07 Zeros of p-series

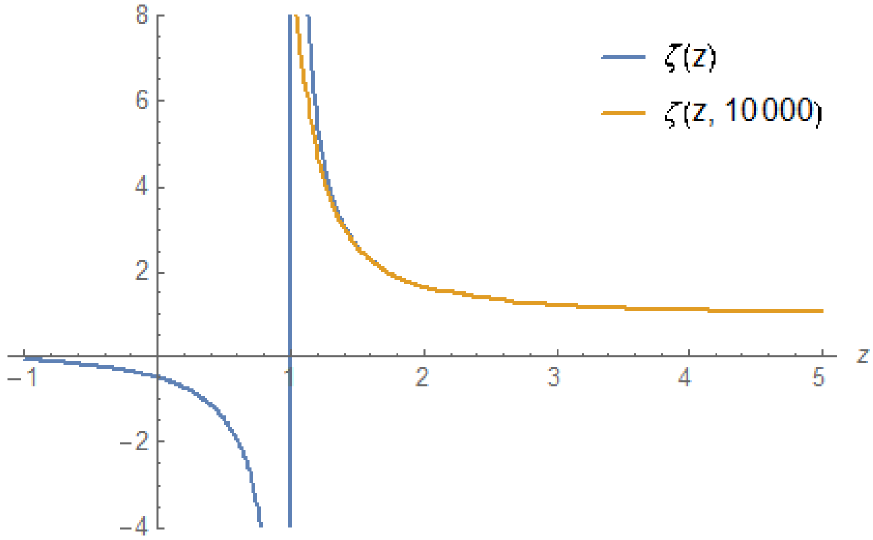
7.1 Riemann Zeta Function & p-series

Riemann zeta function $\zeta(z)$ $(z = x + iy)$ is often defined by the following expression.

$$\zeta(z) = \sum_{s=1}^{\infty} \frac{1}{s^z} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots \qquad x > 1 \tag{1.ζ}$$

The right side is called **p-series**, which is a kind of Dirichlet series.

Both sides are drawn as follows. Blue is the left side and orange is the right side. Althogh the upper limit of Σ is taken 10000, both sides do not match near the singular point $z = 1$. Moreover, p-series is not drawn on the left of the singular point $z = 1$. This is because the p-series diverges at $z < 1$.



The region where the zeros of $\zeta(z)$ exists is called the critical strip, but p-series cannot describe $\zeta(z)$ in this strip. At least, as long as we follow the theory of series.

Dirichlet series representation of $\zeta(z)$ in the critical strip

Then, what is the Dirichlet series that can describe $\zeta(z)$ in the critical strip ? The following equation has been used since ***Euler*** to date to describe $\zeta(z)$ in the critical strip.

$$\zeta(z) = \frac{1}{1-2^{1-z}} \eta(z) = \frac{1}{1-2^{1-z}} \left(\frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + - \cdots \right) \qquad x > 0$$

Although it is certainly a Dirichlet series in ( ), it is doubtful whether the whole is a Dirichlet series. I have been struggling with this problem for many years, but recently I have noticed that this right hand side is also p-series. So, I state this as a formula.

Formula 7.1.1

The following expression holds for the Riemann Zeta Function $\zeta(z)$ $(z = x + iy)$.

$$\zeta(z) = \sum_{s=1}^{\infty} \frac{1}{s^z} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots \qquad x > 0\ ,\ x \neq 1 \tag{1.1}$$

Proof

The following expression holds for the Dirichlet Eeta Function $\eta(z)$ $(z=x+iy)$.

$$\eta(z) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s^z} = \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} +- \cdots \quad x>0 \tag{1.η}$$

Here, $\eta(z)$ is known to have the following η-specific zeros.

$$x=1\,,\ y=\frac{2k\pi}{\log 2} \qquad k=0,1,2,\cdots$$

The function $\eta s(z)$ having such zeros is given by the following Dirichlet type polynomial.

$$\eta s(z) = 1 - 2^{1-z} = \frac{1}{1^z} - \frac{2}{2^z} \qquad \left(= \frac{1}{1^{z-1}} - \frac{1}{2^{z-1}} \right)$$

Since $\zeta(z)$ can be obtained by removing a factor having η-specific zeros from $\eta(z)$,

$$\zeta(z) = \eta(z) \Big/ \left(\frac{1}{1^z} - \frac{2}{2^z} \right) = \left(\frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \frac{1}{5^z} - \frac{1}{6^z} +- \cdots \right) \Big/ \left(\frac{1}{1^z} - \frac{2}{2^z} \right)$$

When this is calculated by hand,

$$\begin{array}{r|l}
 & \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots \\
\hline
\frac{1}{1^z} - \frac{2}{2^z} & \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \frac{1}{5^z} - \frac{1}{6^z} +- \cdots \\
 & \frac{1}{1^z} - \frac{2}{2^z} \\
\hline
 & \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \frac{1}{5^z} - \frac{1}{6^z} +- \cdots \\
 & \frac{1}{2^z} - \frac{2}{4^z} \\
\hline
 & \frac{1}{3^z} + \frac{1}{4^z} + \frac{1}{5^z} - \frac{1}{6^z} + \frac{1}{7^z} - \frac{1}{8^z} +- \cdots \\
 & \frac{1}{3^z} - \frac{2}{6^z} \\
\hline
 & \frac{1}{4^z} + \frac{1}{5^z} + \frac{1}{6^z} + \frac{1}{7^z} - \frac{1}{8^z} + \frac{1}{9^z} - \frac{1}{10^z} +- \cdots \\
 & \frac{1}{4^z} - \frac{2}{8^z} \\
\hline
 & \frac{1}{5^z} + \frac{1}{6^z} + \frac{1}{7^z} + \frac{1}{8^z} + \frac{1}{9^z} - \frac{1}{10^z} + \frac{1}{11^z} - \frac{1}{12^z} +- \cdots \\
\hline
 & \vdots
\end{array}$$

Q.E.D.

Real and imaginary parts

Thus, it was shown that p-series must also hold in the critical strip $0 < x < 1$. So, let us look at this by the real and imaginary parts. They are presented in the following formula.

Formula 7.1.1ri

When the real and imaginary parts of Riemann zeta function $\zeta(x,y)$ are ζ_r , ζ_i respectively, the following expressions hold for $x > 0$, $x \neq 1$.

$$\zeta_r(x,y) = \sum_{s=1}^{\infty} \frac{\cos(y \log s)}{s^x} \tag{1.1r}$$

$$\zeta_i(x,y) = -\sum_{s=1}^{\infty} \frac{\sin(y \log s)}{s^x} \tag{1.1i}$$

Proof

From Formula 7.1.1 ,

$$\zeta(x,y) = \sum_{s=1}^{\infty} \frac{1}{s^{x+iy}} = \sum_{s=1}^{\infty} e^{-(x+iy)\log s} \qquad for\ \ x>0\ ,\ x \neq 1$$

$$= \sum_{s=1}^{\infty} e^{-x\log s} \cdot e^{-iy\log s} = \sum_{s=1}^{\infty} \frac{1}{s^x}\{\cos(y\log s) - i\sin(y\log s)\}$$

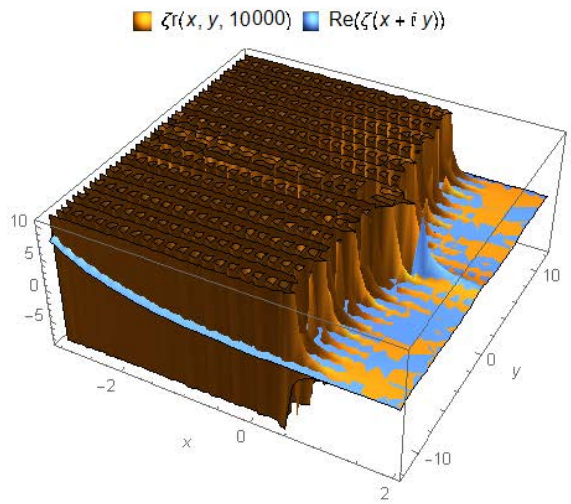
i.e.

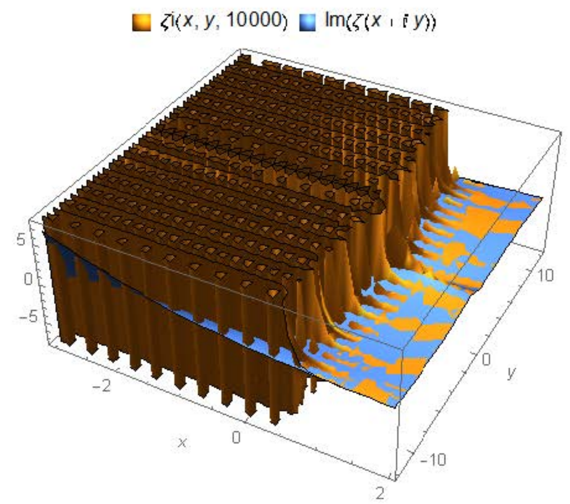
$$\zeta(x,y) = \sum_{s=1}^{\infty} \frac{\cos(y \log s)}{s^x} - i\sum_{s=1}^{\infty} \frac{\sin(y \log s)}{s^x} \qquad for\ \ x>0\ ,\ x \neq 1$$

From this, we obtain the desired expressions.

3D figure

(1.1r), (1.1i) are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, blue is the left side and orange is the right side.

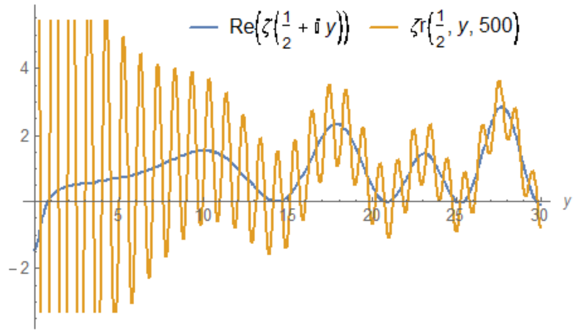


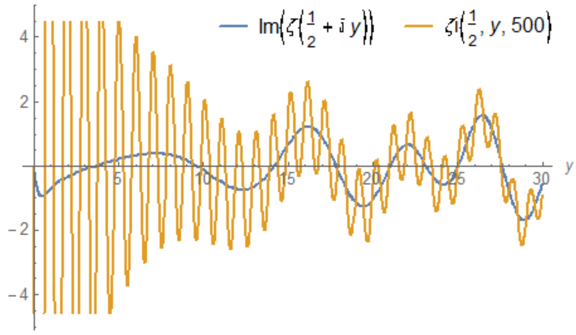


In both figures, a line of convergence is observed at $x = 1$. The p-series (orange) converges on the right side, and diverges on the left side.

2D figure on Critical Line

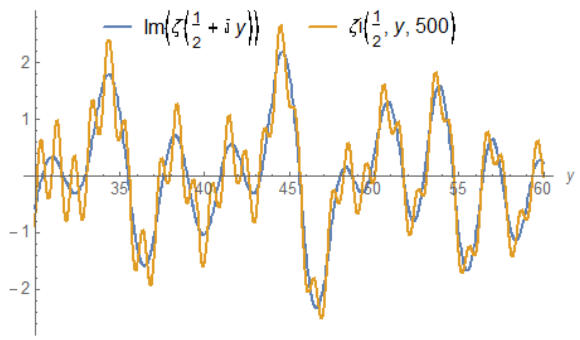
The 2D figure on the critical line $x=1/2$ at $0 \le y \le 30$ is drawn as follows. The left is the real part and the right is the imaginary part. In both figures, blue is the left side and orange is the right side.

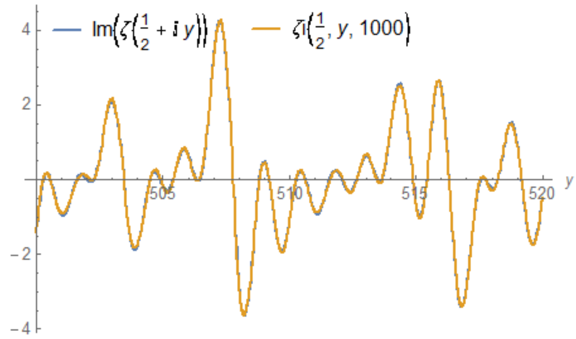




The p-series oscillates violently, and the center value of the oscillation is $Re\{\zeta(1/2+iy)\}$ or $Im\{\zeta(1/2+iy)\}$.

However, its amplitude decreases as y increases. Regarding the imaginary part, 2D figures at $30 \le y \le 60$ and $500 \le y \le 520$ are drawn as follows. The left is $30 \le y \le 60$ and the right is $500 \le y \le 520$.





In $500 \le y \le 520$, it seems that both sides are almost overlapped. But, p-series oscillates and does not converge. That is, **even if y is given, the value of p-series is not determined.**

However, it is when the upper limit of Σ is ∞. In numerical calculations, the p-series is truncated at a finite term, and a certain value is obtained. For example, when the series is censored with 1000 terms and the zero point of the uphill near $y=515$ is calculated numerically, it is as follows

```
FindRoot[ξi[1/2, y, 1000], {y, 515.5}]
{ y → 515.437 }

N[Im[ZetaZero[281]]]
515.435
```

This is quite close to the 281-st non-trivial zero of $\zeta(z)$. When y is this size, if lucky, an approximate value will be obtained in such accuracy.

cf.

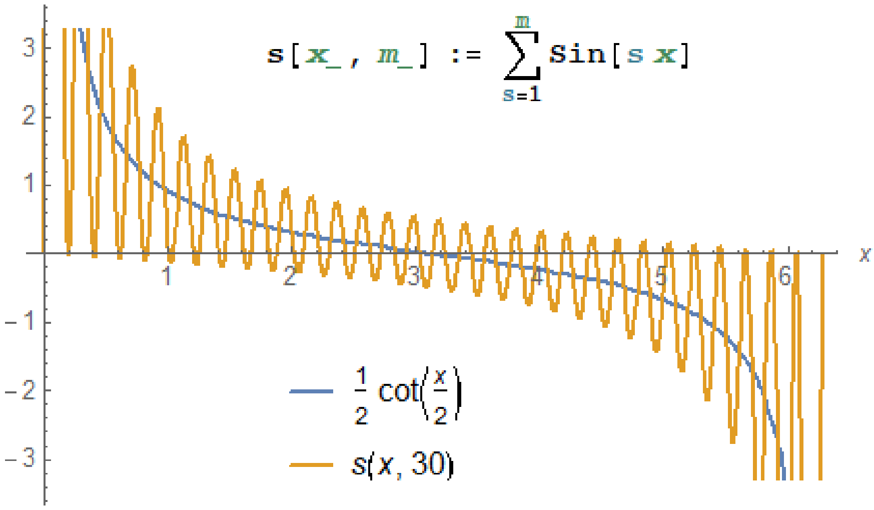
Let Fourier series be

$$-log\left(2sin\frac{x}{2}\right) = \sum_{s=1}^{\infty}\frac{cos(sx)}{s} \qquad 0<x<2\pi$$

Differentiating both sides with respect to x ,

$$\frac{1}{2}cot\frac{x}{2} = \sum_{s=1}^{\infty} sin(sx) \qquad 0 < x < 2\pi \tag{1.2}$$

Since this right side does not converge, (1.2) does not hold actually. The 2D figure of both sides is drawn as follows. Blue is the left side and orange is the right side.



The series $s(x)$ oscillates violently, and the center value of the oscillation is $\frac{1}{2}cot\frac{x}{2}$.

The 2D figure on the critical line mentioned above is similar to this figure. However, while the amplitude of the p-series decreases as the y increases, the amplitude of the serie $s(x)$ does not decrease below a certain level.

Necessity of Summation Method

As far as the 3D and 2D diagrams are concerned, the p-series in the critical strip is not likely to be used as it is. For the interpretation of Formula 7.1.1 , it seems necessary to have a powerful summation method that can justify (1.2) .

7.2 Convergence Acceleration of p-series

In this section, we try to apply the summation method of **Knopp transformation** to the p-sries. Since Knopp transformation is not valid for positive series, we cannot expect the acceleration of (1.ζ) . However, since (1.1r) and (1.1i) are similar to alternating series, we can expect the effect of the summation method on them.

In addition, about Knopp transformation, see " **10 Convergence Acceleration & Summation Method by Double Series of Functions "** (A la carte).

Formula 7.2.1ri (Knopp Transformation)

When the real and imaginary parts of Riemann zeta function $\zeta(x,y)$ are ζ_r , ζ_i respectively, the following expressions hold for $x > 0$, $x \neq 1$.

$$\zeta_r(x,y,q) = \sum_{k=1}^{\infty} \sum_{s=1}^{k} \frac{q^{k-s}}{(q+1)^{k+1}} \binom{k}{s} \frac{\cos(y \log s)}{s^x} \tag{2.1r}$$

$$\zeta_i(x,y,q) = -\sum_{k=1}^{\infty} \sum_{s=1}^{k} \frac{q^{k-s}}{(q+1)^{k+1}} \binom{k}{s} \frac{\sin(y \log s)}{s^x} \tag{2.1i}$$

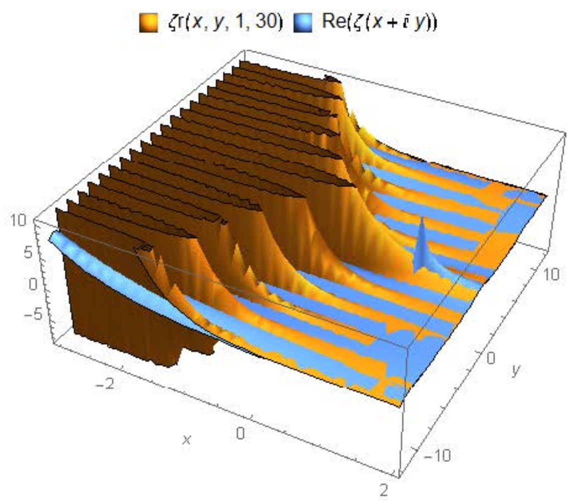
Where, q is an arbitrary positive number.

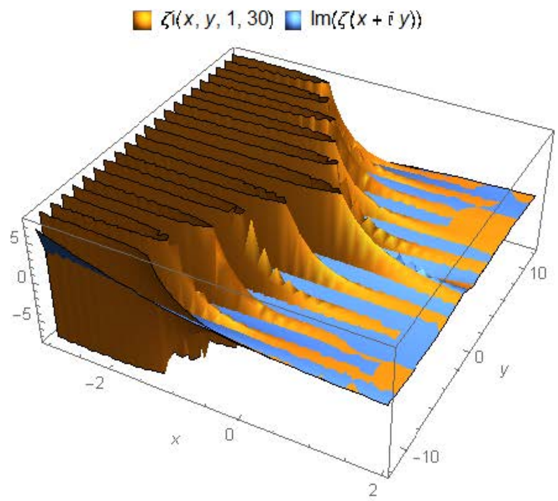
Proof

Applying Knopp transformation to Formula 7.1.1ri , we obtain the desired expressions.

3D figure

(2.1r), (2.1i) are drawn as follows. The left is the real part and the right is the imaginary part. In both figures, blue is the left side and orange is the right side.

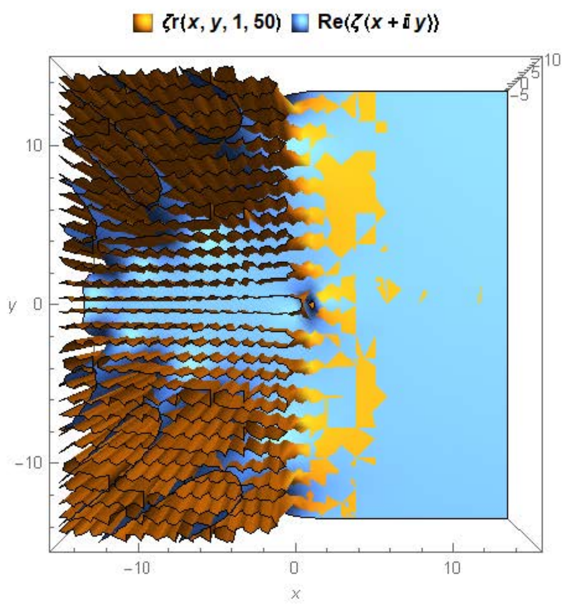


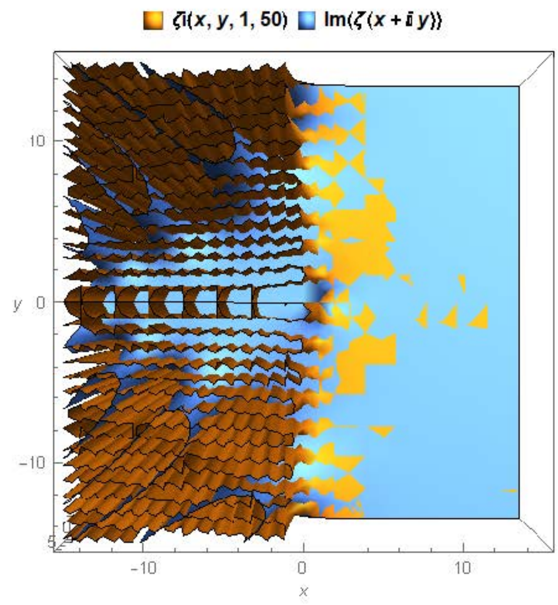


In both figures, attention should be paid to the line of convergence. The line of convergence appears to tilt in the negative direction in proportion to the absolute value of y . At least, it doesn't look perpendicular to the x-axis.

3D figure with an aspect ratio of 1

To observe this in detail, let us draw figures with the same aspect ratio and look at them from above.
Then, it is as follows.

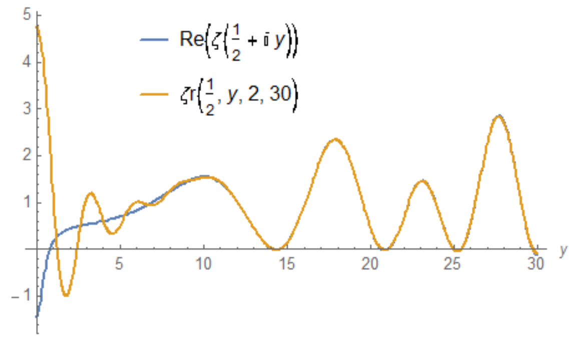


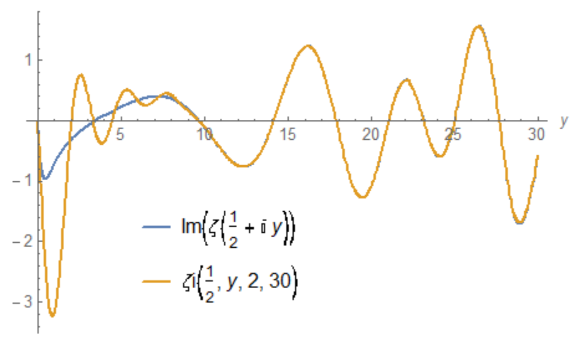


In these figures as well, we can see that the line of convergence is retreating to > shape and the convergence area is expanded. This is considered to be an analytic continuation effect of the Knopp transformation. As the result, it is likely that **most of the critical strip $0 < x < 1$ will fall within this expanded convergence region** .

2D figure on Critical Line

The 2D figure on the critical line $x = 1/2$ at $0 \le y \le 30$ is drawn as follows. The left is the real part and the right is the imaginary part. In both figures, blue is the left side and orange is the right side.





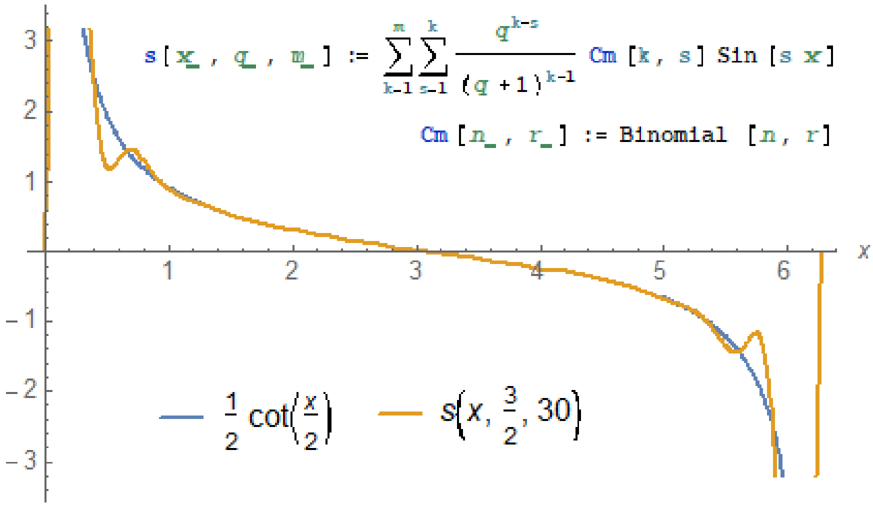
The differences between the p-series and $Re\{\zeta(1/2+iy)\}, Im\{\zeta(1/2+iy)\}$ are large where y is small, but the p-series approache $Re\{\zeta(1/2+iy)\}, Im\{\zeta(1/2+iy)\}$ quickly where y is large. What is important here is that p-series are not diverging. That is, **if y is given, the values of the p-series are determined**. And, they approach $Re\{\zeta(1/2+iy)\}, Im\{\zeta(1/2+iy)\}$ indefinitely as $|y|$ increases.

cf.

Applying Knopp transformation to (1.2) in the previous section, we obtain

$$\frac{1}{2}\cot\frac{x}{2} = \sum_{k=1}^{\infty}\sum_{s=1}^{k}\frac{q^{k-s}}{(q+1)^{k+1}}\binom{k}{s}\sin(sx) \qquad 0 < x < 2\pi \tag{2.2}$$

The 2D figure of both sides is drawn as follows. Blue is the left side and orange is the right side.



The series $s(x)$ is vary close to the left side $\frac{1}{2}cot\frac{x}{2}$ near $x=\pi$. The 2D figure on the critical line mentioned above is similar to this figure.

Effect of summation method by Knopp Transformation

As far as the 3D and 2D diagrams are concerned, it seems that p-series can be used well when y is large. But, when y is small, it is difficult to equate p-series with $\zeta(z)$. This is probably because the summation method of Knopp transformation is not so powerful.

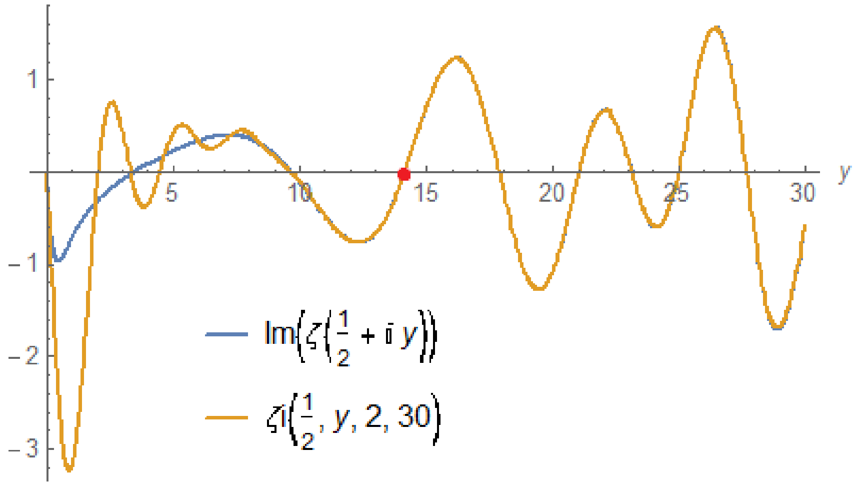
7.3 Zeros of p-series

In this section, we calculate the zeros of the p-series on the critical line by the following formula.

$$\zeta_i(x,y,q) = -\sum_{k=1}^{\infty}\sum_{s=1}^{k}\frac{q^{k-s}}{(q+1)^{k+1}}\binom{k}{s}\frac{\sin(y\log s)}{s^x} \qquad (2.1i)$$

The 1-st zero point

The imaginary part on the critical line $x=1/2$ at $0 \le y \le 30$ is drawn as follows. Blue is the left side and orange is the right side.

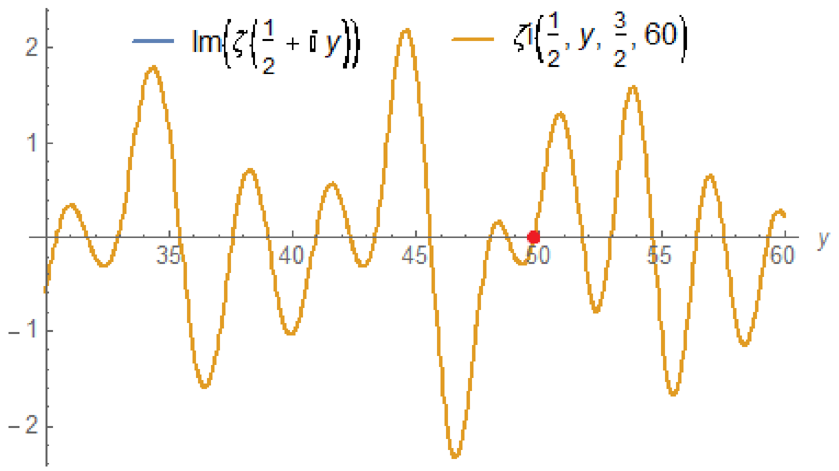


The 1-st zero of $\zeta(1/2+iy)$ coincides with the 2-nd uphill zero (red dot) of the imaginary part. So, when the zeros on both sides near $y=14$ were calculated, the following values were obtained. Although significant 4 digits were obtained, no better approximation was obtained.

```
FindRoot[ζi[1/2, y, 1, 30], {y, 14}]        N[Im[ZetaZero[1]]]
{ y → 14.1394 }                              14.1347
```

The 10-th zero point

The imaginary part on the critical line $x=1/2$ at $30 \le y \le 60$ is drawn as follows. Blue is the left side and orange is the right side, but both sides overlap exactly and blue (left side) is not visible.



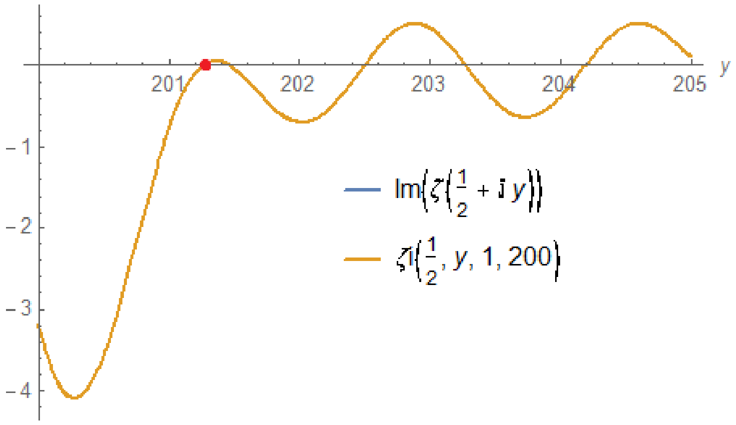
The 10-th zero of $\zeta(1/2+iy)$ coincides with the 11-th uphill zero (red dot) of the imaginary part. So, when the zeros on both sides near $y=50$ were calculated, the following significant 10 digits were obtained.

```
SetPrecision[FindRoot[ξi[1/2, y, 1, 60], {y, 50}], 11]        {y → 49.773832470}

SetPrecision[Im[ZetaZero[10]], 11]                               49.773832478
```

The 80-th zero point

The imaginary part on the critical line $x=1/2$ at $200 \le y \le 205$ is drawn as follows. Blue is the left side and orange is the right side, but both sides overlap exactly and blue (left side) is not visible.



The 80-th zero of $\zeta(1/2+iy)$ coincides with (perhaps) the 81-th uphill zero (red dot) of the imaginary part. So, when the zeros on both sides near $y=201$ were calculated, the following significant 15 digits were obtained.

```
SetPrecision[FindRoot[ξi[1/2, y, 1, 200], {y, 201}], 16]
{y → 201.2647519437039}

SetPrecision[Im[ZetaZero[80]], 16]
     201.2647519437038
```

Conclusion

At present, we must consider Formula 7.2.1ri as an asymptotic expansion in the critical strip $0<x<1$. However, the calculation accuracy increases as the imaginary part $|y|$ of the independent variable increases. As of 2004, the first 10 trillions on the critical line are known to satisfy the Riemann hypothesis. Since the imaginary part $|y_r|$ after the 10th trillion are very large, Formula 7.2.1ri is sufficiently useful even in the critical strip. If a stronger summation method was discovered in the future, p-series could be equated with $\zeta(z)$ in the whole critical strip.

2019.10.01
2020.09.01 Reuped

Kano Kono

Alien's Mathematics