11 Zeros of Dirichlet Eta and System of Transcendental Equations

Abstract

(1) The problem of the zeros of Dirichlet eta function is reduced to overdetermined system of Transcendental equations, by functional equations.

(2) On the critical line, this system of transcendental equations has a solution.

(3) Except on the critical line, this system of transcendental equations is unlikely to have a solution.

11.1 Series of $\eta(1/2 \pm z)$

In this section, we first prepare the following two formulas.

Formula 11.1.1

When the Dirichlet Eta function is $\eta(z)$ ($z = x+iy$) and $\eta(1/2+z) = u_+(z) + i\, v_+(z)$, the following expressions hold for $x > -1/2$.

$$\eta_+(z) := \eta\left(\frac{1}{2}+z\right) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} e^{-z \log s} \tag{1.1$_+$}$$

$$u_+(x,y) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} e^{-x \log s} \cos(y \log s)$$

$$v_+(x,y) = -\sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} e^{-x \log s} \sin(y \log s)$$

Proof

$$\eta(z) = \sum_{s=1}^{\infty} (-1)^{s-1} e^{-z \log s}$$

Replacing z with $1/2+z$,

$$\eta\left(\frac{1}{2}+z\right) = \sum_{s=1}^{\infty} (-1)^{s-1} e^{-\left(\frac{1}{2}+z\right)\log s} = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} e^{-z \log s}$$

Let $z = x+iy$. Then

$$\eta_+(x,y) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} e^{-x \log s} e^{-iy \log s}$$

$$= \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} e^{-x \log(s)} \{\cos(y \log s) - i \sin(y \log s)\}$$

From this, we obtain u_+ , v_+ .

Note

$x = -1/2$ is the line of convergence of this Dirichlet series.

Formula 11.1.2

When the Dirichlet Eta function is $\eta(z)$ ($z=x+iy$) and $\eta(1/2-z) = u_-(z)+i\,v_-(z)$, the following expressions hold for $x<1/2$.

$$\eta_-(z) := \eta\left(\frac{1}{2}-z\right) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}e^{z\log s} \tag{1.1_}$$

$$u_-(x,y) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}e^{x\log s}\cos(y\log s)$$

$$v_-(x,y) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}e^{x\log s}\sin(y\log s)$$

Proof

$$\eta(z) = \sum_{s=1}^{\infty}(-1)^{s-1}e^{-z\log s}$$

Replacing z with $1/2-z$,

$$\eta\left(\frac{1}{2}-z\right) = \sum_{s=1}^{\infty}(-1)^{s-1}e^{-\left(\frac{1}{2}-z\right)\log s} = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}e^{z\log s}$$

Let $z=x+iy$. Then

$$\eta_-(x,y) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}e^{x\log s}e^{iy\log s}$$

$$= \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}e^{x\log s}\{\cos(y\log s) + i\sin(y\log s)\}$$

From this, we obtain u_- , v_- .

Note

$x = 1/2$ is the line of convergence of this Dirichlet series.

11.2 Hyperbolic Function Series

In the previous section, the following two formulas were obtained .

$$\eta_{-}(z) := \eta\left(\frac{1}{2}-z\right) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}e^{z\log s} \tag{1.1$_-$}$$

$$\eta_{+}(z) := \eta\left(\frac{1}{2}+z\right) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}e^{-z\log s} \tag{1.1$_+$}$$

In this section, these are rearranged into two hyperbolic function series. And, this is further expanded into series by real part and imaginary part.

Formula 11.2.1 (cosh function series)

When the set of real numbers is R and $z=x+iy\ (x,y\in R)$, the following formulas hold on the whole complex plane.

$$\eta_c(z) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\cosh(z\log s) \qquad (=u_c+i\,v_c)$$

$$u_c(x,y) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\cosh(x\log s)\cos(y\log s)$$

$$v_c(x,y) = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\sinh(x\log s)\sin(y\log s)$$

Proof

From (1.1$_-$) , (1.1$_+$) ,

$$\frac{1}{2}\left\{\eta\left(\frac{1}{2}-z\right)+\eta\left(\frac{1}{2}+z\right)\right\} = \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\frac{e^{z\log s}+e^{-z\log s}}{2}$$

$$= \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\cosh(z\log s) =: \eta_c(z)$$

Here,

$$\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y$$

So, replacing x with $x\log s$ and y with $y\log s$ respectively,

$$\cosh(z\log s) = \cosh(x\log s)\cos(y\log s) + i\sinh(x\log s)\sin(y\log s)$$

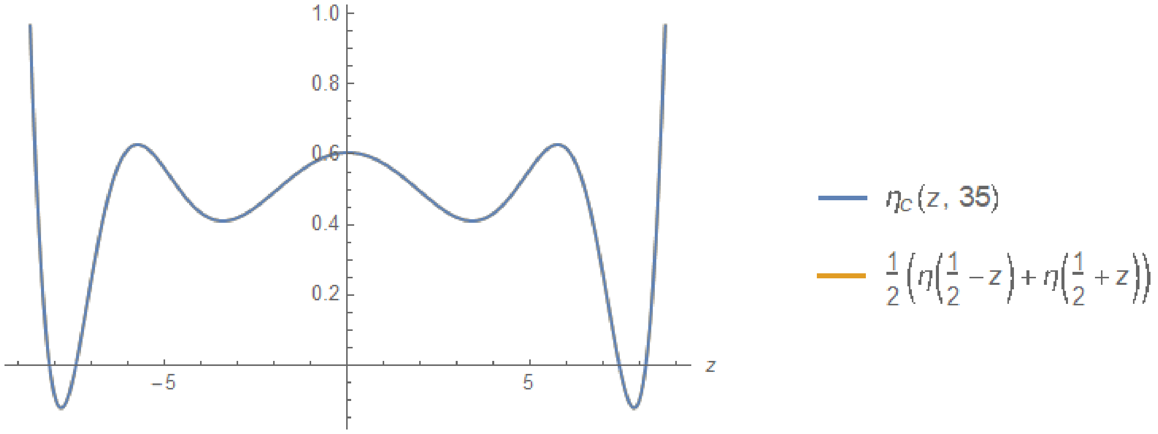
Substituting this for $\eta_c(z)$, we obtain u_c, v_c.

$\eta_c(z)$ which consists of the sum of Dirichlet series is no longer a Dirichlet series. And the convergence area is $[-\infty, 1/2)\cup(-1/2, \infty] = [-\infty, \infty]$. That is, the whole complex plane.

Since the convergence speed of $\eta_c(z)$ is slow, a decent figure can not be drawn. So, applying the Euler transformation to this,

$$\eta_c(z) = \sum_{k=1}^{\infty}\sum_{s=1}^{k}\frac{1}{2^{k+1}}\binom{k}{s}\frac{(-1)^{s-1}}{\sqrt{s}}\cosh(z\log s) \qquad (=u_c+i\,v_c)$$

Using this acceleration formula, $\eta_c(z)$ is drawn as follows. In addition, $\{\eta(1/2-z)+\eta(1/2+z)\}/2$ is also drawn together, but both are exactly overlapped and the latter (orange) cannot be seen.



Formula 11.2.2 (sinh function series)

When the set of real numbers is R and $z=x+iy\ \ (x,y\in R)$, the following formulas hold on the whole complex plane.

$$\eta_s(z)=\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\sinh(z\log s)\qquad (=u_s+i\,v_s)$$

$$u_s(x,y)=\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\sinh(x\log s)\cos(y\log s)$$

$$v_s(x,y)=\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\cosh(x\log s)\sin(y\log s)$$

Proof

From (1.1.-) , (1.1.+) ,

$$\frac{1}{2}\left\{\eta\left(\frac{1}{2}-z\right)-\eta\left(\frac{1}{2}+z\right)\right\}=\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\frac{e^{z\log s}-e^{-z\log s}}{2}$$

$$=\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\sinh\{z\log s\}=:\eta_s(z)$$

Here,

$$\sinh(x+iy)=\sinh x\cos y+i\cosh x\sin y$$

So, replacing x with $x\log s$ and y with $y\log s$ respectively,

$$\sinh(z\log s)=\sinh(x\log s)\cos(y\log s)+i\cosh(x\log s)\sin(y\log s)$$

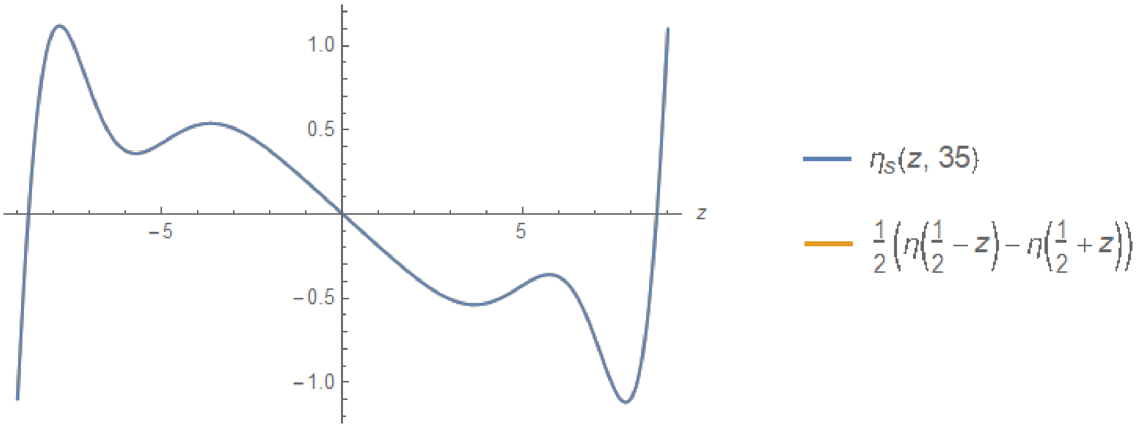
Substituting this for $\eta_s(z)$, we obtain u_s, v_s.

$\eta_s(z)$ which consists of the difference between Dirichlet series, is no longer a Dirichlet series. And the convergence area is $[-\infty,1/2)\cup(-1/2,\infty]=[-\infty,\infty]$. That is, the whole complex plane.

Since the convergence speed of $\eta_s(z)$ is slow, a decent figure can not be drawn. So, applying the Euler transformation to this,

$$\eta_s(z) = \sum_{k=1}^{\infty}\sum_{s=1}^{k}\frac{1}{2^{k+1}}\binom{k}{s}\frac{(-1)^{s-1}}{\sqrt{s}}\sinh(z\log s) \qquad (= u_s + i\, v_s)$$

Using this acceleration formula, $\eta_s(z)$ is drawn as follows. In addition, $\{\eta(1/2-z)-\eta(1/2+z)\}/2$ is also drawn together, but both are exactly overlapped and the latter (orange) cannot be seen.



11.3 Necessary and Sufficient Condition for Zeros

Theorem 11.3.1

When the set of real numbers is R and Dirichlet eta functions is $\eta(z)$ $(z=x+iy\,,\ x,y\in R)$,
$\eta(1/2\pm z)=0$ $(-1/2<x<1/2)$ if and only if the following system of equations has a solution.

$$\begin{cases} u_c(x,y) = \displaystyle\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\cosh(x\log s)\cos(y\log s) = 0 \\ v_c(x,y) = \displaystyle\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\sinh(x\log s)\sin(y\log s) = 0 \\ u_s(x,y) = \displaystyle\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\sinh(x\log s)\cos(y\log s) = 0 \\ v_s(x,y) = \displaystyle\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\cosh(x\log s)\sin(y\log s) = 0 \end{cases}$$

Proof

(1) Necessity

The following functional equation holds for the Dirichlet Eta function $\eta(z)$.

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left(1-2^{z}\right)\eta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\left(1-2^{1-z}\right)\eta(1-z) \qquad 0<Re(z)<1$$

Gamma function and powers of π have no zeros, and $1-2^{z}, 1-2^{1-z}$ have no zeros in $0<Re(z)<1$

Therefore, at the zero of $\eta(z)$,

$$\eta(z) = \eta(1-z) = 0 \qquad 0<Re(z)<1$$

Replacing z with $1/2+z$,

$$\eta\left(\frac{1}{2}+z\right) = \eta\left(\frac{1}{2}-z\right) = 0 \qquad -\frac{1}{2}<Re(z)<\frac{1}{2}$$

From Formula 11.1.1 and Formula 11.1.2,

$$\begin{cases} \eta\left(\dfrac{1}{2}-z\right) = \displaystyle\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}e^{z\log s} = 0 & (3.1_-) \\ \eta\left(\dfrac{1}{2}+z\right) = \displaystyle\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}e^{-z\log s} = 0 & (3.1_+) \end{cases}$$

Further, by Formula 11.2.1 and Formula 11.2.2, these are rearranged as follows.

$$\begin{cases} \dfrac{1}{2}\left\{\eta\left(\dfrac{1}{2}-z\right)+\eta\left(\dfrac{1}{2}+z\right)\right\} = \displaystyle\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\cosh(z\log s) = 0 & (3.1c) \\ \dfrac{1}{2}\left\{\eta\left(\dfrac{1}{2}-z\right)-\eta\left(\dfrac{1}{2}+z\right)\right\} = \displaystyle\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\sqrt{s}}\sinh(z\log s) = 0 & (3.1s) \end{cases}$$

Putting $z=x+iy$ and denoting these by real and imaginary parts, we obtain the desired expression.

(2) Sufficiency

By adding and subtracting (3.1c) and (3.1s) ,

$$\begin{cases} \eta\left(\frac{1}{2}-z\right) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} \left[\cosh(z \log s) + \sinh(z \log s) \right] = 0 \\ \eta\left(\frac{1}{2}+z\right) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} \left[\cosh(z \log s) - \sinh(z \log s) \right] = 0 \end{cases}$$

Substituting $\cosh z = \left(e^{z} + e^{-z}\right)/2$, $\sinh z = \left(e^{z} - e^{-z}\right)/2$ for these,

$$\begin{cases} \eta\left(\frac{1}{2}-z\right) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} e^{z \log s} = 0 \\ \eta\left(\frac{1}{2}+z\right) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} e^{-z \log s} = 0 \end{cases}$$

Q.E.D.

Convergence Acceleration by Euler Transformation

Hereafter, the formulas of Theorem 11.3.1 are drawn. However, since these series are slow to converge, they can not be drawn cleanly, especially when y is small. So, hereafter, for $y \geq 100$, we use the original formula of the theorem as it is, and for $y < 100$, we use the following acceleration formulas by Euler transformation.

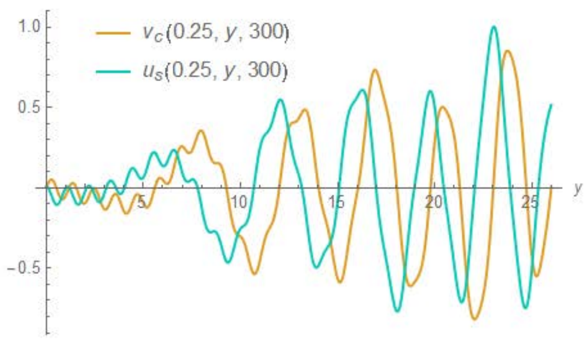
$$u_c(x,y) = \sum_{k=1}^{\infty} \sum_{s=1}^{k} \frac{1}{2^{k+1}} \binom{k}{s} \frac{(-1)^{s-1}}{\sqrt{s}} \cosh(x \log s) \cos(y \log s)$$

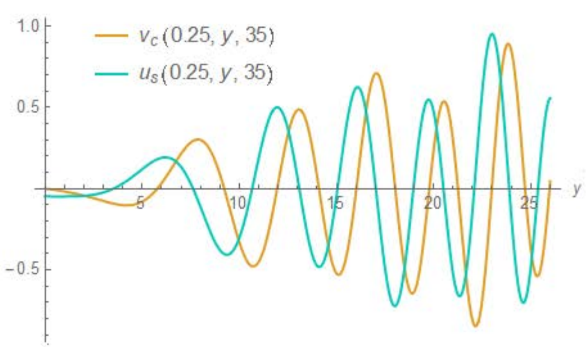
$$v_c(x,y) = \sum_{k=1}^{\infty} \sum_{s=1}^{k} \frac{1}{2^{k+1}} \binom{k}{s} \frac{(-1)^{s-1}}{\sqrt{s}} \sinh(x \log s) \sin(y \log s)$$

$$u_s(x,y) = \sum_{k=1}^{\infty} \sum_{s=1}^{k} \frac{1}{2^{k+1}} \binom{k}{s} \frac{(-1)^{s-1}}{\sqrt{s}} \sinh(x \log s) \cos(y \log s)$$

$$v_s(x,y) = \sum_{k=1}^{\infty} \sum_{s=1}^{k} \frac{1}{2^{k+1}} \binom{k}{s} \frac{(-1)^{s-1}}{\sqrt{s}} \cosh(x \log s) \sin(y \log s)$$

As a sample, drawing v_c , u_s at $x = 0.25$ with the original formula and the accelerated formula, it is as follows. The left figure is the original formula and the right figure is the accelerated formula. The acceleration effect is obvious where y is small.





Overdetermined System

Since there are four equations for two variables in Theorem 11.3.1 , this system of equations is an over-determined system. Such a system of equations generally has no solution.

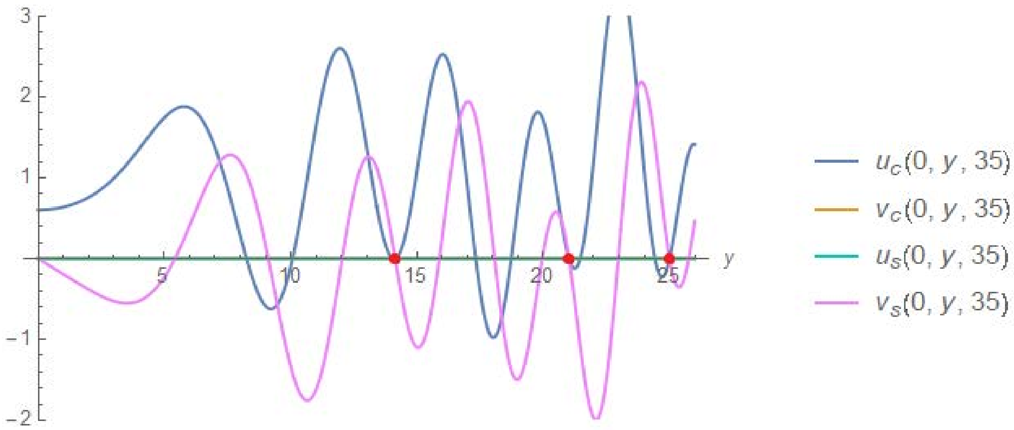
Zeros on the Critical Line

However, such a system of equations may exceptionally has solution. That is the case when $x=0$. Note that $x=0$ is the critical line of function $\eta(1/2+z)$. Substituting $x=0$ for the above,

$$\begin{cases} u_c(0,y) = \sum_{s=1}^{\infty} \dfrac{(-1)^{s-1}}{\sqrt{s}} \cos(y \log s) & = 0 \\ v_c(0,y) = 1 \times 0 & = 0 \\ u_s(0,y) = 1 \times 0 & = 0 \\ v_s(0,y) = \sum_{s=1}^{\infty} \dfrac{(-1)^{s-1}}{\sqrt{s}} \sin(y \log s) & = 0 \end{cases}$$

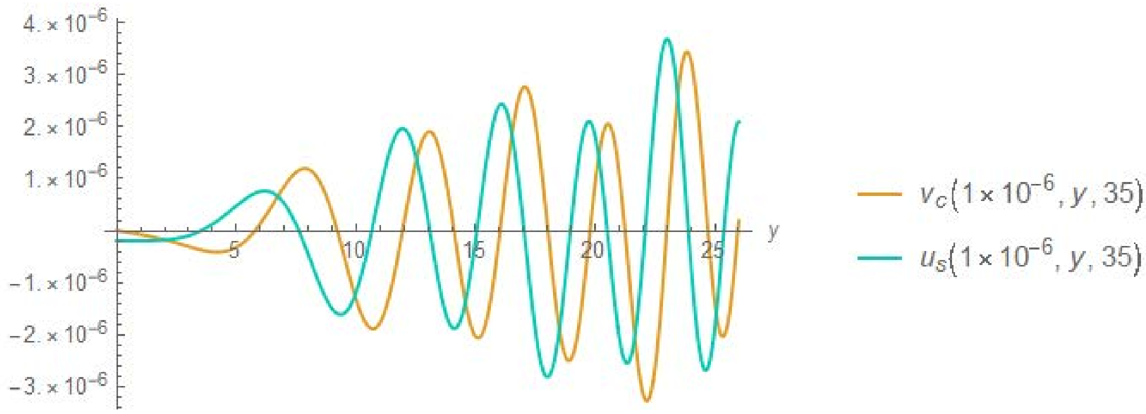
v_c, u_s are each become 0 and the property of overdetermination disappears. And $u_c(0,y)$, $v_s(0,y)$ result in $u_+(0,y), v_+(0,y)$ in Formula 11.1.1, respectively. So, this system of equations has a solution.

When $x=0$, $u_c \sim v_s$ are drawn as follows. Blue is u_c and magenta is v_s. The point (red) where these intersect on the y -axis is the zero point of $\eta(1/2 \pm z)$. Orange is v_c and light blue is u_s. They overlap on the y -axis. Of course, these two straight lines also pass through the red points.

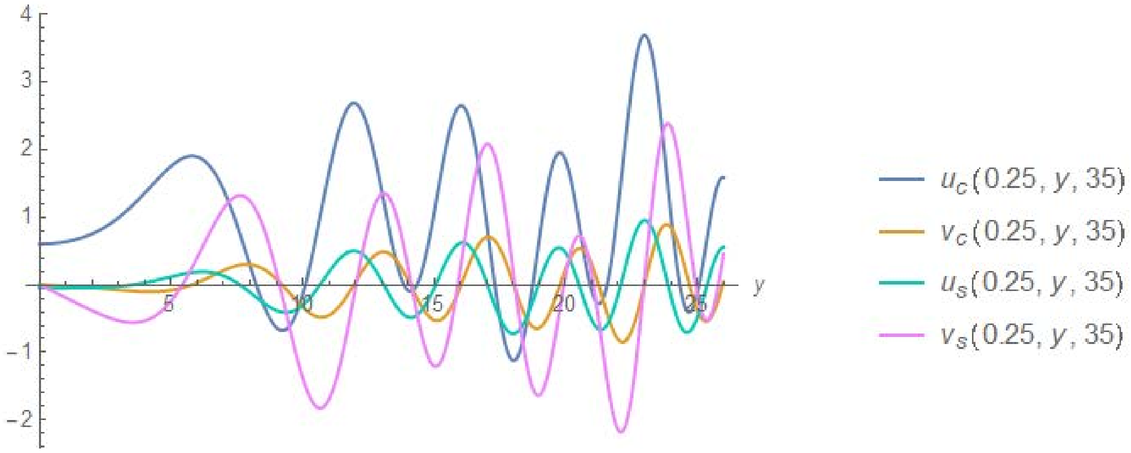


Outside the Critical Line

If x deviates even slightly from 0, v_c, u_s cease to be straight lines. For example, when $x=0.000001$,



As the result, the property of overdetermination is restored. For example, when $x=0.25$, $u_c \sim v_s$ are drawn as follows. It seems unlikely that the four curves would intersect at one point on the y-axis.



Therefore, we can present the following hypothesis, which is equivalent to the Riemann hypothesis.

Hypothesis 11.3.2

When y is a real number and x is a real number s.t. $-1/2 < x < 1/2$. the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} u_c(x,y) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} \cosh(x \log s)\cos(y \log s) = 0 \\ v_c(x,y) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} \sinh(x \log s)\sin(y \log s) = 0 \\ u_s(x,y) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} \sinh(x \log s)\cos(y \log s) = 0 \\ v_s(x,y) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} \cosh(x \log s)\sin(y \log s) = 0 \end{cases}$$

11.4 Necessary Condition for Zeros

The system of equations in Hypothesis11.3.2 is equivalent to the following six sets of system of equations. Each system of equations is necessary condition for zeros.

$$\begin{cases} u_c = 0 \\ v_c = 0 \end{cases}, \quad \begin{cases} u_c = 0 \\ u_s = 0 \end{cases}, \quad \begin{cases} u_c = 0 \\ v_s = 0 \end{cases}, \quad \begin{cases} v_c = 0 \\ u_s = 0 \end{cases}, \quad \begin{cases} v_c = 0 \\ v_s = 0 \end{cases}, \quad \begin{cases} u_s = 0 \\ v_s = 0 \end{cases}$$

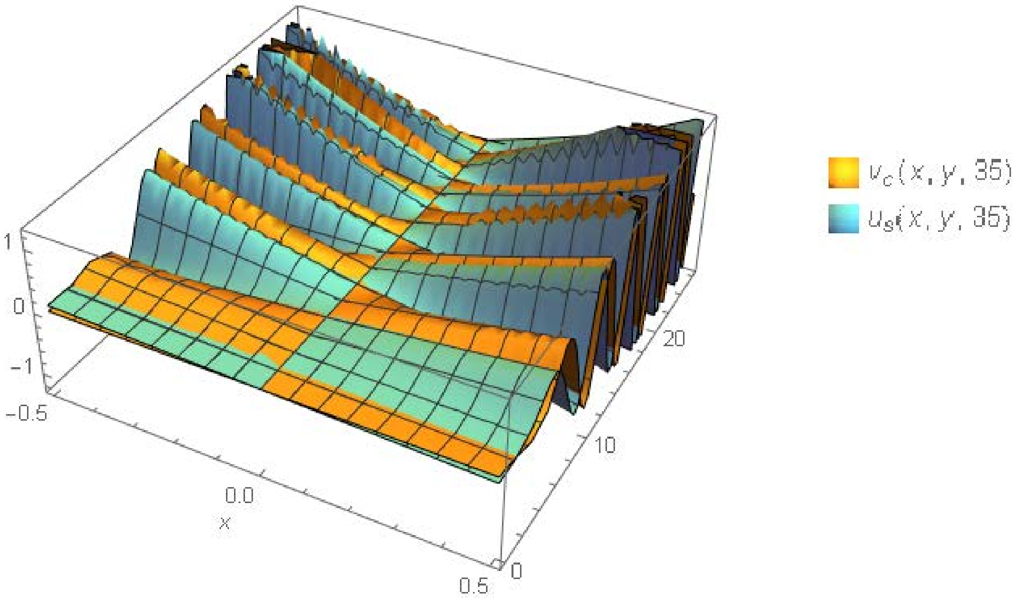
In order to prove Hypothesis11.3.2, it is sufficient to show that any one of these pairs does not have a solution such as $x \neq 0$.

$v_c(x,y)$ and $u_s(x,y)$

The most interesting of these is the following pair.

$$\begin{cases} v_c(x,y) = \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} \sinh\{x \log s\} \sin\{y \log s\} = 0 \\ u_s(x,y) = \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} \sinh\{x \log s\} \cos\{y \log s\} = 0 \end{cases}$$

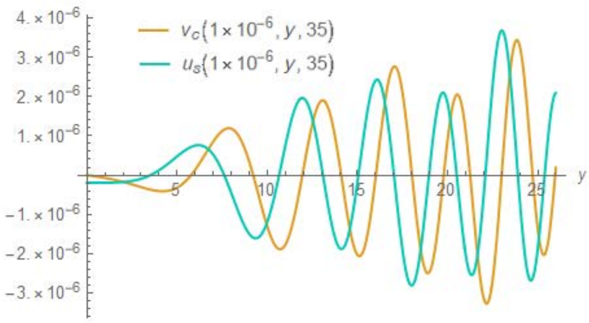
These 3D views are as follows.

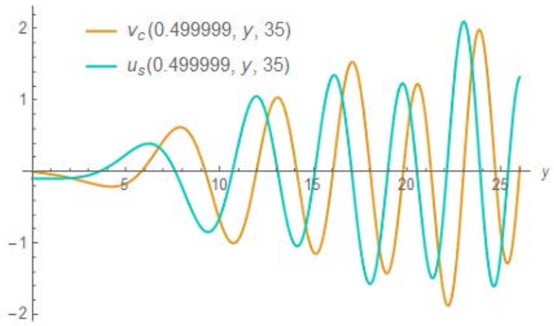


These functions have the following properties.

(1) Amplitude increase and period densification

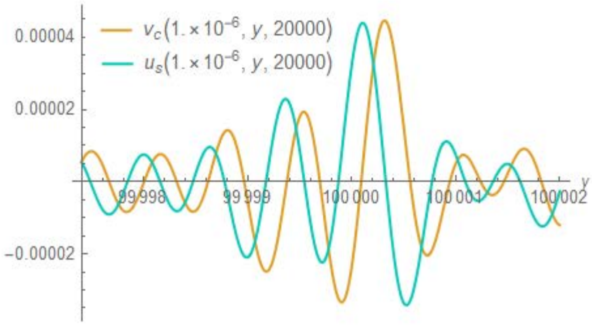
At $x=0$, both v_c and u_s are horizontal lines with a height of 0, but at $x=0.000001$ it becoms the left figure, and at $x=0.499999$ it becoms the right figure. .

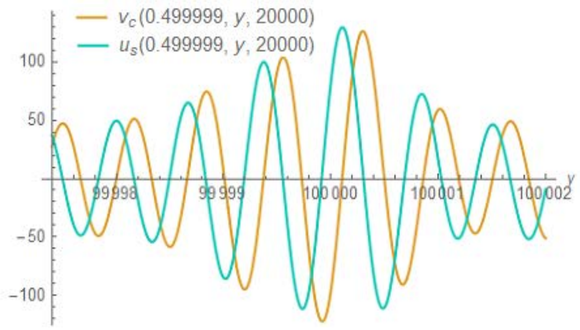




Both figures look like the same waveform, but if the aspect ratio is set to 1, v_c, u_s in the left figure almost overlap with the y axis. The periods of v_c, u_s around here ($y=10 \sim 26$) are about $4 \sim 3$. Here, the period is assumed to be the y coordinate distance from one downhill inflection point to the next downhill inflection point.

On the other hand, The 2D figures around $y=100000$ are as follows. The left figure is at $x=0.000001$ and the right figure is at $x=0.499999$.





We can be see that the amplitudes are larger than in the previous two figures. The periods of v_c, u_s around here ($y=99997 \sim 100002$) are about $0.8 \sim 0.6$, which is considerably shorter than the previous two figures.

From the above four figures, it is understood that the amplitude increases and the frequency becomes dense as $y \to \infty$. These suggest that the possibility that v_c and u_s intersect on the y-axis is extremely low.

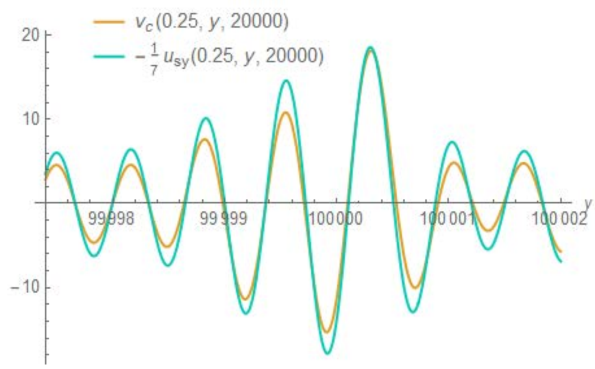
(2) Relationships like cosine and sine

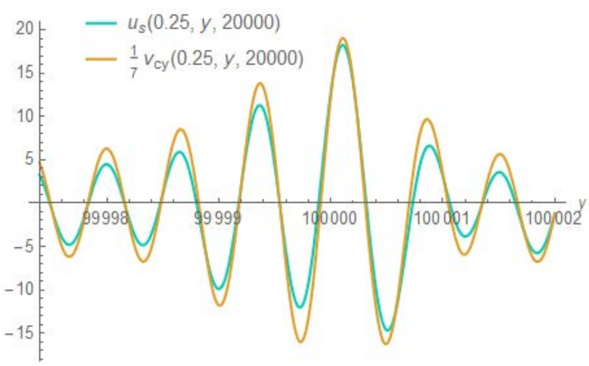
In any figure of (1), the peaks and valleys of v_c are close to the zeros of u_s, and the peaks and valleys of u_s are close to the zeros of v_c. In fact, the partial derivatives of v_c, u_s with respect to y are

$$v_{cy}(x,y) = \sum_{s=1}^{\infty} (-1)^{s-1} \frac{\log s}{\sqrt{s}} \sinh\{x \log s\} \cos\{y \log s\}$$

$$u_{sy}(x,y) = -\sum_{s=1}^{\infty} (-1)^{s-1} \frac{\log s}{\sqrt{s}} \sinh\{x \log s\} \sin\{y \log s\}$$

When $x=0.25$, $y=99997 \sim 100002$, the left figure shows v_c and $-u_{sy}/7$, and the right figure shows u_s and $v_{cy}/7$. The divisions of the derivative by 7 are for amplitude scaling.

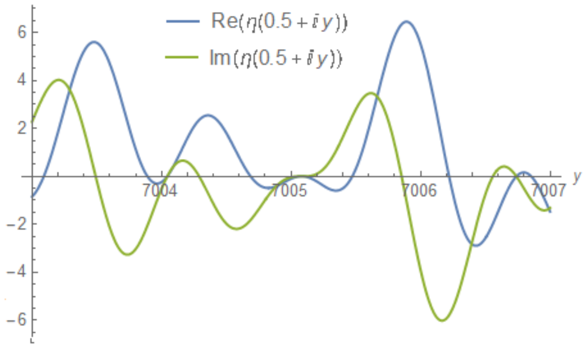


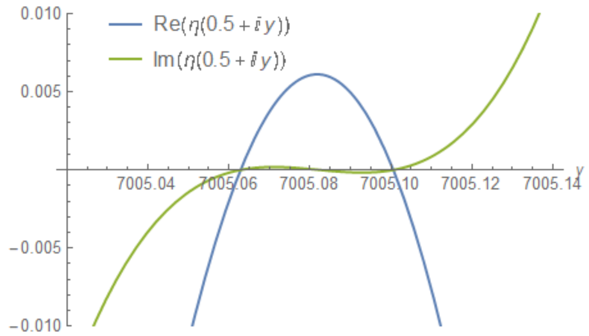


The zeros of v_c and $-u_{sy}/7$ are almost the same in the left figure, and the zeros of u_s and $v_{cy}/7$ are almost the same in the right figure. These show that the relationship between v_c and u_s closely resembles the relationship between cosine and sine. Hence, v_c and u_s are unlikely to intersect on the y-axis.

(3) Vicinity of Lehmer phenomenon

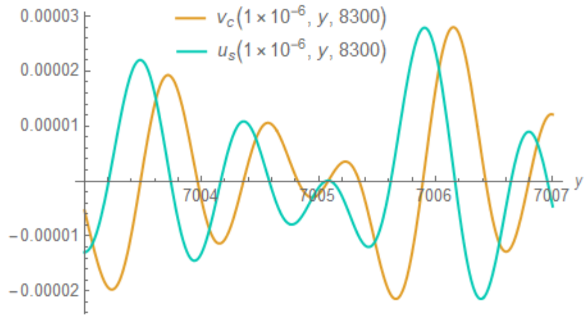
For example, when $x = 1/2$, the real and imaginary parts of η for $y = 7003 \sim 7007$ are drawn in the left figure. Around $y = 7005$, both the real and imaginary parts seem to intersect at one point on the y -axis.

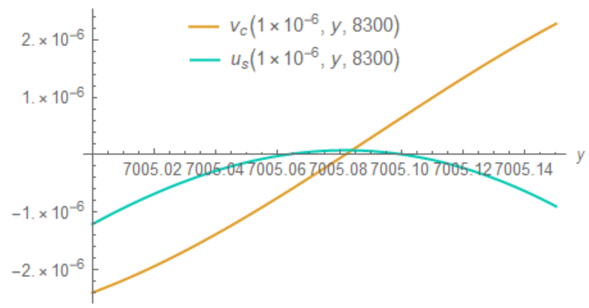




However, when the area around $y = 7005$ is enlarged, it becomes the right figure. The real and imaginary parts intersect at two points on the y -axis. This is known as the Lehmer phenomenon.

When $x = 0.000001$, v_c and u_s for $y = 7003 \sim 7007$ are drawn in the left. v_c and u_s seem to intersect at one point on the y -axis near $y = 7005$.





However, even in this case, when the area around $y = 7005$ is enlarged, it becomes the right figure. We can see that v_c and u_s have no intersection on the y -axis, And the zero of v_c and the maximum of u_s are very close. That is, in the vicinity of the Lehmer phenomenon, the property of (1) is lost, but the property of (2) is kept. In Addition, as $x \to \pm 1/2$, the amplitude increases and the Lehmer phenomenon disappears.

Hypothesis equivalent to the Riemann hypothesis

As seen above, the system of equations consisting of $v_c(x, y)$ and $u_s(x, y)$ has interesting and well-behaved properties. So, I present the following hypothesis, which is equivalent to the Riemann hypothesis.

Hypothesis 11.4.1

When y is a real number and x is a real number s.t. $-1/2 < x < 1/2$. the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} v_c(x,y) = \sum_{s=1}^{\infty} \dfrac{(-1)^{s-1}}{\sqrt{s}} \sinh(x \log s) \sin(y \log s) = 0 \\ u_s(x,y) = \sum_{s=1}^{\infty} \dfrac{(-1)^{s-1}}{\sqrt{s}} \sinh(x \log s) \cos(y \log s) = 0 \end{cases}$$

c.f.

Focusing on the x -axis, the following pair might also be promising.

$$\begin{cases} v_c(x,y) = \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} \sinh\{x \log s\} \sin\{y \log s\} = 0 \\ v_s(x,y) = \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\sqrt{s}} \cosh(x \log s) \sin(y \log s) = 0 \end{cases}$$

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Kano Kono

Hiroshima, Japan

Alien's Mathematics