14 Zeros and Phases in Dirichlet Series

Abstract

(1) Zeros of the Dirichlet series are fixed points, which are independent on the phases.

(2) Non-zers of the Dirichlet series are moving points, which are dependent on the phases.

14.1 Zeros and Phases in General Dirichlet Series

Definition of General Dirichlet Series

Let R be the set of real numbers , $z=x+iy \quad x,y \in R$, r be a natural number, $\lambda_r \in R$, $\lambda_r < \lambda_{r+1}$ and c_r be a complex number. Then the General Dirichlet Series $f(z)$ is defined by the following expression.

$$f(z) = \sum_{r=1}^{\infty} c_r e^{-\lambda_r z} \qquad Re(z) > 0 \tag{1.0}$$

Theorem 14.1.1

When (a,b) is a zero point of the General Dirichlet series $f(x,y)$, the following expression holds.

$$\sum_{r=1}^{\infty} c_r e^{-a\lambda_r} \cdot e^{-i(b\lambda_r+\theta)} = 0 \qquad \forall\theta \in R \tag{1.1}$$

Proof

On the zero point (a,b) of $f(x,y)$,

$$\sum_{r=1}^{\infty} c_r e^{-a\lambda_r - ib\lambda_r} = 0$$

Multiplying both sides by $e^{-i\theta}$ $(\forall\theta \in R)$,

$$\sum_{r=1}^{\infty} c_r e^{-a\lambda_r} \cdot e^{-i(b\lambda_r+\theta)} = 0 \tag{1.1}$$

If Theorem 14.1.1 is expressed by trigonometric functions, it is as follows.

Corollary 14.1.1'

When (a,b) is a zero point of the General Dirichlet series $f(x,y)$, the following expression holds.

$$\sum_{r=1}^{\infty} c_r e^{-a\lambda_r} \cos(b\lambda_r+\theta) = -\sum_{r=1}^{\infty} c_r e^{-a\lambda_r} \sin(b\lambda_r+\theta) = 0 \qquad \forall\theta \in R \tag{1.1'}$$

Note

The corollary can also be proved directly using the trigonometric addition formulas.

Since the phase θ may be arbitrary, the following is further derived.

Corollary 14.1.1"

When (a,b) is a zero point of the general Dirichlet series $f(x,y)$, the following expression holds.

$$\sum_{r=1}^{\infty} c_r e^{-a\lambda_r} \cos(b\lambda_r+\theta_c) = -\sum_{r=1}^{\infty} c_r e^{-a\lambda_r} \sin(b\lambda_r+\theta_s) = 0 \qquad \forall\theta_c, \theta_s \in R \tag{1.1"}$$

Remark 1

The above theorem and corollaries show that the zeros (a, b) of the general Dirichlet series $f(x, y)$ are independent on the phases (θ, θ_c, θ_s). Conversely, it is clear from the above proof that the non-zeros (x, y) are dependent on the phases (θ, θ_c, θ_s) .

Remark 2

The above theorem and corollaries also hold for Fourier Series.

14.2 Zeros and Phases in Ordinary Dirichlet Series

Definition of Ordinary Dirichlet Series

Giving $\lambda_r = \log r$ in the General Dirichlet Series, we obtain Ordinary Dirichlet Series.

$$f(z) = \sum_{r=1}^{\infty} \frac{c_r}{r^z} \qquad Re(z) > 0 \tag{2.0}$$

Ordinary Dirichlet Series is simply called the **Dirichlet Series**. In this chapter as well, hereafter we will call it as such.

Substituting $\lambda_r = logr$ into Theorem 14.1.1 , we obtain the following.

Theorem 14.2.1

When (a,b) is a zero point of the Dirichlet series $f(x,y)$, the following expression holds.

$$\sum_{r=1}^{\infty} \frac{c_r}{r^a} e^{-i(b\log r+\theta)} = 0 \qquad \forall\theta \in R \tag{2.1}$$

If Theorem 14.2.1 is expressed by trigonometric functions, it is as follows.

Corollary 14.2.1'

When (a,b) is a zero point of the Dirichlet series $f(x,y)$, the following expression holds.

$$\sum_{r=1}^{\infty} \frac{c_r}{r^a} \cos(b\log r+\theta) = -\sum_{r=1}^{\infty} \frac{c_r}{r^a} \sin(b\log r+\theta) = 0 \qquad \forall\theta \in R \tag{2.1'}$$

Since the phase θ may be arbitrary, the following is further derived.

Corollary 14.2.1"

When (a,b) is a zero point of the Dirichlet series $f(x,y)$, the following expression holds.

$$\sum_{r=1}^{\infty} \frac{c_r}{r^a} \cos(b\log r+\theta_c) = -\sum_{r=1}^{\infty} \frac{c_r}{r^a} \sin(b\log r+\theta_s) = 0 \qquad \forall\theta_c,\theta_s \in R \tag{2.1"}$$

Note

These show that the zeros (a,b) of the Dirichlet series $f(x,y)$ are independent on the phases (θ, θ_c , θ_s). Conversely, the non-zero points (x,y) are dependent on the phases (θ, θ_c , θ_s) .

14.3 Zeros and Non-Zeros (Examples)

In this section, Dirichlet Eta Function is used for illustration of the previous section.

Dirichlet Eta Function

When $c_r = (-1)^r$, (2.0) in the previous section becomes the Dirichlet Eta Function. That is

$$\eta(z) = \sum_{r=1}^{\infty} \frac{(-1)^r}{r^z} \qquad Re(z) > 0 \tag{3.0}$$

Let $z = x + iy$, $\eta(x,y) = u(x,y) + i\,v(x,y)$, then

$$u(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^r}{r^x} \cos(y \log r)$$

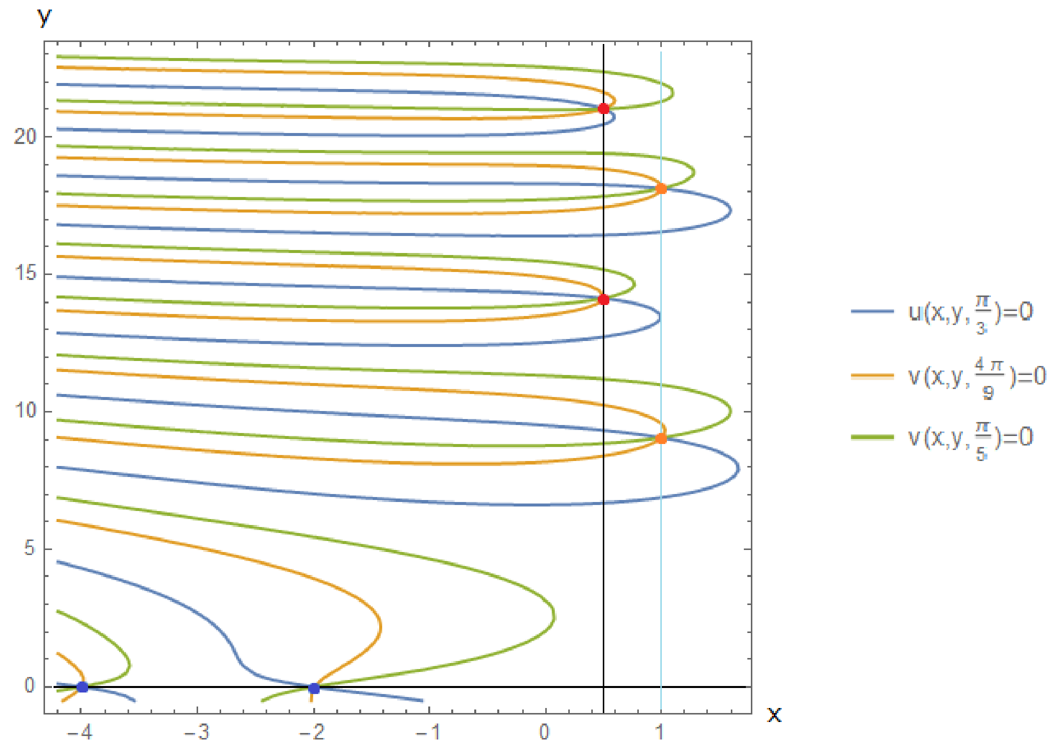
$$v(x,y) = -\sum_{r=1}^{\infty} \frac{(-1)^r}{r^x} \sin(y \log r)$$

14.3.1 Zeros and Phases

According to Theorem 14.2.1" , the zero point (a,b) of the Dirichlet function $\eta(x,y)$ is the solution of the following system of equations.

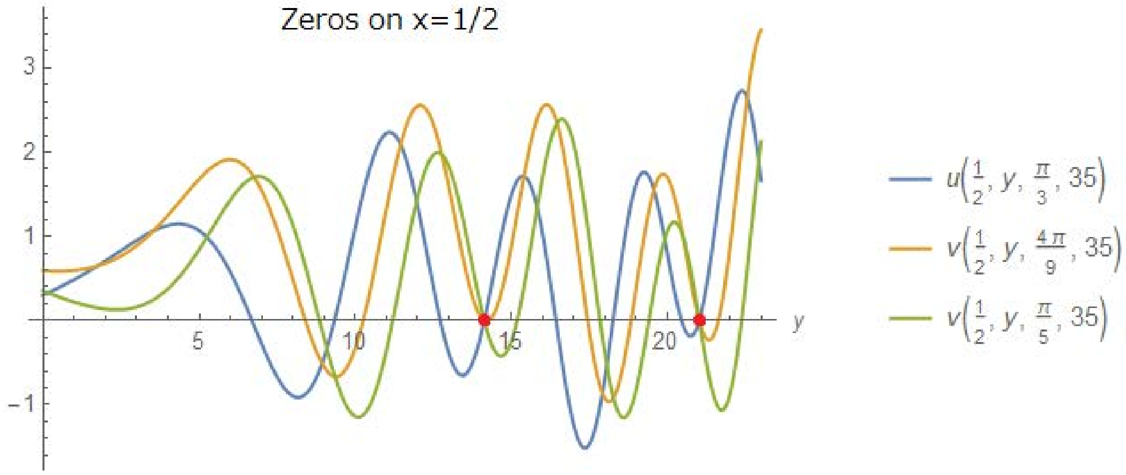
$$\begin{cases} u(x,y,\theta_c) = \sum_{r=1}^{\infty} \frac{(-1)^r}{r^x} \cos(y \log r + \theta_c) = 0 \\ v(x,y,\theta_s) = -\sum_{r=1}^{\infty} \frac{(-1)^r}{r^x} \sin(y \log r + \theta_s) = 0 \end{cases} \qquad \forall\, \theta_c, \theta_s \in R \tag{3.0}$$

Example 1 When $\theta_c = \pi/3$, $\theta_s = 4\pi/9$, $\pi/5$, the contour plot at height 0 is as follows.



The red points are the non-trivial zeros of $\eta(x,y)$ on $x=1/2$, the orange points are the η-specific zeros on $x=1$, and the blue points are the trivial zeros on $y=0$.

Furthermore, when $\theta_c = \pi/3$, $\theta_s = 4\pi/9$, $\pi/5$, the 2D figure on $x=1/2$ is drown as follows.



Similar 2D figures can be drawn on $x=1$ and $y=0$, but they are omitted here.

It should be noted in both figures that these zeros are fixed points (that do not move) for any change in phases θ_c , θ_s .

14.3.2 Non-Zeros and Phases

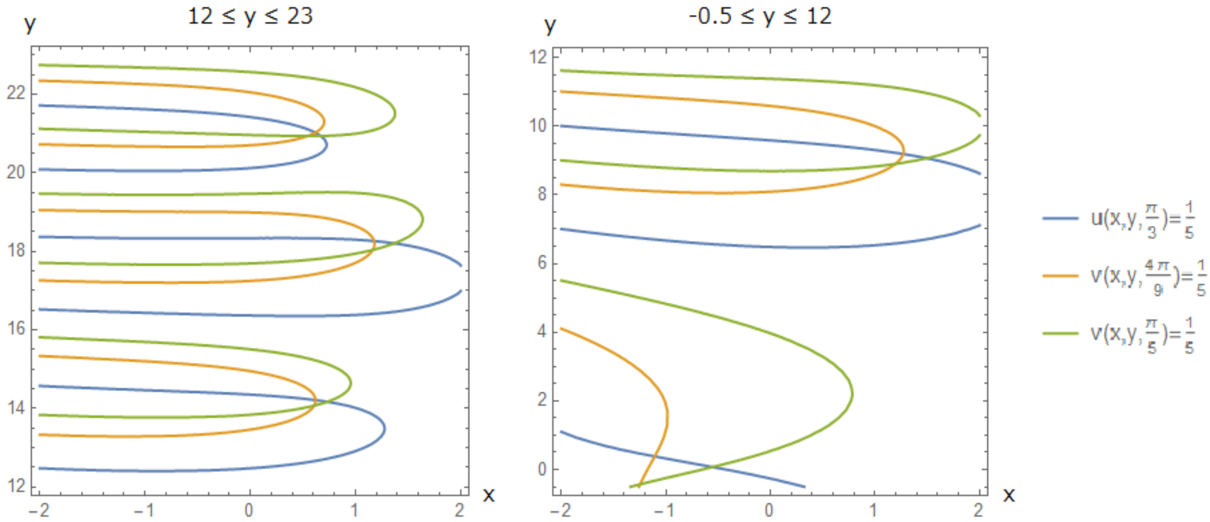
Consider the following system of equations.

$$\begin{cases} u(x,y,\theta_c) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^r}{r^x} \cos(y \log r + \theta_c) = \rho \\ v(x,y,\theta_s) = -\displaystyle\sum_{r=1}^{\infty} \frac{(-1)^r}{r^x} \sin(y \log r + \theta_s) = \rho \end{cases} \qquad \theta_c , \theta_s , \rho \in R \qquad (3.2)$$

When $\rho \neq 0$, the solution is a non-zero point of the Dirichlet Eta Function $\eta(x,y)$.

Example 2 When $\theta_c = \pi/3$, $\theta_s = 4\pi/9$, $\pi/5$, the contour plot at height **1/5** is as follows.

For clarity of intersections, the upper half is drown on the left and the lower half is drown on the right.



In both figures, the points of intersection of the two curves are non-zeros of $\eta(x,y)$, but none of the points of intersection coincide. This shows that the non-zeros of $\eta(x, y)$ are moving according to the phases θ_c, θ_s.

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