12 Zeros of Riemann Zeta and System of Infinite Degree Equations

Abstract

(1) The problem of the zeros of Riemann zeta function is reduced to overdetermined system of infinite degree equations, by functional equations.

(2) On the critical line, this system of infinite degree equations has a solution.

(3) Except on the critical line, this system of infinite degree equations is unlikely to have a solution.

12.1 Laurent Series of $\zeta(z)$ and $\zeta(1-z)$

According to Formula 9.3.1 in " 09 Power Series of Riemann Zeta etc by Real & Imaginary Parts " (Dirichler Series) , Laurent series of Riemann Zeta $\zeta(z)$ is as follows.

Formula 12.1.1 (Formula 9.3.1 Reprint)

When the Riemann zeta function is $\zeta(z)$ ($z=x+iy$) and Stieltjes constansts are γ_s $s=0,1,2,\cdots$, the following expressions hold on the whole complex plane except $z=1$.

$$\zeta(z) = \frac{1}{z-1} + \sum_{s=0}^{\infty} \gamma_s \frac{(-1)^s (z-1)^s}{s!} = u(z) + i\,v(z)$$

$$u(x,y) = \frac{x-1}{(x-1)^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \gamma_{2r+s} \frac{(-1)^s (x-1)^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = -\frac{y}{(x-1)^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \gamma_{2r+s+1} \frac{(-1)^s (x-1)^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$

This immediately gives the following formula.

Formula 12.1.2

When Riemann zeta function is $\zeta(1-z)$ ($z=x+iy$) and Stieltjes constansts are γ_s $s=0,1,2,\cdots$, the following expressions hold on the whole complex plane except $z=0$.

$$\zeta(1-z) = -\frac{1}{z} + \sum_{s=0}^{\infty} \gamma_s \frac{z^s}{s!} = u(z) + i\,v(z)$$

$$u(x,y) = -\frac{x}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \gamma_{2r+s} \frac{x^s}{s!} \frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \frac{y}{x^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \gamma_{2r+s+1} \frac{x^s}{s!} \frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$

Replacing z with $1/2+z$ in the above two formulas, we obtain the following formulas.

Formula 12.1.3

When the Riemann zeta function is $\zeta(1/2+z)$ ($z=x+iy$) and Stieltjes constansts are γ_s $s=0,1,2,\cdots$, the following expressions hold on the whole complex plane except $z=1/2$.

$$\zeta\left(\frac{1}{2}+z\right) = -\frac{1}{1/2-z} + \sum_{s=0}^{\infty}\gamma_s\frac{(1/2-z)^s}{s\,!} = u_+(z)+i\,v_+(z)$$

$$u_+(x,y) = -\frac{1/2-x}{(1/2-x)^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\gamma_{2r+s}\frac{(1/2-x)^s}{s\,!}\frac{(-1)^r y^{2r}}{(2r)\,!}$$

$$v_+(x,y) = -\frac{y}{(1/2-x)^2+y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\gamma_{2r+s+1}\frac{(1/2-x)^s}{s\,!}\frac{(-1)^r y^{2r+1}}{(2r+1)\,!}$$

Where, $0^0 = 1$

Formula 12.1.4

When the Riemann zeta function is $\zeta(1/2-z)$ ($z=x+iy$) and Stieltjes constansts are γ_s $s=0,1,2,\cdots$, the following expressions hold on the whole complex plane except $z=-1/2$.

$$\zeta\left(\frac{1}{2}-z\right) = -\frac{1}{1/2+z} + \sum_{s=0}^{\infty}\gamma_s\frac{(1/2+z)^s}{s\,!} = u_-(z)+i\,v_-(z)$$

$$u_-(x,y) = -\frac{1/2+x}{(1/2+x)^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\gamma_{2r+s}\frac{(1/2+x)^s}{s\,!}\frac{(-1)^r y^{2r}}{(2r)\,!}$$

$$v_-(x,y) = \frac{y}{(1/2+x)^2+y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\gamma_{2r+s+1}\frac{(1/2+x)^s}{s\,!}\frac{(-1)^r y^{2r+1}}{(2r+1)\,!}$$

Where, $0^0 = 1$

12.2 Even and Odd Functions

In this section, we rearrange the Riemann zeta functions $\zeta(1/2 \pm z)$ obtained in the previous section into even and odd functions. And, we further expand these into separate series for real and imaginary parts.

Theorem 12.2.0

When the Riemann zeta functions are $\zeta(1/2 \pm z)$ and Stieltjes constansts are γ_s $s=0,1,2,\cdots$, $\zeta(1/2 \pm z) = 0$ if and only if the following system of equations has a solution.

$$\begin{cases} \zeta_e(z) = -\dfrac{1/2}{1/4 - z^2} + \displaystyle\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{\gamma_{2s+t}}{t!}\left(\frac{1}{2}\right)^t \frac{z^{2s}}{(2s)!} = 0 \\ \zeta_o(z) = -\dfrac{z}{1/4 - z^2} - \displaystyle\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{\gamma_{2s+1+t}}{t!}\left(\frac{1}{2}\right)^t \frac{z^{2s+1}}{(2s+1)!} = 0 \end{cases}$$

Where, $0^0 = 1$

Proof

The following two formulas were obtained in the previous section.

$$\zeta\left(\frac{1}{2}+z\right) = -\frac{1}{1/2 - z} + \sum_{s=0}^{\infty} \gamma_s \frac{(1/2 - z)^s}{s!} \qquad (=: \zeta_+(z))$$

$$\zeta\left(\frac{1}{2}-z\right) = -\frac{1}{1/2 + z} + \sum_{s=0}^{\infty} \gamma_s \frac{(1/2 + z)^s}{s!} \qquad (=: \zeta_-(z))$$

By functional equation, $\zeta(z) = \zeta(1-z) = 0$ on the zeros of $\zeta(z)$.

Therfore, on zeros of $\zeta(1/2+z)$, the following has to hold.

$$\zeta\left(\frac{1}{2}+z\right) = \zeta\left(\frac{1}{2}-z\right) = 0$$

Expressed as series,

$$\begin{cases} -\dfrac{1}{1/2 - z} + \displaystyle\sum_{s=0}^{\infty} \frac{\gamma_s}{s!}\left(\frac{1}{2}-z\right)^s = 0 \\ -\dfrac{1}{1/2 + z} + \displaystyle\sum_{s=0}^{\infty} \frac{\gamma_s}{s!}\left(\frac{1}{2}+z\right)^s = 0 \end{cases}$$

Here,

$$(a+b)^s = \sum_{t=0}^{s} \binom{s}{t} a^{s-t} b^t$$

Using this,

$$\left(\frac{1}{2}-z\right)^s = \sum_{t=0}^{s} (-1)^t \binom{s}{t} \left(\frac{1}{2}\right)^{s-t} z^t$$

$$\left(\frac{1}{2}+z\right)^s = \sum_{t=0}^{s} \binom{s}{t} \left(\frac{1}{2}\right)^{s-t} z^t$$

Substituting these for the above,

$$-\frac{1}{1/2-z}+\sum_{s=0}^{\infty}\frac{\gamma_s}{s!}\sum_{t=0}^{s}(-1)^t\binom{s}{t}\left(\frac{1}{2}\right)^{s-t}z^t=0 \tag{2.1}$$

$$-\frac{1}{1/2+z}+\sum_{s=0}^{\infty}\frac{\gamma_s}{s!}\sum_{t=0}^{s}\binom{s}{t}\left(\frac{1}{2}\right)^{s-t}z^t=0 \tag{2.2}$$

(2.1) + (2.2) is

$$\frac{1}{1/2-z}+\frac{1}{1/2+z}=\frac{1/2+z+1/2-z}{(1/2-z)(1/2+z)}=\frac{1}{1/4-z^2}$$

$$\sum_{t=0}^{s}(-1)^t\binom{s}{t}\left(\frac{1}{2}\right)^{s-t}z^t+\sum_{t=0}^{s}\binom{s}{t}\left(\frac{1}{2}\right)^{s-t}z^t=2\sum_{t=0}^{s}\binom{s}{2t}\left(\frac{1}{2}\right)^{s-2t}z^{2t}$$

So,

$$-\frac{1}{1/4-z^2}+2\sum_{s=0}^{\infty}\frac{\gamma_s}{s!}\sum_{t=0}^{s}\binom{s}{2t}\left(\frac{1}{2}\right)^{s-2t}z^{2t}=0$$

(2.1) - (2.2) is

$$\frac{1}{1/2-z}-\frac{1}{1/2+z}=\frac{1/2+z-(1/2-z)}{(1/2-z)(1/2+z)}=\frac{2z}{1/4-z^2}$$

$$\sum_{t=0}^{s}(-1)^t\binom{s}{t}\left(\frac{1}{2}\right)^{s-t}z^t-\sum_{t=0}^{s}\binom{s}{t}\left(\frac{1}{2}\right)^{s-t}z^t=-2\sum_{t=0}^{s}\binom{s}{2t+1}\left(\frac{1}{2}\right)^{s-2t-1}z^{2t+1}$$

So,

$$-\frac{2z}{1/4-z^2}-2\sum_{s=0}^{\infty}\frac{\gamma_s}{s!}\sum_{t=0}^{s}\binom{s}{2t+1}\left(\frac{1}{2}\right)^{s-2t-1}z^{2t+1}=0$$

That is,

$$\zeta_e(z)=-\frac{1/2}{1/4-z^2}+\sum_{s=0}^{\infty}\frac{\gamma_s}{s!}\sum_{t=0}^{s}\binom{s}{2t}\left(\frac{1}{2}\right)^{s-2t}z^{2t}=0$$

$$\zeta_o(z)=-\frac{z}{1/4-z^2}-\sum_{s=0}^{\infty}\frac{\gamma_s}{s!}\sum_{t=0}^{s}\binom{s}{2t+1}\left(\frac{1}{2}\right)^{s-2t-1}z^{2t+1}=0$$

Both are correct, but somewhat inefficient. Therefore, after rearranging these, s and t are exchanged as follows.

$$\zeta_e(z)=-\frac{1/2}{1/4-z^2}+\sum_{s=0}^{\infty}\sum_{t=2s}^{\infty}\frac{\gamma_t}{t!}\binom{t}{2s}\left(\frac{1}{2}\right)^{t-2s}z^{2s}=0$$

$$\zeta_o(z)=-\frac{z}{1/4-z^2}-\sum_{s=0}^{\infty}\sum_{t=2s+1}^{\infty}\frac{\gamma_t}{t!}\binom{t}{2s+1}\left(\frac{1}{2}\right)^{t-2s-1}z^{2s+1}=0$$

Here,

$$\sum_{t=s}^{\infty}\frac{\gamma_t}{t!}\binom{t}{s}\left(\frac{1}{2}\right)^{t-s}=\frac{1}{s!}\sum_{t=s}^{\infty}\frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s}=\frac{1}{s!}\sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^{t}$$

applying this for the above,

$$\zeta_e(z) = -\frac{1/2}{1/4 - z^2} + \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{\gamma_{2s+t}}{t!}\left(\frac{1}{2}\right)^t \frac{z^{2s}}{(2s)!} = 0$$

$$\zeta_o(z) = -\frac{z}{1/4 - z^2} - \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{\gamma_{2s+1+t}}{t!}\left(\frac{1}{2}\right)^t \frac{z^{2s+1}}{(2s+1)!} = 0$$

Conversely, if $\zeta_e(z)=\zeta_o(z)=0$, it will result in $\zeta(1/2+z)=\zeta(1/2-z)=0$ by tracing the above inverse. Q.E.D.

Function $\zeta_e(z)$, $\zeta_o(z)$ are further expanded into series by real and imaginary parts.

Formula 12.2.1 (Even function)

$$\zeta_e(z) = -\frac{1/2}{1/4 - z^2} + \sum_{s=0}^{\infty} f^{(2s)}(0)\frac{z^{2s}}{(2s)!}$$

$$u_e(x,y) = -\frac{1/4 - x^2 + y^2}{2\left\{\left(1/4 - x^2 + y^2\right)^2 + 4x^2y^2\right\}} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v_e(x,y) = -\frac{xy}{\left(1/4 - x^2 + y^2\right)^2 + 4x^2y^2} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where,

$$f^{(s)}(0) = \sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t \quad \left\{ = \sum_{t=s}^{\infty}\frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s}\right\}, \qquad 0^0 = 1$$

Proof

From Theorem 12.2.0 ,

$$\zeta_e(z) = -\frac{1/2}{1/4 - z^2} + \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{\gamma_{2s+t}}{t!}\left(\frac{1}{2}\right)^t \frac{z^{2s}}{(2s)!}$$

Here, let

$$f^{(s)}(0) := \sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t$$

Then,

$$f^{(2s)}(0) = \sum_{t=0}^{\infty}\frac{\gamma_{2s+t}}{t!}\left(\frac{1}{2}\right)^t$$

So,

$$\zeta_e(z) = -\frac{1/2}{1/4 - z^2} + \sum_{s=0}^{\infty} f^{(2s)}(0)\frac{z^{2s}}{(2s)!}$$

According to Formula 14.1.2" in " 14 Taylor Expansion by Real Part & Imaginary Part " (Alacarte) ,

$$f(z) = \sum_{s=0}^{\infty} f^{(2s)}(0)\frac{z^{2s}}{(2s)!} = u(x,y) + i\,v(x,y)$$

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where, $0^0 = 1$

Separating the first term of $\zeta_e(z)$ by real and imaginary parts and applying this formula to the second term, we obtain the desired expressions.

Formula 12.2.2 (Odd function)

$$\zeta_o(z) = -\frac{z}{1/4 - z^2} - \sum_{s=0}^{\infty} f^{(2s+1)}(0)\frac{z^{2s+1}}{(2s+1)!}$$

$$u_o(x,y) = -\frac{x\left(1/4 - x^2 - y^2\right)}{\left(1/4 - x^2 + y^2\right)^2 + 4x^2y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

$$v_o(x,y) = -\frac{y\left(1/4 + x^2 + y^2\right)}{\left(1/4 - x^2 + y^2\right)^2 + 4x^2y^2} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

Where,

$$f^{(s)}(0) = \sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t \quad \left\{ = \sum_{t=s}^{\infty}\frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s} \right\} , \quad 0^0 = 1$$

Proof

By Theorem 12.2.0 & Formula 14.1.2" in " 14 Taylor Expansion by Real Part & Imaginary Part " (Alacarte), it is proved in a similar way to Formula 12.2.1 .

Note

$$f^{(s)}(0) = \sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t \quad \left\{ = \sum_{t=s}^{\infty}\frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s} \right\}$$

In this chapter, { } is used for drawing. Because the drawing speed is $6 \sim 9$ times different in *Mathematica*. (cause unknown)

12.3 Necessary and Sufficient Condition for Zeros

Theorem 12.3.1

When the Riemann zeta functions are $\zeta(1/2\pm z)$ and Stieltjes constanst are γ_s $s=0,1,2,\cdots$, $\zeta(1/2\pm z)=0$ if and only if the following system of equations has a solution.

$$\begin{cases} u_e = -\dfrac{1/4-x^2+y^2}{2\left\{\left(1/4-x^2+y^2\right)^2+4x^2y^2\right\}} + \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_e = -\dfrac{xy}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} + \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \\ u_o = -\dfrac{x\left(1/4-x^2-y^2\right)}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} - \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_o = -\dfrac{y\left(1/4+x^2+y^2\right)}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} - \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \end{cases}$$

Where,

$$f^{(s)}(0) = \sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t \quad \left\{ = \sum_{t=s}^{\infty}\frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s} \right\}, \qquad 0^0=1$$

Proof

Representing Theorem 12.2.0 with Formula 12.2.1 and Formula 12.2.2, we obtain the desired expressions.

Overdetermined System

Since there are four equations for two variables, this system of equations is an overdetermined system. Such a system of equations generally has no solution.

Zeros on the Critical Line

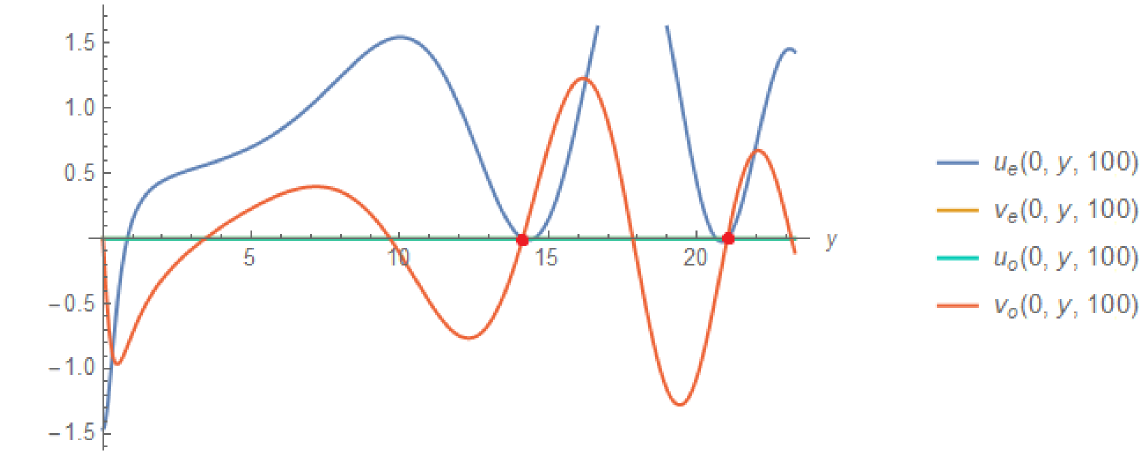
However, such a system of equations may exceptionally has solution. That is the case when $x=0$. Note that $x=0$ is the critical line of function $\zeta(1/2+z)$. Substituting $x=0$ for the above,

$$\begin{cases} u_e = -\dfrac{1/2}{1/4+y^2} + \displaystyle\sum_{r=0}^{\infty} f^{(2r)}(0)\frac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_e = 0+0=0 \\ u_o = 0-0=0 \\ v_o = -\dfrac{y}{1/4+y^2} - \displaystyle\sum_{r=0}^{\infty} f^{(2r+1)}(0)\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \end{cases}$$

Where, $$f^{(s)}(0) = \sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t , \qquad 0^0=1$$

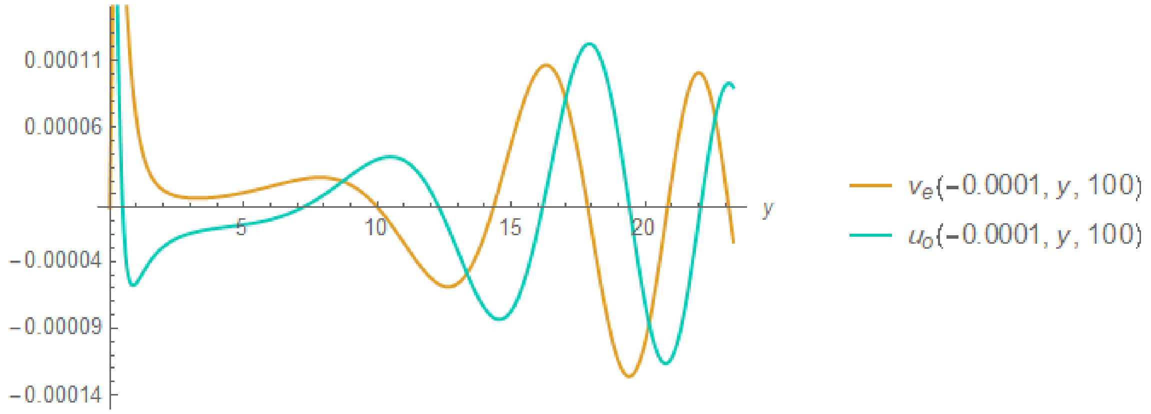
v_e, u_o are each become 0 and the property of overdetermination disappears. Substituting $f^{(2r)}, f^{(2r+1)}$ for v_e, u_o and calculating, these result in $u_+(0,y), v_+(0,y)$ in Formula 12.1.3, respectively. Therefore, this system of equations has a solution.

When $x=0$, $u_e \sim v_o$ are drawn as follows. Blue is u_e and red tea is v_o. The point (red) where these intersect on the y -axis is the zero point of $\zeta(1/2 \pm z)$. Orange is v_e and light blue is u_o. They overlap on the y -axis. Of course, these two straight lines also pass through the red points.

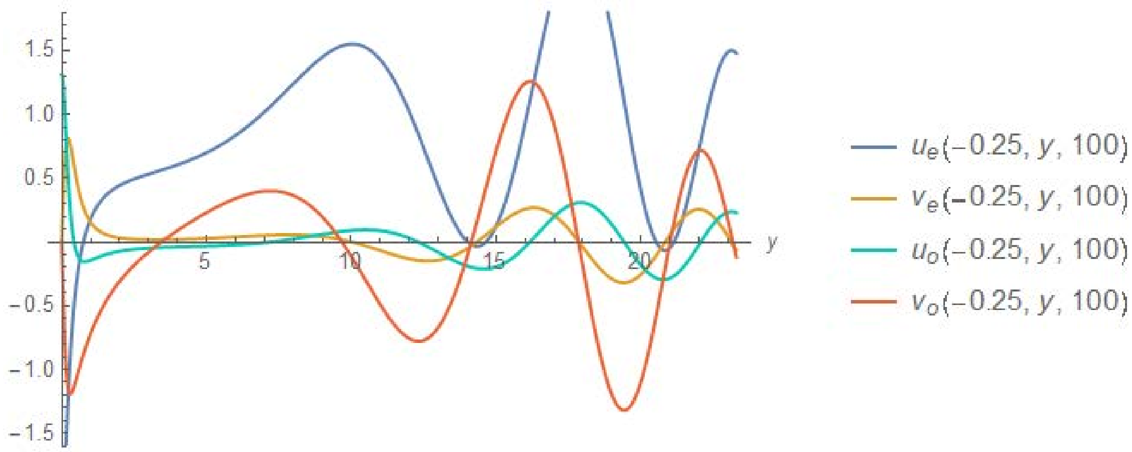


Zeros outside the Critical Line

If x deviates even slightly from 0 , v_e, u_o cease to be straight lines. For example, when $x=-0.0001$,



As the result, the property of overdetermination is restored. For example, when $x=-0.25$, $u_e \sim v_o$ are drawn as follows. It seems unlikely that the four curves would intersect at one point on the y -axis.



Therefore, we can present the following hypothesis, which is equivalent to the Riemann hypothesis.

Hypothesis 12.3.2

When $\gamma_s \; s=0,1,2,\cdots$ are Stieltjes constansts and x, y are real numbers, The following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} u_e = -\dfrac{1/4-x^2+y^2}{2\left\{\left(1/4-x^2+y^2\right)^2+4x^2y^2\right\}} + \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_e = -\dfrac{xy}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} + \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \\ u_o = -\dfrac{x\left(1/4-x^2-y^2\right)}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} - \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_o = -\dfrac{y\left(1/4+x^2+y^2\right)}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} - \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \end{cases}$$

Where, $f^{(s)}(0) = \displaystyle\sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t \quad \left\{ = \sum_{t=s}^{\infty}\frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s} \right\}, \qquad 0^0 = 1$

12.4 Necessary Condition for Zeros

The system of equations in Hypothesis12.3.2 is equivalent to the following six sets of system of equations. Each system of equations is necessary condition for zeros.

$$\begin{cases} u_e = 0 \\ v_e = 0 \end{cases}, \quad \begin{cases} u_e = 0 \\ u_o = 0 \end{cases}, \quad \begin{cases} u_e = 0 \\ v_o = 0 \end{cases}, \quad \begin{cases} v_e = 0 \\ u_o = 0 \end{cases}, \quad \begin{cases} v_e = 0 \\ v_o = 0 \end{cases}, \quad \begin{cases} v_o = 0 \\ u_o = 0 \end{cases}$$

Where, $f^{(s)}(0) = \sum_{t=0}^{\infty} \frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t \quad \left\{ = \sum_{t=s}^{\infty} \frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s} \right\}, \quad 0^0 = 1$

In order to prove Hypothesis12.3.2, it is sufficient to show that any one of these pairs does not have a solution such as $x \neq 0$. As a result of considering the partial derivatives of u_e ,v_e, u_o, v_o with respect to y , we present two system of equations that are almost equivalent to the Riemann hypothesis.

Hypothesis 12.4.1

When γ_s $s=0,1,2,\cdots$ are Stieltjes constansts and x , y are real numbers, the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} v_o = -\dfrac{y\left(1/4+x^2+y^2\right)}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} - \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \\ u_o = -\dfrac{x\left(1/4-x^2-y^2\right)}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} - \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r}}{(2r)!} = 0 \end{cases}$$

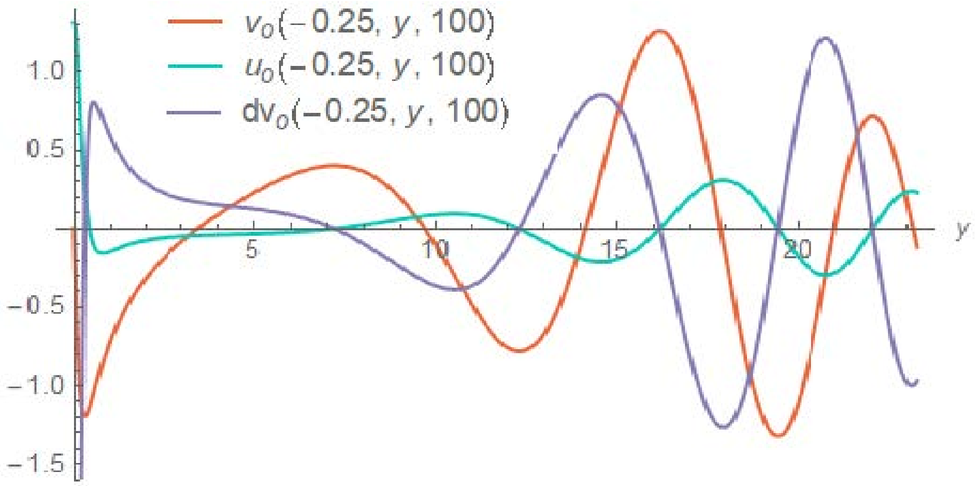
Where, $f^{(s)}(0) = \sum_{t=0}^{\infty} \frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t \quad \left\{ = \sum_{t=s}^{\infty} \frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s} \right\}, \quad 0^0 = 1$

Remark

The partial derivatives of v_o with respect to y is

$$\frac{\partial v_o}{\partial y} = \frac{y\left(1/4+x^2+y^2\right)\left\{4y\left(1/4-x^2+y^2\right)+8x^2y\right\}}{\left\{\left(1/4-x^2+y^2\right)^2+4x^2y^2\right\}^2} - \frac{1/4-x^2+3y^2}{\left(1/4-x^2+y^2\right)^2+4x^2y^2}$$
$$- \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

When $x = -0.25$, these are drawn as follows. Red tea is v_o , light blue is u_o and purple is $\partial v_o / \partial y$.



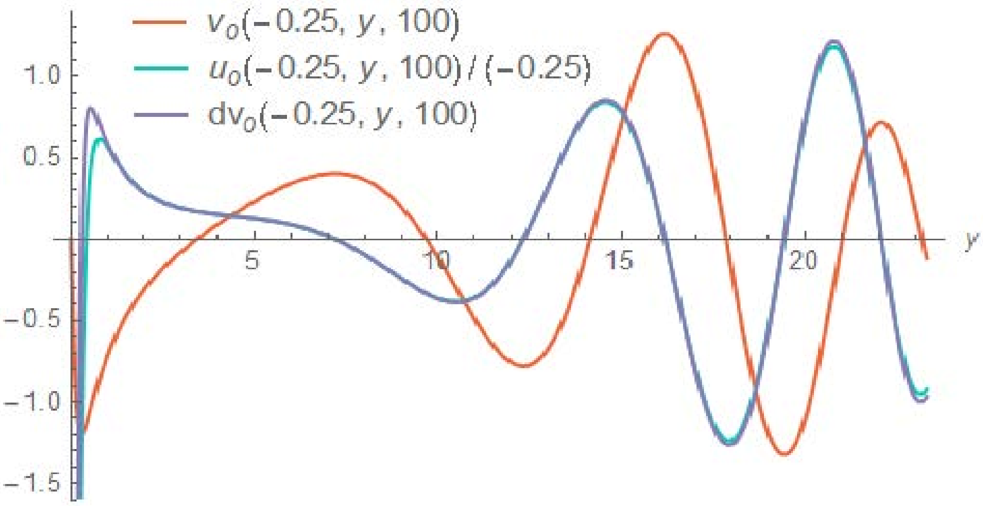
Zeros of $\partial v_o/\partial y$ (purple) correspond to peaks or bottoms of v_o (red tea) . And we can be seen that the zeros of u_o (light blue) and the zeros of $\partial v_o/\partial y$ (purple) are quite close. That is, v_o (red) and u_o (light blue) are unlikely to share zeros .

To observe this in more detail, let us consider the following system of equations, which is equivalent to the above at $x \neq 0$ ..

$$\begin{cases} v_o = -\dfrac{y\left(1/4+x^2+y^2\right)}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} - \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \\ \dfrac{u_o}{x} = -\dfrac{1/4-x^2-y^2}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} - \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{f^{(2r+2s+1)}(0)}{2s+1}\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!} = 0 \end{cases}$$

$$\frac{\partial v_o}{\partial y} = \frac{y\left(1/4+x^2+y^2\right)\left\{4y\left(1/4-x^2+y^2\right)+8x^2y\right\}}{\left\{\left(1/4-x^2+y^2\right)^2+4x^2y^2\right\}^2} - \frac{1/4-x^2+3y^2}{\left(1/4-x^2+y^2\right)^2+4x^2y^2}$$
$$- \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

When $x=-0.25$, these are drawn as follows. Red tea is v_o , light blue is u_o/x and purple is $\partial v_o/\partial y$.



Zeros of $\partial v_o/\partial y$ (purple) correspond to peaks or bottoms of v_o (red tea) . And u_o/x (light blue) and $\partial v_o/\partial y$ (purple) overlap, and u_o/x (light blue) is almost invisible. Actually, the zeros points of u_o/x (light blue) and $\partial v_o/\partial y$ (purple) and their difference are as follows. In addition, zeros near 22 may be inaccurate due to the underflow.

v_o/x	7.18500	12.3133	16.1699	19.4249	22.0703
$\partial v_o/\partial y$	7.19453	12.3223	16.1755	19.4255	22.0685
difference	-0.00953	-0.0090	-0.0056	-0.0006	0.0018

We do not have to worry about the complexity of the fractional functions of v_o , u_o/x , $\partial v_0/\partial y$. Because, they approach 0 as $y \to \infty$. So, if we can prove that the zeros of the following two series are close when $y \to \infty$, then we can say that v_o and u_o do not have common zeros. .

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{f^{(2r+2s+1)}(0)}{2s+1}\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!} \quad , \quad \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+1)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!}$$

These double series differ only in their coefficients. Moreover, the differential coefficients $f^{(2r+2s+1)}(0)$ are the same. This means that there is no effect of irregular fluctuations in the Stiltjes constants when comparing both double series. This is the reason why this system of equations was chosen. An analytical comparison of these zeros does not seem impossible.

Hypothesis 12.4.2

When $\gamma_s \; s=0,1,2,\cdots$ are Stieltjes constansts and x, y are real numbers,
the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} u_e = -\dfrac{1/4-x^2+y^2}{2\left\{\left(1/4-x^2+y^2\right)^2+4x^2y^2\right\}} + \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r}}{(2r)!} = 0 \\ v_e = -\dfrac{xy}{\left(1/4-x^2+y^2\right)^2+4x^2y^2} + \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s+1}}{(2s+1)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!} = 0 \end{cases}$$

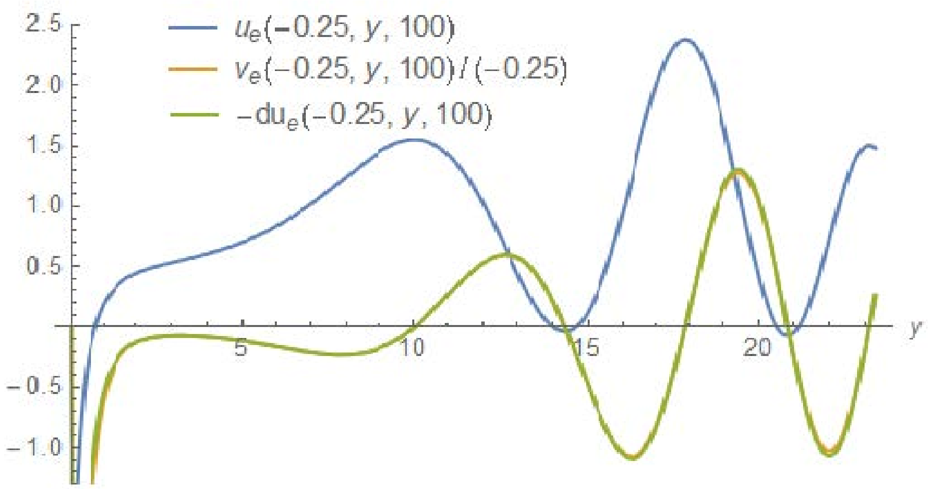
Where, $f^{(s)}(0) = \displaystyle\sum_{t=0}^{\infty}\frac{\gamma_{s+t}}{t!}\left(\frac{1}{2}\right)^t \quad \left\{ = \sum_{t=s}^{\infty}\frac{\gamma_t}{(t-s)!}\left(\frac{1}{2}\right)^{t-s}\right\}, \quad 0^0=1$

Remark

The partial derivatives of u_e with respect to y is

$$\frac{\partial u_e}{\partial y} = \frac{\left(1/4-x^2+y^2\right)\left\{4y\left(1/4-x^2+y^2\right)+8x^2y\right\}}{2\left\{\left(1/4-x^2+y^2\right)^2+4x^2y^2\right\}^2} - \frac{y}{\left(1/4-x^2+y^2\right)^2+4x^2y^2}$$
$$- \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

And, when $x=-0.25$, u_e, v_e/x, $-\partial u_e/\partial y$ are drawn as follows.



v_e/x (orange) and $-\partial u_e/\partial y$ (green) overlap, and v_e/x (orange) is invisible. Also in this case, if we can prove that the zeros of the following two series are close when $y \to \infty$, then we can say that u_e and v_e do not have common zeros..

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{f^{(2r+2s+2)}(0)}{2s+1}\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}, \quad \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f^{(2r+2s+2)}(0)\frac{x^{2s}}{(2s)!}\frac{(-1)^r y^{2r+1}}{(2r+1)!}$$

The differential coefficients $f^{(2r+2s+2)}(0)$ are the same in both series. So, there is no effect of irregular fluctuations in the Stiltjes constants on the comparison of both double series. Then, an analytical comparison of these zeros is also likely.

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Alien's Mathematics