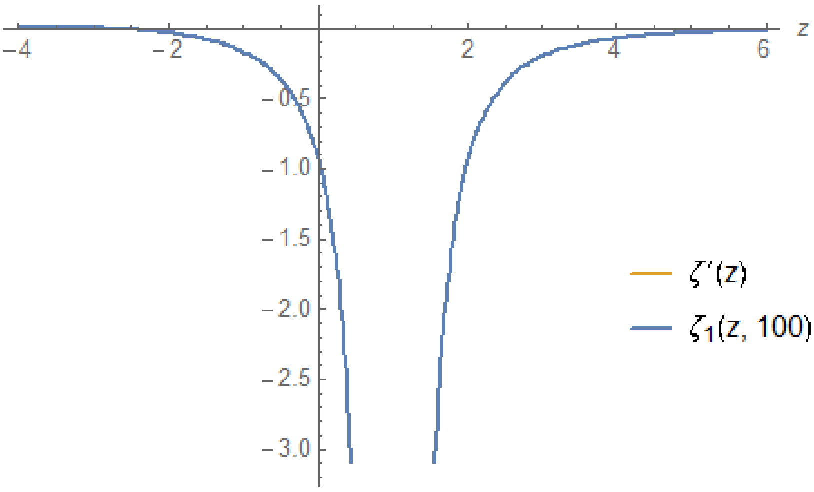
27 Zeros of Super Derivative of Riemann Zeta

27.1 Zeros of $\zeta'(z)$

27.1.0 Laurent Expansion of $\zeta'(z)$

Accordint to " **26 Higher and Super Calculus of Zeta Function etc** " Formula 26.1.2h , when γ_r is a Stieltjes constant, the1st order derivative $\zeta^{(1)}(z)$ of Rieamnn zeta fanction is expanded to Laurent series around 1 as follows

$$\zeta^{(1)}(z) = -\frac{1}{(z-1)^2} + \sum_{r=0}^{\infty}(-1)^r \gamma_r \frac{(z-1)^{r-1}}{\Gamma(1+r-1)} \tag{1.0}$$



As is clear from this formula and the figure, $\zeta^{(1)}(z)$ has a pole of order 2 at $z=1$.

27.1.1 Non-Trivial Zeros of $\zeta'(z)$

According to " *New zero-free regions for the derivatives of the Riemann Zeta Function* " (T. Binder etc.) , the followings are known regarding non-trivial zeros of $\zeta^{(1)}(z)$

(i) The Riemann hypothesis is equivalent to that the zeros of $\zeta^{(1)}(x+iy)$ do not exist in $0<x<1/2$.

(ii) The zeros of $\zeta^{(1)}(x+iy)$ exist in $x<2.93938$.

(iii) In $x \ge 1/2$, the number of zeros of $\zeta^{(1)}(z)$ is less than the number of zeros of $\zeta(z)$.

In this section, we actually ask for the zeros of $\zeta^{(1)}(z)$ bearing these things in mind.

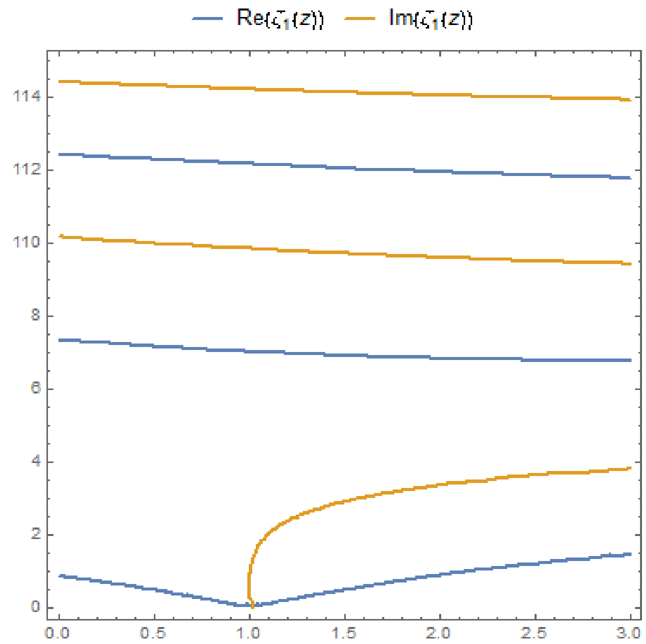
At the zeros of $\zeta^{(1)}(z)$, both the real part $Re\{\zeta^{(1)}(z)\}$ and the imaginary part $Im\{\zeta^{(1)}(z)\}$ have to be 0. That is, solutions of the following simultaneous equations must be obtained.

$$\begin{cases} Re\left[\zeta^{(1)}(x+iy)\right] = 0 & \text{(1.1r)} \\ Im\left[\zeta^{(1)}(x+iy)\right] = 0 & \text{(1.1i)} \end{cases}$$

Since it is impossible to solve this, we are obliged to use the numerical solution method like the Newton-Raphson method. Since the function $FindRoot[\,]$ for it is implemented in formula manipulation software *Mathematica,* we use this. However, for that purpose, approximate position of the zeros must be known. In order to know this, we draw contour plots of (1.1r) and (1.1i) and find the intersection. Since the function *ContourPlot* [ ] for it is implemented in formula manipulation software *Mathematica,* we use this. At this time, x can be up to 3 at most from the above **(ii)** .

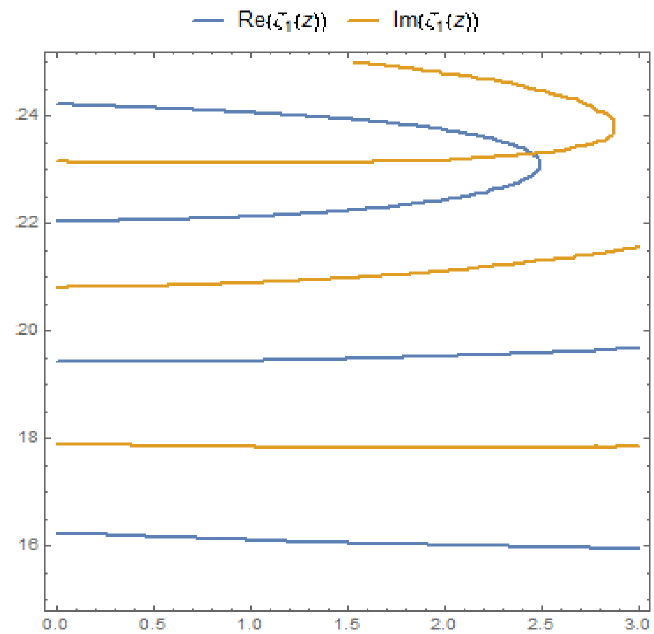
(1) $0 \leq y \leq 15$

The contour plot in this interval is drawn as follows. Although $(1,0)$ looks like the intersection of the real part and the imaginary part, this is not a zero but a singular point (pole) . Therefore, there is no zero of $\zeta^{(1)}(z)$ in this interval.



(2) $15 \leq y \leq 25$

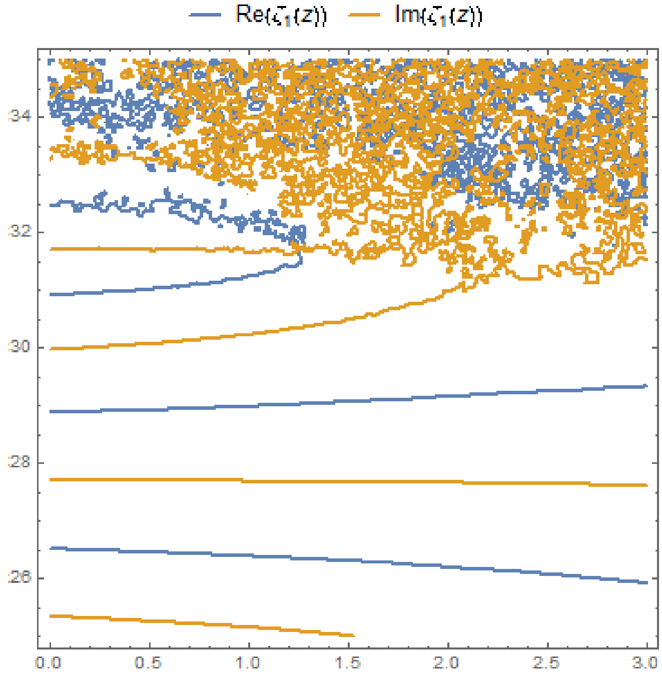
Zero is found near $(2.5, 23)$. The result of the numerical calculation is written on the right. This is the first zero of $\zeta^{(1)}(z)$.




```
FindRoot[ζ1[z, 200], {z, 2 + 23 i}]
{z → 2.46316 + 23.2983 i}
```

(3) $25 \le y \le 35$

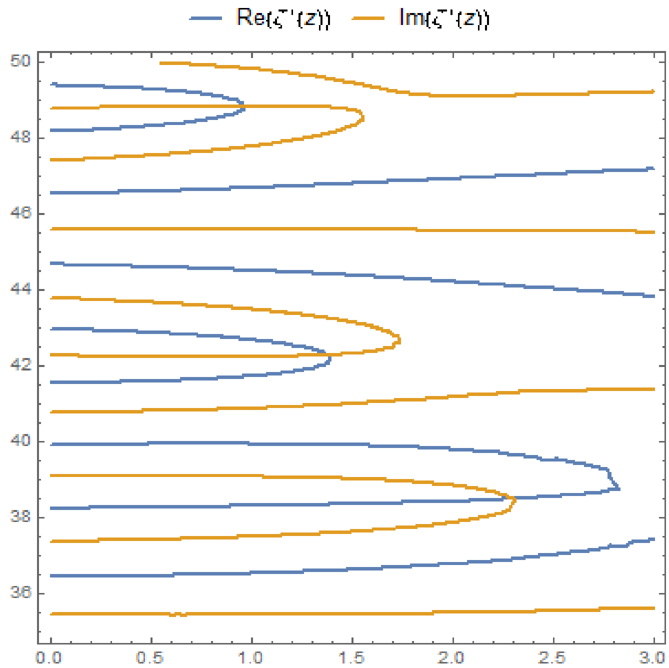
Zero is found near $(1,32)$. However, an exact figure cannot be drawn by the series of (1.0) and the upper one becomes a ghost. Of course, accurate numerical calculation is also impossible. So, it was calculated by the function $Zeta'[\,]$ of *Mathematica*. It is written on the right.




```
FindRoot[Zeta'[z], {z, 1 + 32 i}]
{z → 1.2865 + 31.7083 i}
```

(4) $35 \le y \le 50$

For $y \ge 35$, we draw and calculate by the left side ($Zeta'[\,]$ of *Mathematica*) rather than the right side of (1.0). Then, the contour plot is drawn as follows. 3 zeros are seen. The results of the numerical calculation are written on the right.




```
FindRoot[Zeta'[z], {z, 1 + 49 i}]
{z → 0.964686 + 48.8472 i}

FindRoot[Zeta'[z], {z, 1 + 42 i}]
{z → 1.38276 + 42.291 i}

FindRoot[Zeta'[z], {z, 2 + 38 i}]
{z → 2.30757 + 38.49 i}
```

As mentioned above, 5 zeros were obtained in the Interval $0 \le y \le 50$. If these are recalculated with significant 16 digits, it is as follows.

2.46316186945432 + 23.29832049276286 ⅈ

1.28649682226905 + 31.70825008311591 ⅈ

2.30757006372263 + 38.48998317307893 ⅈ

1.38276360571167 + 42.29096455459673 ⅈ

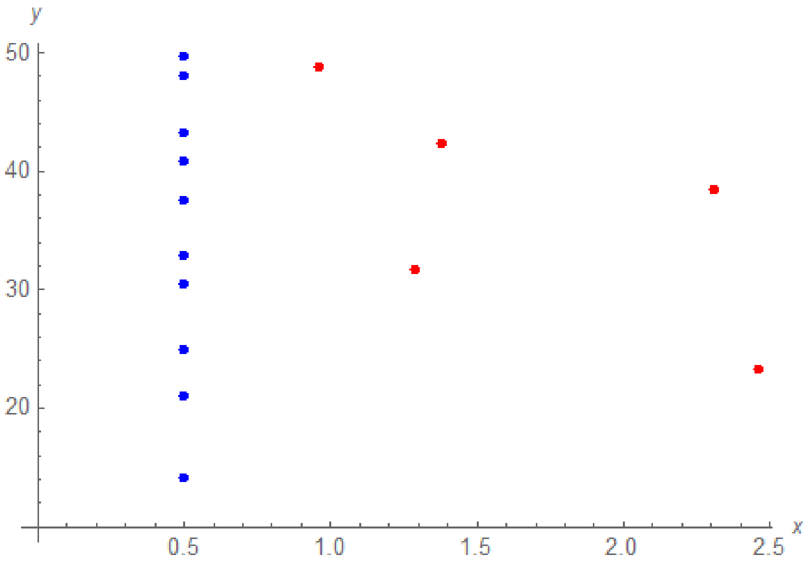
0.96468562270569 + 48.84715990506848 ⅈ

For trial, if the first zero is substituted for $Zeta'[z]$, it is as follows.

```
N[Zeta'[2.46316186945432 + 23.29832049276286 ⅈ]]
```

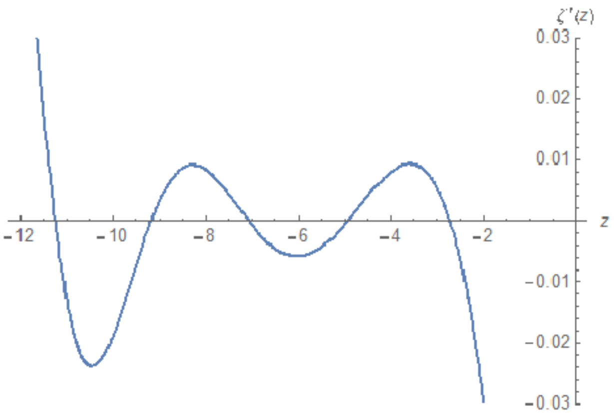
$-6.25143\times10^{-17} + 8.60553\times10^{-17}\,ⅈ$

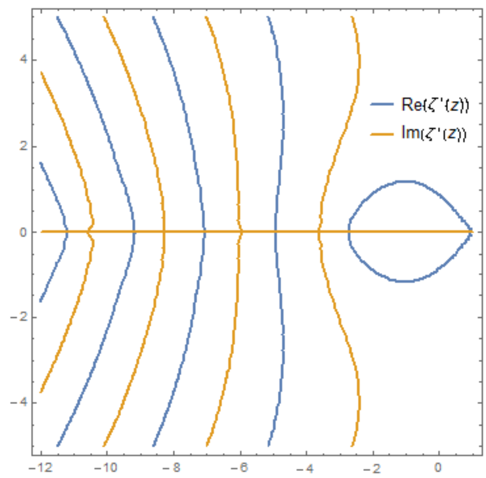
If these are plotted on the complex plane with the zeros of $\zeta(z)$, it is as follows. Blue is a zero of $\zeta(z)$ and Red is a zero of $\zeta^{(1)}(z)$. The latter number is reduced by half from the former. Further, each of the latter exists on the right side ($y > 1/2$) of the former.



27.1.2 Trivial Zeros of $\zeta'(z)$

I have not seen trivial zeoros of $\zeta^{(1)}(z)$. So, first, let us draw the 2D diagram and the contour plot. Then,





Observing these diagrams, we can see that trivial zeros of $\zeta^{(1)}(z)$ are distributed in the negative region of the x-axis. If the first few are calculated with significant 16 digits, it is as follows.

-2.717262829204574
-4.936762108594946
-7.074597145007147
-9.170493162785828
-11.24121232537534

Observing these, we can see that

(1) There is no zero near -2.

(2) There is no zero near $-4n \quad (n=1,2,3,\cdots)$.

(3) There are zeros near $-(4n-1) \quad (n=1,2,3,\cdots)$.

(4) The number of contours of the real part and the number of contours of the imaginary part are the same.

27.2 Zeros of $\zeta^{(1/2)}(z)$

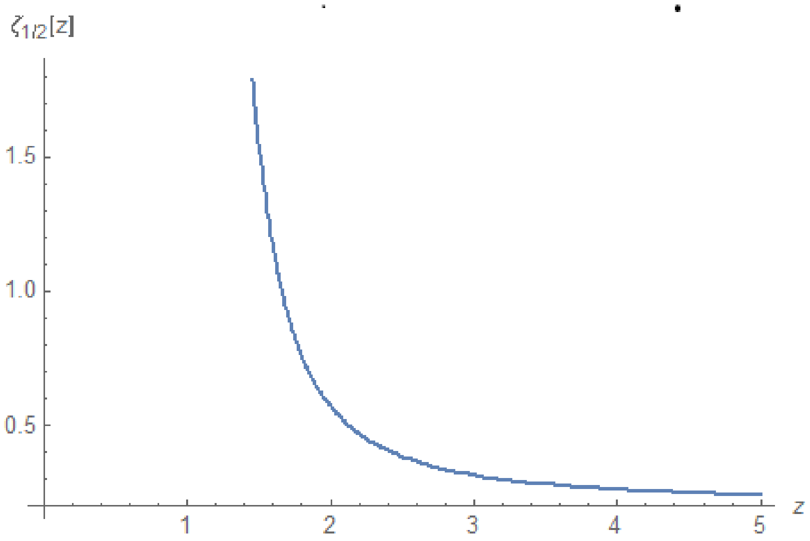
27.2.0 Laurent Expansion of Super Derivative of $\zeta(z)$

Accordint to " **26 Higher and Super Calculus of Zeta Function etc** " Formula 26.1.2s , when p is a positive number, $\Gamma(p)$ is the Gamma Function, $\psi(p)$ is the Digamma Function and γ_r is a Stieltjes constant, the p th order derivative $\zeta^{(p)}(z)$ of Rieamnn zeta fanction is expanded to Laurent series around 1 as follows

$$\zeta^{(p)}(z)=\frac{\log(z-1)-\psi(-p)-\gamma_0}{\Gamma(-p)}(z-1)^{-p-1}+\sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r-p}}{\Gamma(1+r-p)} \tag{2.0}$$

Where, $\dfrac{\psi(-p)}{\Gamma(-p)}=(-1)^{1+p}p!\quad for\ p=0,1,2,\cdots$

When $p=1/2$, this is illustrated as follows. As is clear from this formula and the figure, $\zeta^{(p)}(z)$ has a pole of order $(1+p)$ at $z=1$.



Though the super differentiation for arbitrary $p>0$ is possible using (2.0) , we deal with the super derivative between $\zeta^{(0)}(z)$ and $\zeta^{(1)}(z)$ in this chapter. And, in order to overlook the zeros of the super derivative in this interval, it seems good to observe the zeros of $\zeta^{(1/2)}(z)$ which is in the middle.

From (2.0) , the function used for this is as follows.

$$\zeta^{(1/2)}(z)=\frac{\log(z-1)-\psi(-1/2)-\gamma_0}{\Gamma(-1/2)}(z-1)^{-3/2}+\sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r-1/2}}{\Gamma(r+1/2)} \tag{2.1}$$

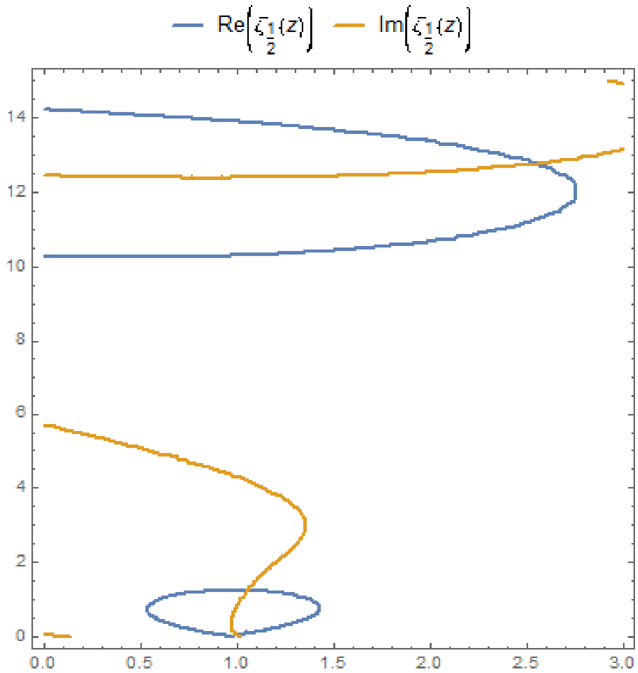
27.2.1 Non-Trivial Zeros of $\zeta^{(1/2)}(z)$

The method of finding the zeros follows the method of the previous section. However, the interval of the real part x of z should be slightly wider.

(1) $0\le y\le 15$

The contour plot in this interval is drawn as follows. The intersection of the real part and the imaginary part (zero) is found in two places. The results of the numerical calculation are written on the right. The bottom intersection is not a zero but a singularity point.

```
ContourPlot[{Re[ζ1/2[x + ⅈ y, 200]] == 0, Im[ζ1/2[x + ⅈ y, 200]] == 0},
        {x, 0, 3}, {y, 0, 15},
        PlotLegends → Placed[{Re[ζ1/2[z]], Im[ζ1/2[z]]}, Above]]
```

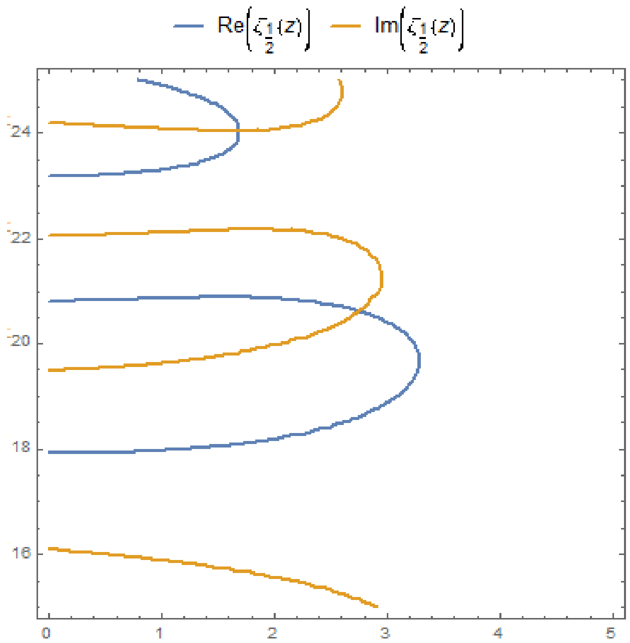



```
FindRoot[ζ1/2[z, 500], {z, 2 + 12 ⅈ}]
{z → 2.55716 + 12.7817 ⅈ}
```

```
FindRoot[ζ1/2[z, 500], {z, 1 + ⅈ}]
{z → 1.05107 + 1.24731 ⅈ}
```

(2) 15 $\leq y \leq$ 25

2 zeros are seen also in this interval. The results of the numerical calculation are written on the right. However, The first zero is an estimate.

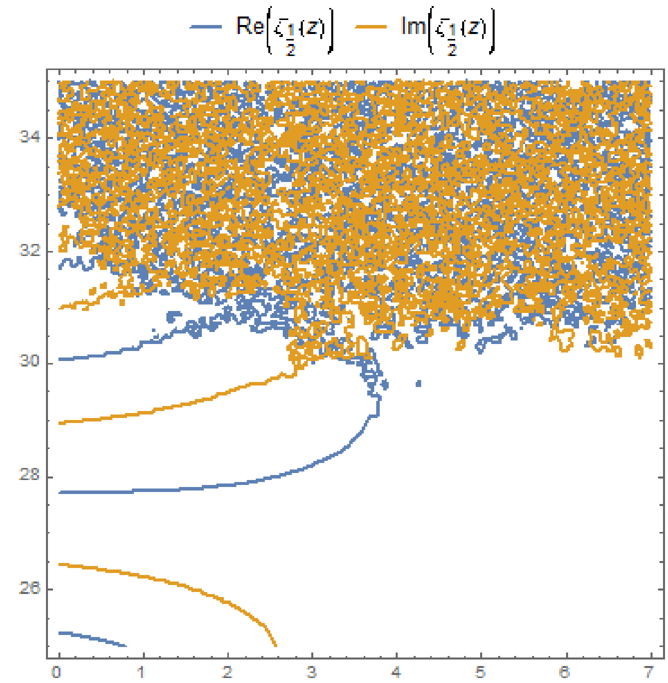



```
FindRoot[ζ1/2[z, 800], {z, 2 + 24 ⅈ}]
{z → 1.69601 + 24.0402 ⅈ}   (estimate)
```

```
FindRoot[ζ1/2[z, 500], {z, 3 + 20 ⅈ}]
{z → 2.75378 + 20.6323 ⅈ}
```

(3) 25 $\leq y \leq$ 35

The contour plot is drawn as follows. Most are ghosts and the location of the intersection can not be seen. This figure shows that it is impossible to obtain zeros in wider interval, by the above formula (2.1) and my personal computer & the software.



As mentioned above, 4 zeros were obtained in the Interval $0 \le y \le 25$. If the first 3 are recalculated with significant 16 digits, it is as follows.

```
1.051072530522430 + 1.247313910052378 i
2.557158597919913 + 12.781692663345648 i
2.753780639391152 + 20.63230715264229 i
```

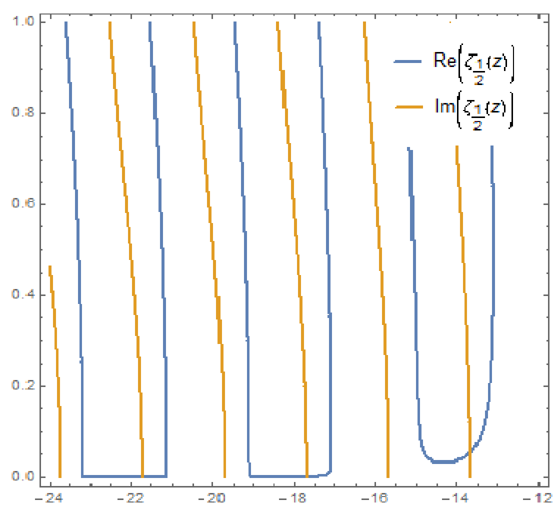
For trial, if the first zero is substituted for (2.1), it is as follows.

```
N[ζ1/2[1.051072530522430 + 1.247313910052378 i , 800]]
1.66533×10^-16 - 5.55112×10^-17 i
```

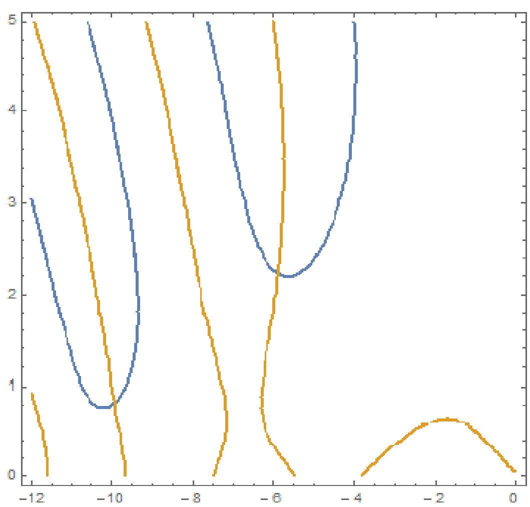
27.2.2 Trivial Zeros of $\zeta^{(1/2)}(z)$

The contour plots in $-24 \le x \le 12$ and $-12 \le x \le 0$ are respectively drawn as follows. The left is the former and the right is the latter. The blue line is the real part and the orange line is the imaginary part.

-24 ≤ x ≤ -12



-12 ≤ x ≤ 0



If these are calculated with significant 16 digits, it is as follows.

$-5.864198759417357 + 2.229323948865088\,i$

$-9.918588597642838 + 0.821020955865908\,i$

$-13.65599352145248 + 0.054348228602989\,i$

$-17.68830911654609 + 0.000815036808679\,i$

$-21.71469646228788 + 0.000005378781454\,i$

Observing these, we can see that

(1) There is no zero near $x = -2$.

(2) There is no zero near $x = -4n \quad (n=1,2,3,\cdots)$.

(3) There are zeros near $x = -(4n+2) \quad (n=1,2,3,\cdots)$.

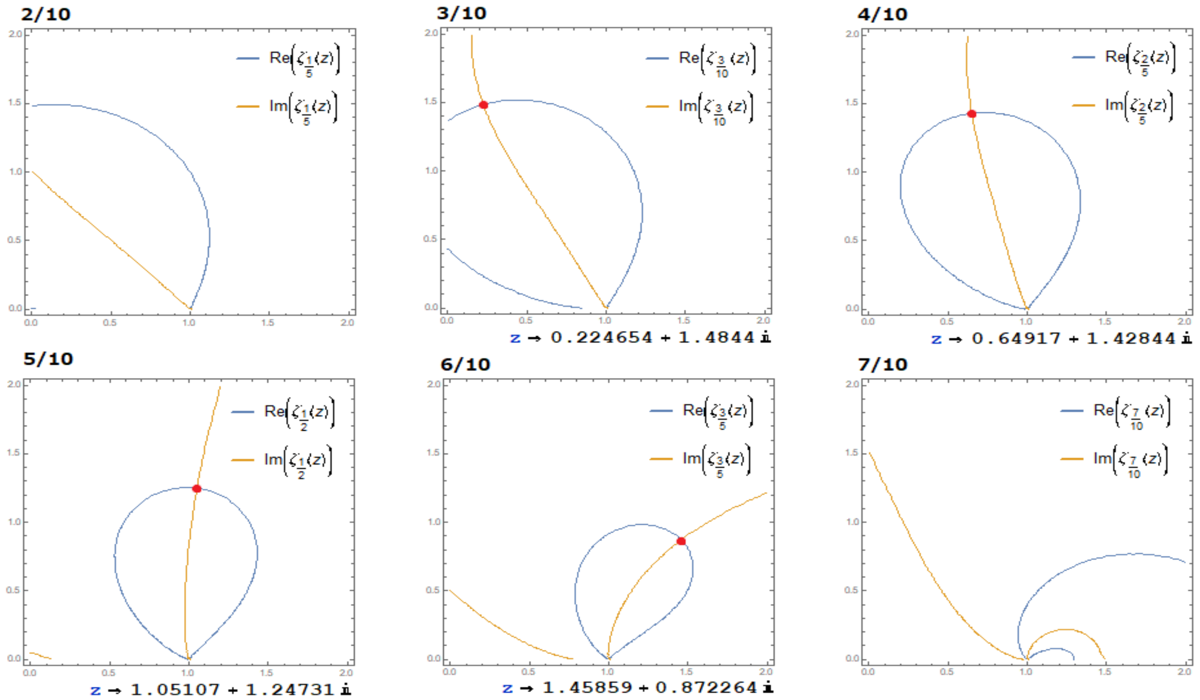
(4) The number of contours of the real part is half the number of contours of the imaginary part.

27.3 Transition of Non-Trivial Zeros associated with the Super Dderivative.

In this section, we investigate the transition of non-trivial zeros associated with the Super Derivative using formula (2.0) .

27.3.1 Non-Trivial Zeros of $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$ $(0 \le y \le 2)$

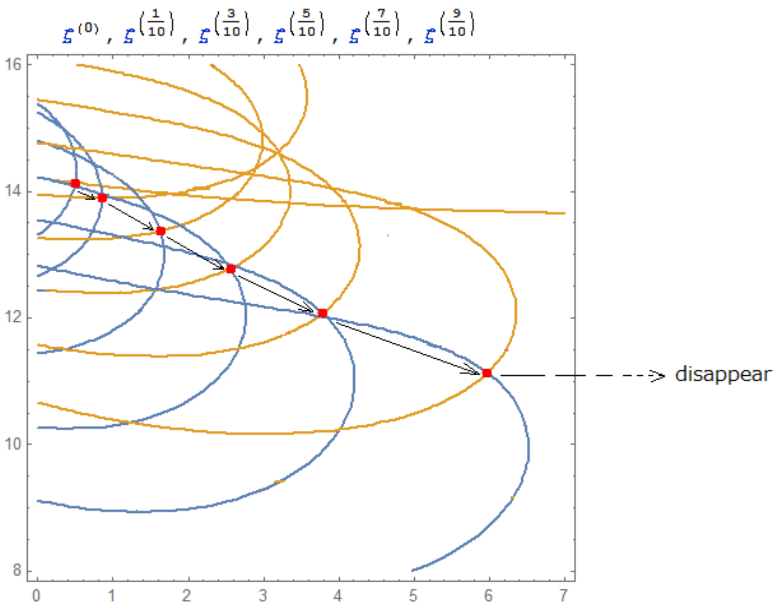
Here, we investigate the transition of the zeros of $\zeta^{(2/10)}(z) \sim \zeta^{(7/10)}(z)$ in $0 \le y \le 2$. The interval of x is set to $0 \le x \le 2$. Drawing these contour lines and looking for intersections, we obtain the followings.



The zero of $\zeta^{(p)}(z)$ appears near $p=3/10$, moves to the right at $p=4/10$, $5/10$, $6/10$, and disappears at $p=7/10$. This is a zero peculiar to super derivative that appears only on non-integer order.

27.3.2 Non-Trivial Zeros of $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$ $(8 \le y \le 16)$

Next, we investigate the transition of the zeros of $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$ in $8 \le y \le 16$. The interval of x is set to $0 \le x \le 7$. if these contour lines and intersections are drawn on 1 figure, it is as follow.



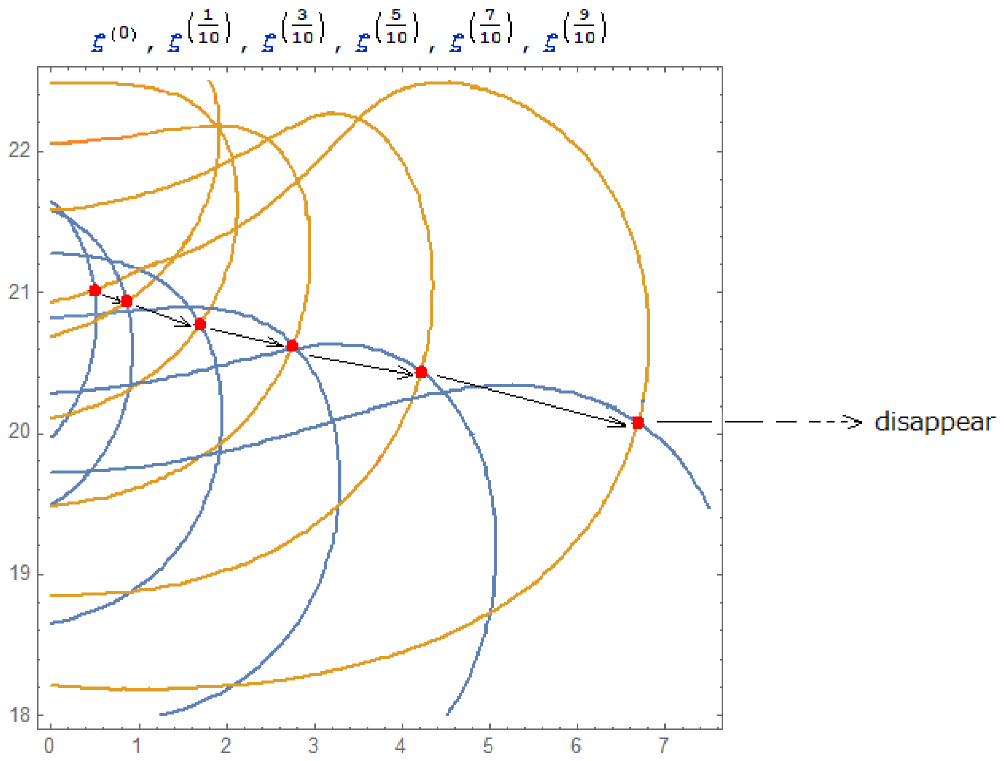
The blue line is the real part and the orange line is the imaginary part. The red dot is the intersection of these namely the zero. The red dot on the left is the first zero $1/2+i\,14.1347\cdots$ of $\zeta(z)$. This zero moves to the right at $p=1/10 \sim 9/10$ and exists as long as $p<1$. And the zero disappears at the moment of $p=1$. Because, at this time, the real part and the imaginary part become parallel and do not cross.

By reference, the zero of $\zeta^{(0.99999)}(z)$ is $z=20.59398\cdots+i\,9.68836\cdots$.

For the above animation, click here Anim2732.gif

27.3.3 Non-Trivial Zeros of $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$ $(18 \le y \le 22.5)$

Next, we investigate the transition of the zeros of $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$ in $18 \le y \le 22.5$. The interval of x is set to $0 \le x \le 7.5$. if these contour lines and intersections are drawn on 1 figure, it is as follow.



The blue line is the real part and the orange line is the imaginary part. The red dot is the intersection of these namely the zero. The red dot on the left is the 2nd zero $1/2+i\,21.0220\cdots$ of $\zeta(z)$. This zero moves to the right at $p=1/10 \sim 9/10$ and exists as long as $p<1$. And the zero disappears at the moment of $p=1$. Because, at this time, the real part and the imaginary part become parallel and do not cross.

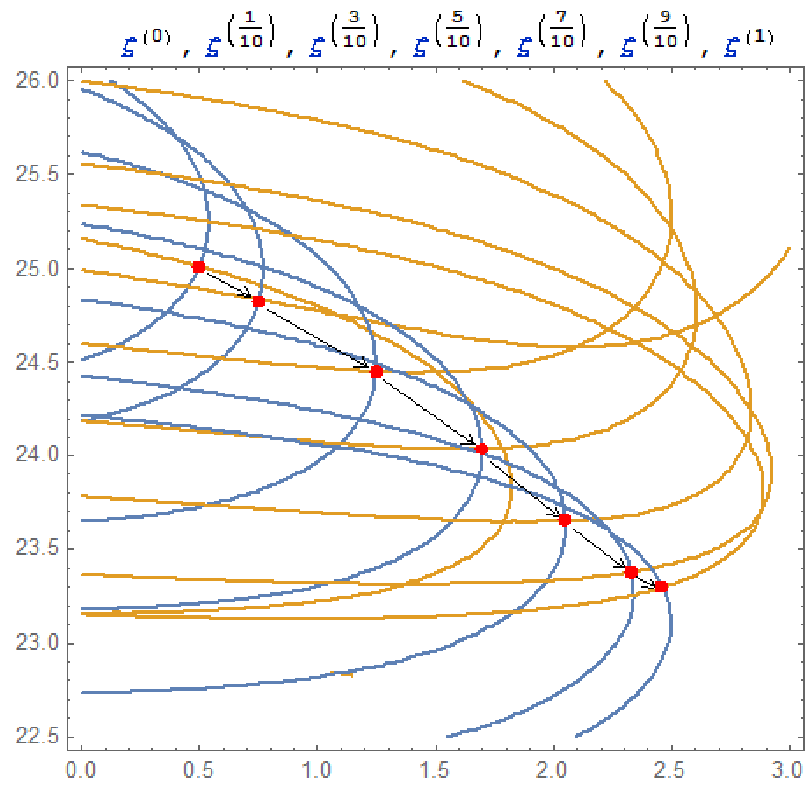
By reference, zero of $\zeta^{(0.999)}(z)$ is about $z=13.984+i\,19.522$.

27.3.4 Non-Trivial Zeros of $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$ $(22.5 \le y \le 26)$

Last, we investigate the transition of the zeros of $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$ in $22.5 \le y \le 26$. The interval of x is set to $0 \le x \le 3$. if these contour lines and intersections are drawn on 1 figure, it is as follow.

The blue line is the real part and the orange line is the imaginary part. The red dot is the intersection of these namely zero.

The red dot on the left is the 3rd zero $1/2+i\,25.0108\cdots$ of $\zeta(z)$. This zero moves to the right at $p=1/10 \sim 9/10$ and finally reaches the zero $2.4631\cdots+i\,23.2983\cdots$ of $\zeta^{(1)}(z)$.



For the above animation, click here Anim2734.gif

Summary

Investigating the transition of the non-triviall zeros of $\zeta^{(p)}(z)$ $p=0 \sim 1$ in interval $0 \le y \le 26$, we obtained the following results.

(1) There exist non-trivial zeros peculiar to super derivative that appear only on non-integer order $0<p<1$.

(2) The first two zeros $1/2+i\,14.1347\cdots$ and $1/2+i\,21.0220\cdots$ of $\zeta(z)$ move to the lower right with increasing p , and disappear suddenly at $\zeta^{(1)}(z)$. This is consistent with the fact that the number of zeros of $\zeta^{(1)}(z)$ is less than the number of zeros of $\zeta(z)$ (**27.1.1 (iii)**) .

(3) The 3rd zero $1/2+i\,25.0108\cdots$ of $\zeta(z)$ moves to the lower right with increasing p , and becomes the 1st zero $2.4631\cdots+i\,23.2983\cdots$ of $\zeta^{(1)}(z)$. This assists the assertion that the zeros of $\zeta^{(1)}(z)$ do not exist in $0<x<1/2$ (**27.1.1 (i)**) .

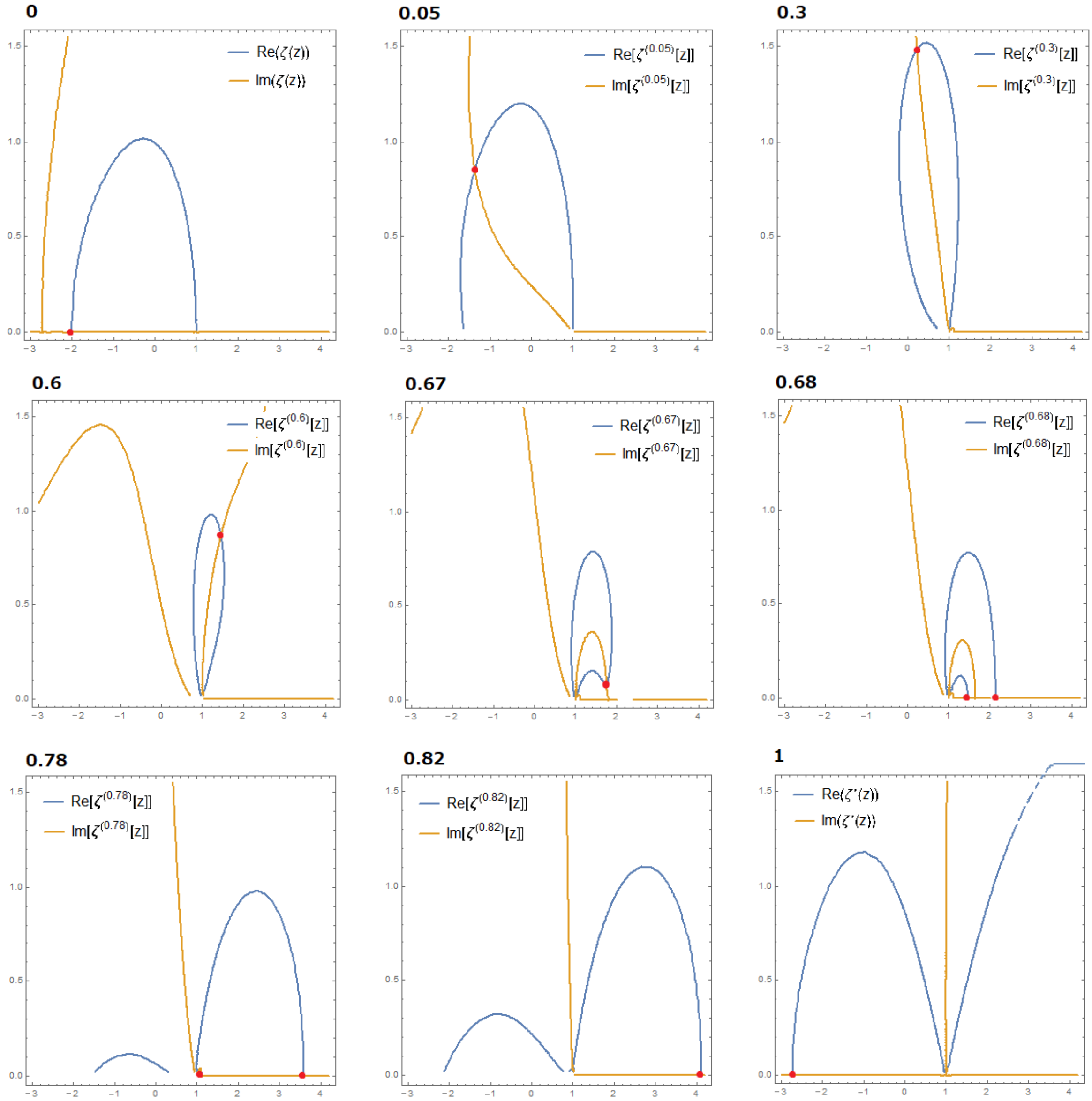
(4) It is impossible to obtain zeros in $y \ge 26$ by the formula (2.0) and my personal computer & the software. We need a faster convergent formula, a faster personal computer, a dedicated program, or all of them.

27.4 Transition of Trivial Zeros associated with the Super Derivative.

In this section, we investigate the transition of trivial zeros associated with the Super Derivative using formula (2.0) .

27.4.1 Transition of the trivial zero $z = -2$ at $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$

Here, we investigate the transition of the trivial zero $z = -2$ at $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$. The interval of x is set to $-3 \le x \le 4.2$. When the zero is traced with nine contour plots, it is as follows. The blue line is the real part, the orange line is the imaginary part, and the red dot is a zero.

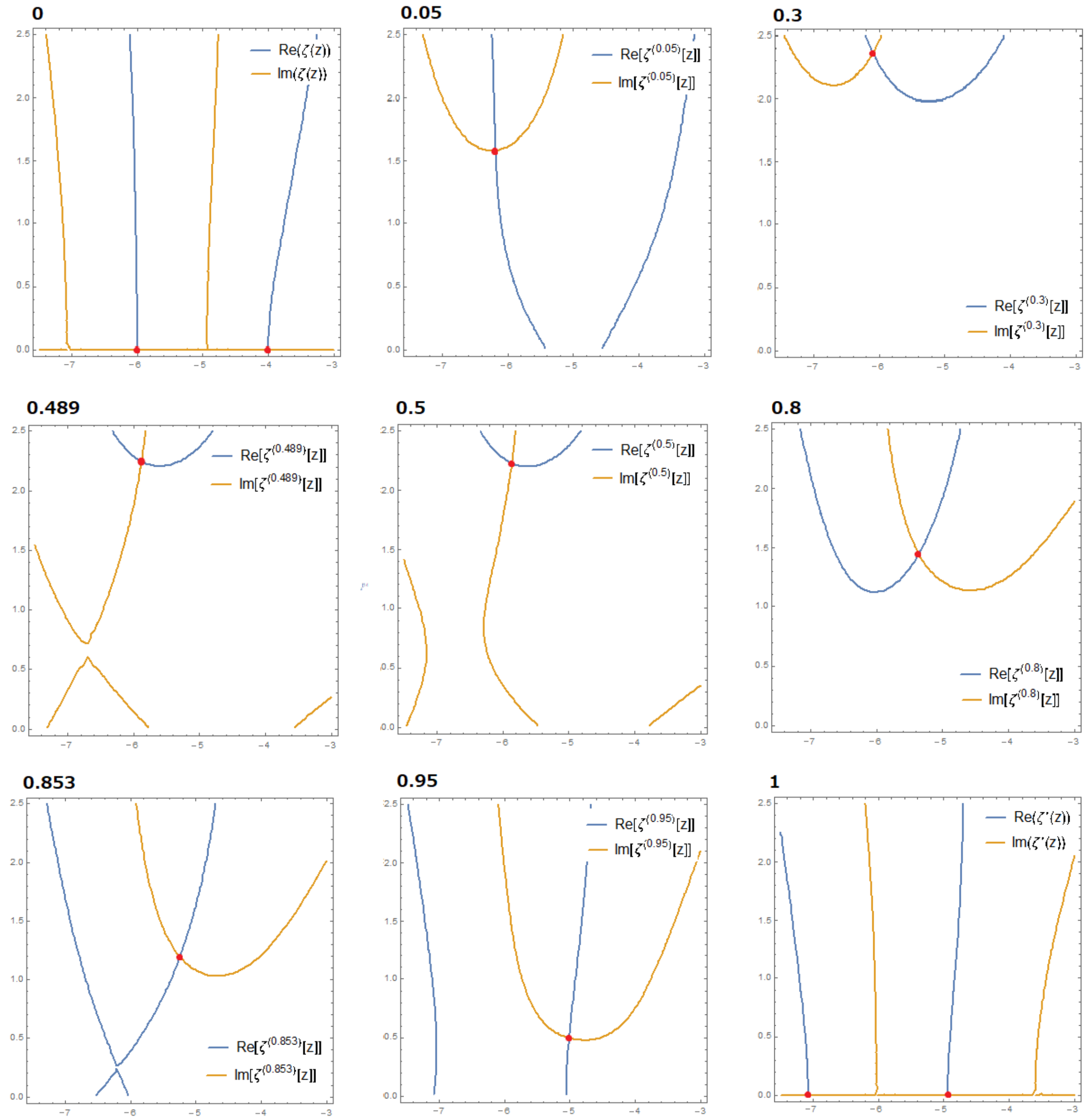


$\zeta(z)$'s zero -2 draws a ballistic trajectory on a complex plane with increasing p , and falls near 1.77 . $\mathbf{0} \sim \mathbf{0.67}$ show the transition. In $\mathbf{0.68}$, this zero is separated into two zeros on the x -axis. In $\mathbf{0.78}$, two zeros move to the right and the left respectively. In $\mathbf{0.82}$, the left zero is swallowed in the singular point 1 and disappears. In $\mathbf{1}$, the right zero disappears. Simultaneously, $\zeta^{(1)}(z)$'s zero $-2.71726\cdots$ appears on the left side. This looks like the transmigration from $x=1$. Additionally, the zeros existing on the half line $x>1, y=0$ at $0.670597 \le p < 1$ are trivial zeros peculiar to the non-integer order derivative.

For the above animation, click here Anim2741.gif

27.4.2 Transition of the trivial zeros $z = -4, -6$ at $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$

Here, we investigate the transition of the trivial zeros $z = -4, -6$ at $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$. The interval of x is set to $-7.5 \le x \le -3$. When the zeros are traced with nine contour plots, it is as follows.
The blue line is the real part, the orange line is the imaginary part, and the red dot is the zero.



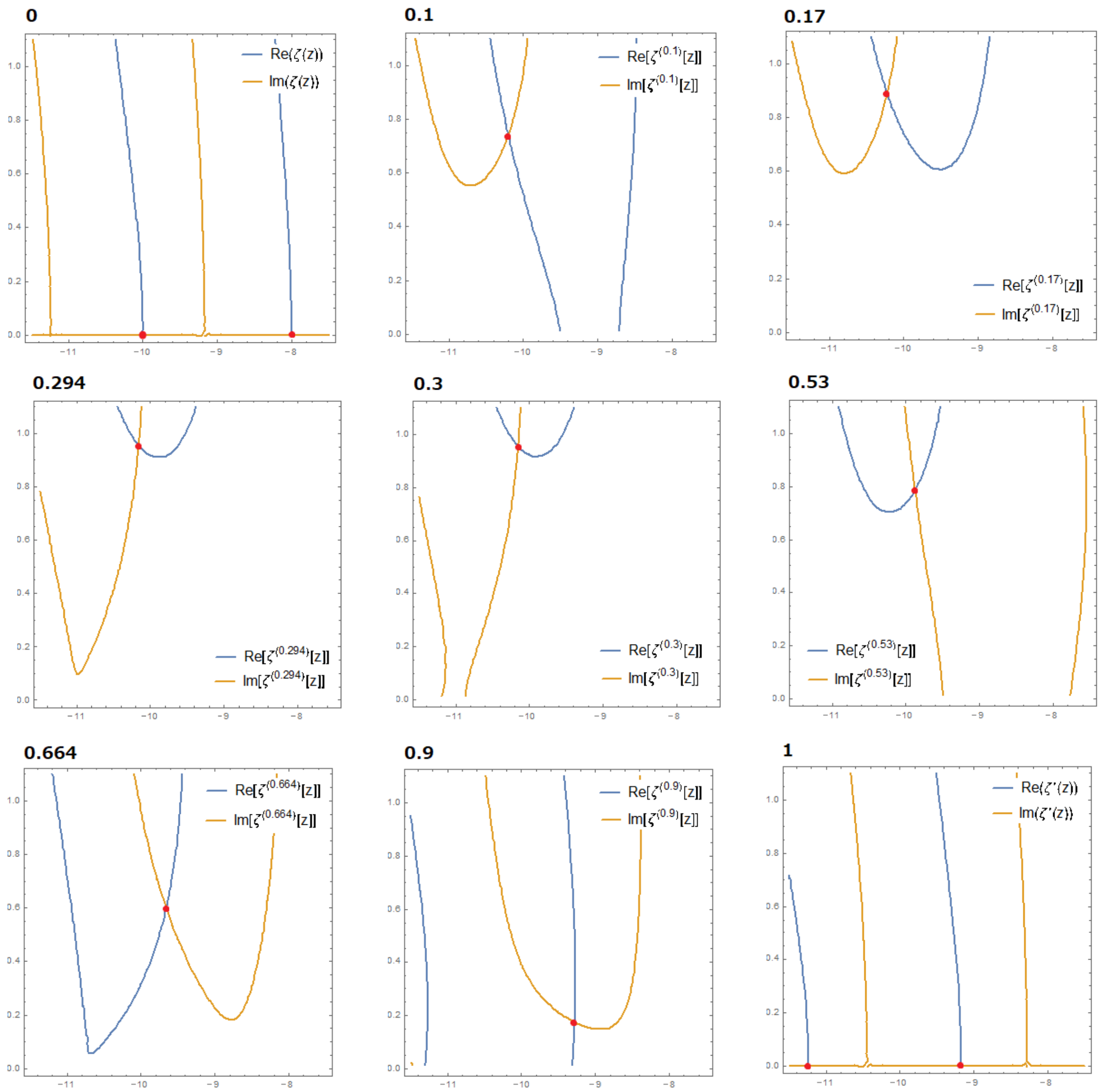
In $\mathbf{0}$, there are two zeros $-4, -6$ of $\zeta(z)$. Zero -4 disappears at the moment p increases. Zero -6 draws a lofted trajectory on a complex plane with increasing p, and falls to $\zeta^{(1)}(z)$'s zero $-4.93676\cdots$. Simultaneously, another $\zeta^{(1)}(z)$'s zero $-7.07459\cdots$ appears on the left side. This looks like the trans-migration from -6.

For the above animation, click here Anim2742.gif

27.4.3 Transition of the trivial zeros $z = -8, -10$ at $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$

Here, we investigate the transition of the trivial zeros $z = -8, -10$ at $\zeta^{(0)}(z) \sim \zeta^{(1)}(z)$. The interval of x

is set to $-11.5 \le x \le -7.5$. When the zeros are traced with nine contour plots, it is as follows.
The blue line is the real part, the orange line is the imaginary part, and the red dot is a zero .



In $\mathbf{0}$, there are two zeros $-8, -10$ of $\zeta(z)$. Zero -8 disappears at the moment p increases. Zero -10 draws a lofted trajectory on a complex plane with increasing p , and falls to $\zeta^{(1)}(z)$'s zero $-9.17049\cdots$. Simultaneously, another $\zeta^{(1)}(z)$'s zero $-11.24121\cdots$ appears on the left side. This looks like the trans-migration from -10 .

Summary

Investigating the transitions of $\zeta(z)$'s trivial zeros $-2, -4, \sim, -10$ associated $\zeta^{(p)}(z)$ $p = 0 \sim 1$, we obtained the following results.

(1) There exist trivial zeros peculiar to super derivative that appear only on non-integer order $0.670597 \le p < 1$. And, these are real numbers greater than 1 .

(2) Half of $\zeta(z)$'s zero -2 transmigrates from the singular point 1 to $\zeta^{(1)}(z)$'s zero $-2.71726\cdots$, and. the other half disappears at the moment of $p=1$. The above (1) occurs in this process.

(3) $\zeta(z)$'s zero -6 moves to $\zeta^{(1)}(z)$'s zero $-4.93676\cdots$, and transmigrates to $\zeta^{(1)}(z)$'s zero $-7.07459\cdots$ simultaneously. $\zeta(z)$'s zero -4 disappears at the moment of $p>0$.

(4) $\zeta(z)$'s zero -10 moves to $\zeta^{(1)}(z)$'s zero $-9.17049\cdots$, and transmigrates to $\zeta^{(1)}(z)$'s zero $-11.24121\cdots$ simultaneously. $\zeta(z)$'s zero -8 disappears at the moment of $p>0$.

2018.12.07
2018.12.20 Renewed
2021.01.29 Added animations

Kano Kono

Alien's Mathematics