinguished. Although
this is expanded in the right figure and from $y=30$ to $y=45$ is drawn, the zeros of this vicinity cannot be
distinguished, either. If $y=917$, *Mathematica* can somehow find zeros, but if $y>917$, *Mathematica* can
not plot. That is, although $\Xi(z)$ has analytically excellent properties, it is unsuitable for numerical calculation.

11.1.1 Calculation by Sign Function

The reason for this is that the function value of $\Xi_h(y)$ at large y is too small. If so, the idea of increasing
the function value is very natural. So, we normalize $\Xi_h(y)$ and call it **sign function** . That is,

$$sgn(y) \;=\; -\frac{\Xi_h(y)}{|\Xi_h(y)|}$$

According to " **07 Completed Riemann Zeta** " 7.4.1 (4) , $Im\{\Xi_h(y)\}=0$ for arbitrary real number y .
Then,

$$sgn(y) \;=\; -\frac{\Xi_h(y)}{|\Xi_h(y)|}$$

$$= \frac{\left(\frac{1}{2}+iy\right)\left(\frac{1}{2}-iy\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+iy\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\varsigma\left(\frac{1}{2}+iy\right)}{\left|\left(\frac{1}{2}+iy\right)\left(\frac{1}{2}-iy\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+iy\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\varsigma\left(\frac{1}{2}+iy\right)\right|}$$

Here,

$$\left(\frac{1}{2}+iy\right)\left(\frac{1}{2}-iy\right)\Big/\left|\left(\frac{1}{2}+iy\right)\left(\frac{1}{2}-iy\right)\right| = \left(\frac{1}{4}+y^2\right)\Big/\left|\frac{1}{4}+y^2\right| = 1$$

$$(\because \ y: \ real \ number)$$

$$\pi^{-\frac{1}{2}\left(\frac{1}{2}+iy\right)}\Big/\left|\pi^{-\frac{1}{2}\left(\frac{1}{2}+iy\right)}\right| = \pi^{-\frac{1}{4}}\pi^{-\frac{iy}{2}}\Big/\left|\pi^{-\frac{1}{4}}\pi^{-\frac{iy}{2}}\right| = \pi^{-\frac{iy}{2}}$$

$$\left(\because \ \left|\pi^{-iy/2}\right| = 1\right)$$

Then,

$$sgn(y) = \pi^{-\frac{iy}{2}}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\varsigma\left(\frac{1}{2}+iy\right)}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\varsigma\left(\frac{1}{2}+iy\right)\right|} \qquad (1.1)$$

Note: Attention when drawing

   Although the numerator of (1.1) is a real number theoretically, small imaginary number is included in numerical calculation. In numerical calculation, there is no problem if the imaginary part is ignored. However, at the time of drawing, $sgn(y)$ may be regarded as a complex number and may not be drawn. In that case, we only need to draw $Re[sgn(y)]$.

   Since this function has an absolute value of 1, it must be easy to find the zero of $\varXi_h(y)$. In fact, when this is drawn, it becomes as follows.
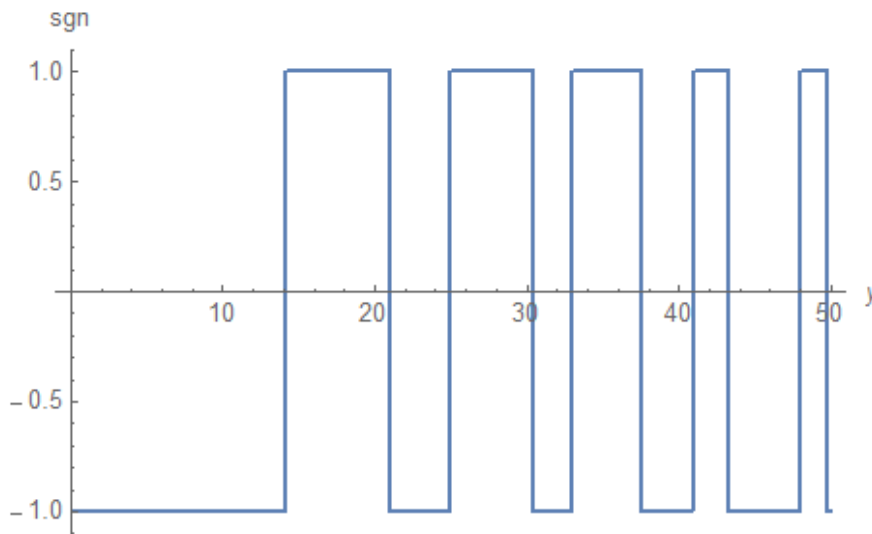

With this function, it seems that the zeros of $\varXi_h(y)$ can be found with certainty. The zero point is where

the value of $sgn\left(y\right)$ becomes indeterminate as a result of approaching by the dichotomy method from the place considered as the zero point. However, the dichotomy method is not efficient. It is better to calculate the value of zero poit directly from $\zeta\left(1/2+iy\right)$ using this function only to find the approximate position of the zero point. For example, for a zero point near $y=48$, we use the function *FindRoot* of *Mathematica* as follows. (Imaginary part ignored.)

$$\text{FindRoot}\left[\text{Zeta}\left[\frac{1}{2}+iy\right],\ \{y,\ 48\}\right]\qquad\left\{y\to 48.0052-1.72833\times 10^{-15}\,i\right\}$$

Lehmer's Phenominon

As for finding the position of the zeros, $sgn\left(y\right)$ seems to be certain. However, the following case exists.
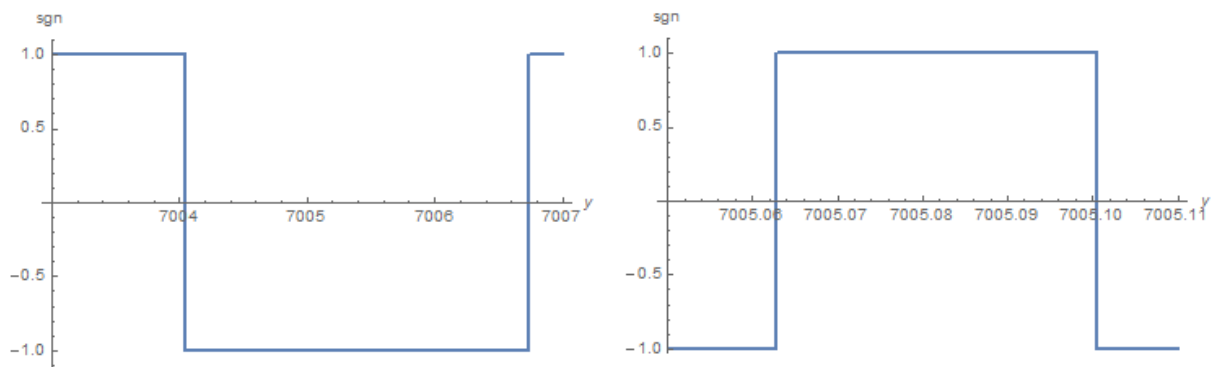

In the left figure, no zero is found near $y=7005$. However, if this vicinity is enlarged, as shown in the right figure, two zeros exist. Because this interval is as narrow as 0.037, it is ignored in the left figure. Such an example is called the *Lehmer's Phenominon*. Sign function $sgn\left(y\right)$ has the disadvantage that it is easy to miss this phenomenon.

11.2 Invert the sign of |ζ(1/2+iy)|

The absolute value of the Riemann zeta function $\zeta(1/2+iy)$ is as drawn on the left figure.

If the sign function $sgn(y)$ of the previous section is superimposed on this, it becoms a right figure.
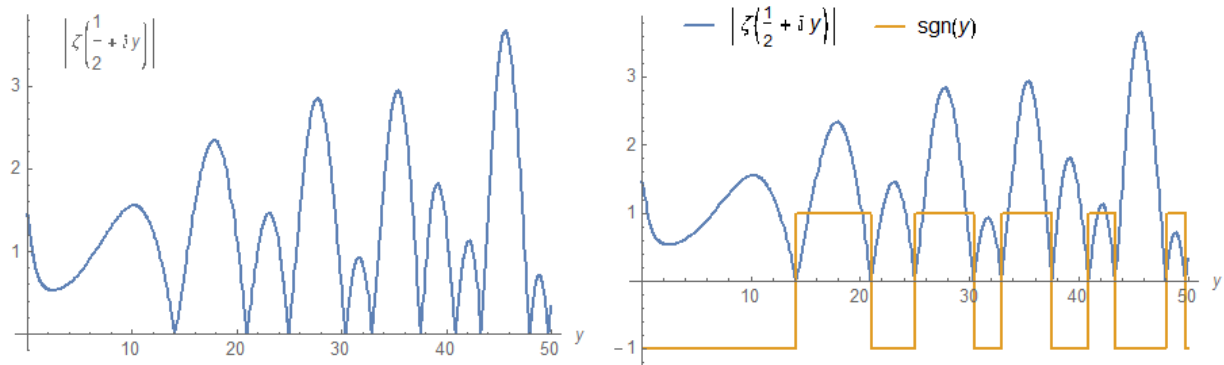
To be noted is the right figure. The values of $sgn(y)$ in the intervals we want to be negative are exactly negative. The product of the two can be expected to be a smooth function. In fact, the product of the two is drawn as follows.
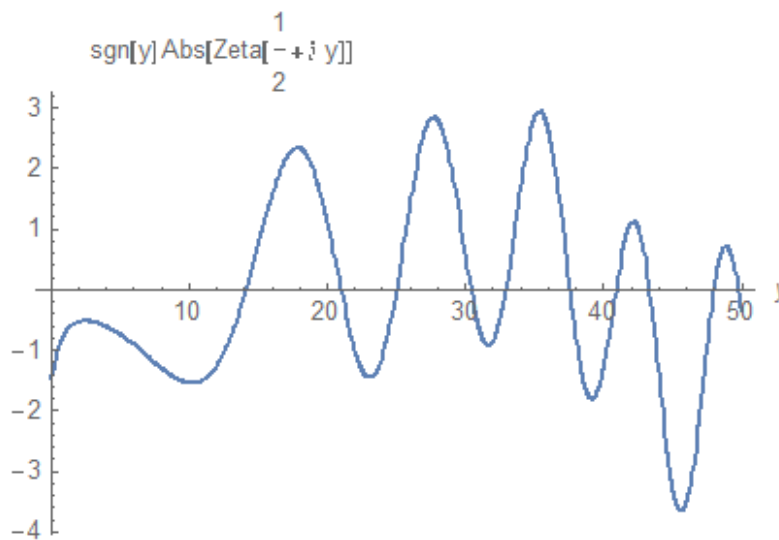
Naturally, all the zeros of this function coincide with the zeros on the critical line of the Riemann zeta $\zeta(z)$. Moreover, the absolute value is much larger than $\Xi_h(y)$.

Z function

We call this product the Z **function**. Then, it is expressed as follows.

$$Z(y) = sgn(y)\left|\zeta\left(\frac{1}{2}+iy\right)\right|$$

From the previous section,

$$sgn(y) = \pi^{-\frac{iy}{2}}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right)}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right)\right|} \tag{1.1}$$

Substituting this for the above,

$$Z(y) = \pi^{-\frac{iy}{2}} \frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right)}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right)\right|}\left|\zeta\left(\frac{1}{2}+iy\right)\right|$$

From this, we obtain

$$Z(y) = \pi^{-\frac{iy}{2}} \frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\right|}\zeta\left(\frac{1}{2}+iy\right) \qquad (2.1)$$

Using this function, we can find the zeros on the critical line of $\zeta(z)$ by the intersection of the curve and the y-axis.

Note: Attention when drawing

Although the numerator of (2.1) is a real number theoretically, small imaginary number is included in numerical calculation. In numerical calculation, there is no problem if the imaginary part is ignored. However, at the time of drawing, $Z(y)$ may be regarded as a complex number and may not be drawn. In that case, we only need to draw $Re[Z(y)]$.

Lehmer's Phenominon

When zeros near $y=18882504$ are drawn using this function, it is as follws. In the left figure, it is difficult to distinguish the zeros. In the z function, the Lehmer's phenomenon appears in this way. However, if this figure is enlarged, it becomes possible to distinguish zeros as shown in the right figure. As regards this point, z function is less dangerous than the function $sgn(y)$ in the previous section
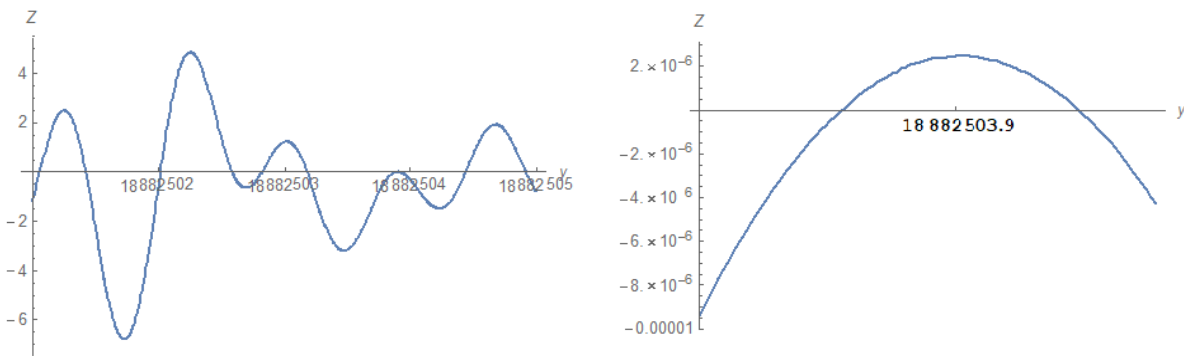
Zeros near y = 26 trillion

When this is drawn using (2.1) and $Mathematica$, it is as follows.
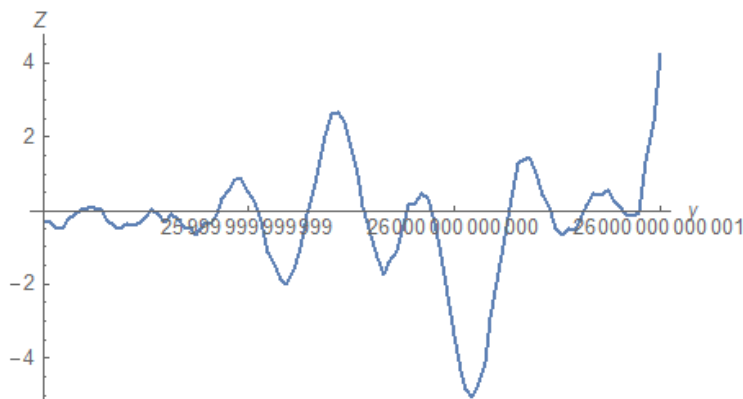
The zero point near the right end of this figure is calculated as follows.

$\text{N[Z[26\,000\,000\,000\,000.883]]}$ $\text{N[Z[26\,000\,000\,000\,000.8831]]}$

$0.0714604 - 0.00599445\,i$ Indeterminate

$\text{N}\left[\text{Zeta}\left[\dfrac{1}{2} + i\,26\,000\,000\,000\,000.8831\right]\right]$

$0.0000988717 + 0.000250425\,i$

Although *Mathematica* is implemented with a dedicated function $RiemannSiegel\,Z$ which incorporates Riemann-Siegel asymptotic formula, here is the limit of calculation even if using this. That is, (2.1) is comparable to the dedicated function $RiemannSiegel\,Z$. This is a surprise. Although we ingtended to use (2.1), we can only think that *Mathematica* used $RiemannSiegel\,Z$.

11.3 Riemann-Siegel Z function

The Z function in the previous section is different from the Riemann-Siegel Z function in both origin and form. However, both are the same function. In fact, we can derive the latter from the former. For the purpose, we prepare a Lemma.

Lemma 11.3.1

When $f(z)$ is a complex function defined on the domain D, the following expression holds.

$$e^{i\,Im\,log\,f(z)} = \frac{f(z)}{|f(z)|}$$

Proof

Expressing the function $f(z)$ in polar form,

$$f(z) = |f(z)|\,e^{i\,Arg\,f(z)}$$

On the other hand,

$$log\,f(z) = |f(z)| + i\,Arg\,f(z)$$

From this,

$$Im\,log\,f(z) = Arg\,f(z)$$

Substituting this for the above,

$$f(z) = |f(z)|\,e^{i\,Im\,log\,f(z)}$$

From this, we obtain the desired expression.

Riemann-Siegel Z function

From the previous section,

$$Z(y) = \pi^{-\frac{iy}{2}} \frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\right|}\,\zeta\left(\frac{1}{2}+iy\right) \tag{2.1}$$

Applying Lemma 11.3.1 to the gamma function,

$$Z(y) = \pi^{-\frac{iy}{2}}\,e^{i\,Im\,log\,\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}}\,\zeta\left(\frac{1}{2}+iy\right)$$

$$= e^{i\left[Im\,log\,\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\} - \frac{iy}{2}log\,\pi\right]}\,\zeta\left(\frac{1}{2}+iy\right)$$

Here, let

$$\theta(y) = Im\,log\,\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\} - \frac{y}{2}log\,\pi$$

Then,

$$Z(y) = e^{i\theta(y)}\,\zeta\left(\frac{1}{2}+iy\right) \tag{3.1}$$

This is a definitional equation of Riemann-Siegel Z function.

Note

The calculation speed by (3.1) and *Mathematica* is also abnormal. It is impossible speed without using the dedicated function $RiemannSiegel\,Z$.

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<div align="center">**Alien's Mathematics**</div>