1 Zeta Generating Functions

Both of hyperbolic functions and trigonometric functions can be expanded to Fourier series and Taylor series. And if the termwise higher order integration of these is carried out, Riemann Zeta at natural number can be obtained. Where, these are automorphisms which are expressed by lower zetas. However, in this chapter, we stop those so far .

The work that obtain the non-automorphism formulas by removing lower zetas from these are performed after **chapter 2** .

On the other hand, if the termwise higher order differentiation of these is carried out, we obtain the negative integer Zeta. these are the non-automorphism formulas.

1.1 Generating function of coth x family

The idea of obtaining Zeta by integrating the following expression is the most natural.

$$\frac{1}{e^x-1} = e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + \cdots \qquad \left(= \frac{1}{2}coth\frac{x}{2} - \frac{1}{2} \right)$$

1.1.1 Termwise Higher Integral of Fourier Series of $1/(e^x-1)$

Lemma 1.1.1

When $\zeta(x) = \frac{1}{r^x}$, the following expressions hold for $x>0$.

$$\int_0^x \frac{1}{e^x-1}dx = -\sum_{r=1}^{\infty}\frac{e^{-rx}}{r^1} + \frac{x^0}{0!}\zeta(1) \tag{1.1}$$

$$\int_0^x\int_0^x \frac{1}{e^x-1}dx^2 = \sum_{r=1}^{\infty}\frac{e^{-rx}}{r^2} - \frac{x^0}{0!}\zeta(2) + \frac{x^1}{1!}\zeta(1) \tag{1.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{1}{e^x-1}dx^3 = -\sum_{r=1}^{\infty}\frac{e^{-rx}}{r^2} + \frac{x^0}{0!}\zeta(3) - \frac{x^1}{1!}\zeta(2) + \frac{x^2}{2!}\zeta(1) \tag{1.3}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \frac{1}{e^x-1}dx^n = (-1)^n\sum_{r=1}^{\infty}\frac{e^{-rx}}{r^n} - \sum_{s=0}^{n-1}(-1)^{n-s}\frac{x^s}{s!}\zeta(n-s) \tag{1.n}$$

Proof

$1/(e^x-1)$ can be expanded to Fourier series as follows.

$$\frac{1}{e^x-1} = \frac{e^{-x}}{1-e^{-x}} = e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + \cdots$$

$$= cos1ix + cos2ix + cos3ix + cos4ix + \cdots$$
$$- i(sin1ix + sin2ix + sin3ix + sin4ix + \cdots)$$

Integrating both sides of this with respect to x from 0 to x ,

$$\int_0^x \frac{1}{e^x-1}dx = \left[-\left(\frac{e^{-x}}{1^1} + \frac{e^{-2x}}{2^1} + \frac{e^{-3x}}{3^1} + \frac{e^{-4x}}{4^1} + \cdots \right) \right]_0^x$$

$$= -\left(\frac{e^{-x}}{1^1} + \frac{e^{-2x}}{2^1} + \frac{e^{-3x}}{3^1} + \frac{e^{-4x}}{4^1} + \cdots \right) + \frac{e^0}{1^1} + \frac{e^0}{2^1} + \frac{e^0}{3^1} + \frac{e^0}{4^1} + \cdots$$

Here, let $\zeta(x) = \dfrac{1}{r^x}$, then

$$\int_0^x \frac{1}{e^x-1}dx = -\sum_{r=1}^{\infty}\frac{e^{-rx}}{r^1} + \frac{x^0}{0!}\zeta(1)$$

Next, integrating both sides of this with respect to x from 0 to x ,

$$\int_0^x\int_0^x \frac{1}{e^x-1}dx^2 = \left[\sum_{r=1}^{\infty}\frac{e^{-rx}}{r^2}\right]_0^x + \frac{x^1}{1!}\zeta(1)$$

$$= \sum_{r=1}^{\infty}\frac{e^{-rx}}{r^2} - \sum_{r=1}^{\infty}\frac{e^{r0}}{r^2} + \frac{x^1}{1!}\zeta(1)$$

$$= \sum_{r=1}^{\infty}\frac{e^{-rx}}{r^2} - \frac{x^0}{0!}\zeta(2) + \frac{x^1}{1!}\zeta(1)$$

Hereafter, in a similar way we obtein the desired expressions. In addition, these are the collateral integrals.

1.1.2 Termwise Higher Integral of Taylor Series of $1/(e^x-1)$

Lemma 1.1.2

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and $H_s = \sum_{t=1}^{s} 1/t$ are Harmonic numbers, the following expressions hold for $0<x<2\pi$.

$$\int_0^x \frac{1}{e^x-1}dx = \log x - \frac{1}{2}\frac{x^1}{1!} + \sum_{r=1}^{\infty}\frac{B_{2r}x^{2r}}{2r(2r)!} + \frac{x^0}{0!}\zeta(1) \tag{2.1}$$

$$\int_0^x\int_0^x \frac{1}{e^x-1}dx^2 = \frac{x^1}{1!}(\log x - H_1) - \frac{1}{2}\frac{x^2}{2!} + \sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+1}}{2r(2r+1)!} + \frac{x^1}{1!}\zeta(1) \tag{2.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{1}{e^x-1}dx^3 = \frac{x^2}{2!}(\log x - H_2) - \frac{1}{2}\frac{x^3}{3!} + \sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+2}}{2r(2r+2)!} + \frac{x^2}{2!}\zeta(1) \tag{2.3}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \frac{1}{e^x-1}dx^n = \frac{x^{n-1}}{(n-1)!}(\log x - H_{n-1}) - \frac{x^n}{2n!} + \sum_{r=1}^{\infty}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!} + \frac{x^{n-1}}{(n-1)!}\zeta(1) \tag{2.n}$$

Proof

$$x\coth x = 1 + \sum_{r=1}^{\infty}\frac{2^{2r}B_{2r}}{(2r)!}x^{2r} \qquad |x| < \pi$$

From this,

$$\frac{1}{2}\coth\frac{x}{2} = \frac{1}{x} + \frac{1}{2}\sum_{r=1}^{\infty}\frac{2^{2r}B_{2r}}{2r(2r-1)!}\left(\frac{x}{2}\right)^{2r-1} = \frac{1}{x} + \sum_{r=1}^{\infty}\frac{B_{2r}}{2r(2r-1)!}x^{2r-1}$$

Since

$$\frac{1}{e^x-1} = \frac{1}{2}\coth\frac{x}{2} - \frac{1}{2}$$

Then

$$\frac{1}{e^x-1} = \frac{1}{x} - \frac{1}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{2r(2r-1)!} x^{2r-1} \qquad |x| < 2\pi$$

Integrating both sides of this with respect to x from 0 to x,

$$\int_0^x \frac{1}{e^x-1} dx = \left[\log x - \frac{x}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{2r(2r)!} x^{2r} \right]_0^x$$

$$= \log x - \frac{x^1}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{2r(2r)!} x^{2r} - \log 0$$

Here, let $-\log 0 = \zeta(1)$.

Because,

$$\log(1-x) = -\sum_{r=1}^{\infty} \frac{x^r}{r} \qquad -1 \leqq x < 1$$

Disregarding the convergence condition, let $x \to 1$. Then $\log 0 = -\zeta(1)$.

Thus, using this, we obtain

$$\int_0^x \frac{1}{e^x-1} dx = \log x - \frac{1}{2} \frac{x^1}{1!} + \sum_{r=1}^{\infty} \frac{B_{2r} x^{2r}}{2r(2r)!} + \frac{x^0}{0!} \zeta(1)$$

Next, integrating both sides of this with respect to x from 0 to x,

$$\int_0^x \int_0^x \frac{1}{e^x-1} dx^2 = \left[\frac{x^1}{1!} (\log x - H_1) - \frac{1}{2} \frac{x^2}{2!} + \sum_{r=1}^{\infty} \frac{B_{2r} x^{2r+1}}{2r(2r+1)!} \right]_0^x + \frac{x}{1!} \zeta(1)$$

$$= \frac{x^1}{1!} (\log x - H_1) - \frac{1}{2} \frac{x^2}{2!} + \sum_{r=1}^{\infty} \frac{B_{2r} x^{2r+1}}{2r(2r+1)!} + \frac{x}{1!} \zeta(1)$$

Hereafter, in a similar way we obtein the desired expressions.

1.1.3 Riemann Zeta

Two lemmas mentioned above are not useful in itself. However, comparing these, we can obtain the following Riemann Zeta.

Formula 1.1.3

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42, \cdots$, are Bernoulli numbers and $H_s = \sum_{t=1}^{s} 1/t$ are Harmonic numbers, the following expressions hold for $0 < x < 2\pi$.

$$0 = \log x - \frac{1}{2} \frac{x^1}{1!} + \sum_{r=1}^{\infty} \frac{e^{-rx}}{r^1} + \sum_{r=1}^{\infty} \frac{B_{2r}}{2r(2r)!} x^{2r}$$

$$\zeta(2) = -\frac{x^1}{1!} (\log x - H_1) + \frac{1}{2} \frac{x^2}{2!} + \sum_{r=1}^{\infty} \frac{e^{-rx}}{r^2} - \sum_{r=1}^{\infty} \frac{B_{2r} x^{2r+1}}{2r(2r+1)!}$$

$$\zeta(3) = \frac{x^2}{2!} (\log x - H_2) - \frac{1}{2} \frac{x^3}{3!} + \sum_{r=1}^{\infty} \frac{e^{-rx}}{r^2} + \sum_{r=1}^{\infty} \frac{B_{2r} x^{2r+2}}{2r(2r+2)!} + \frac{x^1}{1!} \zeta(2)$$

$$\zeta(4) = -\frac{x^3}{3!} (\log x - H_3) + \frac{1}{2} \frac{x^4}{4!} + \sum_{r=1}^{\infty} \frac{e^{-rx}}{r^4} - \sum_{r=1}^{\infty} \frac{B_{2r} x^{2r+3}}{2r(2r+3)!} - \frac{x^2}{2!} \zeta(2) + \frac{x^1}{1!} + \zeta(3)$$

$$\vdots$$

$$\zeta(n) = \frac{(-x)^{n-1}}{(n-1)!}\left(\log x - H_{n-1}\right) + \frac{(-1)^n}{2}\frac{x^n}{n!} + \sum_{r=1}^{\infty}\frac{e^{-rx}}{r^n}$$

$$-(-1)^n\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+n-1}}{2r(2r+n-1)!} - \sum_{s=1}^{n-2}\frac{(-1)^s x^s}{s!}\zeta(n-s)$$

Proof

Comparing (1.1) and (2.1), we obtain

$$0 = \log x - \frac{1}{2}\frac{x^1}{1!} + \sum_{r=1}^{\infty}\frac{e^{-rx}}{r^1} + \sum_{r=1}^{\infty}\frac{B_{2r}}{2r(2r)!}x^{2r}$$

Comparing (1.2) and (2.2) , we obtain

$$\zeta(2) = -\frac{x^1}{1!}\left(\log x - H_1\right) + \frac{1}{2}\frac{x^2}{2!} + \sum_{r=1}^{\infty}\frac{e^{-rx}}{r^2} - \sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+1}}{2r(2r+1)!}$$

Comparing (1.3) and (2.3) , we obtain

$$\zeta(3) = \frac{x^2}{2!}\left(\log x - H_2\right) - \frac{1}{2}\frac{x^3}{3!} + \sum_{r=1}^{\infty}\frac{e^{-rx}}{r^2} + \sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+2}}{2r(2r+2)!} + \frac{x^1}{1!}\zeta(2)$$

Hereafter, in a similar way we obtein the desired expressions.

1.1.4 Lineal Higher Interal of $1/(e^x-1)$

Mentioned above was a result of the collateral higher integral of $1/(e^x-1)$. If this is integrated in the lineal line, it becomes the following.

First of all,

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x}\frac{1}{e^x-1}dx^n = (-1)^n\sum_{r=1}^{\infty}\frac{e^{-rx}}{r^n}$$

According to Cauchy formula for repeated integration ,

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x}\frac{1}{e^x-1}dx^n = \frac{1}{\Gamma(n)}\int_{\infty}^{x}\frac{(x-t)^{n-1}}{e^t-1}dt$$

From these

$$(-1)^n\sum_{r=1}^{\infty}\frac{e^{-rx}}{r^n} = \frac{1}{\Gamma(n)}\int_{\infty}^{x}\frac{(x-t)^{n-1}}{e^t-1}dt$$

Giving this $x=0$,

$$(-1)^n\zeta(n) = \frac{1}{\Gamma(n)}\int_{\infty}^{0}\frac{(-t)^{n-1}}{e^t-1}dt = \frac{(-1)^n}{\Gamma(n)}\int_{0}^{\infty}\frac{t^{n-1}}{e^t-1}dt$$

From this, we obtain the well-known formula

$$\zeta(n) = \frac{1}{\Gamma(n)}\int_{0}^{\infty}\frac{t^{n-1}}{e^t-1}dt$$

n does not need to be a natural number any longer.

Note

It turns that $(-1)^n\zeta(n)$ is in fact the following higher order lineal definite integral.

$$\int_{\infty}^{0}\int_{\infty}^{x}\cdots\int_{\infty}^{x}\frac{1}{e^{x}-1}dx^{n} = (-1)^{n}\zeta(n)$$

By-product

$$-\log 0 = \zeta(1)$$

1.2 Generating function of tanh x family

The idea of the obtaining Zeta by integrating the following expression is also natural.

$$\frac{1}{e^x+1} = e^{-x} - e^{-2x} + e^{-3x} - e^{-4x} + \cdots \qquad \left(= -\frac{1}{2}\tanh\frac{x}{2} + \frac{1}{2} \right) \tag{1.0}$$

1.2.1 Termwise Higher Integral of Fourier Series of $1/(e^x+1)$

Formula 1.2.1

When $\eta(x) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^x}$, the following expressions hold for $x>0$.

$$\int_0^x \frac{1}{e^x+1}dx = -\sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{-rx}}{r^1} + \frac{x^0}{0!}\eta(1) \tag{1.1}$$

$$\int_0^x\int_0^x \frac{1}{e^x+1}dx^2 = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{-rx}}{r^2} - \frac{x^0}{0!}\eta(2) + \frac{x^1}{1!}\eta(1) \tag{1.2}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \frac{1}{e^x+1}dx^n = (-1)^n\sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{-rx}}{r^n} - \sum_{s=0}^{n-1}(-1)^{n-s}\frac{x^s}{s!}\eta(n-s) \tag{1.n}$$

Proof

$1/(e^x+1)$ can be expanded to Fourier series as follows.

$$\frac{1}{e^x+1} = \frac{e^{-x}}{1+e^{-x}} = e^{-x} - e^{-2x} + e^{-3x} - e^{-4x} +- \cdots$$

$$= \cos 1ix - \cos 2ix + \cos 3ix - \cos 4ix +- \cdots$$

$$- i(\sin 1ix - \sin 2ix + \sin 3ix - \sin 4ix +- \cdots)$$

Integrating both sides of this with respect to x from 0 to x ,

$$\int_0^x \frac{1}{e^x+1}dx = -\left[\frac{e^{-x}}{1^1} - \frac{e^{-2x}}{2^1} + \frac{e^{-3x}}{3^1} - \frac{e^{-4x}}{4^1} +- \cdots \right]_0^x$$

$$= -\left(\frac{e^{-x}}{1^1} - \frac{e^{-2x}}{2^1} + \frac{e^{-3x}}{3^1} - \frac{e^{-4x}}{4^1} +- \cdots \right) + \left(\frac{e^{0}}{1^1} - \frac{e^{0}}{2^1} + \frac{e^{0}}{3^1} - \frac{e^{0}}{4^1} +- \cdots \right)$$

Here, let $\eta(x) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^x}$, then

$$\int_0^x \frac{1}{e^x+1}dx = -\sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{-rx}}{r^1} + \frac{x^0}{0!}\eta(1)$$

Next, integrating both sides of this with respect to x from 0 to x ,

$$\int_0^x\int_0^x \frac{1}{e^x+1}dx^2 = \left[\sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{-rx}}{r^2} \right]_0^x + \frac{x^1}{1!}\eta(1)$$

$$= \sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{-rx}}{r^2} - \sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{-r0}}{r^2} + \frac{x^1}{1!}\eta(1)$$

$$=\sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{-rx}}{r^2} - \frac{x^0}{0!}\eta(2)+\frac{x^1}{1!}\eta(1)$$

Hereafter, in a similar way we obtein the desired expressions. In addition, these are the collateral integrals.

1.2.2 Termwise Higher Integral of Taylor Series of $1/(e^x+1)$

Formula 1.2.2

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers, the following expressions hold for $|x|<\pi$.

$$\int_0^x \frac{1}{e^x+1}dx = -\sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}}{2r(2r)!}x^{2r} + \frac{1}{2}\frac{x^1}{1!} \tag{2.1}$$

$$\int_0^x\int_0^x \frac{1}{e^x+1}dx^2 = -\sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}}{2r(2r+1)!}x^{2r+1} + \frac{1}{2}\frac{x^2}{2!} \tag{2.2}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \frac{1}{e^x+1}dx^n = -\sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}}{2r(2r+n-1)!}x^{2r+n-1} + \frac{1}{2}\frac{x^n}{n!} \tag{2.n}$$

Proof

$$tanh\,x = \sum_{r=1}^{\infty}\frac{2^{2r}(2^{2r}-1)B_{2r}}{2r(2r-1)!}x^{2r-1} \qquad |x| < \frac{\pi}{2}$$

From this,

$$\frac{1}{2}tanh\frac{x}{2} = \frac{1}{2}\sum_{r=1}^{\infty}\frac{2^{2r}(2^{2r}-1)B_{2r}}{2r(2r-1)!}\left(\frac{x}{2}\right)^{2r-1} = \sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}}{2r(2r-1)!}x^{2r-1}$$

Since

$$\frac{1}{e^x+1} = -\left(\frac{1}{2}tanh\frac{x}{2}-\frac{1}{2}\right)$$

Then

$$\frac{1}{e^x+1} = -\sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}}{2r(2r-1)!}x^{2r-1} + \frac{1}{2} \qquad |x|<\pi \tag{2.0}$$

Integrating both sides of this with respect to x from 0 to x,

$$\int_0^x \frac{1}{e^x+1}dx = -\left[\sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}}{2r(2r)!}x^{2r}\right]_0^x + \frac{x}{2}$$

$$= -\sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}}{2r(2r)!}x^{2r} + \frac{1}{2}\frac{x^1}{1!}$$

Next, integrating both sides of this with respect to x from 0 to x,

$$\int_0^x\int_0^x \frac{1}{e^x+1}dx^2 = -\left[\sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}}{2r(2r+1)!}x^{2r+1}\right]_0^x + \frac{x^2}{2\cdot 2!}$$

$$= -\sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}}{2r(2r+1)!}x^{2r+1} + \frac{1}{2}\frac{x^2}{2!}$$

Hereafter, in a similar way we obtein the desired expressions.

1.2.3 Dirichlet Eta

Comparing the two formulas mentioned above , we obtain the following Dirichlet Eta

Formula 1.2.3

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and n is two or more natural number, the following expressions hold for $0<x<\pi$.

$$\eta(1) = \frac{1}{2}\frac{x^1}{1!} - \sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^1} - \sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}x^{2r}}{2r(2r)!}$$

$$\eta(2) = -\frac{1}{2}\frac{x^2}{2!} - \sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^2} + \sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}x^{2r+1}}{2r(2r+1)!} + \frac{x^1}{1!}\eta(1)$$

$$\eta(3) = \frac{1}{2}\frac{x^3}{3!} - \sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^3} - \sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}x^{2r+2}}{2r(2r+2)!} - \frac{x^2}{2!}\eta(1) + \frac{x^1}{1!}\eta(2)$$

$$\vdots$$

$$\eta(n) = -\frac{(-1)^n}{2}\frac{x^n}{n!} - \sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^n}$$

$$+ (-1)^n\sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}x^{2r+n-1}}{2r(2r+n-1)!} - \sum_{s=1}^{n-1}\frac{(-1)^s x^s}{s!}\eta(n-s)$$

Proof

Comparing (1.1) and (2.1) , we obtain

$$\eta(1) = \frac{1}{2}\frac{x^1}{1!} - \sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^1} - \sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}x^{2r}}{2r(2r)!}$$

Comparing (1.2) and (2.2) , we obtain

$$\eta(2) = -\frac{1}{2}\frac{x^2}{2!} - \sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^2} + \sum_{r=1}^{\infty}\frac{(2^{2r}-1)B_{2r}x^{2r+1}}{2r(2r+1)!} + \frac{x^1}{1!}\eta(1)$$

Hereafter, in a similar way we obtein the desired expressions. In addition, $\zeta(n)$ can be obtained immediately by multiplying this by $1/(1-2^{1-n})$.

1.2.4 Lineal Higher Interal of $1/(e^x+1)$

Mentioned above was a result of the collateral higher integral of $1/(e^x+1)$. If this is integrated in the lineal line, it becomes the following.

First of all,

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x}\frac{1}{e^x+1}dx^n = (-1)^n\sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^n}$$

According to Cauchy formula for repeated integration ,

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x}\frac{1}{e^x+1}dx^n = \frac{1}{\Gamma(n)}\int_{\infty}^{x}\frac{(x-t)^{n-1}}{e^t+1}dt$$

From these

$$(-1)^n\sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^n} = \frac{1}{\Gamma(n)}\int_{\infty}^{x}\frac{(x-t)^{n-1}}{e^t+1}dt$$

Giving this $x=0$,

$$(-1)^n\eta(n) = \frac{1}{\Gamma(n)}\int_{\infty}^{0}\frac{(-t)^{n-1}}{e^t+1}dt = \frac{(-1)^n}{\Gamma(n)}\int_{0}^{\infty}\frac{t^{n-1}}{e^t+1}dt$$

Then

$$\eta(n) = \frac{1}{\Gamma(n)}\int_{0}^{\infty}\frac{t^{n-1}}{e^t+1}dt$$

Using $\zeta(n) = \dfrac{2^{n-1}}{2^{n-1}-1}\eta(n)$ for this, we obtain

$$\zeta(n) = \frac{2^{n-1}}{2^{n-1}-1}\frac{1}{\Gamma(n)}\int_{0}^{\infty}\frac{t^{n-1}}{e^t+1}dt$$

n does not need to be a natural number any longer.

1.2.5 Higher order differentiation of $1/(e^x+1)$

(1) Higher order derivatives of $1/(e^x+1)$

Formula 1.2.5

Let us define the coefficients that make powers constant as follows.

$$1^n + 2^n x + 3^n x^2 + 4^n x^3 + \cdots = \frac{\sum_{r=1}^{n} {}_nD_r x^{r-1}}{(1-x)^{n+1}} \qquad |x| \leqq 1 \tag{5.0}$$

Then, the following expressions hold.

$$\frac{e^x}{(e^x+1)^2} = 1^1e^{-x} - 2^1e^{-2x} + 3^1e^{-3x} - 4^1e^{-4x} +- \cdots$$

$$\frac{e^x(e^x-1)}{(e^x+1)^3} = 1^2e^{-x} - 2^2e^{-2x} + 3^2e^{-3x} - 4^2e^{-4x} +- \cdots$$

$$\frac{e^x(e^{2x}-4e^x+1)}{(e^x+1)^4} = 1^3e^{-x} - 2^3e^{-2x} + 3^3e^{-3x} - 4^3e^{-4x} +- \cdots$$

$$\frac{e^x(e^{3x}-11e^{2x}+11e^x-1)}{(e^x+1)^5} = 1^4e^{-x} - 2^4e^{-2x} + 3^4e^{-3x} - 4^4e^{-4x} +- \cdots$$

$$\vdots$$

$$\frac{e^x \sum_{r=1}^{n} (-1)^{r-1} {}_nD_r e^{(n-r)x}}{\left(e^x+1\right)^{n+1}} = 1^n e^{-x} - 2^n e^{-2x} + 3^n e^{-2x} - 4^n e^{-4x} + - \cdots \tag{5.n}$$

Proof

Replacing x with $-e^{-x}$ in the difinition (5.0) ,

$$1^n - 2^n e^{-x} + 3^n e^{-2x} - 4^n e^{-3x} + - \cdots = \frac{\sum_{r=1}^{n} (-1)^{r-1} {}_nD_r e^{-(r-1)x}}{\left(1+e^{-x}\right)^{n+1}}$$

Multiplying by e^{-x} the both sides,

$$1^n e^{-x} - 2^n e^{-2x} + 3^n e^{-3x} - 4^n e^{-4x} + - \cdots = \frac{\sum_{r=1}^{n} (-1)^{r-1} {}_nD_r e^{-rx}}{\left(1+e^{-x}\right)^{n+1}}$$

Multiplying by $e^{x(n+1)}$ the numerator and the denominator in the right side, we obtain (5.n) immediately. This is corresponding to the n th order derivative of (1.0) .

Note

Although these coefficients are mentioned later , it is given by the following formula.

$${}_nD_r = \sum_{k=0}^{r} (-1)^k \binom{n+1}{k} (r-k)^n \qquad n=1,2,3,\cdots$$

(2) Higher order difference quotients of $1/(e^x+1)$

Substituting $x=0$ for the above, we obtain the higher order difference quationts immediately.

Formula 1.2.5'

$$\frac{1}{2^2} = 1^1-2^1+3^1-4^1+-\cdots = \eta(-1)$$

$$\frac{1-1}{2^3} = 1^2-2^2+3^2-4^2+-\cdots = \eta(-2)$$

$$\frac{1-4+1}{2^4} = 1^3-2^3+3^3-4^3+-\cdots = \eta(-3)$$

$$\frac{1-11+11-1}{2^5} = 1^4-2^4+3^4-4^4+-\cdots = \eta(-4)$$

$$\vdots$$

$$\frac{\sum_{r=1}^{n} (-1)^{r-1} {}_nD_r}{2^{n+1}} = 1^n - 2^n + 3^n - 4^n + - \cdots = \eta(-n) \tag{5.n'}$$

(3) Riemann Zeta at negative integers

Formula 1.2.5"

When ${}_nD_r$ are the coefficients that make powers constant, B_n are the Bernoulli Numbers, and T_{n-1} are the Tangent Numbers such that

$$T_{n-1} = 2^n\left(2^n-1\right)\frac{|B_n|}{n} \qquad n=2,3,4,\cdots$$

the following expressions hold.

$$\zeta(-n) = \frac{1}{2^{n+1}\left(1-2^{n+1}\right)}\sum_{r=1}^{n}(-1)^{r-1}{}_nD_r \qquad n=1,2,3,\cdots \qquad (5.z0)$$

$$\zeta(1-2n) = \frac{(-1)^{n-1}}{2^{2n}\left(1-2^{2n}\right)}T_{2n-1} \qquad n=1,2,3,\cdots \qquad (5.z1)$$

$$= (-1)^{2n-1}\frac{B_{2n}}{2n} \qquad n=1,2,3,\cdots \qquad (5.z1')$$

Proof

From (5.n') ,

$$\eta(-n) = \frac{\sum_{r=1}^{n}(-1)^{r-1}{}_nD_r}{2^{n+1}} \qquad n=1,2,3,\cdots$$

Applying $\zeta(x) = \dfrac{2^{x-1}}{2^{x-1}-1}\eta(x)$ to this,

$$\zeta(-n) = \frac{2^{-n-1}}{2^{-n-1}-1}\frac{\sum_{r=1}^{n}(-1)^{r-1}{}_nD_r}{2^{n+1}} = \frac{1}{2^{n+1}\left(1-2^{n+1}\right)}\sum_{r=1}^{n}(-1)^{r-1}{}_nD_r$$

Replacing n with $2n-1$,

$$\zeta(1-2n) = \frac{1}{2^{2n}\left(1-2^{2n}\right)}\sum_{r=1}^{2n-1}(-1)^{r-1}{}_{2n-1}D_r$$

Here, the following relation exists between this numerator and the tangent number. (See **1.7.5**).

$$\sum_{r=1}^{2n-1}(-1)^{r-1}{}_{2n-1}D_r = (-1)^{n-1}T_{2n-1} \qquad \left\{T_{2n-1} = 2^{2n}\left(2^{2n}-1\right)\frac{|B_{2n}|}{2n}\right\}$$

Using this,

$$\zeta(1-2n) = \frac{1}{2^{2n}\left(1-2^{2n}\right)}\sum_{r=1}^{2n-1}(-1)^{r-1}{}_{2n-1}D_r$$

$$= \frac{(-1)^{n-1}}{2^{2n}\left(1-2^{2n}\right)}T_{2n-1} = \frac{(-1)^{n-1}}{2^{2n}\left(1-2^{2n}\right)}2^{2n}\left(2^{2n}-1\right)\frac{|B_{2n}|}{2n}$$

$$= -(-1)^{n-1}\frac{|B_{2n}|}{2n} = (-1)^n\frac{(-1)^{n-1}B_{2n}}{2n} = (-1)^{2n-1}\frac{B_{2n}}{2n}$$

Thus, the well-known relational expression was obtained.

Note

It was found out by Mr.Sugioka that Zeta at minus integers can be obtained by the higher order differentiation of (1.0) etc.

1.3 Generating function of csch x family

Zeta can be obtained also when the following expression is integrated.

$$\frac{e^x}{e^{2x}-1} = e^{-1x} + e^{-3x} + e^{-5x} + e^{-7x} + \cdots \qquad \left(= \frac{csch\,x}{2} \right)$$

1.3.1 Termwise Higher Integral of Fourier Series of $e^x/(e^{2x}-1)$

Lemma 1.3.1

When $\lambda(x) = \sum_{r=1}^{\infty} \frac{1}{(2r-1)^x}$, the following expressions hold for $x>0$.

$$\int_0^x \frac{e^x}{e^{2x}-1}dx = -\sum_{r=1}^{\infty} \frac{e^{-(2r-1)x}}{(2r-1)^1} + \frac{x^0}{0!}\lambda(1) \tag{1.1}$$

$$\int_0^x\int_0^x \frac{e^x}{e^{2x}-1}dx^2 = \sum_{r=1}^{\infty} \frac{e^{-(2r-1)x}}{(2r-1)^2} - \frac{x^0}{0!}\lambda(2)+\frac{x^1}{1!}\lambda(1) \tag{1.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{e^x}{e^{2x}-1}dx^3 = -\sum_{r=1}^{\infty} \frac{e^{-(2r-1)x}}{(2r-1)^3} + \frac{x^0}{0!}\lambda(3) - \frac{x^1}{1!}\lambda(2)+\frac{x^2}{2!}\lambda(1) \tag{1.3}$$

$$\vdots$$

$$\int_0^x \cdots \int_0^x \frac{e^x}{e^{2x}-1}dx^n = (-1)^n \sum_{r=1}^{\infty} \frac{e^{-(2r-1)x}}{(2r-1)^n} - \sum_{s=0}^{n-1} (-1)^{n-s}\frac{x^s}{s!}\lambda(n-s) \tag{1.n}$$

Proof

$e^x/(e^{2x}-1)$ can be expanded to Fourier series as follows.

$$\frac{e^x}{e^{2x}-1} = \frac{e^{-x}}{1-e^{-2x}} = e^{-x} + e^{-3x} + e^{-5x} + e^{-7x} + \cdots \tag{1.0}$$

$$= cos\,1ix + cos\,3ix + cos\,5ix + cos\,7ix + \cdots$$
$$+ i(sin\,1ix + sin\,3ix + sin\,5ix + sin\,7ix + \cdots)$$

Integrating both sides of this with respect to x from 0 to x ,

$$\int_0^x \frac{e^x}{e^{2x}-1}dx = \left[-\left(\frac{e^{-x}}{1^1} + \frac{e^{-3x}}{3^1} + \frac{e^{-5x}}{5^1} + \frac{e^{-7x}}{7^1} + \cdots \right) \right]_0^x$$

$$= -\left(\frac{e^{-x}}{1^1} + \frac{e^{-3x}}{3^1} + \frac{e^{-5x}}{5^1} + \frac{e^{-7x}}{7^1} + \cdots \right) + \frac{e^0}{1^1} + \frac{e^0}{3^1} + \frac{e^0}{5^1} + \frac{e^0}{7^1} + \cdots$$

Here, let $\lambda(x) = \sum_{r=1}^{\infty} \frac{1}{(2r-1)^x}$, then

$$\int_0^x \frac{e^x}{e^{2x}-1}dx = -\sum_{r=1}^{\infty} \frac{e^{-(2r-1)x}}{(2r-1)^1} + \frac{x^0}{0!}\lambda(1)$$

Next, integrating both sides of this with respect to x from 0 to x ,

$$\int_0^x\int_0^x \frac{e^x}{e^{2x}-1}dx^2 = \left[\sum_{r=1}^{\infty} \frac{e^{-(2r-1)x}}{(2r-1)^2} \right]_0^x + \frac{x^1}{1!}\lambda(1)$$

$$= \sum_{r=1}^{\infty} \frac{e^{-(2r-1)x}}{(2r-1)^2} - \frac{x^0}{0!}\lambda(2) + \frac{x^1}{1!}\lambda(1)$$

Hereafter, in a similar way we obtein the desired expressions. In addition, these are the collateral integrals.

1.3.2 Termwise Higher Integral of Taylor Series of $e^x/(e^{2x}-1)$

Lemma 1.3.2

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42, \cdots$ are Bernoulli numbers and $H_s=\sum_{t=1}^{s} 1/t$ are Harmonic numbers, the following expressions hold for $|x|<\pi$.

$$\int_0^x \frac{e^x}{e^{2x}-1}dx = \frac{1}{2}\frac{x^0}{0!}\left(\log\frac{x}{2}-H_0\right) - \frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)B_{2r}x^{2r}}{2r(2r)!} + \frac{x^0}{0!}\lambda(1) \tag{2.1}$$

$$\int_0^x\int_0^x \frac{e^x}{e^{2x}-1}dx^2 = \frac{1}{2}\frac{x^1}{1!}\left(\log\frac{x}{2}-H_1\right) - \frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)B_{2r}x^{2r+1}}{2r(2r+1)!} + \frac{x^1}{1!}\lambda(1) \tag{2.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{e^x}{e^{2x}-1}dx^3 = \frac{1}{2}\frac{x^2}{2!}\left(\log\frac{x}{2}-H_2\right) - \frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)B_{2r}x^{2r+2}}{2r(2r+2)!} + \frac{x^2}{2!}\lambda(1) \tag{2.3}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \frac{e^x}{e^{2x}-1}dx^n = \frac{1}{2}\frac{x^{n-1}}{(n-1)!}\left(\log\frac{x}{2}-H_{n-1}\right) - \frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!} + \frac{x^{n-1}}{(n-1)!}\lambda(1) \tag{2.n}$$

Proof

$$\operatorname{csch} x = \frac{1}{x} - \sum_{r=1}^{\infty}\frac{(2^{2r}-2)B_{2r}}{2r(2r-1)!}x^{2r-1} \quad , \quad \frac{e^x}{e^{2x}-1} = \frac{\operatorname{csch} x}{2}0$$

From this,

$$\frac{e^x}{e^{2x}-1} = \frac{1}{2x} - \frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)B_{2r}}{2r(2r-1)!}x^{2r-1} \qquad |x|<\pi$$

Integrating both sides of this with respect to x from 0 to x,

$$\int_0^x \frac{e^x}{e^{2x}-1}dx = \frac{1}{2}\left[\log x - \sum_{r=1}^{\infty}\frac{(2^{2r}-2)B_{2r}}{2r(2r)!}x^{2r}\right]_0^x$$

$$= \frac{1}{2}\log x - \frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)B_{2r}}{2r(2r)!}x^{2r} - \frac{1}{2}\log 0$$

Here,

$$-\frac{1}{2}\log 0 = -\frac{\log(2\cdot 0)}{2} = -\frac{\log 2}{2} - \frac{\log 0}{2}$$

and from $\lambda(n) = \dfrac{2^n-1}{2^n}\zeta(n)$ and $\zeta(1) = -\log 0$ (See **1.1.2**)

$$\lambda(1) = \frac{\zeta(1)}{2} = -\frac{\log 0}{2}$$

Then

$$-\frac{1}{2}\log 0 = -\frac{\log 2}{2} - \frac{\log 0}{2} = -\frac{\log 2}{2} + \lambda(1)$$

Substituting this for the above,

$$\int_0^x \frac{e^x}{e^{2x}-1}dx = \frac{1}{2}\log\frac{x}{2} - \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(2r)!}x^{2r} + \frac{x^0}{0!}\lambda(1)$$

Next, integrating both sides of this with respect to x from 0 to x,

$$\int_0^x\int_0^x \frac{e^x}{e^{2x}-1}dx^2 = \frac{1}{2}\left[\frac{x^1}{1!}\left(\log\frac{x}{2}-H_1\right) - \sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+1}}{2r(2r+1)!}\right]_0^x + \frac{x^1}{1!}\lambda(1)$$

$$= \frac{1}{2}\frac{x^1}{1!}\left(\log\frac{x}{2}-H_1\right) - \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+1}}{2r(2r+1)!} + \frac{x^1}{1!}\lambda(1)$$

Hereafter, in a similar way we obtein the desired expressions.

1.3.3 Dirichlet Lambda

Two lemmas mentioned above are not useful in itself. However, comparing these, we can obtain the following Dirichlet Lambda.

Formula 1.3.3

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and $H_s=\sum_{t=1}^{s}1/t$ are Harmonic numbers, the following expressions hold for $0<x<\pi$.

$$0 = \frac{1}{2}\frac{x^0}{0!}\left(\log\frac{x}{2}-H_0\right) + \sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^1} - \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(2r)!}x^{2r}$$

$$\lambda(2) = -\frac{1}{2}\frac{x^1}{1!}\left(\log\frac{x}{2}-H_1\right) + \sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^2} + \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+1}}{2r(2r+1)!}$$

$$\lambda(3) = \frac{1}{2}\frac{x^2}{2!}\left(\log\frac{x}{2}-H_2\right) + \sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^3} - \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+2}}{2r(2r+2)!} + \frac{x^1}{1!}\lambda(2)$$

$$\lambda(4) = -\frac{1}{2}\frac{x^3}{3!}\left(\log\frac{x}{2}-H_3\right) + \sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^4} + \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+3}}{2r(2r+3)!} - \frac{x^2}{2!}\lambda(2) + \frac{x^1}{1!}\lambda(3)$$

$$\vdots$$

$$\lambda(n) = \frac{(-1)^{n-1}}{2}\frac{x^{n-1}}{(n-1)!}\left(\log\frac{x}{2}-H_{n-1}\right) + \sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^n}$$
$$+ \frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+n-1}}{2r(2r+n-1)!} - \sum_{s=1}^{n-2}\frac{(-1)^s x^s}{s!}\lambda(n-s)$$

Proof

Comparing (1.1) and (2.1), we obtain

$$0 = \frac{1}{2}\frac{x^0}{0!}\left(\log\frac{x}{2}-H_0\right) + \sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^1} - \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(2r)!}x^{2r}$$

Comparing (1.2) and (2.2) , we obtain

$$\lambda(2)= -\frac{1}{2}\frac{x^1}{1!}\left(log\frac{x}{2}-H_1\right)+\sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^2}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+1}}{2r(2r+1)!}$$

Hereafter, in a similar way we obtein the desired expressions. In addition, $\zeta(n)$ can be obtained immediately by multiplying this by $1/\left(1-2^{-n}\right)$.

1.3.4 Lineal Higher Interal of $e^x/\left(e^{2x}-1\right)$

Mentioned above was a result of the collateral higher integral of $e^x/\left(e^{2x}-1\right)$. If this is integrated in the lineal line, it becomes the following.

First of all,

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x}\frac{e^x}{e^{2x}-1}dx^n = (-1)^n\sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{-(2r-1)x}}{(2r-1)^n}$$

According to Cauchy formula for repeated integration ,

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x}\frac{e^x}{e^{2x}-1}dx^n = \frac{1}{\Gamma(n)}\int_{\infty}^{x}\frac{e^t(x-t)^{n-1}}{e^{2t}-1}dt$$

From these,

$$(-1)^n\sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{-(2r-1)x}}{(2r-1)^n} = \frac{1}{\Gamma(n)}\int_{\infty}^{x}\frac{e^t(x-t)^{n-1}}{e^{2t}-1}dt$$

Giving this $x=0$,

$$(-1)^n\lambda(n) = \frac{1}{\Gamma(n)}\int_{\infty}^{0}\frac{e^t(-t)^{n-1}}{e^{2t}-1}dt = \frac{(-1)^n}{\Gamma(n)}\int_{0}^{\infty}\frac{e^t t^{n-1}}{e^{2t}-1}dt$$

Then,

$$\lambda(n) = \frac{1}{\Gamma(n)}\int_{0}^{\infty}\frac{e^t t^{n-1}}{e^{2t}-1}dt = \frac{1}{2\Gamma(n)}\int_{0}^{\infty}t^{n-1}csch\,t\,dt$$

Using $\zeta(n) = \frac{2^n}{2^n-1}\lambda(n)$ for this, we obtain

$$\zeta(n) = \frac{2^n}{2^n-1}\frac{1}{\Gamma(n)}\int_{0}^{\infty}\frac{e^t t^{n-1}}{e^{2t}-1}dt = \frac{2^{n-1}}{2^n-1}\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{n-1}csch\,t\,dt$$

n does not need to be a natural number any longer.

By-product

$$-log\,0 = 2\lambda(1)$$

1.4 Generating function of cot family

Zeta can be obtained also when the following expression is integrated.

$$\frac{ie^{ix}}{1-e^{ix}} = i\left(e^{ix}+e^{2ix}+e^{3ix}+e^{4ix}+\cdots\right) \qquad \left(= -\frac{1}{2}cot\frac{x}{2}-\frac{i}{2}\right) \tag{1.0}$$

1.4.1 Termwise Higher Integral of Fourier Series of $ie^{ix}/(1-e^{ix})$

Lemma 1.4.1

When $\zeta(x)=\sum_{r=1}^{\infty}\frac{1}{r^x}$, the following expressions hold for $0<x<2\pi$.

$$\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx = i^0\sum_{r=1}^{\infty}\frac{e^{irx}}{r^1} - i^0\zeta(1) \tag{1.1}$$

$$\int_0^x\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx^2 = i^{-1}\sum_{r=1}^{\infty}\frac{e^{irx}}{r^2} - i^{-1}\frac{x^0}{0!}\zeta(2) - i^0\frac{x^1}{1!}\zeta(1) \tag{1.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx^3 = i^{-2}\sum_{r=1}^{\infty}\frac{e^{irx}}{r^3} - \frac{i^{-2}x^0}{0!}\zeta(3) - \frac{i^{-1}x^1}{1!}\zeta(2) - \frac{i^0x^2}{2!}\zeta(1) \tag{1.3}$$

$$\int_0^x\cdots\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx^4 = i^{-3}\sum_{r=1}^{\infty}\frac{e^{irx}}{r^4} - \frac{i^{-3}x^0}{0!}\zeta(4) - \frac{i^{-2}x^1}{1!}\zeta(3) - \frac{i^{-1}x^2}{2!}\zeta(2) - \frac{i^0x^3}{3!}\zeta(1)$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx^n = i^{-(n-1)}\sum_{r=1}^{\infty}\frac{e^{irx}}{r^n} - \sum_{s=0}^{n-1}\frac{i^{-(n-1-s)}x^s}{s!}\zeta(n-s) \tag{1.n}$$

Proof

$ie^{ix}/(1-e^{ix})$ can be expanded to Fourier series as follows.

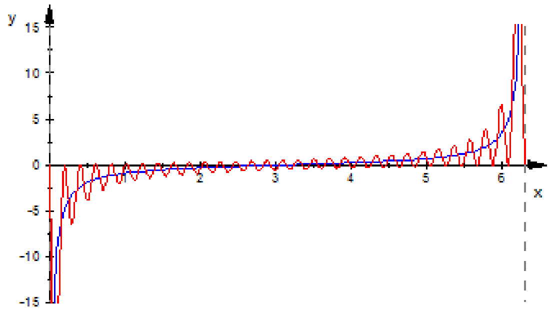
$$\frac{ie^{ix}}{1-e^{ix}} = i\left(e^{ix}+e^{2ix}+e^{3ix}+e^{4ix}+\cdots\right) \tag{1.0}$$

i.e.

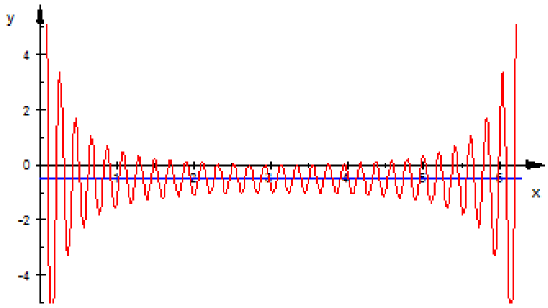
$$-\frac{1}{2}cot\frac{x}{2}-\frac{i}{2} = -(sin1x+sin2x+sin3x+sin4x+\cdots)$$
$$+i(cos1x+cos2x+cos3x+cos4x+\cdots) \tag{1.0'}$$

When (1.0') is illustrated, it is as follows.

real number part



imaginary number part



The blue is the left side (function) and the red is the right side (series). Left side is the locus of the median (central line) of the right side. So, it is clear that (1.0) does not hold as an equation.

However, the higher order integral of both sides of (1.0) holds as an equation.

Integrating both sides of (1.0) with respect to x from 0 to x ,

$$\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx = i^0\left[\frac{e^{ix}}{1^1}+\frac{e^{2ix}}{2^1}+\frac{e^{3ix}}{3^1}+\frac{e^{4ix}}{4^1}+\cdots\right]_0^x$$

$$= i^0\left(\frac{e^{ix}}{1^1}+\frac{e^{2ix}}{2^1}+\frac{e^{3ix}}{3^1}+\frac{e^{4ix}}{4^1}+\cdots\right) - i^0\left(\frac{e^0}{1^1}+\frac{e^0}{2^1}+\frac{e^0}{3^1}+\frac{e^0}{4^1}+\cdots\right)$$

Here, let $\zeta(x)=\sum_{r=1}^{\infty}\frac{1}{r^x}$, then

$$\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx = i^0\sum_{r=1}^{\infty}\frac{e^{irx}}{r^1} - i^0\zeta(1)$$

Next, integrating both sides of this with respect to x from 0 to x ,

$$\int_0^x\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx^2 = i^{-1}\left[\sum_{r=1}^{\infty}\frac{e^{irx}}{r^2}\right]_0^x - i^0\frac{x^1}{1!}\zeta(1)$$

$$= i^{-1}\sum_{r=1}^{\infty}\frac{e^{irx}}{r^2} - i^{-1}\sum_{r=1}^{\infty}\frac{e^0}{r^2} - i^0\frac{x^1}{1!}\zeta(1)$$

$$= i^{-1}\sum_{r=1}^{\infty}\frac{e^{irx}}{r^2} - i^{-1}\frac{x^0}{0!}\zeta(2) - i^0\frac{x^1}{1!}\zeta(1)$$

Hereafter, in a similar way we obtein the desired expressions. In addition, these are the collateral integrals.

1.4.2 Termwise Higher Integral of Taylor Series of $ie^{ix}/(1-e^{ix})$

Lemma 1.4.2

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and $H_s=\sum_{t=1}^{s}1/t$ are Harmonic numbers, the following expressions hold for $0<x<2\pi$.

$$\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx = -\log x - \frac{i}{2}\frac{x^1}{1!} + \sum_{r=1}^{\infty}\frac{|B_{2r}|}{2r(2r)!}x^{2r} + \frac{x^0}{0!}\left\{\frac{i\pi}{2}-\zeta(1)\right\} \tag{2.1}$$

$$\int_0^x\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx^2 = -\frac{x^1}{1!}(\log x - H_1) - \frac{i}{2}\frac{x^2}{2!} + \sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+1}}{2r(2r+1)!} + \frac{x^1}{1!}\left\{\frac{i\pi}{2}-\zeta(1)\right\} \tag{2.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx^3 = -\frac{x^2}{2!}(\log x - H_2) - \frac{i}{2}\frac{x^3}{3!} + \sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+2}}{2r(2r+2)!} + \frac{x^2}{2!}\left\{\frac{i\pi}{2}-\zeta(1)\right\} \tag{2.3}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \frac{ie^{ix}}{1-e^{ix}}dx^n = -\frac{x^{n-1}}{(n-1)!}(\log x - H_{n-1}) - \frac{i}{2}\frac{x^n}{n!} + \sum_{r=1}^{\infty}\frac{|B_{2r}|x^{n-1+2r}}{2r(n-1+2r)!} + \frac{x^{n-1}}{(n-1)!}\left\{\frac{i\pi}{2}-\zeta(1)\right\} \tag{2.n}$$

Proof

$$x\cot x = 1+\sum_{r=1}^{\infty}(-1)^{r-1}\frac{2^{2r}B_{2r}}{(2r)!}x^{2r} \qquad 0<x<\pi$$

From this,

$$\frac{x}{2}\cot\frac{x}{2} = 1+\sum_{r=1}^{\infty}(-1)^{r-1}\frac{2^{2r}B_{2r}}{(2r)!}\frac{x^{2r}}{2^{2r}} = 1-\sum_{r=1}^{\infty}\frac{|B_{2r}|}{(2r)!}x^{2r}$$

Dividing both sides by x,

$$\frac{1}{2}\cot\frac{x}{2} = \frac{1}{x} - \sum_{r=1}^{\infty}\frac{|B_{2r}|}{2r(2r-1)!}x^{2r-1} \qquad 0<x<2\pi$$

Here,

$$\frac{ie^{ix}}{1-e^{ix}} = -\frac{1}{2}\cot\frac{x}{2}-\frac{i}{2}$$

Then,

$$\frac{ie^{ix}}{1-e^{ix}} = -\frac{1}{x}-\frac{i}{2}+\sum_{r=1}^{\infty}\frac{|B_{2r}|}{2r(2r-1)!}x^{2r-1} \qquad 0<x<2\pi$$

Integrating both sides of this with respect to x from 0 to x,

$$\int_0^x\frac{ie^{ix}}{1-e^{ix}}dx = \int_0^x\left[-\log x-\frac{ix}{2}+\sum_{r=1}^{\infty}\frac{|B_{2r}|}{2r(2r)!}x^{2r}\right]_0^x$$

$$= -\log x-\frac{i}{2}\frac{x^1}{1!}+\sum_{r=1}^{\infty}\frac{|B_{2r}|}{2r(2r)!}x^{2r}+\log 0$$

Here,

$$\log 0 = \log\left(e^{\frac{i\pi}{2}}\cdot 0\right) = \log e^{\frac{i\pi}{2}}+\log 0 = \frac{i\pi}{2}+\log 0$$

And using also $\log 0 = -\zeta(1)$, we obtain

$$\int_0^x\frac{ie^{ix}}{1-e^{ix}}dx = -\log x-\frac{i}{2}\frac{x^1}{1!}+\sum_{r=1}^{\infty}\frac{|B_{2r}|}{2r(2r)!}x^{2r}+\frac{i\pi}{2}-\zeta(1)$$

Next, integrating both sides of this with respect to x from 0 to x,

$$\int_0^x\int_0^x\frac{ie^{ix}}{1-e^{ix}}dx^2 = \int_0^x\left[-\frac{x^1}{1!}(\log x-H_1)-\frac{i}{2}\frac{x^2}{2!}+\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+1}}{2r(2r+1)!}\right]_0^x+\frac{x^1}{1!}\left\{\frac{i\pi}{2}-\zeta(1)\right\}$$

$$= -\frac{x^1}{1!}(\log x-H_1)-\frac{i}{2}\frac{x^2}{2!}+\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+1}}{2r(2r+1)!}+\frac{x^1}{1!}\left\{\frac{i\pi}{2}-\zeta(1)\right\}$$

Hereafter, in a similar way we obtein the desired expressions.

1.4.3 Riemann Zeta Polynomial (Trigonometric form)

Two lemmas mentioned above are not useful in itself. However, comparing these, we can obtain the following Riemann zeta polynomials.

Formula 1.4.3

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and $H_s=\sum_{t=1}^{s}1/t$ are

Harmonic numbers, the following expressions hold for $0<x<2\pi$.

(1) Even Zeta

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^{1}}+\frac{1}{2}\frac{x^{1}}{1!}-\frac{\pi}{2}\frac{x^{0}}{0!}=0 \tag{3.1es}$$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^{3}}-\frac{1}{2}\frac{x^{3}}{3!}+\frac{\pi}{2}\frac{x^{2}}{2!}=\frac{x^{1}}{1!}\zeta(2) \tag{3.2es}$$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^{5}}+\frac{1}{2}\frac{x^{5}}{5!}-\frac{\pi}{2}\frac{x^{4}}{4!}=\frac{x^{1}}{1!}\zeta(4)-\frac{x^{3}}{3!}\zeta(2) \tag{3.3es}$$

$\vdots$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^{2n-1}}-\frac{(-1)^{n}}{2}\left\{\frac{x^{2n-1}}{(2n-1)!}-\frac{\pi x^{2n-2}}{(2n-2)!}\right\}=-\sum_{s=1}^{n-1}\frac{(-1)^{s}x^{2s-1}}{(2s-1)!}\zeta(2n-2s) \tag{3.nes}$$

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^{2}}-\frac{1}{2}\frac{x^{2}}{2!}+\frac{\pi}{2}\frac{x^{1}}{1!}=\frac{x^{0}}{0!}\zeta(2) \tag{3.1ec}$$

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^{4}}+\frac{1}{2}\frac{x^{4}}{4!}-\frac{\pi}{2}\frac{x^{3}}{3!}=\frac{x^{0}}{0!}\zeta(4)-\frac{x^{2}}{2!}\zeta(2) \tag{3.2ec}$$

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^{6}}-\frac{1}{2}\frac{x^{6}}{6!}+\frac{\pi}{2}\frac{x^{5}}{5!}=\frac{x^{0}}{0!}\zeta(6)-\frac{x^{2}}{2!}\zeta(4)+\frac{x^{4}}{4!}\zeta(2) \tag{3.3ec}$$

$\vdots$

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^{2n}}+\frac{(-1)^{n}}{2}\left\{\frac{x^{2n}}{(2n)!}-\frac{\pi x^{2n-1}}{(2n-1)!}\right\}=\sum_{s=0}^{n-1}\frac{(-1)^{s}x^{2s}}{(2s)!}\zeta(2n-2s) \tag{3.nec}$$

(2) Odd Zeta

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^{1}}+\log x-\sum_{r=1}^{\infty}\frac{|B_{2r}|}{2r(2r)!}x^{2r}=0 \tag{3.1oc}$$

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^{3}}-\frac{x^{2}}{2!}\left(\log x-H_{2}\right)+\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+2}}{2r(2r+2)!}=\frac{x^{0}}{0!}\zeta(3) \tag{3.2oc}$$

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^{5}}+\frac{x^{4}}{4!}\left(\log x-H_{4}\right)-\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+4}}{2r(2r+4)!}=\frac{x^{0}}{0!}\zeta(5)-\frac{x^{2}}{2!}\zeta(3) \tag{3.3oc}$$

$\vdots$

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^{2n-1}}-\frac{(-1)^{n}x^{2n-2}}{(2n-2)!}\left(\log x-H_{2n-2}\right)+(-1)^{n}\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+2n-2}}{2r(2r+2n-2)!}$$

$$=\sum_{s=0}^{n-2}\frac{(-1)^{s}x^{2s}}{(2s)!}\zeta(2n-1-2s) \tag{3.noc}$$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^{2}}+\frac{x^{1}}{1!}\left(\log x-H_{1}\right)-\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+1}}{2r(2r+1)!}=0 \tag{3.1os}$$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^4}-\frac{x^3}{3!}\left(\log x-H_3\right)+\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+3}}{2r(2r+3)!}=\frac{x^1}{1!}\zeta(3) \tag{3.2os}$$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^6}+\frac{x^5}{5!}\left(\log x-H_5\right)-\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+5}}{2r(2r+5)!}=\frac{x^1}{1!}\zeta(5)-\frac{x^3}{3!}\zeta(3) \tag{3.3os}$$

$$\vdots$$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^{2n}}-\frac{(-1)^n x^{2n-1}}{(2n-1)!}\left(\log x-H_{2n-1}\right)+(-1)^n\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+2n-1}}{2r(2r+2n-1)!}$$

$$=-\sum_{s=1}^{n-1}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\zeta(2n+1-2s) \tag{3.nos}$$

Proof

Comparing (1.1) and (2.1), we obtain

$$\sum_{r=1}^{\infty}\frac{e^{irx}}{r^1}=-\log x-\frac{i}{2}\frac{x^1}{1!}+\sum_{r=1}^{\infty}\frac{|B_{2r}|}{2r(2r)!}x^{2r}+\frac{x^0}{0!}\frac{i\pi}{2}$$

Substituting $e^{irx}=\cos rx+i\sin rx$ for this,

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^1}+i\sum_{r=1}^{\infty}\frac{\sin rx}{r^1}=-\log x-\frac{i}{2}\frac{x^1}{1!}+\sum_{r=1}^{\infty}\frac{|B_{2r}|}{2r(2r)!}x^{2r}+\frac{x^0}{0!}\frac{i\pi}{2}$$

From this,

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^1}+\log x-\sum_{r=1}^{\infty}\frac{|B_{2r}|}{2r(2r)!}x^{2r}=0 \tag{3.1oc}$$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^1}+\frac{1}{2}\frac{x^1}{1!}-\frac{\pi}{2}\frac{x^0}{0!}=0 \tag{3.1es}$$

Comparing (1.2) and (2.2) , we obtain

$$\sum_{r=1}^{\infty}\frac{e^{irx}}{r^2}-\zeta(2)=-\frac{ix^1}{1!}\left(\log x-H_1\right)+\frac{1}{2}\frac{x^2}{2!}+i\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+1}}{2r(2r+1)!}-\frac{\pi}{2}\frac{x^1}{1!}$$

Substituting $e^{irx}=\cos rx+i\sin rx$ for this,

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^2}+i\sum_{r=1}^{\infty}\frac{\sin rx}{r^2}-\zeta(2)=-\frac{ix^1}{1!}\left(\log x-H_1\right)+\frac{1}{2}\frac{x^2}{2!}+i\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+1}}{2r(2r+1)!}-\frac{\pi}{2}\frac{x^1}{1!}$$

From this,

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^2}-\frac{1}{2}\frac{x^2}{2!}+\frac{\pi}{2}\frac{x^1}{1!}=\frac{x^0}{0!}\zeta(2) \tag{3.1ec}$$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^2}+\frac{x^1}{1!}\left(\log x-H_1\right)-\sum_{r=1}^{\infty}\frac{|B_{2r}|\,x^{2r+1}}{2r(2r+1)!}=0 \tag{3.1os}$$

Hereafter, in a similar way we obtein the desired expressions.

1.4.4 Riemann Zeta (Exponential form)

If the above formula is expressed by an exponential function, it is as follows.

Formula 1.4.4

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and $H_s=\sum_{t=1}^{s}1/t$ are Harmonic numbers, the following expressions hold for $0<x<2\pi$.

$$0 = \log x + \frac{1}{2}\frac{(ix)^1}{1!} - \frac{i\pi}{2}\frac{(ix)^0}{0!} + \sum_{r=1}^{\infty}\frac{e^{irx}}{r^1} - \sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}(ix)^0}{2r(2r)!}$$

$$\zeta(2) = \frac{(ix)^1}{1!}(\log x - H_1) + \frac{1}{2}\frac{(ix)^2}{2!} - \frac{i\pi}{2}\frac{(ix)^1}{1!} + \sum_{r=1}^{\infty}\frac{e^{irx}}{r^2} - \sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}(ix)^1}{2r(2r+1)!}$$

$$\zeta(n) = \frac{(ix)^{n-1}}{(n-1)!}(\log x - H_{n-1}) + \frac{1}{2}\frac{(ix)^n}{n!} - \frac{i\pi}{2}\frac{(ix)^{n-1}}{(n-1)!} + \sum_{r=1}^{\infty}\frac{e^{irx}}{r^n}$$

$$- \sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r}(ix)^{n-1}}{2r(2r+n-1)!} - \sum_{s=1}^{n-2}\frac{(ix)^s}{s!}\zeta(n-s)$$

1.4.5 Higher order differentiation of $ie^{ix}/(1-e^{ix})$

Although the higher order differentiation of (1.0) does not hold as a equation, it is useful to generate Riemann Zeta at negative integers. And using the coefficients that make powers constant, we obtain the same results as **1.2.5** .

By-products

$$-\log 0 = \zeta(1) - \frac{\pi i}{2}$$

1.5 Generating function of tan x family

Zeta can be obtained also when the following expression is integrated.

$$\frac{ie^{ix}}{1+e^{ix}} = i\left(e^{ix} - e^{2ix} + e^{3ix} - e^{4ix} +- \cdots\right) \qquad \left(= -\frac{1}{2}\tan\frac{x}{2} + \frac{i}{2}\right) \tag{1.0}$$

1.5.1 Termwise Higher Integral of Fourier Series of $ie^{ix}/(1+e^{ix})$

Lemma 1.5.1

When $\eta(x) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^x}$, the following expressions hold for $|x| < \pi$.

$$\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx = i^0 \sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{irx}}{r^1} - i^0\frac{x^0}{0!}\eta(1) \tag{1.1}$$

$$\int_0^x\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx^2 = i^{-1} \sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{irx}}{r^2} - i^{-1}\frac{x^0}{0!}\eta(2) - i^0\frac{x^1}{1!}\eta(1) \tag{1.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx^3 = i^{-2} \sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{irx}}{r^3} - \frac{i^{-2}x^0}{0!}\eta(3) - \frac{i^{-1}x^1}{1!}\eta(2) - \frac{i^0x^2}{2!}\eta(1) \tag{1.3}$$

$$\int_0^x\!\!\cdots\!\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx^4 = i^{-3} \sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{irx}}{r^4} - \frac{i^{-3}x^0}{0!}\eta(4) - \frac{i^{-2}x^1}{1!}\eta(3) - \frac{i^{-1}x^2}{2!}\eta(2) - \frac{i^0x^3}{3!}\eta(1)$$

$$\vdots$$

$$\int_0^x\!\!\cdots\!\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx^n = i^{-(n-1)} \sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{irx}}{r^n} - \sum_{s=0}^{n-1}\frac{i^{-(n-1-s)}x^s}{s!}\eta(n-s) \tag{1.n}$$

Proof

$ie^{ix}/(1+e^{ix})$ can be expanded to Fourier series as follows.

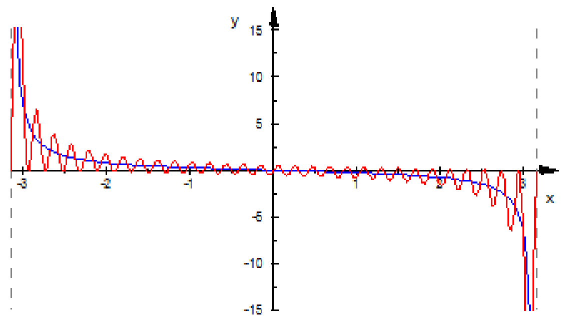
$$\frac{ie^{ix}}{1+e^{ix}} = i\left(e^{ix} - e^{2ix} + e^{3ix} - e^{4ix} +- \cdots\right) \tag{1.0}$$

i.e.

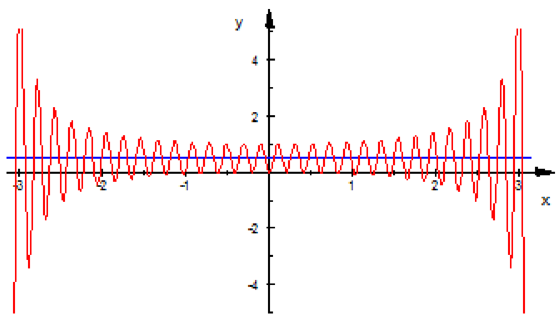
$$-\frac{1}{2}\tan\frac{x}{2} + \frac{i}{2} = -(\sin 1x - \sin 2x + \sin 3x - \sin 4x +- \cdots)$$
$$+ i(\cos 1x - \cos 2x + \cos 3x - \cos 4x +- \cdots) \tag{1.0'}$$

When (1.0') is illustrated, it is as follows.

real number part



imaginary number part



The blue is the left side (function) and the red is the right side (series). Left side is the locus of the median (central line) of the right side. So, it is clear that (1.0) does not hold as an equation.

However, the higher order integral of both sides of (1.0) holds as an equation.

Integrating both sides of (1.0) with respect to x from 0 to x ,

$$\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx = i^0\left[\frac{e^{ix}}{1^1}-\frac{e^{2ix}}{2^1}+\frac{e^{3ix}}{3^1}-\frac{e^{4ix}}{4^1}+\cdots\right]_0^x$$

$$= i^0\left(\frac{e^{ix}}{1^1}-\frac{e^{2ix}}{2^1}+\frac{e^{3ix}}{3^1}-\frac{e^{4ix}}{4^1}+\cdots\right) - i^0\left(\frac{e^0}{1^1}-\frac{e^0}{2^1}+\frac{e^0}{3^1}-\frac{e^0}{4^1}+\cdots\right)$$

Here, let $\eta(x) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^x}$, then

$$\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx = i^0\sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{irx}}{r^1} - i^0\frac{x^0}{0!}\eta(1)$$

Next, integrating both sides of this with respect to x from 0 to x ,

$$\int_0^x\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx^2 = i^{-1}\left[\sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{irx}}{r^2}\right]_0^x - i^0\frac{x^1}{1!}\eta(1)$$

$$= i^{-1}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{e^{irx}}{r^2} - i^{-1}\frac{x^0}{0!}\eta(2) - i^0\frac{x^1}{1!}\eta(1)$$

Hereafter, in a similar way we obtein the desired expressions. Of course, these are the collateral integrals.

1.5.2 Termwise Higher Integral of Taylor Series of $ie^{ix}/(1+e^{ix})$

Lemma 1.5.2

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers, the following expressions hold for $|x|<\pi$.

$$\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx = -\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r)!}x^{2r} + \frac{i}{2}\frac{x^1}{1!} \tag{2.1}$$

$$\int_0^x\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx^2 = -\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+1)!}x^{2r+1} + \frac{i}{2}\frac{x^2}{2!} \tag{2.2}$$

$$\int_0^x\int_0^x\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx^3 = -\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+2)!}x^{2r+2} + \frac{i}{2}\frac{x^3}{3!} \tag{2.3}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x \frac{ie^{ix}}{1+e^{ix}}dx^n = -\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+n-1)!}x^{2r+n-1} + \frac{i}{2}\frac{x^n}{n!} \tag{2.n}$$

Proof

$$tan\ x = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{2^{2r}\left(2^{2r}-1\right)B_{2r}}{(2r)!}x^{2r-1}$$

From this,

$$\frac{1}{2}\tan\frac{x}{2} = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{2^{2r}\left(2^{2r}-1\right)B_{2r}}{(2r)!}\frac{x^{2r-1}}{2^{2r}} = \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)\left|B_{2r}\right|}{2r(2r-1)!}x^{2r-1}$$

Here,

$$\frac{ie^{ix}}{1+e^{ix}} = -\frac{1}{2}\tan\frac{x}{2}+\frac{i}{2}$$

Then

$$\frac{ie^{ix}}{1+e^{ix}} = -\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)\left|B_{2r}\right|}{2r(2r-1)!}x^{2r-1} + \frac{i}{2} \qquad |x| < \pi \tag{2.0}$$

Integrating both sides of this with respect to x from 0 to x repeatedly, we obtain the desired expressions.

1.5.3 Dirichlet Eta Polynomial (Trigonometric form)

Comparing the two formulas mentioned above, we obtain the following Dirichlet eta polynomials

Formula 1.5.3

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers, the following expressions hold for $0<|x|<\pi$.

(1) Even Eta

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^1} - \frac{1}{2}\frac{x^1}{1!} = 0 \tag{3.1es}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^3} + \frac{1}{2}\frac{x^3}{3!} = \frac{x^1}{1!}\eta(2) \tag{3.2es}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^5} - \frac{1}{2}\frac{x^5}{5!} = \frac{x^1}{1!}\eta(4)-\frac{x^3}{3!}\eta(2) \tag{3.3es}$$

$$\vdots$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^{2n-1}} + \frac{(-1)^n x^{2n-1}}{2(2n-1)!} = -\sum_{s=1}^{n-1}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\eta(2n-2s) \tag{3.nes}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^2} + \frac{1}{2}\frac{x^2}{2!} = \frac{x^0}{0!}\eta(2) \tag{3.1ec}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^4} - \frac{1}{2}\frac{x^4}{4!} = \frac{x^0}{0!}\eta(4)-\frac{x^2}{2!}\eta(2) \tag{3.2ec}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^6} + \frac{1}{2}\frac{x^6}{6!} = \frac{x^0}{0!}\eta(6)-\frac{x^2}{2!}\eta(4)+\frac{x^4}{4!}\eta(2) \tag{3.3ec}$$

$$\vdots$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^{2n}} - \frac{(-1)^n x^{2n}}{2(2n)!} = \sum_{s=0}^{n-1}\frac{(-1)^s x^{2s}}{(2s)!}\eta(2n-2s) \tag{3.nec}$$

(2) Odd Eta

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^1} + \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)\left|B_{2r}\right|}{2r(2r)!}x^{2r} = \frac{x^0}{0!}\eta(1) \tag{3.1oc}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^3} - \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+2)!}x^{2r+2} = \frac{x^0}{0!}\eta(3) - \frac{x^2}{2!}\eta(1) \qquad (3.2oc)$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^5} + \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+4)!}x^{2r+4} = \frac{x^0}{0!}\eta(5) - \frac{x^2}{2!}\eta(3) + \frac{x^4}{4!}\eta(1)$$

$$\vdots$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^{2n-1}} - (-1)^n\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r+2n-2}}{2r(2r+2n-2)!}$$

$$= \sum_{s=0}^{n-1}\frac{(-1)^s x^{2s}}{(2s)!}\eta(2n-1-2s) \qquad (3.noc)$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^2} + \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+1)!}x^{2r+1} = \frac{x^1}{1!}\eta(1) \qquad (3.1os)$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^4} - \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+3)!}x^{2r+3} = \frac{x^1}{1!}\eta(3) - \frac{x^3}{3!}\eta(1) \qquad (3.2os)$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^6} + \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|}{2r(2r+5)!}x^{2r+5} = \frac{x^1}{1!}\eta(5) - \frac{x^3}{3!}\eta(3) + \frac{x^5}{5!}\eta(1)$$

$$\vdots$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^{2n}} - (-1)^n\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r+2n-1}}{2r(2r+2n-1)!}$$

$$= -\sum_{s=1}^{n}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\eta(2n+1-2s) \qquad (3.nos)$$

Proof

Comparing Lemma 1.5.1 and Lemma 1.5.2 and using $e^{irx} = \cos rx + i\sin rx$, we obtain the desired expressions. In addition, $\zeta(n)$ can be obtained immediately by multiplying this by $1/\left(1-2^{1-n}\right)$.

1.5.4 Dirichlet Eta (Exponential form)

If the above formula is expressed by an exponential function, it is as follows.

Formula 1.5.4

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers, the following expressions hold for $0<|x|<\pi$.

$$\eta(1) = -\frac{1}{2}\frac{(ix)^1}{1!} - \sum_{r=1}^{\infty}(-1)^r\frac{e^{irx}}{r^1} + \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}(ix)^0}{2r(2r)!}$$

$$\eta(n) = -\frac{1}{2}\frac{(ix)^n}{n!} - \sum_{r=1}^{\infty}(-1)^r\frac{e^{irx}}{r^n} + \sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}(ix)^{n-1}}{2r(2r+n-1)!} - \sum_{s=1}^{n-1}\frac{(ix)^s}{s!}\eta(n-s)$$

1.5.5 Higher order differentiation of $ie^{ix}/(1+e^{ix})$

Although the higher order differentiation of (1.0) does not hold as a equation, it is useful to generate Riemann Zeta at negative integers. And using the coefficients that make powers constant, we obtain the same results as **1.2.5** .

1.6 Generating function of csc x family family

Zeta can be obtained also when the following expression is integrated.

$$\frac{ie^{ix}}{1-e^{2ix}} = i\left(e^{ix}+e^{3ix}+e^{5ix}+e^{-7ix}+\cdots\right) \qquad \left(=-\frac{csc\,x}{2}\right) \tag{1.0}$$

1.6.1 Termwise Higher Integral of Fourier Series of $ie^{ix}/(1-e^{2ix})$

Lemma 1.6.1

When $\lambda(x)=\sum_{r=1}^{\infty}\frac{1}{(2r-1)^x}$, the following expressions hold for $|x|<\pi$.

$$\int_0^x \frac{ie^{ix}}{1-e^{2ix}}dx = i^0\sum_{r=1}^{\infty}\frac{e^{(2r-1)ix}}{(2r-1)^1} - i^0\lambda(1) \tag{1.1}$$

$$\int_0^x\!\!\int_0^x \frac{ie^{ix}}{1-e^{2ix}}dx^2 = i^{-1}\sum_{r=1}^{\infty}\frac{e^{(2r-1)ix}}{(2r-1)^2} - i^{-1}\frac{x^0}{0!}\lambda(2) - i^0\frac{x^1}{1!}\lambda(1) \tag{1.2}$$

$$\int_0^x\!\!\int_0^x\!\!\int_0^x \frac{ie^{ix}}{1-e^{2ix}}dx^3 = i^{-2}\sum_{r=1}^{\infty}\frac{e^{(2r-1)ix}}{(2r-1)^3} - \frac{i^{-2}x^0}{0!}\lambda(3) - \frac{i^{-1}x^1}{1!}\lambda(2) - \frac{i^0x^2}{2!}\lambda(1) \tag{1.3}$$

$$\vdots$$

$$\int_0^x\!\!\cdots\!\int_0^x \frac{ie^{ix}}{1-e^{2ix}}dx^n = i^{-(n-1)}\sum_{r=1}^{\infty}\frac{e^{(2r-1)ix}}{(2r-1)^n} - \sum_{s=0}^{n-1}\frac{i^{-(n-1-s)}x^s}{s!}\lambda(n-s) \tag{1.n}$$

Proof

$ie^{ix}/(1-e^{2ix})$ can be expanded to Fourier series as follows.

$$\frac{ie^{ix}}{1-e^{2ix}} = i\left(e^{ix}+e^{3ix}+e^{5ix}+e^{-7ix}+\cdots\right) \qquad \left(=-\frac{csc\,x}{2}\right) \tag{1.0}$$

i.e.

$$-\frac{csc\,x}{2} = -(sin\,x+sin\,3x+sin\,5x+\cdots)+i(cos\,x+cos\,3x+cos\,5x+\cdots) \tag{1.0'}$$

(1.0) also does not hold as an equation. However, the higher order integral of both sides of (1.0) holds as an equation. Integrating both sides of (1.0) with respect to x from 0 to x ,

$$\int_0^x \frac{ie^{ix}}{1-e^{2ix}}dx = i^0\left[\frac{e^{ix}}{1^1}+\frac{e^{3ix}}{3^1}+\frac{e^{5ix}}{5^1}+\frac{e^{7ix}}{7^1}+\cdots\right]_0^x$$

$$= i^0\left(\frac{e^{ix}}{1^1}+\frac{e^{3ix}}{3^1}+\frac{e^{5ix}}{5^1}+\frac{e^{7ix}}{7^1}+\cdots\right) - i^0\left(\frac{e^{0}}{1^1}+\frac{e^{0x}}{3^1}+\frac{e^{0}}{5^1}+\frac{e^{0}}{7^1}+\cdots\right)$$

Here, let $\lambda(x)=\sum_{r=1}^{\infty}\frac{1}{(2r-1)^x}$, then

$$\int_0^x\!\!\int_0^x \frac{ie^{ix}}{1-e^{2ix}}dx^2 = i^{-1}\left[\frac{e^{(2r-1)ix}}{(2r-1)^2}\right]_0^x - i^0\frac{x^1}{1!}\lambda(1)$$

$$= i^{-1}\sum_{r=1}^{\infty}\frac{e^{(2r-1)ix}}{(2r-1)^2} - i^{-1}\sum_{r=1}^{\infty}\frac{e^0}{(2r-1)^2} - i^0\frac{x^1}{1!}\lambda(1)$$

$$= i^{-1}\sum_{r=1}^{\infty}\frac{e^{(2r-1)ix}}{(2r-1)^2} - i^{-1}\frac{x^0}{0!}\lambda(2) - i^0\frac{x^1}{1!}\lambda(1)$$

Hereafter, in a similar way we obtein the desired expressions. Of course, these are the collateral integrals.

1.6.2 Termwise Higher Integral of Taylor Series of $ie^{ix}/(1-e^{2ix})$

Lemma 1.6.2

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and $H_s=\sum_{t=1}^{s}1/t$ are Harmonic numbers, the following expressions hold for $|x|<\pi$.

$$\int_0^x\frac{ie^{ix}}{1-e^{2ix}}dx = -\frac{1}{2}\frac{x^0}{0!}\left(\log\frac{x}{2}-H_0\right)-\frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)|B_{2r}|x^{2r}}{2r(2r)!}+\frac{x^0}{0!}\left\{\frac{i\pi}{4}-\lambda(1)\right\}$$

$$\int_0^x\int_0^x\frac{ie^{ix}}{1-e^{2ix}}dx^2 = -\frac{1}{2}\frac{x^1}{1!}\left(\log\frac{x}{2}-H_1\right)-\frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)|B_{2r}|x^{2r+1}}{2r(2r+1)!}+\frac{x^1}{1!}\left\{\frac{i\pi}{4}-\lambda(1)\right\}$$

$$\int_0^x\int_0^x\int_0^x\frac{ie^{ix}}{1-e^{2ix}}dx^3 = -\frac{1}{2}\frac{x^2}{2!}\left(\log\frac{x}{2}-H_2\right)-\frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)|B_{2r}|x^{2r+2}}{2r(2r+2)!}+\frac{x^2}{2!}\left\{\frac{i\pi}{4}-\lambda(1)\right\}$$

$$\vdots$$

$$\int_0^x\cdots\int_0^x\frac{ie^{ix}}{1-e^{2ix}}dx^n = -\frac{x^{n-1}}{2(n-1)!}\left(\log\frac{x}{2}-H_{n-1}\right)-\frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)|B_{2r}|x^{n-1+2r}}{2r(n-1+2r)!}$$

$$+\frac{x^{n-1}}{2(n-1)!}\left\{\frac{i\pi}{4}-\lambda(1)\right\}$$

Proof

$$\csc x = \frac{1}{x}+\sum_{r=1}^{\infty}\frac{(2^{2r}-2)|B_{2r}|}{2r(2r-1)!}x^{2r-1} \quad , \qquad \frac{ie^{ix}}{1-e^{2ix}} = -\frac{\csc x}{2}$$

From this,

$$\frac{ie^{ix}}{1-e^{2ix}} = -\frac{1}{2x}-\frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)|B_{2r}|}{2r(2r-1)!}x^{2r-1} \qquad |x|<\pi$$

Integrating both sides of this with respect to x from 0 to x,

$$\int_0^x\frac{ie^{ix}}{1-e^{2ix}}dx = -\frac{1}{2}\left[\log x+\sum_{r=1}^{\infty}\frac{(2^{2r}-2)|B_{2r}|}{2r(2r)!}x^{2r}\right]_0^x$$

$$= -\frac{1}{2}\log x-\frac{1}{2}\sum_{r=1}^{\infty}\frac{(2^{2r}-2)|B_{2r}|}{2r(2r)!}x^{2r}+\frac{1}{2}\log 0$$

Here,

$$\frac{1}{2}\log 0 = \frac{1}{2}\log\left(2\cdot e^{\frac{i\pi}{2}}\cdot 0\right) = \frac{\log 2}{2}+\frac{1}{2}\log e^{\frac{i\pi}{2}}+\frac{\log 0}{2} = \frac{\log 2}{2}+\frac{i\pi}{4}+\frac{\log 0}{2}$$

And using $\lambda(1) = -\dfrac{\log 0}{2}$ (See **1.3.2**)

$$\frac{1}{2}\log 0 = \frac{1}{2}\log 2 + \frac{i\pi}{4} - \lambda(1)$$

Substitutng this for the above,

$$\int_0^x \frac{ie^{ix}}{1-e^{2ix}}dx = -\frac{1}{2}\log\frac{x}{2} - \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)\left|B_{2r}\right|}{2r(2r)!}x^{2r} + \frac{i\pi}{4} - \lambda(1)$$

Next, integrating both sides of this with respect to x from 0 to x,

$$\int_0^x\int_0^x \frac{ie^{ix}}{1-e^{2ix}}dx^2 = -\frac{1}{2}\left[\frac{x^1}{1!}\left(\log\frac{x}{2} - H_1\right) + \sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)\left|B_{2r}\right|x^{2r+1}}{2r(2r+1)!}\right]_0^x + \frac{x^1}{1!}\left\{\frac{i\pi}{4} - \lambda(1)\right\}$$

$$= -\frac{1}{2}\frac{x^1}{1!}\left(\log\frac{x}{2} - H_1\right) - \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)\left|B_{2r}\right|x^{2r+1}}{2r(2r+1)!} + \frac{x^1}{1!}\left\{\frac{i\pi}{4} - \lambda(1)\right\}$$

Hereafter, in a similar way we obtein the desired expressions. Of course, these are the collateral integrals.

1.6.3 Dirichlet Lambda Polynomial (Trigonometric form)

Two lemmas mentioned above are not useful in itself. However, comparing these, we can obtain the following Dirichlet lambda polynomials.

Formula 1.6.3

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and $H_s = \sum_{t=1}^{s} 1/t$ are Harmonic numbers, the following expressions hold for $0 < |x| < \pi$.

(1) Even Lambda

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^1} - \frac{x^0}{0!}\frac{\pi}{4} = 0$$

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^3} + \frac{x^2}{2!}\frac{\pi}{4} = \frac{x^1}{1!}\lambda(2)$$

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^5} - \frac{x^4}{4!}\frac{\pi}{4} = \frac{x^1}{1!}\lambda(4) - \frac{x^3}{3!}\lambda(2)$$

$$\vdots$$

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n-1}} + \frac{(-1)^n}{4}\frac{\pi x^{2n-2}}{(2n-2)!} = -\sum_{s=1}^{n-1}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\lambda(2n-2s)$$

$$\sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^2} + \frac{x^1}{1!}\frac{\pi}{4} = \frac{x^0}{0!}\lambda(2)$$

$$\sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^4} - \frac{x^3}{3!}\frac{\pi}{4} = \frac{x^0}{0!}\lambda(4) - \frac{x^2}{2!}\lambda(2)$$

$$\sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^6} + \frac{x^5}{5!}\frac{\pi}{4} = \frac{x^0}{0!}\lambda(6) - \frac{x^2}{2!}\lambda(4) + \frac{x^4}{4!}\lambda(2)$$

$$\vdots$$

$$\sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n}}-\frac{(-1)^n}{4}\frac{\pi x^{2n-1}}{(2n-1)!}=\sum_{s=0}^{n-1}\frac{(-1)^s x^{2s}}{(2s)!}\lambda(2n-2s)$$

(2) Odd Lambda

$$\sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^{1}}+\frac{1}{2}\frac{x^0}{0!}\left(\log\frac{x}{2}-H_0\right)+\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r}}{2r(2r)!}=0$$

$$\sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^{3}}-\frac{1}{2}\frac{x^2}{2!}\left(\log\frac{x}{2}-H_2\right)-\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r+2}}{2r(2r+2)!}=\frac{x^0}{0!}\lambda(3)$$

$$\sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^{5}}+\frac{1}{2}\frac{x^4}{4!}\left(\log\frac{x}{2}-H_4\right)+\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r+4}}{2r(2r+4)!}=\frac{x^0}{0!}\lambda(5)-\frac{x^2}{2!}\lambda(3)$$

$$\vdots$$

$$\sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n-1}}-\frac{(-1)^n x^{2n-2}}{2(2n-2)!}\left(\log\frac{x}{2}-H_{2n-2}\right)-\frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r+2n-2}}{2r(2r+2n-2)!}$$

$$=\sum_{s=0}^{n-2}\frac{(-1)^s x^{2s}}{(2s)!}\lambda(2n-1-2s)$$

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2}}+\frac{1}{2}\frac{x^1}{1!}\left(\log\frac{x}{2}-H_1\right)+\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r+1}}{2r(2r+1)!}=0$$

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^{4}}-\frac{1}{2}\frac{x^3}{3!}\left(\log\frac{x}{2}-H_3\right)-\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r+3}}{2r(2r+3)!}=\frac{x^1}{1!}\lambda(3)$$

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^{6}}+\frac{1}{2}\frac{x^5}{5!}\left(\log\frac{x}{2}-H_5\right)+\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r+5}}{2r(2r+5)!}=\frac{x^1}{1!}\lambda(5)-\frac{x^3}{3!}\lambda(3)$$

$$\vdots$$

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n}}-\frac{(-1)^n x^{2n-1}}{2(2n-1)!}\left(\log\frac{x}{2}-H_{2n-1}\right)-\frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r+2n-1}}{2r(2r+2n-1)!}$$

$$=-\sum_{s=1}^{n-1}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\lambda(2n+1-2s)$$

Proof

Comparing Lemma 1.6.1 and Lemma 1.6.2 and using $e^{irx}=\cos rx+i\sin rx$, we obtain the desired expressions. In addition, $\zeta(n)$ can be obtained immediately by multiplying this by $1/\left(1-2^{-n}\right)$.

1.6.4 Dirichlet Lambda (Exponential form)

If the above formula is expressed by an exponential function, it is as follows.

Formula 1.6.4

When $B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ \cdots$ are Bernoulli numbers and $H_s=\sum_{t=1}^{s}1/t$ are Harmonic numbers, the following expressions hold for $0<|x|<\pi$.

$$0=\frac{1}{2}\frac{(ix)^0}{0!}\left(\log\frac{x}{2}-H_0\right)-\frac{i\pi}{4}\frac{(ix)^0}{0!}+\sum_{r=1}^{\infty}\frac{e^{(2r-1)ix}}{(2r-1)^1}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|x^{2r}(ix)^0}{2r(2r)!}$$

$$\lambda(2) = \frac{1}{2}\frac{(ix)^1}{1!}\left(\log\frac{x}{2} - H_1\right) - \frac{i\pi}{4}\frac{(ix)^1}{1!} + \sum_{r=1}^{\infty}\frac{e^{(2r-1)ix}}{(2r-1)^2} + \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)\left|B_{2r}\right| x^{2r}(ix)^1}{2r(2r+1)!}$$

$$\lambda(n) = \frac{1}{2}\frac{(ix)^{n-1}}{(n-1)!}\left(\log\frac{x}{2} - H_{n-1}\right) - \frac{i\pi}{4}\frac{(ix)^{n-1}}{(n-1)!} + \sum_{r=1}^{\infty}\frac{e^{(2r-1)ix}}{(2r-1)^n}$$

$$+ \frac{1}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)\left|B_{2r}\right| x^{2r}(ix)^{n-1}}{2r(2r+n-1)!} - \sum_{s=1}^{n-2}\frac{(ix)^s}{s!}\lambda(n-s)$$

By-products

$$-\log\frac{0}{2} = 2\lambda(1) - \frac{\pi i}{2}$$

1.7 Coefficients that make powers constant

1.7.1 Definition

We call "***Polynomial that makes powers of geometric series constant***" the polynomial that satisfies the following equation. And we call "***Coefficients that make powers constant***" the coefficients ${}_nD_r$.

$$1^n + 2^n x + 3^n x^2 + 4^n x^3 + \cdots = \frac{\sum_{r=1}^{n} {}_nD_r x^{r-1}}{(1-x)^{n+1}} \qquad |x| \leqq 1 \qquad (1.0)$$

These are the following numbers concretely.

$$\begin{matrix} {}_1D_1 \\ {}_2D_1 \ {}_2D_2 \\ {}_3D_1 \ {}_3D_2 \ {}_3D_3 \\ {}_4D_1 \ {}_4D_2 \ {}_4D_3 \ {}_4D_4 \\ {}_5D_1 \ {}_5D_2 \ {}_5D_3 \ {}_5D_4 \ {}_5D_5 \\ {}_6D_1 \ {}_6D_2 \ {}_6D_3 \ {}_6D_4 \ {}_6D_5 \ {}_6D_6 \\ {}_7D_1 \ {}_7D_2 \ {}_7D_3 \ {}_7D_4 \ {}_7D_5 \ {}_7D_6 \ {}_7D_7 \\ \vdots \end{matrix} = \begin{matrix} 1 \\ 1 \quad 1 \\ 1 \quad 4 \quad 1 \\ 1 \quad 11 \quad 11 \quad 1 \\ 1 \quad 26 \quad 66 \quad 26 \quad 1 \\ 1 \quad 57 \quad 302 \quad 302 \quad 57 \quad 1 \\ 1 \quad 120 \quad 1191 \quad 2416 \quad 1191 \quad 120 \quad 1 \\ \vdots \end{matrix}$$

Explanation

Assume a geometric series as follows.

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \qquad |x| \leqq 1 \qquad (0)$$

Raising (0) to the 2nd power (= differentiate 1time and divide by 1!) ,

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \frac{1}{(1-x)^2} \qquad |x| \leqq 1 \qquad (1)$$

The coefficients of this series are all the 1st power.

Raising (0) to the 3rd power (= differentiate 2times and divide by 2!) ,

$$1 + 3x + 6x^2 + 10x^3 + 15x^4 + \cdots = \frac{1}{(1-x)^3} \qquad |x| \leqq 1 \qquad (2')$$

The coefficients of this series are not the 2nd power.
However, multipling (2') by a polynomial $1+x$,

$$1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + 5^2 x^4 + \cdots = \frac{1+x}{(1-x)^3} \qquad |x| \leqq 1 \qquad (2)$$

Then, the coefficients of this series become all the 2nd power.

Raising (0) to the 4th power (= differentiate 3times and divide by 3!) ,

$$1 + 4x + 10x^2 + 20x^3 + 35x^4 + \cdots = \frac{1}{(1-x)^4} \qquad |x| \leqq 1 \qquad (3')$$

The coefficients of this series are not the 3nd power.
However, multipling (3') by a polynomial $1+4x+x^2$,

$$1^3 + 2^3 x + 3^3 x^2 + 4^3 x^3 + 5^3 x^4 + \cdots = \frac{1+4x+x^2}{(1-x)^4} \qquad |x| \leqq 1 \qquad (3)$$

Then, the coefficients of this series become all the 3rd power.

In this way, the polynomial $\sum_{r=1}^{n} {}_nD_r\, x^{r-1}$ can make the power of coefficients of a geometric series constant such that $1^n, 2^n, 3^n, 4^n, \cdots$.

1.7.2 Properties of the coefficients

When ${}_nD_r$ are Coefficients that make powers constant and Tangent Numbers are T_n

$$T_1=1\ ,\ T_3=2\ ,\ T_5=16\ ,\ T_7=272\ ,\ T_9=7936\ ,\ \cdots\ ,\qquad T_0=T_2=T_4=T_6=\ \cdots\ =0$$

the followswing expressions hold.

$$\sum_{r=1}^{n} {}_nD_r = n\,! \tag{2.1}$$

$${}_nD_2 = {}_{n-1}D_2 + 2^{n-1} - 1 \qquad\qquad n=3,4,5,\cdots \tag{2.2}$$

$$T_{2n-1} = (-1)^{n-1}\sum_{r=1}^{2n-1}(-1)^{r-1}\ {}_{2n-1}D_r \qquad\qquad n=1,2,3,\cdots \tag{2.3}$$

(2.1) is illustrated as follows.

$$\begin{array}{cc}
1 & = 1! \\
1 + 1 & = 2! \\
1 + 4 + 1 & = 3! \\
1 + 11 + 11 + 1 & = 4! \\
1 + 26 + 66 + 26 + 1 & = 5! \\
1 + 57 + 302 + 302 + 57 + 1 & = 6! \\
1+120+1191+2416+1191+120+1 & = 7! \\
\vdots & \vdots
\end{array}$$

(2.2) is shown as follows.

n	2	3	4	5	6	7	$\cdots$
${}_nD_2$	1	4	11	26	57	120	$\cdots$
difference	2^2-1	2^3-1	2^4-1	2^5-1	2^6-1		$\cdots$

(2.3) is illustrated as follows.

$$\begin{array}{ccc}
1 & = & 1 = T_1 \\
1 - 1 & = & 0 = T_2 \\
1 - 4 + 1 & = & -2 = -T_3 \\
1 - 11 + 11 - 1 & = & 0 = T_4 \\
1 - 26 + 66 - 26 + 1 & = & 16 = T_5 \\
1 - 57 + 302 - 302 + 57 - 1 & = & 0 = T_6 \\
1-120+1191-2416+1191-120+1 & = & -272 = -T_7 \\
\vdots & & \vdots
\end{array}$$

$$\sum_{r=1}^{n-1}(-1)^{r-1}\ {}_{n-1}D_r = i^{n-2}T_{n-1} \qquad i=\sqrt{-1}\quad ,\quad n=2,3,4,\cdots$$

1.7.3 Calculation method one by one for the Coefficients that make powers constant.

The calculation method that the author devised is shown as follows.

		Calculating forumla
2	1 1	
	2,2	
3	1 4 1	4 = 1×2 + 1×2
	3,2 2,3	
4	1 11 11 1	11 = 1×3 + 4×2
	4,2 3,3 2,4	
5	1 26 66 26 1	26 = 1×4 +11×2 , 66 = 11×3 + 11×3
	5,2 4,3 3,4 2,5	
6	1 57 302 302 57 1	57 = 1×5 +26×2 , 302 = 26×4 + 66×3
	6,2 5,3 4,4 3,5 2,6	
7	1 120 1191 2416 1191 120 1	120= 1×6 +57×2 , 1191 = 57×5 + 302×3 , 2416 = 302×4 + 302×4
	⋮	⋮

1.7.4 Direct calculation method for the Coefficients that make powers constant.

Formula1.7.4

$$ {}_nD_r = \sum_{k=0}^{r-1}(-1)^k\binom{n+1}{k}(r-k)^n \qquad n=1,2,3,\cdots \tag{4.1}$$

Proof

From the definition (1.0) ,

$$(1-x)^{n+1}\left(1^n+2^nx^1+3^nx^2+4^nx^3+\cdots\right)=\sum_{r=1}^{n}{}_nD_rx^{r-1}$$

Here,

$$(1-x)^{n+1}=\binom{n+1}{0}x^0-\binom{n+1}{1}x^1+\cdots+(-1)^{n+1}\binom{n+1}{n+1}x^{n+1}$$

Substitute this for the above. Then, the left side is

$$\left\{\binom{n+1}{0}x^0-\binom{n+1}{1}x^1+\binom{n+1}{2}x^2-\binom{n+1}{3}x^3+\cdots+(-1)^{n+1}\binom{n+1}{n+1}x^{n+1}\right\}$$

$$\times\left(1^nx^0+2^nx^1+3^nx^2+4^nx^3+\cdots\right)$$

$$=1^n\binom{n+1}{0}x^0+\left\{\binom{n+1}{0}2^n-\binom{n+1}{1}1^n\right\}x^1$$

$$+\left\{\binom{n+1}{0}3^n-\binom{n+1}{1}2^n+\binom{n+1}{2}1^n\right\}x^2$$

$$+\left\{\binom{n+1}{0}4^n-\binom{n+1}{1}3^n+\binom{n+1}{2}2^n-\binom{n+1}{3}1^n\right\}x^3$$

$$\vdots$$

The right side is ${}_nD_1 x^0 + {}_nD_2 x^1 + \cdots + {}_nD_n x^{n-1}$. In order for the both sides to be equal for arbitrary x, (4.1) is necessary.

Example

$${}_3D_2 = \sum_{k=0}^{1} (-1)^k \binom{4}{k} (2-k)^3 = \binom{4}{0} 2^3 - \binom{4}{1} 1^3 = 4$$

$${}_5D_3 = \sum_{k=0}^{2} (-1)^k \binom{6}{k} (3-k)^6 = \binom{6}{0} 3^5 - \binom{6}{1} 2^5 + \binom{6}{2} 1^5 = 66$$

1.7.5 The coefficient , Eulerian Number and Tangent Number

Mr.Sugioka instituted the following hypothesis on his site「数学の研究」in July, 2006.

$$\left(1-2^n\right)\zeta(1-n) = \frac{\text{odd number}}{2^m} \qquad n = 2, 4, 6, \cdots \quad m : \text{natural number}$$

Sugimoto and Kono sent him the following answers that affirm the hypothesis respectively.

$$\left|\left(1-2^n\right)\zeta(1-n)\right| = \frac{T_{n-1}}{2^n} \qquad n=2,3,4,\cdots \tag{a}$$

$$\left(1-2^n\right)\zeta(1-n) = \frac{\sum_{r=1}^{n-1} (-1)^{r-1} {}_{n-1}D_r}{2^n} \qquad n=2,3,4,\cdots \tag{b}$$

From (a) and (b) , we obtained

$$T_{n-1} = \left|\sum_{r=1}^{n-1} (-1)^{r-1} {}_{n-1}D_r\right| \qquad n=2,3,4,\cdots$$

i.e.

$$T_{2n-1} = (-1)^{n-1} \sum_{r=1}^{2n-1} (-1)^{r-1} {}_{2n-1}D_r \qquad n=1,2,3,\cdots \tag{2.3}$$

In addition, at this time, it turned out that the coefficient is in fact Eulerian Number.
That is ,

$${}_nD_r = \left\langle {n \atop r-1} \right\rangle = \sum_{k=0}^{r} (-1)^k \binom{n+1}{k} (r-k)^n \qquad \left(\left\langle {n \atop r} \right\rangle \text{ is an Eulerian Number} \right)$$

2012.04.04

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Alien's Mathematics