28 Zeros of Super Integral of Riemann Zeta

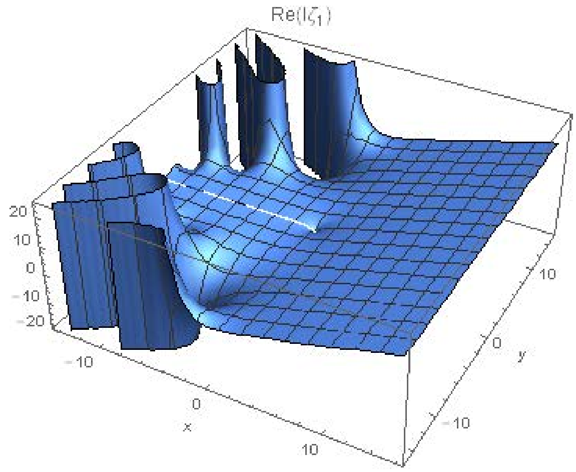
28.1 Zeros of $\zeta^{<1>}(z)$

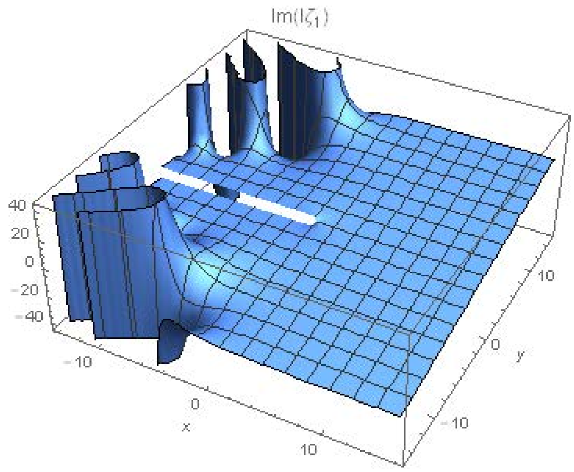
27.1.0 Series Expansion of $\zeta^{<1>}(z)$

Accordint to " **26 Higher and Super Calculus of Zeta Function etc** " Formula 26.1.1h , when γ_r is a Stieltjes constant, the1st order lineal primitive $\zeta^{<1>}(z)$ of Rieamnn zeta fanction is expessed as follows

$$\zeta^{<1>}(z) = \log(z-1) + \sum_{r=0}^{\infty} (-1)^r \gamma_r \frac{(z-1)^{r+1}}{(r+1)!} \tag{1.0}$$

The 3D figures of the real part and the imaginary part are as follows. The left is the real part and the right is the imaginary part. As is clear from both figures, there is no zero on the half line $x<0, y=0$.





28.1.1 Zeros of $\zeta^{<1>}(z)$

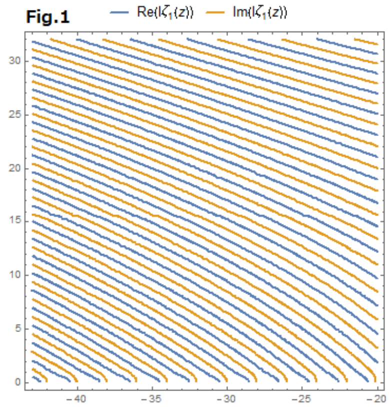
At the zeros of $\zeta^{<1>}(z)$, both the real part $Re\{\zeta^{<1>}(z)\}$ and the imaginary part $Im\{\zeta^{<1>}(z)\}$ have to be 0. That is, solutions of the following simultaneous equations must be obtained.

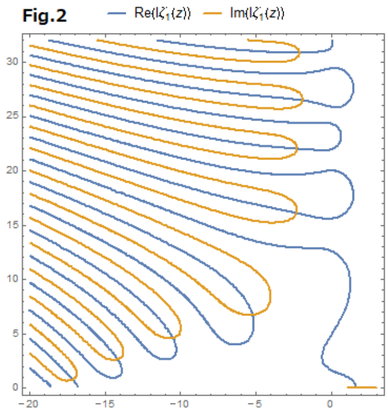
$$\begin{cases} Re\left[\zeta^{<1>}(x+iy)\right] = 0 & (1.1r) \\ Im\left[\zeta^{<1>}(x+iy)\right] = 0 & (1.1i) \end{cases}$$

Since it is impossible to solve this, we are obliged to use the numerical solution method like the Newton-Raphson method. Since the function $FindRoot[\,]$ for it is implemented in formula manipulation software *Mathematica,* we use this. However, for that purpose, approximate position of the zeros must be known. In order to know this, we draw contour plots of (1.1r) and (1.1i) and find the intersection. Since the function $ContourPlot[\,]$ for it is implemented in *Mathematica,* we use this.

(1) Wide area figures

In order to explore the location of the zeros, let us draw 3 contour plots. The interval of y is set to $0 \le x \le 32$ Then, it is as follows. The blue line is the real part, the orange line is the imaginary part, and the intersection is a zero.





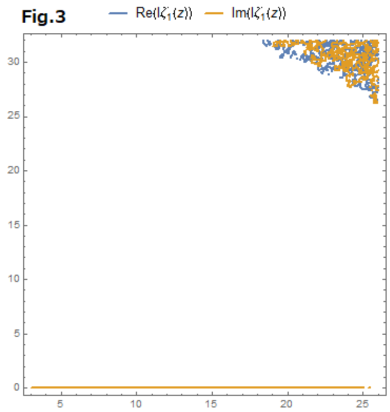


Fig.1 is $-43 \le x \le -20$. There is no zero here. Fig.2 is $-20 \le x \le 3$. There are nine zeros here. Fig.3 is $3 \le x \le -26$. There is no zero here.

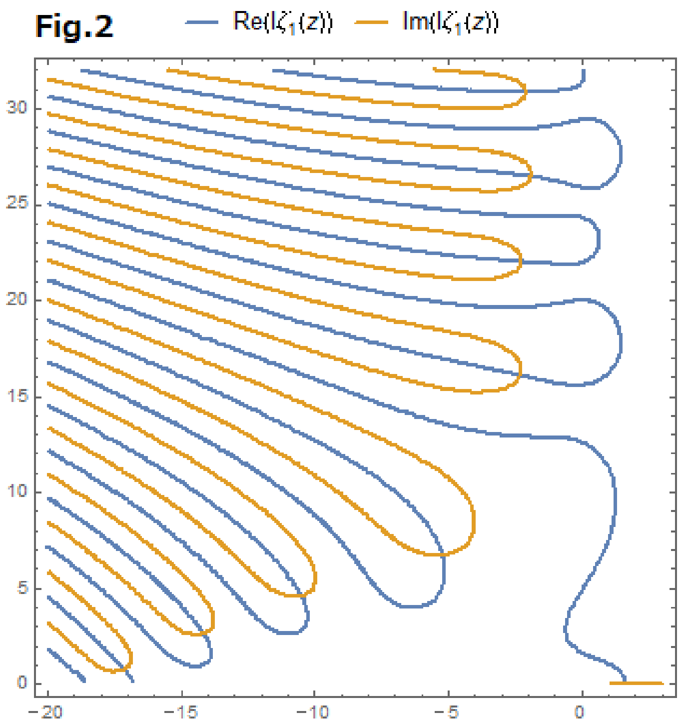
When these upper ($y \ge 32$) contour plots were drawn, the three plots became ghosts. It is difficult to draw any more by the formula (1.0) and my personal computer & the software. However, the followings may be said.

(i) There seems to be no zero in the upper part of Fig.1 and Fig.3.

(ii) There seems to be innumerous zeros in the upper part of Fig.2.

(2) Enlarged figure

When Fig.2 is enlarged and the zeros are calculated with significant 16 digits using the $FindRoot[\,]$ of *Mathematica*, it is as follows.



$-2.30074531502972 + 16.08542457590717\,i$

$-2.25748998985921 + 22.00901540490299\,i$

$-1.9080 + 26.4561\,i$ estimated

$-2.0862 + 30.8579\,i$ estimated

1.669008212478207

$-5.207610164448825 + 6.722221913809304\,i$

$-10.491624715391694 + 4.611112879246793\,i$

$-14.14832472367627 + 2.63517233968627\,i$

$-17.1410646110841 + 0.77114740836467\,i$

28.2 Transition of Non-Trivial Zeros associated with the Super Integral.

In this section, we investigate the transition of non-trivial zeros associated with the Super Integral.

28.2.0 Laurent Expansion of Super Integral of $\zeta(z)$

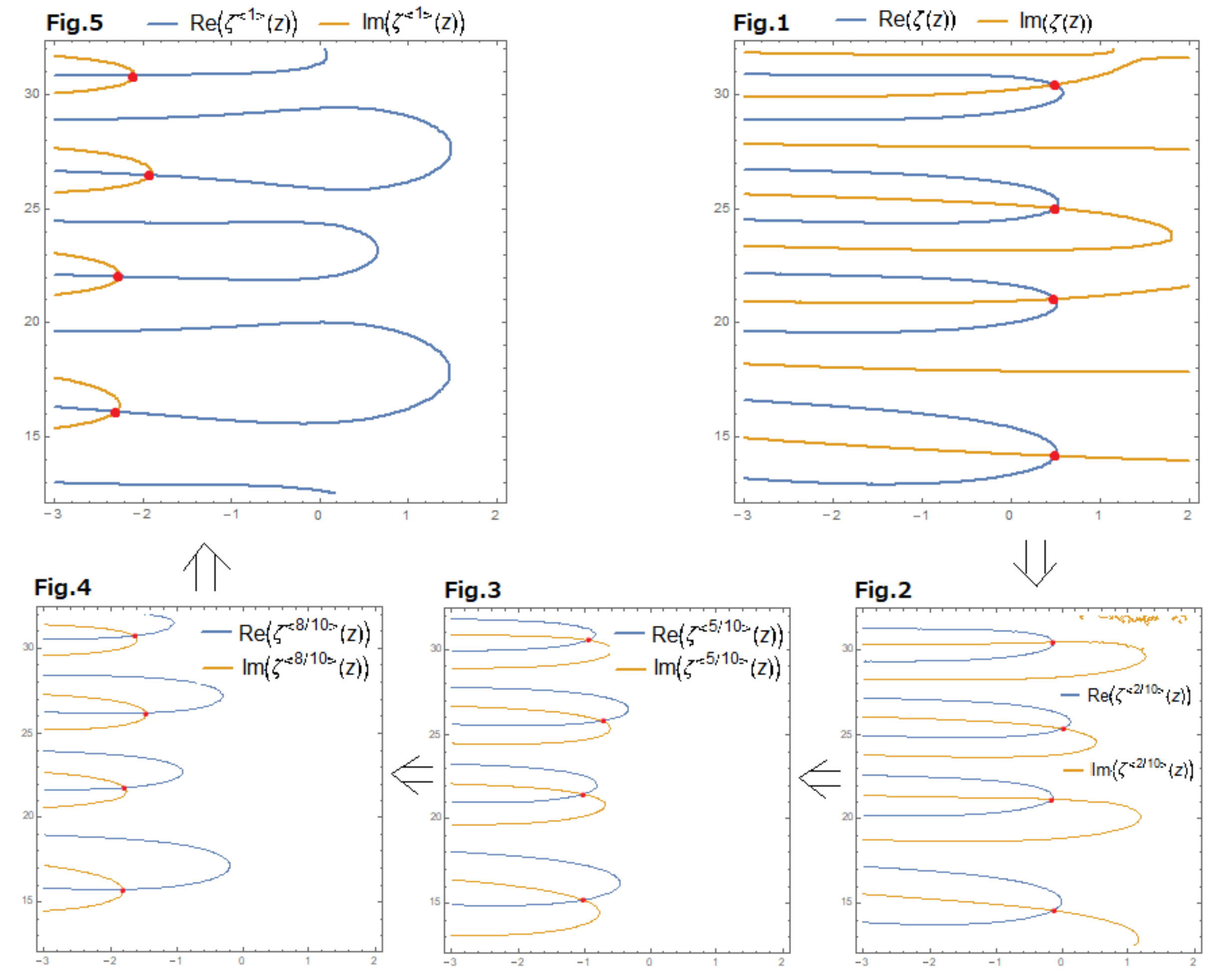
Accordint to " **26 Higher and Super Calculus of Zeta Function etc** " Formula 26.1.1s , when p is a positive number, $\Gamma(p)$ is the Gamma Function, $\psi(p)$ is the Digamma Function and γ_r is a Stieltjes constant, the p th order lineal primitive $\zeta^{<p>}(z)$ of Rieamnn zeta fanction is expessed as follows

$$\zeta^{\langle p\rangle}(z) = \frac{\log(z-1)-\psi(p)-\gamma_0}{\Gamma(p)}(z-1)^{p-1} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r+p}}{\Gamma(1+r+p)} \qquad (2.0)$$

As is clear from this formula, when $0<p<1$, $\zeta^{<p>}(z)$ has a pole of order $(1-p)$ at $z=1$.

28.2.1 Non-Trivial Zeros of $\zeta^{<0>}(z) \sim \zeta^{<1>}(z)$ $(12.5 \le y \le 32)$

Here, we investigate the transition of the zeros of $\zeta^{<0>}(z) \sim \zeta^{<1>}(z)$ in $12.5 \le y \le 32$. The interval of x is set to $-3 \le x \le 2$. When the zeros are traced with 5 contour plots, it is as follows. The blue line is the real part, the orange line is the imaginary part, and the red dot is a zero.



In Fig.1, the non-trivial zeros of $\zeta^{<0>}(z)$ are drawn. In Fig.2, the zeros of $\zeta^{<0.2>}(z)$ are drawn. Looking at these figures, we can see that the zeros are moving to the left with increasing p . And finally they reach Fig.5. If Fig.5 and Fig.1 are shown numerically, it is as follows.

$$-2.3007\cdots + i\,16.0854\cdots \quad \Longleftarrow \quad 1/2 + i\,14.1347\cdots$$

$$-2.2574\cdots + i\,22.0090\cdots \quad \Longleftarrow \quad 1/2 + i\,21.0220\cdots$$

$$-1.9080\cdots + i\,26.4561\cdots \quad \Longleftarrow \quad 1/2 + i\,25.0108\cdots$$

$$-2.0862\cdots + i\,30.8578\cdots \quad \Longleftarrow \quad 1/2 + i\,30.4248\cdots$$

For the above animation, click here Anim2821.gif

Summary

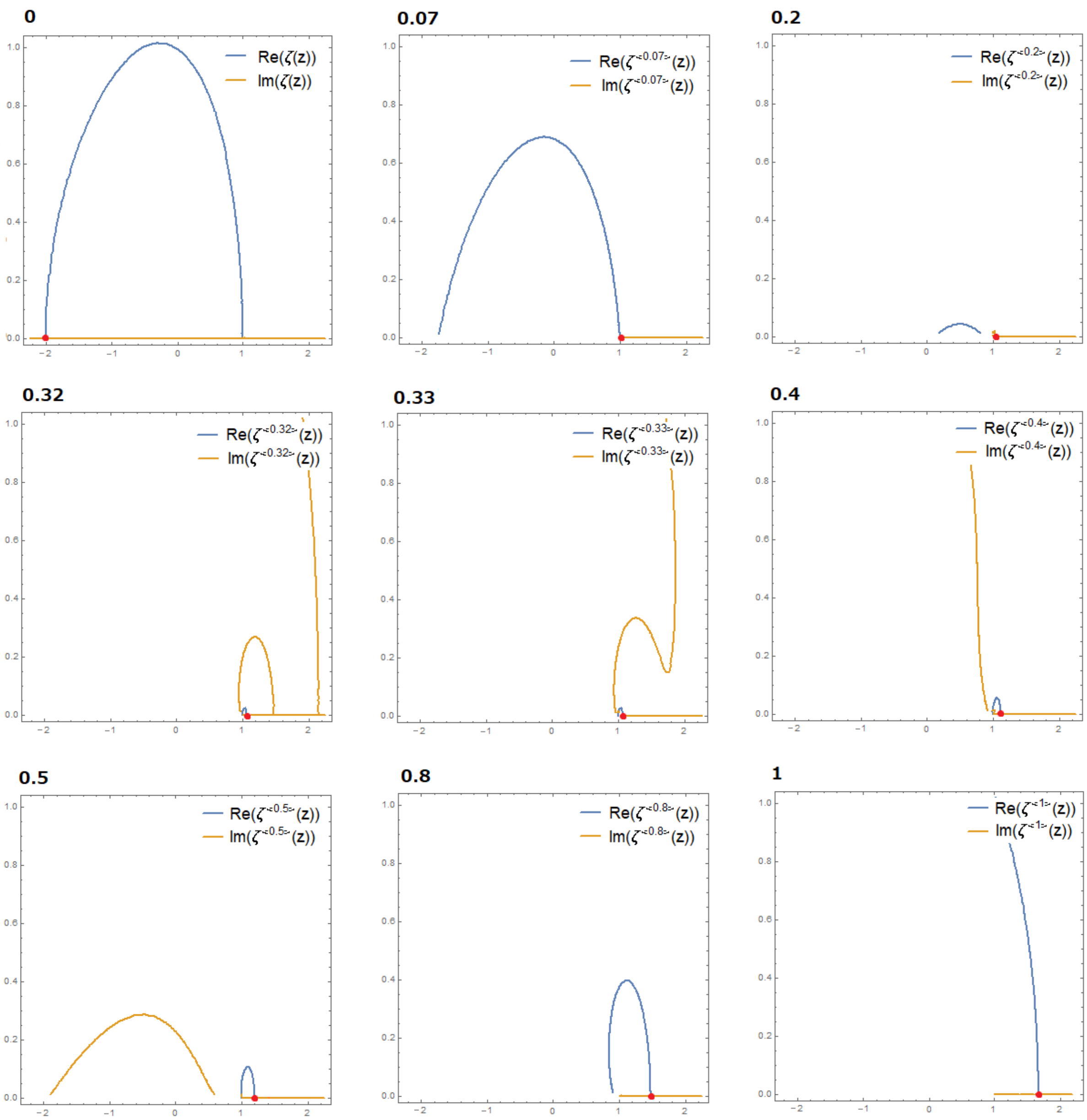
The non-trivial zeros of $\zeta(z)$ move to the upper left on the complex plane with increasing p .

28.3 Transition of Trivial Zeros associated with the Super Integral.

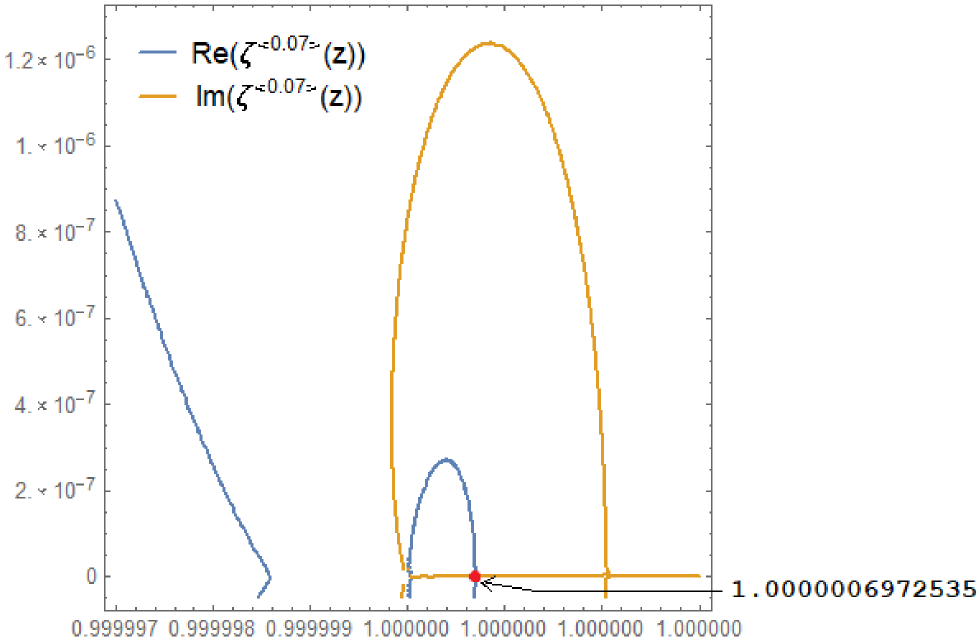
In this section, we investigate the transition of trivial zeros associated with the Super Integral using formula (2.0) .

28.3.1 Transition of the trivial zero $z = -2$ at $\zeta^{<0>}(z) \sim \zeta^{<1>}(z)$

Here, we investigate the transition of the trivial zero $z = -2$ at $\zeta^{<0>}(z) \sim \zeta^{<1>}(z)$. The interval of x is set to $-2.25 \le x \le 2.25$. When the zero is traced with nine contour plots, it is as follows. The blue line is the real part, the orange line is the imaginary part, and the red dot is a zero.



In $\mathbf{0}$, $\zeta(z)$'s trivial zero -2 is drawn. Another intersection 1 is a singular point. This zero point transmigrates to the immediate right of singular point 1 at the moment of $p > 0$. $\mathbf{0.07}$ show the transition.
Although this transmigration also occurs at a very small order $p \approx 0$, the transmigration at $p < 0.07$ is not able to be detected by the formula (2.0) and my personal computer & the software. Incidentally, an enlarged figure of $\mathbf{0.07}$ near 1 is as follows.

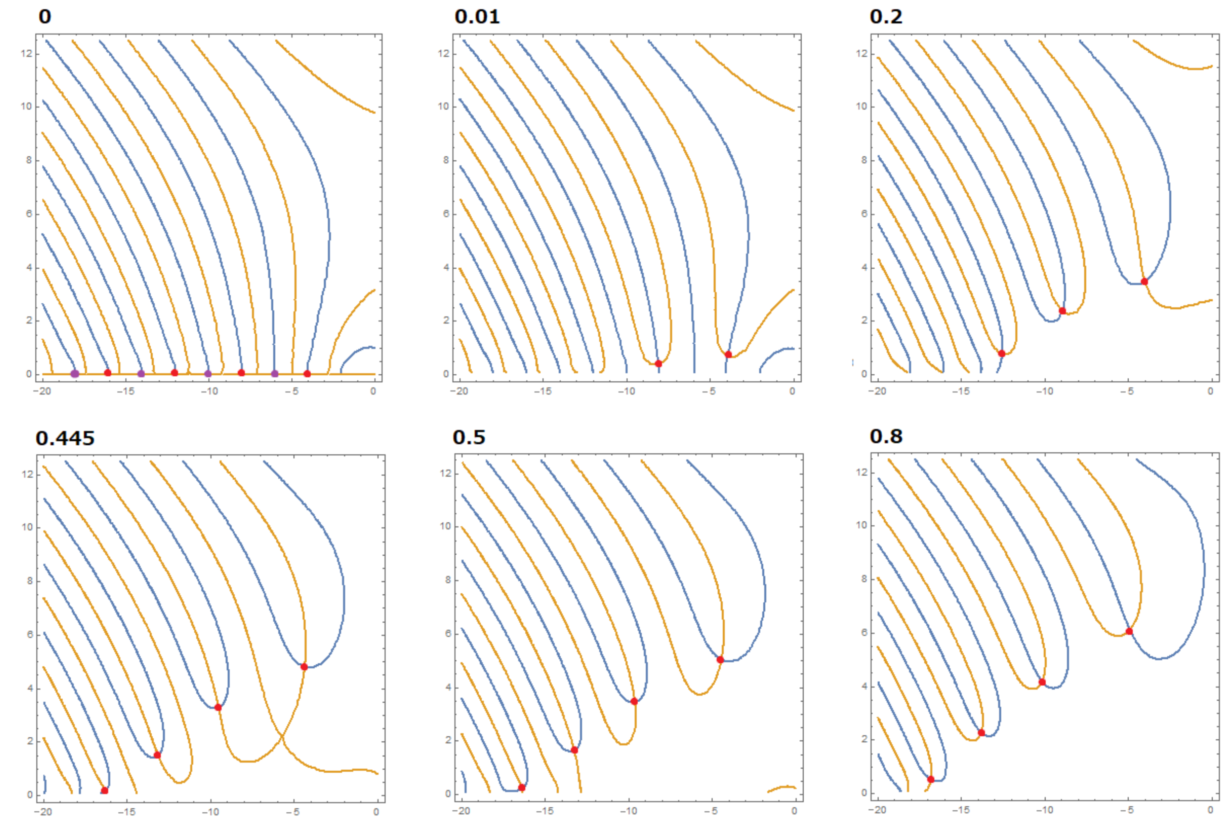


After this, the zero point moves to the right with increasing p , and reaches $1.6690082124782\cdots$ at 1.

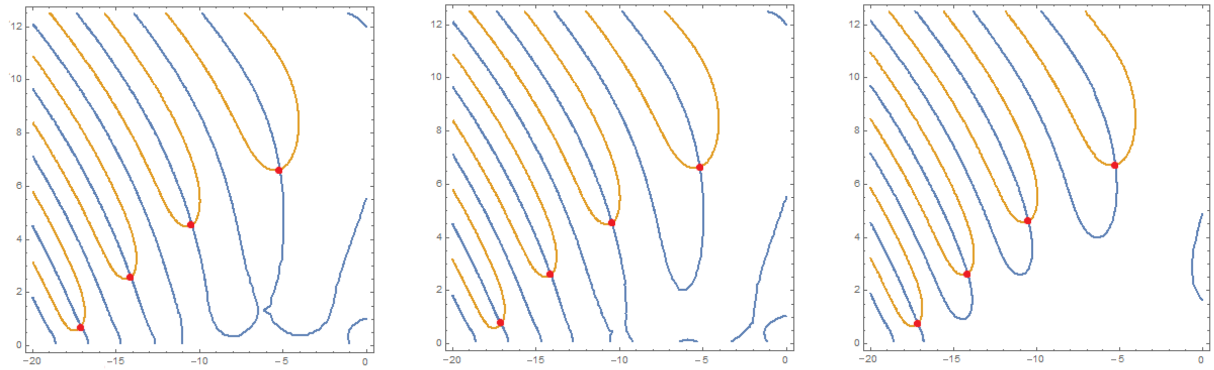
For the above animation, click here Anim2831.gif

28.3.2 Transition of the trivial zeros $z = -4, -6, \cdots$ at $\zeta^{<0>}(z) \sim \zeta^{<1>}(z)$

Here, we investigate the transition of the trivial zeros $z = -4, -6, \cdots$ at $\zeta^{<0>}(z) \sim \zeta^{<0>}(z)$. The interval of x is set to $-20 \le x \le 0$. When the zeros are traced with nine contour plots, it is as follows. The blue line is the real part, and the orange line is the imaginary part. The zeros are drawn with a red dot and a purple dot.



0.9711 0.972 1



In $\mathbf{0}$, $\zeta(z)$'s trivial 8 zeros are drawn. $-4, -8, -12, -16$ are drawn with a red dot and $-6, -10, -14, -18$ are drawn with a purple dot.

Although all zeros disappear at the moment of $p>0$, among them, $-4, -8, -12, -16$ revive in the order near the origin with increasing p, and rises on the complex plane. On the contrary, $-6, -10, -14, -18$ do not revive. $\mathbf{0.01, 0.2, 0.445}$ show the transitions. After $\mathbf{0.5}$, these zeros rise further in the complex plane.

Although partner changes of the contour plots are also observed In $\mathbf{0.445} \sim \mathbf{0.972}$, the four zeros finally reach $\mathbf{1}$. The results of these transitions are numerically shown as follows.

$$
\begin{array}{lcl}
-4 & \Longrightarrow & -5.2076\cdots + i\,6.7222\cdots \\
-8 & \Longrightarrow & -10.4916\cdots + i\,4.6111\cdots \\
-12 & \Longrightarrow & -14.1483\cdots + i\,2.6351\cdots \\
-16 & \Longrightarrow & -17.1410\cdots + i\,0.7711\cdots
\end{array}
$$

For the above animation, click here Anim2832.gif

Summary

Investigating the transitions of $\zeta(z)$'s trivial zeros $-2, -4, -6, \cdots$ associated $\zeta^{<p>}(z)$ $p=0 \sim 1$, we obtained the following results.

(1) $\zeta(z)$'s zero -2 transmigrates to the immediate right of singular point 1 at the moment of $p>0$, and moves to the right on the x axis with increasing p

(2) Although $\zeta(z)$'s trivial zeros $-4, -6, -8, \cdots$ disappear at the moment of $p>0$, among them, zeros $-4, -8, -12, -16$ revive in the order near the origin with increasing p, and move to the upper left on the complex plane. In addition, zeros -18 or less can not revive at $0<p\le 1$.

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2021.01.29 Added animations

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Alien's Mathematics