02 Dirichlet Series & Logarithmic Power Series

2.1 Definition & Theorems

Definition 2.1.1 (Ordinary Dirichlet Series)

When s, a_n $(n=1,2,3,\cdots)$ are complex numbers, we call the following **Ordinary Dirichlet Series**.

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \cdots$$

Note

Giving $\lambda_n = \log n$ in the general Dirichlet series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ $(\lambda_n \in R\ ,\ \lambda_n < \lambda_{n+1}\quad n=1,2,3,\cdots)$, we obtain the ordinary Dirichlet series. When we say merely Dirichlet series, it means ordinary Dirichlet series in many cases. So, in this chapter, we follow this custom.

Theorem 2.1.2

When σ, t are real numbers and $f(s) = \sum_{n=1}^{\infty} a_n / n^s$ $(s = \sigma + t\,i)$ is Dirichlet series,
One of the followings holds.

1. $f(s)$ converges for arbitrary s .

2. $f(s)$ diverges for arbitrary s .

3. There exist a certain real number σ_c such that $f(s)$ converges for s $s.t.$ $\sigma > \sigma_c$ and $f(s)$ diverges for s $s.t.$ $\sigma < \sigma_c$.

This σ_c is called the **line of convergence**. By convention, $\sigma_c = \infty$ if $f(s)$ converges nowhere and $\sigma_c = -\infty$ if $f(s)$ converges everywhere on the complex plane.

How to calculate σ_c

1. When $\sum_{k=1}^{n} a_k$ is divergent,

$$\sigma_c = \lim_{n\to\infty} sup \frac{\log\left|a_1 + a_2 + \cdots + a_n\right|}{\log n}$$

2. When $\sum_{k=1}^{n} a_k$ is convergent,

$$\sigma_c = \lim_{n\to\infty} sup \frac{\log\left|a_{n+1} + a_{n+2} + a_{n+3} \cdots\right|}{\log n}$$

Example 1.1 p - series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

Then,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} 1 = n$$

Since this is divergent,

$$\sigma_c = \lim_{n \to \infty} sup \frac{log\,|n|}{log\,n} = 1$$

Example 1.2 Dirichlet Eta series

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + - \cdots$$

Then,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (-1)^{k-1} = 1 \; or \; 0$$

Since this is divergent,

$$\sigma_c = \lim_{n \to \infty} sup \frac{log\,|1 \; or \; 0|}{log\,n} = \lim_{n \to \infty} \frac{log\,|1|}{log\,n} = 0$$

Absolute Convergence

A Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n / n^s$ is **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n| / n^s$ is convergent. There exist a σ_a $(0 \le \sigma_a - \sigma_c \le 1)$ such that $f(s)$ converges absolutely for $\sigma > \sigma_a$ and converges non-absolutely for $\sigma < \sigma_a$. This σ_a is called the **line of absolute convergence**.

How to calculate σ_a

1. When $\sum_{k=1}^{n} |a_k|$ is divergent,

$$\sigma_a = \lim_{n \to \infty} sup \frac{log\left(|a_1| + |a_2| + \cdots + |a_n| \right)}{log\,n}$$

2. When $\sum_{k=1}^{n} |a_k|$ is convergent,

$$\sigma_a = \lim_{n \to \infty} sup \frac{log\left(|a_{n+1}| + |a_{n+2}| + |a_{n+3}| \cdots \right)}{log\,n}$$

Example 2.1 p- series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

Then,

$$\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} |1| = n$$

Since this is divergent,

$$\sigma_a = \lim_{n\to\infty} sup \frac{\log n}{\log n} = 1$$

Example 2.2 Dirichlet Eta series

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + - \cdots$$

Then,

$$\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} \left| (-1)^{k-1} \right| = n$$

Since this is divergent,

$$\sigma_a = \lim_{n\to\infty} sup \frac{\log n}{\log n} = 1$$

Uniform Convergence

As seen in Theorem 1.1.2 (**1.1**) , general Dirichlet serie $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ converges uniformly in a certain domain. This is the same also about the ordinary Dirichlet series.

There exist a σ_u $(0 \le \sigma_u - \sigma_c \le 1/2)$ such that $f(s)$ converges uniformly for $\sigma > \sigma_u$ and converges non-uniformly for $\sigma < \sigma_u$. This σ_u is called the **line of uniform convergence**.

How to calculate σ_u

$$\sigma_u = \lim_{n\to\infty} sup \frac{\log T_n}{\log n}$$

where

$$T_n = \sup_{|t|<\infty} \left| \sum_{k=1}^{n} \frac{a_k}{k^{it}} \right| \tag{3.1}$$

$$= \sup_{|t|<\infty} \sqrt{\left(\sum_{k=1}^{n} a_k \cos(t \log k) \right)^2 + \left(\sum_{k=1}^{n} a_k \sin(t \log k) \right)^2} \tag{3.2}$$

$$= \sup_{|t|<\infty} \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k \cos\left(t \log \frac{k}{j} \right)} \tag{3.3}$$

Proof

We prove the second line and the third line of the proviso.

Since $k = e^{\log k}$,

$$\sum_{k=1}^{n} a_k k^{-it} = \sum_{k=1}^{n} a_k e^{-it \log k}$$

Substitute $e^{-it \log k} = \cos(t \log k) - i \sin(t \log k)$ for the right side,

$$\sum_{k=1}^{n} a_k e^{-it \log k} = \sum_{k=1}^{n} a_k \{\cos(t \log k) - i \sin(t \log k)\}$$

.

i.e.

$$\sum_{k=1}^{n} \frac{a_k}{k^{it}} = \sum_{k=1}^{n} a_k \cos(t \log k) - i \sum_{k=1}^{n} a_k \sin(t \log k)$$

Transforming the right side into a pole form,

$$\sum_{k=1}^{n} \frac{a_k}{k^{it}} = a\,(\cos\phi + i \sin\phi)$$

Where,

$$a = \sqrt{\left(\sum_{k=1}^{n} a_k \cos(t \log k)\right)^2 + \left(\sum_{k=1}^{n} a_k \sin(t \log k)\right)^2}$$

$$\phi = \tan^{-1}\left\{\sum_{k=1}^{n} a_k \cos(t \log k)\ ,\ -\sum_{k=1}^{n} a_k \sin(t \log k)\right\}$$

($\tan^{-1}(x,y)$ is inverse tangent function in consideration of 4 quadrants.)

Taking absolute value,

$$\left|\sum_{k=1}^{n} \frac{a_k}{k^{it}}\right| = |a\,(\cos\phi + i \sin\phi)| = |a|\,|\cos\phi + i \sin\phi| = a$$

i.e.

$$\left|\sum_{k=1}^{n} \frac{a_k}{k^{it}}\right| = \sqrt{\left(\sum_{k=1}^{n} a_k \cos(t \log k)\right)^2 + \left(\sum_{k=1}^{n} a_k \sin(t \log k)\right)^2}$$

Expanding the inner of the $\sqrt{\ }$,

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j a_k \{\cos(t \log j)\cos(t \log k) + \sin(t \log j)\sin(t \log k)\}$$

Here, since

$$\cos(t \log j)\cos(t \log k) + \sin(t \log j)\sin(t \log k) = \cos\left(y \log \frac{k}{j}\right)$$

we obtain (3.3) .

Example 3.1 p - series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

Then, since $a_k = 1$,

$$T_n = \sup_{|t|<\infty} \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} \cos\left(t \log \frac{k}{j}\right)} = \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} 1} = \sqrt{n^2} = n$$

$$\sigma_u = \lim_{n \to \infty} \sup \frac{\log n}{\log n} = 1$$

Example 3.2 Dirichlet Eta series

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + - \cdots$$

Though $a_k = (-1)^{k-1}$, calculation of T_n is difficult even if which formula is used.
So, we put

$$\sigma(n,t) = \frac{\log\left|\sum_{k=1}^{n}(-1)^{k-1}k^{-it}\right|}{\log n}$$

Calculatiing for $n=5,800$, $t=2000\sim2011$, we obtain the follows.

```
Table[σ[5800, t], {t, 2000, 2011}]
```

```
{0.382371 , 0.401028 , 0.498087 , 0.402331 , 0.371634 , 0.368319 ,
 0.41249 , 0.449513 , 0.448168 , 0.387783 , 0.383517 , 0.392516 }
```

When we consider the result of these numerical computation, it seems to be $\sigma_u = 1/2$.

Theorem 2.1.3 (Holomorphy)

If Dirichlet serie $f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ $(s = \sigma + t\,i)$ converges for $\sigma > \sigma_c$,

$f(s)$ is **holomophic** at $\sigma > \sigma_c$. And the derivative of $f(s)$ is given as follows.

$$f^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty}\frac{a_n \log^k n}{n^s}$$

Theorem 2.1.4 (Uniqueness)

Let two Dirichlet series are as follows.

$$f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s} \quad , \quad g(s) = \sum_{n=1}^{\infty}\frac{b_n}{n^s}$$

If both are convergent in a certain domain and $f(s) = g(s)$ holds at there, $a_n = b_n$ for $n = 1, 2, 3, \cdots$.

Bibliography

「数論入門」D.B.ザギヤー 著、片山考次 訳　岩波　1990年, etc.

2.2 Dirichlet Series & Power Series

Theorem 2.2.1 (Mellin transform)

Let Dirichlet series $f(s)$ and Power series $F(z)$ are as follows.

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad , \qquad F(z) = \sum_{n=1}^{\infty} a_n z^n$$

Then, the following expression holds in the domain in which $f(s)$ converges absolutely

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} F(e^{-t}) t^{s-1} dt \tag{1.1}$$

Proof

Substituting $z = e^{-w}$ for $F(z)$,

$$F(e^{-w}) = \sum_{n=1}^{\infty} a_n e^{-nw}$$

Integrating both sides of this m times with respect to w from ∞ to w ,

$$\int_{\infty}^{w} \cdots \int_{\infty}^{w} F(e^{-w}) dw^m = (-1)^m \sum_{n=1}^{\infty} \frac{a_n}{n^m} e^{-nw} \tag{1.m}$$

From Cauchy formula for repeated integration,

$$\int_{\infty}^{w} \cdots \int_{\infty}^{w} F(e^{-w}) dw^m = \frac{1}{\Gamma(m)} \int_{\infty}^{w} F(e^{-t}) (w-t)^{m-1} dt$$

So

$$(-1)^m \sum_{n=1}^{\infty} \frac{a_n}{n^m} e^{-nw} = \frac{1}{\Gamma(m)} \int_{\infty}^{w} F(e^{-t}) (w-t)^{m-1} dt$$

Giving $w = 0$ to both sides,

$$(-1)^m \sum_{n=1}^{\infty} \frac{a_n}{n^m} = \frac{1}{\Gamma(m)} \int_{\infty}^{0} F(e^{-t}) (-t)^{m-1} dt = \frac{(-1)^m}{\Gamma(m)} \int_0^{\infty} F(e^{-t}) t^{m-1} dt$$

Since nolonger m need not be a natural number, replacing this with a complex number s ,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} F(e^{-t}) t^{s-1} dt$$

This is called **Mellin transform .**

Note

In (1.m) , replacing a natural number m with a complex number s , and giving $w = 0$ to both sides,

$$\int_{\infty}^{0} \sim \int_{\infty}^{w} F(e^{-w}) dw^s = (-1)^s \sum_{n=1}^{\infty} \frac{a_n}{n^s} = (-1)^s f(s) \tag{1.s}$$

That is, The super integral of $F(e^{-w})$ with respect to w from ∞ to 0 is equal to the product of $(-1)^s$ and $f(s)$.

Especially, when $a_n = 1$, $n = 1, 2, 3, \cdots$

Left: $$\int_{\infty}^{0} \sim \int_{\infty}^{w} F(e^{-w}) dw^s = \int_{\infty}^{0} \sim \int_{\infty}^{w} \sum_{n=1}^{\infty} e^{-nw} dw^s = \int_{\infty}^{0} \sim \int_{\infty}^{w} \frac{1}{e^w - 1} dw^s$$

Right: $$(-1)^s \sum_{n=1}^{\infty} \frac{1}{n^s} = (-1)^s \zeta(s)$$

Therefore,

$$\int_{\infty}^{0} \sim \int_{\infty}^{w} \frac{1}{e^w - 1} dw^s = (-1)^s \zeta(s)$$

Remark

As mentioned above, the followings were clarified about Mellin transform ..

i The coefficients of Dirichlet series $f(s)$ and the coefficient of Power series $F(z)$ are common.

ii The variable s of $f(s)$ and the variable t of $F(e^{-t})$ are independent. because, the former is the number of times of integration and the latter is a variable of the integrand. And since $z = e^{-t}$, s and z are also independent. Therefore, Dirichlet series $f(s)$ and Power series $F(z)$ are independent.

Conclusion

That is, Power series $F(z)$ is merely medium in the Mellin transform. Regrettably, we cannot use Power series as analysis tools of Dirichlet series.

2.3 Dirichlet Series & Logarithmic Power Series

2.3.1 Relation between Dirichlet Series & Logarithmic Power Series

Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \cdots \qquad -\pi \le Im(s) < \pi \tag{1.0}$$

is transferred to the following series by conversion $e^{-s} = z$.

$$\sum_{n=1}^{\infty} a_n z^{\log n} = a_1 z^{\log 1} + a_2 z^{\log 2} + a_3 z^{\log 3} + \cdots \tag{1.1}$$

Derivation

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} a_n e^{-s \log n} = \sum_{n=1}^{\infty} a_n (e^{-s})^{\log n} = \sum_{n=1}^{\infty} a_n z^{\log n}$$

Since, the conversion $z = e^{-s}$ is many-to-one function, the range from which (1.0) is correctly transferred to (1.1) is limited to $-\pi \le Im(s) < \pi$. Nevertheless, we can use (1.1) as analysis tools of Dirichlet series (1.0) in the range. We will call the series (1.1) **Logarithmic Power Series**.

2.3.2 Circle of convergence & Line of convergence

As seen above, the relation between the variable z of logarithmic power series and the variable s of Dirichlet series was as follows..

$$s = -\log z = -\log|z| - i \arg z$$

Here, let $s = \sigma + it$, $z = x + iy$. Then,

$$\sigma + it = -\log|x + iy| - i \arg(x + iy)$$

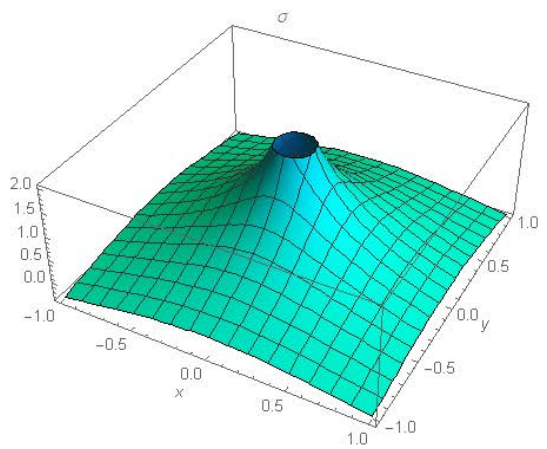
Although coordinates $x - y$ are transferred to coordinates $\sigma - t$ by the conversion, the coordinates $\sigma - t$ are similar to pola coordinates $r - \theta$. The difference among both is only the existence of $\log$ in the real part.

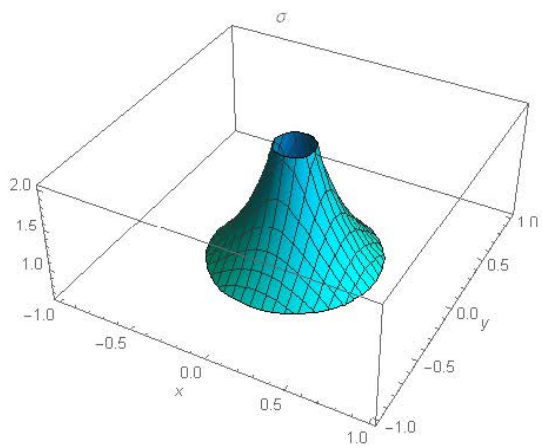
From this,

$$\sigma = -\log\sqrt{x^2 + y^2} \tag{2.σ}$$

$$t = -\arg(x + iy) \tag{2.t}$$

The real part (2.σ) is drawn on the left figure.





All the horizontal sections of the real part (2.σ) are circles centered at the origin. Radius of each circle is given by $\sqrt{x^2+y^2}$. That is, $r=\sqrt{x^2+y^2}=e^{-\sigma}$. Therefore, the convergence radius R of (1.1) is given by the line of convergence σ_c of (1.0) , as follows.

$$R=e^{-\sigma_c}.$$

For example, if the line of convergence is $\sigma_c=1/2$, the convergence radius is $R=e^{-1/2}=0.60653\cdots$. If the left figure is horizontally cut at the height $1/2$, it is the right figure. Radius of the base is $0.60653\cdots$ and the upper part is a convergence region of (1.1) .

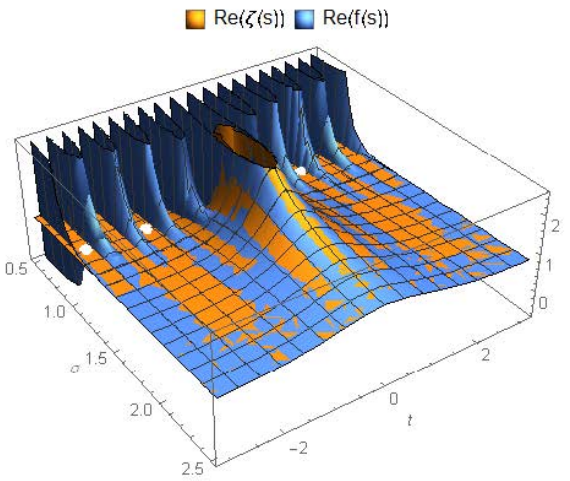
2.4 p - series & Logarithmic Geometric Series

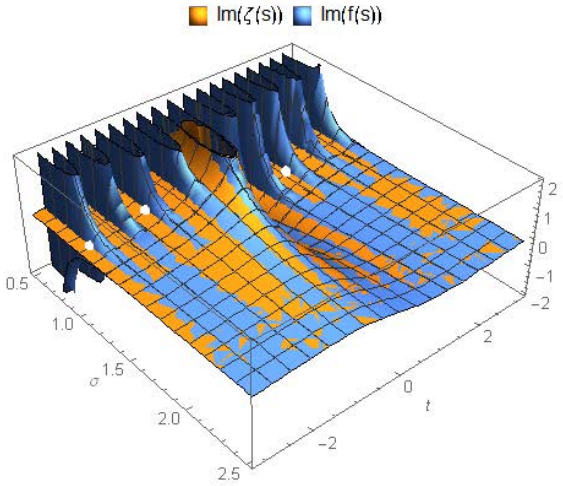
Riemann zeta function $\zeta(s)$ is expanded in a Dirichlet seris at $Re(s)>1$ as follows.

$$\zeta(s) = \sum_{n=1}^{\infty} e^{-s\log n} = \sum_{n=1}^{\infty} \frac{1}{n^s} \equiv f(s) \tag{1.f}$$

The right-hand side Dirichlet series is particularly called $\boldsymbol{p}$ **- series** .

When $s = \sigma + it$, the real part and the imaginary part of both sides are illustrated respectively as follows. The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, we can see that the line of convergence of the right-hand side is 1. Further, for comparison with latter figures, appropriate 3 points on the circumference is marked in white.

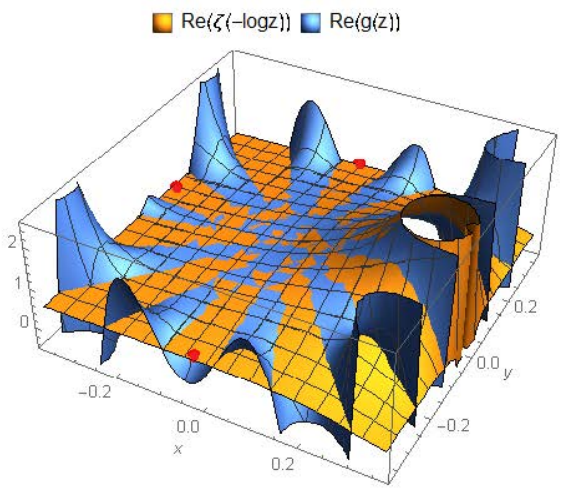


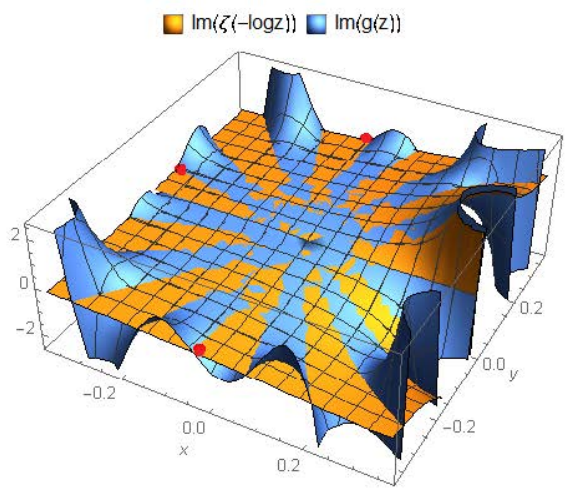


Substituting $e^{-s} = z$ ($i.e.\ s = -\log z$) for (1.f) ,

$$\zeta(-\log z) = \sum_{n=1}^{\infty} z^{\log n} \equiv g(z) \tag{1.g}$$

Though the right side $g(z)$ is a kind of Logarithmic power series, we will call $g(z)$ **Logarithmic Geometric Series** in particular. When $z = x + iy$, the real part and the imaginary part of both sides are illustrated respectively as follows. The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, we can see that the radius R of convergence of the right side is approximately 0.37 . Further, we can see that the 3 points in the former figures are transferred to the red points in both figures by $z = e^{-s}$.





Note

In this convergence circle, $s=\sigma+it$ s.t. $\sigma\le 1$ or $|t|\ge\pi$ can not be expressed . Further, since the line of convergence of $f(s)$ is $\sigma_c=1$, the radius of convergence of $g(z)$ is $R=e^{-1}=0.367879\cdots$.

Expression by summation formula

Applying Euler-Maclaurin summation formula to the logarithmic geometric series $g(z)$, we obtain the approximate value easily.

Theorem 2.4.1

Let B_{2m} be Bernoulli number , $B_{2m}(t)$ be Bernoulli polynomial, $S_1(m,n)$ be Stirling number of the 1st kind. Then the following expressions hold for a complex number z s.t. $|z|\ne e^{-1}$.

$$\sum_{k=1}^{n} z^{\log k}=\frac{n z^{\log n}-1}{\log(ez)}+\frac{z^{\log n}+1}{2}$$

$$+\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left(\frac{z^{\log n}}{n^{2r-1}}-1\right)\sum_{s=1}^{2r-1}S_1(2r-1,s)\log^s z+R_{2m} \tag{2.1}$$

$$R_{2m}=-\frac{1}{(2m)!}\sum_{s=1}^{2m}S_1(2m,s)\log^s z\int_1^n B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^{2m}}dt \tag{2.r}$$

Especially, when $|z|<e^{-1}=0.367879\cdots$

$$\sum_{k=1}^{\infty} z^{\log k}=\frac{1}{2}-\frac{1}{\log(ez)}-\frac{\log z}{12}+\frac{\log z-\log^2 z}{2}\int_1^{\infty}B_2(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^2}dt \tag{2.2}$$

$$\approx\frac{1}{2}-\frac{1}{\log(ez)}-\frac{\log z}{12} \tag{2.2'}$$

Proof

When $f(x)$ is a function of class C^{2m} on a closed interval $[a,b]$, $\lfloor t\rfloor$ is the floor function, B_{2r} is Bernoulli number and $B_{2m}(x)$ is Bernoulli polynomial, Euler-Maclaurin summation formula was as follows.

$$\sum_{k=a}^{b}f(k)=\int_a^b f(t)dt+\frac{f(b)+f(a)}{2}+\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left\{f^{(2r-1)}(b)-f^{(2r-1)}(a)\right\}+R_{2m}$$

$$R_{2m}=-\frac{1}{(2m)!}\int_a^b B_{2m}(t-\lfloor t\rfloor)f^{(2m)}(t)dt$$

So, let

$$\{S_n\}=z^{\log 1},z^{\log 2},z^{\log 3},\ \cdots,z^{\log n}$$

Then,

$$f(t)=z^{\log t}$$

$$\int_1^n f(t)dt=\left[\frac{t z^{\log t}}{1+\log z}\right]_1^n=\frac{n z^{\log n}-1}{\log(ez)}$$

$$f^{(2m)}(t)=\frac{z^{\log t}}{t^{2m}}\sum_{s=1}^{m}S_1(2m,s)\log^s z$$

$$f^{(2r-1)}(t) = \frac{z^{\log t}}{t^{2r-1}} \sum_{s=1}^{2r-1} S_1(2r-1,s)\ \log^s z$$

Substituting these for the above formula ,

$$\sum_{k=1}^{n} z^{\log k} = \frac{n z^{\log n} - 1}{\log(ez)} + \frac{z^{\log n}+1}{2}$$

$$+ \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} \left(\frac{z^{\log n}}{n^{2r-1}} - 1 \right) \sum_{s=1}^{2r-1} S_1(2r-1,s)\ \log^s z + R_{2m} \qquad (2.1)$$

$$R_{2m} = -\frac{1}{(2m)!} \int_1^n B_{2m}(t-\lfloor t \rfloor) \frac{z^{\log t}}{t^{2m}} \sum_{s=1}^{2m} S_1(2m,s)\ \log^s z dt$$

$$= -\frac{1}{(2m)!} \sum_{s=1}^{2m} S_1(2m,s) \log^s z \int_1^n B_{2m}(t-\lfloor t \rfloor) \frac{z^{\log t}}{t^{2m}} dt \qquad (2.r)$$

Substituting $m=1$ for (2.1) and (2.r) ,

$$\sum_{k=1}^{n} z^{\log k} = \frac{n z^{\log n} - 1}{\log(ez)} + \frac{z^{\log n}+1}{2} + \frac{B_2}{2} \left(\frac{z^{\log n}}{n} - 1 \right) S_1(1,1)\ \log^1 z + R_2$$

$$= \frac{n z^{\log n} - 1}{\log(ez)} + \frac{z^{\log n}+1}{2} + \frac{1}{12} \left(\frac{z^{\log n}}{n} - 1 \right) \log z + R_2$$

$$R_2 = -\frac{1}{2} \sum_{s=1}^{2} S_1(2,s) \log^s z \int_1^n B_2(t-\lfloor t \rfloor) \frac{z^{\log t}}{t^2} dt$$

$$= \frac{\log z - \log^2 z}{2} \int_1^n B_2(t-\lfloor t \rfloor) \frac{z^{\log t}}{t^2} dt$$

When $0 < |z| < e^{-1}$, $\lim_{n\to\infty} z^{\log n} = 0$, $\lim_{n\to\infty} n z^{\log n} = 0$. Then,

$$\sum_{k=1}^{\infty} z^{\log k} = \frac{1}{2} - \frac{1}{\log(ez)} - \frac{\log z}{12} + R_2$$

$$R_2 = \frac{\log z - \log^2 z}{2} \int_1^\infty B_2(t-\lfloor t \rfloor) \frac{z^{\log t}}{t^2} dt$$

Substituting the latter for the former, we obtain (2.2) .

Example 4.1 $\sum_{k=1}^{100} (0.4+1.1i)^{\log k}$, $m=2$

```
fl[z_, n_] := Sum[z^Log[k], {k, 1, n}]

fr[z_, n_, m_] := (n z^Log[n] - 1)/Log[e z] + (z^Log[n] + 1)/2
    + Sum[BernoulliB[2 r]/(2 r)! ((z^Log[n])/n^(2 r-1) - 1) Sum[StirlingS1[2 r - 1, s] Log[z]^s, {s, 1, 2 r-1}], {r, 1, m}]

Rm[z_, n_, m_] := -1/(2 m)! Sum[StirlingS1[2 m, s] Log[z]^s, {s, 1, 2 m}]
    × Integrate[BernoulliB[2 m, t - Floor[t]] (z^Log[t])/t^(2 m), {t, 1, n}]
```

```
N[fl[0.4 + 1.1 i, 100]]
13.4771 - 122.309 i
```

```
N[fr[0.4 + 1.1 i, 100, 2]]
13.478 - 122.31 i
```

```
N[Rm[0.4 + 1.1 i, 100, 2]]
-0.00085829 + 0.000674227 i
```

```
13.477984040734217  - 122.3096690302117 i
-0.000858289889215  +   0.0006742274917 i
-------------------------------------------
13.4771             - 122.309 i
```

Example 4.2 $\sum_{k=1}^{\infty} (-0.3+0.21\,i)^{\log k}$

$$\text{fl}[z\_] := \sum_{k=1}^{\infty} z^{\text{Log}[k]}$$

$$\text{fr}[z\_] := \frac{1}{2} - \frac{1}{\text{Log}[e\ z]} - \frac{\text{Log}[z]}{12} + \frac{\text{Log}[z] - \text{Log}[z]^2}{2} \int_1^{100} \text{BernoulliB}[2,\ t - \text{Floor}[t]]\ \frac{z^{\text{Log}[t]}}{t^2}\ dt$$

```
fl[-0.3 + 0.21 i]
0.61337 + 0.206181 i
```

```
N[fr[-0.3 + 0.21 i]]
0.61337 + 0.206181 i
```

Example 4.3 $\sum_{k=1}^{\infty} z^{\log k}$, $z = 0.1, 0.2, 0.3$

$$\text{fl}[z\_] := \sum_{k=1}^{\infty} z^{\text{Log}[k]} \qquad \text{fr}[z\_] := \frac{1}{2} - \frac{1}{\text{Log}[e\ z]} - \frac{\text{Log}[z]}{12}$$

```
N[{fl[0.1], fr[0.1]}]
{1.43104, 1.45959}
```

```
N[{fl[0.2], fr[0.2]}]
{2.26058, 2.27498}
```

```
N[{fl[0.3], fr[0.3]}]
{5.49448, 5.50295}
```

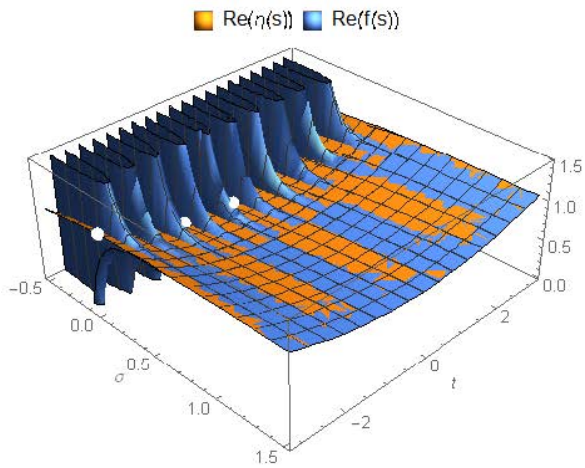
2.5 η - series & Logarithmic Geometric Series

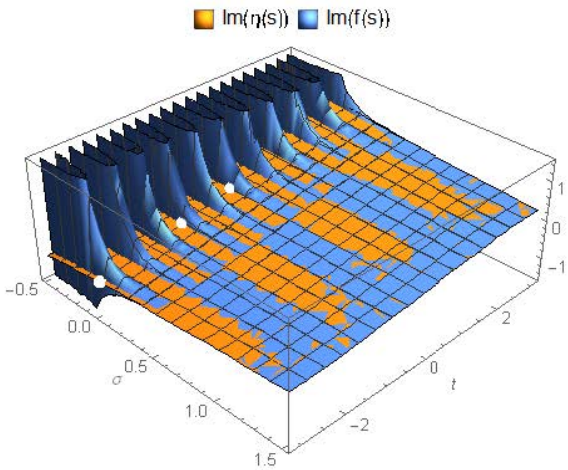
Dirichlet eta function $\eta(s)$ is expanded in a Dirichlet seris at $Re(s) > 0$ as follows.

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} e^{-s \log n} \equiv f(s) \tag{1.f}$$

The right-hand side series is particularly called **Dirichlet η - series** .

When $s = \sigma + i t$, the real part and the imaginary part of both sides are illustrated respectively as follows. The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, we can see that the line of convergence of the right-hand side is 0. Further, for comparison with latter figures, appropriate 3 points on the circumference is marked in white.

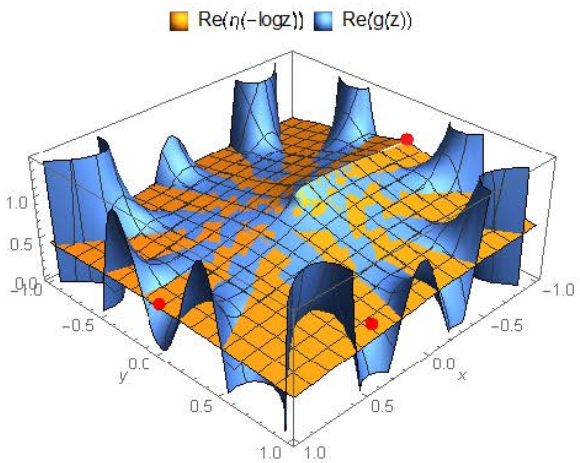


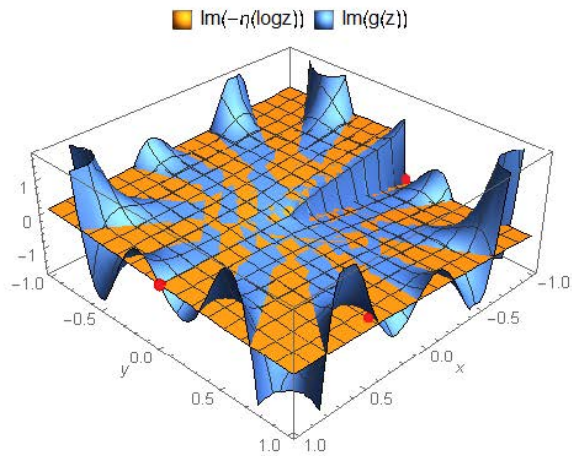


Substituting $e^{-s} = z$ ($i.e.\ s = -\log z$) for (1.f) ,

$$\eta(-\log z) = \sum_{n=1}^{\infty} (-1)^{n-1} z^{\log n} \equiv g(z) \tag{1.g}$$

When $z = x + i y$, the real part and the imaginary part of both sides are illustrated respectively as follows. The left figure is a real part, the right figure is an imaginary part. In both figures, the left-hand side is orange, and the right-hand side is blue. In both figures, we can see that the radius R of convergence of the right side is approximately 1 . Further, we can see that the 3 points in the former figures are transferred to the red points in both figures by $z = e^{-s}$.





Note

In this convergence circle, $s = \sigma + i t$ s.t. $\sigma \leq 0$ or $|t| \geq \pi$ can not be expressed . Further, since the line of convergence is $\sigma_c = 0$, the radius of convergence of $g(z)$ is $R = e^{-0} = 1$.

Expression by summation formula

Applying Euler-Maclaurin summation formula to the logarithmic geometric series $g(z)$, we obtain the approximate value easily.

Theorem 2.5.1

Let B_{2m} be Bernoulli number, $B_{2m}(t)$ be Bernoulli polynomial, $S_1(m,n)$ be Stirling number of the 1st kind. Then the following expressions hold for a complex number z s.t. $|z| \neq e^{-1}$.

$$\sum_{k=1}^{2n}(-1)^{k-1}z^{\log k} = \left(1-2z^{\log 2}\right)\left(\frac{1}{2}-\frac{1}{\log(ez)}\right) - \frac{z^{\log 2n}}{2}$$
$$+\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left\{\frac{\left(1-2^{2r}\right)z^{\log 2n}}{(2n)^{2r-1}} - \left(1-2z^{\log 2}\right)\right\}$$
$$\times\sum_{s=1}^{2r-1}S_1(2r-1,s)\log^s z + R_{2m} \qquad (2.1)$$

$$R_{2m} = -\frac{1}{(2m)!}\sum_{s=1}^{2m}S_1(2m,s)\log^s z$$
$$\times\left\{\int_1^{2n}B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^{2m}}dt - 2z^{\log 2}\int_1^{n}B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^{2m}}dt\right\} \qquad (2.r)$$

Especially, when $|z| < e^{-0} = 1$

$$\sum_{k=1}^{\infty}(-1)^{k-1}z^{\log k} = \left(1-2z^{\log 2}\right)\left(\frac{1}{2}-\frac{1}{\log(ez)}-\frac{\log z}{12}\right)$$
$$+\frac{\left(1-2z^{\log 2}\right)\left(\log z-\log^2 z\right)}{2}\int_1^{\infty}B_2(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^2}dt \qquad (2.2)$$
$$\approx\left(1-2z^{\log 2}\right)\left(\frac{1}{2}-\frac{1}{\log(ez)}-\frac{\log z}{12}\right) \qquad (2.2')$$

Proof

$$S_{2n} \equiv \sum_{k=1}^{2n}z^{\log k}$$
$$S_{2n}^{evn} \equiv \sum_{k=1}^{n}z^{\log(2k)} = z^{\log 2}\sum_{k=1}^{n}z^{\log k} = z^{\log 2}S_n$$
$$S_{2n}^{+-} \equiv \sum_{k=1}^{2n}(-1)^{k-1}z^{\log k} = S_{2n} - 2S_{2n}^{evn}$$

i.e.

$$S_{2n}^{+-} = \sum_{k=1}^{2n}(-1)^{k-1}z^{\log k} = S_{2n} - 2z^{\log 2}S_n$$

Applying Theorem 2.4.1 in previous section to S_{2n}, S_n,

$$\sum_{k=1}^{2n}z^{\log k} = \frac{2n\,z^{\log 2n}-1}{\log(ez)} + \frac{z^{\log 2n}+1}{2}$$
$$+\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left(\frac{z^{\log 2n}}{(2n)^{2r-1}}-1\right)\sum_{s=1}^{2r-1}S_1(2r-1,s)\log^s z + R_{2m}^2$$

$$R_{2m}^{2} = -\frac{1}{(2m)!}\sum_{s=1}^{2m} S_1(2m,s)\log^s z\int_1^{2n} B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^{2m}}dt$$

$$\sum_{k=1}^{n} z^{\log k} = \frac{n z^{\log n}-1}{\log(ez)} + \frac{z^{\log n}+1}{2}$$
$$+\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left(\frac{z^{\log n}}{n^{2r-1}}-1\right)\sum_{s=1}^{2r-1} S_1(2r-1,s)\log^s z + R_{2m}^{1}$$

$$R_{2m}^{1} = -\frac{1}{(2m)!}\sum_{s=1}^{2m} S_1(2m,s)\log^s z\int_1^{n} B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^{2m}}dt$$

Therefore,

$$\sum_{k=1}^{2n}(-1)^{k-1}z^{\log k} = \left(\frac{2n z^{\log 2n}-1}{\log(ez)}+\frac{z^{\log 2n}+1}{2}\right) - 2z^{\log 2}\left(\frac{n z^{\log n}-1}{\log(ez)}+\frac{z^{\log n}+1}{2}\right)$$
$$+\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left(\frac{z^{\log 2n}}{(2n)^{2r-1}}-1\right)\sum_{s=1}^{2r-1} S_1(2r-1,s)\log^s z$$
$$-2z^{\log 2}\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left(\frac{z^{\log n}}{(2n)^{2r-1}}-1\right)\sum_{s=1}^{2r-1} S_1(2r-1,s)\log^s z$$
$$+R_{2m}^{2} - 2z^{\log 2}R_{2m}^{1}$$

As first,

$$\left(\frac{2n z^{\log 2n}-1}{\log(ez)}+\frac{z^{\log 2n}+1}{2}\right) - 2z^{\log 2}\left(\frac{n z^{\log n}-1}{\log(ez)}+\frac{z^{\log n}+1}{2}\right)$$
$$= \left(\frac{1}{2}-\frac{1}{\log(ez)}+\frac{z^{\log 2n}}{2}\right) - 2z^{\log 2}\left(\frac{1}{2}-\frac{1}{\log(ez)}+\frac{z^{\log n}}{2}\right)$$
$$= \left(1-2z^{\log 2}\right)\left(\frac{1}{2}-\frac{1}{\log(ez)}\right) - \frac{z^{\log 2n}}{2}$$

As second,

$$\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left(\frac{z^{\log 2n}}{(2n)^{2r-1}}-1\right)\sum_{s=1}^{2r-1} S_1(2r-1,s)\log^s z$$
$$-2z^{\log 2}\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left(\frac{z^{\log n}}{n^{2r-1}}-1\right)\sum_{s=1}^{2r-1} S_1(2r-1,s)\log^s z$$
$$= \sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left\{\frac{\left(1-2^{2r}\right)z^{\log 2n}}{(2n)^{2r-1}} - \left(1-2z^{\log 2}\right)\right\}\sum_{s=1}^{2r-1} S_1(2r-1,s)\log^s z$$

Last,

$$R_{2m}^{2} - 2z^{\log 2}R_{2m}^{1} = -\frac{1}{(2m)!}\sum_{s=1}^{2m} S_1(2m,s)\log^s z\int_1^{2n} B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^{2m}}dt$$
$$+\frac{2z^{\log 2}}{(2m)!}\sum_{s=1}^{2m} S_1(2m,s)\log^s z\int_1^{n} B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^{2m}}dt$$

$$= -\frac{1}{(2m)!}\sum_{s=1}^{2m} S_1(2m,s)\log^s z$$

$$\times\left\{\int_1^{2n} B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^{2m}}dt - 2z^{\log 2}\int_1^{n} B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^{2m}}dt\right\}$$

Substituting these for the above, weobtain (2.1) and (2.r) .

When $0<z\neq e^{-1}<1$, $\lim_{n\to\infty} z^{\log 2n}=0$. Then,

$$\sum_{k=1}^{\infty}(-1)^{k-1}z^{\log k} = \left(1-2z^{\log 2}\right)\left(\frac{1}{2}-\frac{1}{\log(ez)}\right)$$

$$-\left(1-2z^{\log 2}\right)\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\sum_{s=1}^{2r-1}S_1(2r-1,s)\log^s z + R_{2m}$$

$$R_{2m} = -\frac{1-2z^{\log 2}}{(2m)!}\sum_{s=1}^{2m}S_1(2m,s)\log^s z\int_1^{\infty}B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^{2m}}dt$$

Giving $m=1$,

$$\sum_{k=1}^{\infty}(-1)^{k-1}z^{\log k} = \left(1-2z^{\log 2}\right)\left(\frac{1}{2}-\frac{1}{\log(ez)}\right)$$

$$-\left(1-2z^{\log 2}\right)\frac{B_2}{2}S_1(1,1)\log^1 z + R_2$$

$$= \left(1-2z^{\log 2}\right)\left(\frac{1}{2}-\frac{1}{\log(ez)}\right) - \left(1-2z^{\log 2}\right)\frac{\log z}{12} + R_2$$

$$R_2 = -\frac{1-2z^{\log 2}}{2}\sum_{s=1}^{2}S_1(2,s)\log^s z\int_1^{\infty}B_{2m}(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^2}dt$$

$$= \frac{\left(1-2z^{\log 2}\right)\left(\log z-\log^2 z\right)}{2}\int_1^{n}B_2(t-\lfloor t\rfloor)\frac{z^{\log t}}{t^2}dt$$

Substituting the latter for the former, we obtain (2.2) .

Example 5.1 $\sum_{k=1}^{50}(-1)^{k-1}1.3^{\log k}$, $m=2$

```
fl[z_, n_] := Sum[(-1)^(k-1) z^Log[k], {k, 1, 2 n}]

fr[z_, n_, m_] := (1 - 2 z^Log[2]) (1/2 - 1/Log[e z]) - z^Log[2 n]/2
    + Sum[BernoulliB[2 r]/(2 r)! ((1 - 2^(2 r)) z^Log[2 n]/(2 n)^(2 r-1) - (1 - 2 z^Log[2])),
      {r, 1, m}] × Sum[StirlingS1[2 r - 1, s] Log[z]^s, {s, 1, 2 r-1}]

R2m[z_, n_, m_] := -1/(2 m)! Sum[StirlingS1[2 m, s] Log[z]^s, {s, 1, 2 m}]
```

$$\times\left(\int_1^{2n} \text{BernoulliB}[2m, t-\text{Floor}[t]]\,\frac{z^{\text{Log}[t]}}{t^{2m}}\,dt - 2\,z^{\text{Log}[2]}\int_1^{n} \text{BernoulliB}[2m, t-\text{Floor}[t]]\,\frac{z^{\text{Log}[t]}}{t^{2m}}\,dt\right)$$

```
N[fl[1.3, 50]]          N[fr[1.3, 50, 2]]   N[R2m[1.3, 50, 2]]
-1.23724                -1.23734            0.000104948
```

Example 5.2 $\sum_{k=1}^{\infty}(-1)^{k-1}0.9^{\log k}$

$$\text{fl}[z\_] := \sum_{k=1}^{\infty}(-1)^{k-1}\,z^{\text{Log}[k]}$$

$$\text{fr}[z\_] := \left(1-2\,z^{\text{Log}[2]}\right)\left(\frac{1}{2}-\frac{1}{\text{Log}[e\,z]}-\frac{\text{Log}[z]}{12}\right) + \frac{\left(1-2\,z^{\text{Log}[2]}\right)\left(\text{Log}[z]-\text{Log}[z]^2\right)}{2} \times\int_1^{100}\text{BernoulliB}[2, t-\text{Floor}[t]]\,\frac{z^{\text{Log}[t]}}{t^2}\,dt$$

```
fl[0.9]          fr[0.9]
0.523446         0.523446
```

Example 5.3 $\sum_{k=1}^{\infty}(-1)^{k-1}z^{\log k}$, $z = 0.6, 0.7, 0.8$

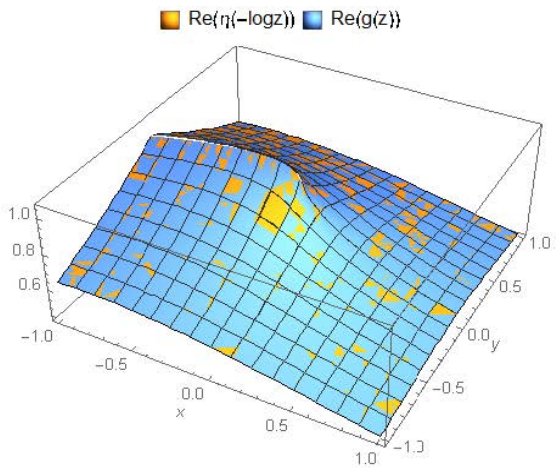
$$\text{fl}[z\_] := \sum_{k=1}^{\infty}(-1)^{k-1}\,z^{\text{Log}[k]} \qquad \text{fr}[z\_] := \left(1-2\,z^{\text{Log}[2]}\right)\left(\frac{1}{2}-\frac{1}{\text{Log}[e\,z]}-\frac{\text{Log}[z]}{12}\right)$$

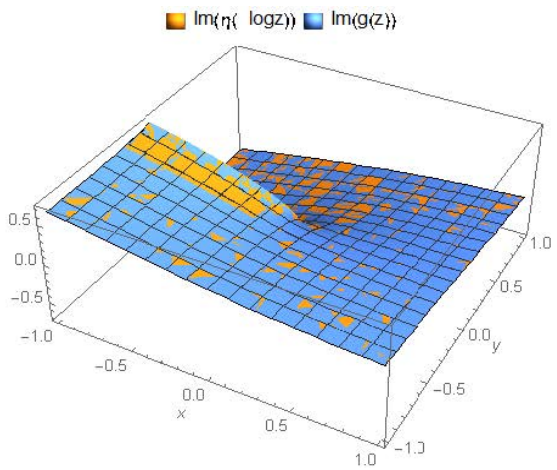
```
N[{fl[0.6], fr[0.6]}]     N[{fl[0.7], fr[0.7]}]     N[{fl[0.8], fr[0.8]}]
{0.606987, 0.606145}      {0.576507, 0.575805}      {0.548826, 0.548345}
```

Acceleration of Logarithmic Power Series (Analytic Continuation)

Applying Knopp Transformation to Logarithmic Power Series $\sum_{k=1}^{\infty}(-1)^{k-1}(x+iy)^{\log k}$,

$$g(x,y,q,m) = \sum_{j=1}^{m}\sum_{k=1}^{j}\frac{q^{j-k}}{(q+1)^{j+1}}\binom{j}{k}(-1)^{k-1}(x+iy)^{\log k} \tag{3.1}$$



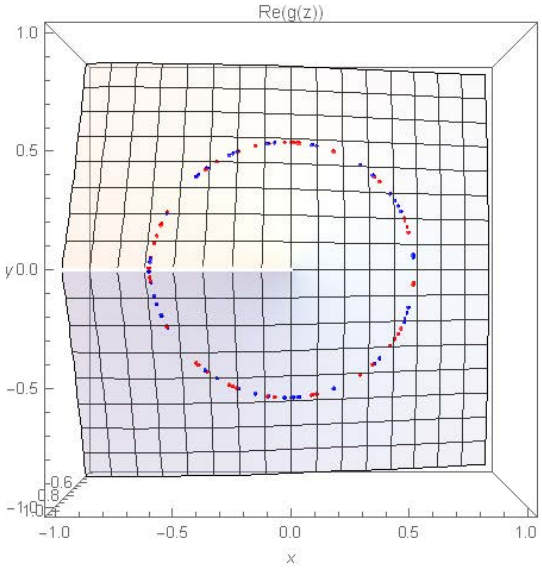


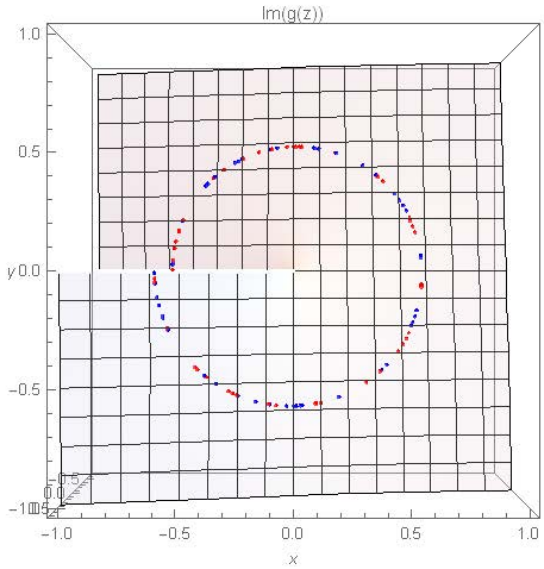
Since q may arbitrary positive number, we give $q=1$, $m=30$. When (3.1) and $\eta(-\log z)$ are drawn together, it is as above. The left figure is a real part, the right figure is an imaginary part. In both figures, orange is $\eta(-\log z)$ and blue is (3.1) . Both are consistent well and are seen in spots. Comparing these with the figures of (1.g) , we see that logarithmic geometric series is analytically continued from the circle to the square.

Virtual zeros of Dirichlet Eta Function

Even if Knopp Transformation is applied, the area transferred to both figures from $\eta(\sigma+it)$ is limited to $\sigma>0$, $|t|<\pi$. Since all known zeros of $\eta(s)$ are out of this range, the zeros of $\eta(s)$ are not contained all over these figures. Regrettably, logarithmic geometric series can not also be used as analysis tool of Dirichlet series.

However, It is possible to draw the virtual image of the zeros of $\eta(s)$ on these figures. For example, by e^{-s} , $s=1/2+i14.134725\cdots$ is reduced $4\pi i$ and is equated with $s=1/2+i1.568354\cdots$.
So, if 100 zeros $s=1/2\pm it_n$ $n=1,2,\cdots,50$ of $\eta(s)$ are mechanically changed into z and (3.1) is drawn, it is as follows. Blue is $-t_n$ and red is t_n . Although these are virtual zeros, we can see that these are distributed on the circumference of radius $e^{-1/2}=0.60653\cdots$.





2016.10.05

Kano. Kono

Alien's Mathematics