09 Li coefficients for the Completed Dirichlet Beta

Abstract

(1) The Li coefficients $\lambda_n \quad n=1,2,3,\cdots$ can also be defined for the Completed Dirichlet Beta Function $\omega(z)$.

They are expressed as components of $\omega(z)$ and their values are obtained by recursive calculation.

(2) The Li coefficients $\lambda_n \quad n=1,2,3,\cdots$ can also be defined in terms of the Hadamard product of $\omega(z)$.

They are expressed by the conjugate zeros $x_r \pm i\, y_r \quad r=1,2,3,\cdots$ of $\omega(z)$.

(3) On the critical line $x_r = 1/2 \quad r=1,2,3,\cdots$, the Li coefficients $\lambda_n \quad n=1,2,3,\cdots$ are expressed as the sum of squares

of real numbers. That is, the Li Coefficients on the critical line satisfy Li's Criterion.

In fact, the LI coefficients calculated using 10000 zeros on the critical line were almost identical to the values in **(1)**.

Introduction

1. Functions Used in This Paper

In this paper, we deal with the Dirichlet Beta Function $\beta(z)$ and the Completed Dirichlet Beta Function $\omega(z)$, which are defined as follows:

$$\beta(z) = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + \cdots = \frac{1}{4^z}\left\{ \varsigma\left(z,\frac{1}{4}\right) - \varsigma\left(z,\frac{3}{4}\right) \right\} \tag{0.0}$$

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1+z}{2}\right)\beta(z) \tag{0.1}$$

In addition, it is known that these zeros are equivalent in the critical strip ($0 < Re(z) < 1$).

2. Notation for sums and products of reciprocals of zeros

The sum of the reciprocals of the zeros of the Dirichlet Beta Function is generally written as

$$\sum_{\rho} \frac{1}{\rho} \quad , \quad \prod_{\rho}\left(1 - \frac{z}{\rho}\right) \qquad \text{where } \rho \text{ runs over all the zeros.}$$

This notation is conceptual and valid to this extent. However, this notation cannot describe the semi-multiple series of this.

For example, if we were to forcefully express λ_1^3 using this notation, it would look like

$$\left(\sum_{\rho}\frac{1}{\rho}\right)^3 = \sum_{\rho}\frac{1}{\rho^3} + 3\left(\sum_{\rho}\frac{1}{\rho}\right)\sum\frac{1}{\rho\rho} - 3\sum\frac{1}{\rho\rho\rho} \tag{0.3}$$

where, in the double sum and triple sum, ρ is not duplicated.

These notations are inconvenient and practical calculations are impossible. Therefore, in this chapter we use the followings.

(1) Complex number notation

$$\sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}}$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}}$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}\rho_{r_3}}$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}\rho_{r_3}\rho_{r_4}}$$

$$\vdots$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \cdots \sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}\cdots\rho_{r_n}}$$

(2) Real and imaginary parts notation

However, even with the notation (1), it is difficult to examine the real and imaginary parts of the zeros ρ_k in detail.

So, considering that the $\omega(z)$ has conjugate zeros, we replace ρ_k $k = 1, 2, 3, \cdots$ as follows.

$$\rho_1 = x_1 - i\,y_1 \ , \ \ \rho_2 = x_1 + i\,y_1 \ , \ \ \rho_3 = x_2 - i\,y_2 \ , \ \ \rho_4 = x_2 + i\,y_2 \ , \ \ \rho_5 = x_3 - i\,y_3 \ , \ \ \rho_6 = x_3 + i\,y_3 \ , \ \cdots$$

Using this, the example (1) can be rewritten as follows.

$$\sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}} = \sum_{r=1}^{\infty} \left(\frac{1}{x_r - i\,y_r} + \frac{1}{x_r + i\,y_r} \right) = \sum_{r=1}^{\infty} \frac{2x_r}{x_r^2 + y_r^2}$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}} = \sum_{r=1}^{\infty} \sum_{s=1+r}^{\infty} \frac{2^2 x_r x_s}{\left(x_r^2 + y_r^2\right)\left(x_s^2 + y_s^2\right)} + \sum_{r=1}^{\infty} \frac{2^0}{x_r^2 + y_r^2}$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}\rho_{r_3}} = \sum_{r=1}^{\infty} \sum_{s=1+r}^{\infty} \sum_{t=1+s}^{\infty} \frac{2^3 x_r x_s x_t}{\left(x_r^2 + y_r^2\right)\left(x_s^2 + y_s^2\right)\left(x_t^2 + y_t^2\right)}$$

$$+ \sum_{r=1}^{\infty} \sum_{s=1+r}^{\infty} \frac{2^1 (x_r + x_s)}{\left(x_r^2 + y_r^2\right)\left(x_s^2 + y_s^2\right)}$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}\rho_{r_3}\rho_{r_4}} = \sum_{r=1}^{\infty} \sum_{s=1+r}^{\infty} \sum_{t=1+s}^{\infty} \sum_{u=1+t}^{\infty} \frac{2^4 x_r x_s x_t x_u}{\left(x_r^2 + y_r^2\right)\left(x_s^2 + y_s^2\right)\left(x_t^2 + y_t^2\right)\left(x_u^2 + y_u^2\right)}$$

$$+ \sum_{r=1}^{\infty} \sum_{s=1+r}^{\infty} \sum_{t=1+s}^{\infty} \frac{2^2 (x_r x_s + x_r x_t + x_s x_t)}{\left(x_r^2 + y_r^2\right)\left(x_s^2 + y_s^2\right)\left(x_t^2 + y_t^2\right)}$$

$$+ \sum_{r=1}^{\infty} \sum_{s=1+r}^{\infty} \frac{2^0}{\left(x_r^2 + y_s^2\right)\left(x_r^2 + y_s^2\right)}$$

$\vdots$

9.1 Li's Criterion and Li Coefficients

Li's criterion for the Dirichlet Beta Function $\beta(z)$ was as follows:

Li's Criterion

The Riemann hypothesis for the Dirichlet Beta Function is equivalent to the inequality;

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{dz^n}\Big[(1-z)^{n-1}\log\omega(z)\Big]\Bigg|_{z=0} \geq 0 \qquad for\ n=1, 2, 3, \ldots$$

where,

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z) = \prod_{\rho}\left(1-\frac{z}{\rho}\right)$$

When ρ is a non-trivial zero of the Dirichlet Beta Function $\beta(z)$, it is known that the Li coefficient λ_n is equivalent to the following:

Li Coefficients λ_n

$$\lambda_n = \sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^n\right)$$

where ρ runs over all the zeros.

This formula is derived from the right-hand side of the above proviso and the definition of λ_n ..

The first few are expanded as follows:

$$\lambda_1 = \sum_{\rho}\frac{1}{\rho}$$

$$\lambda_2 = \sum_{\rho}\left(\frac{2}{\rho}-\frac{1}{\rho^2}\right) = 2\sum_{\rho}\frac{1}{\rho} - \sum_{\rho}\frac{1}{\rho^2}$$

$$\lambda_3 = \sum_{\rho}\left(\frac{3}{\rho}-\frac{3}{\rho^2}+\frac{1}{\rho^3}\right) = 3\sum_{\rho}\frac{1}{\rho} - 3\sum_{\rho}\frac{1}{\rho^2} + \sum_{\rho}\frac{1}{\rho^3}$$

$$\lambda_4 = \sum_{\rho}\left(\frac{4}{\rho}-\frac{6}{\rho^2}+\frac{4}{\rho^3}-\frac{1}{\rho^4}\right) = 4\sum_{\rho}\frac{1}{\rho} - 6\sum_{\rho}\frac{1}{\rho^2} + 4\sum_{\rho}\frac{1}{\rho^3} - \sum_{\rho}\frac{1}{\rho^4}$$

$$\vdots$$

Now, using the notation in the previous section, Li coefficients can be written as follows:

Lemma 9.1.1

If ρ_{r_1} $r_1 = 1, 2, 3, \cdots$ are nontrivial zeros of the Dirichlet Beta Function $\beta(z)$, the Li coefficients λ_n $n = 1, 2, 3, \cdots$ are expressed as follows.

$$\lambda_1 = \sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}}$$

$$\lambda_2 = 2\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}} - \sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^2}$$

$$\lambda_3 = 3\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}} - 3\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^2} + \sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^3}$$

$$\lambda_4 = 4\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}} - 6\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^2} + 4\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^3} - \sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^4}$$

$$\vdots$$

$$\lambda_n = n \sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}} + \sum_{k=2}^{n} (-1)^{k-1} \binom{n}{k} \sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}^{k}}$$

(1.1n)

Semi-multiple series representation of Li coefficients

Now, λ_1, λ_2, λ_3, $\cdots$ can be expressed as the following semi-multiple series.

$$\sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}} \;,\; \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}} \;,\; \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_1}\rho_{r_3}} \;,\; \cdots$$

To do so, we first need the following recurrence relation:

Lemma 9.1.2

When n is natural number s.t. $n \geq 2$, for a convergent infinite series, the following holds.

$$\sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}^{n}} = \left(\sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}} \right)^{n} - 2 \left(\sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}} \right)^{n-2} H_2$$

$$- \sum_{s=0}^{n-3} \left(\sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}} \right)^{s} \left\{ \sum_{t=2}^{n-s-1} (-1)^{t} \left(\sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}^{n-s-t}} \right) H_t + (-1)^{n-s}(n-s) H_{n-s} \right\}$$

(1.2n)

Where,

$$H_2 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}}$$

$$H_3 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}\rho_{r_3}}$$

$$\vdots$$

$$H_n = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \cdots \sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}\rho_{r_3}\cdots\rho_{r_n}}$$

(1.Hn)

When $n \leq 2$, the 3 rd term of (1.2n) is ignored.

Proof

Theorem 5.2.2 on my site " **05 Power Series and Semi Multiple Series** " (Infinite Degree Equations) is as follows.

Theorem 5.2.2 (Reprint)

When n is natural number s.t. $n \geq 2$, for a convergent infinite series, the following holds.

$$\left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^{n} = \sum_{r_1=1}^{\infty} a_r^{n} + 2 \left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^{n-2} H_2$$

$$+ \sum_{s=0}^{n-3} \left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^{s} \left(\sum_{t=2}^{n-s-1} (-1)^{t} \left(\sum_{r_1=1}^{\infty} a_{r_1}^{n-s-t} \right) H_t + (-1)^{n-s}(n-s) H_{n-s} \right)$$

Where,

$$H_2 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} a_{r_1} a_{r_2}$$

$$H_3 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$

$$\vdots$$

$$H_n = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \cdots \sum_{r_n=r_{n-1}+1}^{\infty} a_{r_1} a_{r_2} a_{r_3} \cdots a_{r_n}$$

Where,

When $n \leq 2$, the 3 nd term is ignored.

Replacing a_{r_n} with $1/\rho_{r_n}$ in this theorem, we obtain Lemma 9.1.2

<div align="right">Q.E.D.</div>

On the other hand, the following relationship exists between the semi-multiple series in Lemma 9.1.2 and the coefficients of the Maclaurin series of $\omega(z)$.

Lemma 9.1.3

Let the function $\omega(z)$ and its Maclaurin series be

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1+z}{2}\right)\beta(z) = \sum_{r=0}^{\infty} A_r z^r$$

Then, the following equations hold for the zeros ρ_{r_1} $r_1 = 1, 2, 3, \cdots$ of $\omega(z)$.

$$A_1 = -\sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}} \tag{1.3$_1$}$$

$$A_2 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}}$$

$$A_3 = -\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}\rho_{r_3}}$$

$$A_4 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}\rho_{r_3}\rho_{r_4}}$$

$$\vdots$$

$$A_n = (-1)^n \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \cdots \sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{\rho_{r_1}\rho_{r_2}\cdots\rho_{r_n}} \tag{1.3n}$$

Proof

$\omega(z)$ is completely factored by the zeros ρ_{r_1} $r_1 = 1, 2, 3, \cdots$ as follows.

$$\omega(z) = \left(1-\frac{z}{\rho_1}\right)\left(1-\frac{z}{\rho_2}\right)\left(1-\frac{z}{\rho_3}\right)\left(1-\frac{z}{\rho_4}\right)\cdots$$

According to Formula 3.2.1 in my site " **03 Vieta's Formulas in Infinite-degree Equation** " (Infinite Degree Equations) ,
Vieta's formula also holds for infinite degree equations. Therefore, we obtain the desired expressions.

<div align="right">Q.E.D.</div>

And these values of A_n $n=1, 2, 3, 4, \cdots$ are given by π and the higher derivatives of $\Gamma(z)$, $\beta(z)$, which are the components of $\omega(z)$. That is, the coefficients A_r $r=1, 2, 3, \cdots$ in Lemma 9.1.3 is given by Theorem 8.2.1 on my site " **08 Power Series of Completed Dirichlet Beta** ". If we reprint this, it is as follows.

Theorem 8.2.1 (Reprint)

Let the Completed Dirichlet Beta Function $\omega(z)$ and its Maclaurin series be as follows.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1+z}{2}\right)\beta(z) = \sum_{r=0}^{\infty} A_r z^r$$

Then, these coefficients A_r $r=0, 1, 2, 3, \cdots$ are given by,

$$A_r = \frac{1}{\pi}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^{-s}}{(r-s)!}\left(log\,2+\frac{1}{2}log\,\pi\right)^{r-s}\frac{(-1)^{s-t}}{(s-t)!}\left\{\gamma_{s-t}\left(\frac{1}{4}\right)-\gamma_{s-t}\left(\frac{3}{4}\right)\right\}\frac{g_t(1)}{2^t\,t!}$$

$$g_r(1) = \begin{cases} 1 & r = 0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0(1),\,\psi_1(1),\,...,\,\psi_{r-1}(1)\right) & r = 1, 2, 3, \cdots \end{cases}$$

Where, $\psi_n(z)$ is the Polygamma function, $B_{n,k}(f_1, f_2, \dots)$ is the Bell polynomial and $\gamma_r(a)$ is Generalized Stieltjes constant defined by

$$\gamma_r(a) = \lim_{m \to \infty} \left\{ \sum_{k=0}^{m} \frac{\log^r(k+a)}{k+a} - \frac{\log^{r+1}(m+a)}{r+1} \right\} \qquad \begin{array}{l} r = 0, 1, 2, \cdots \\ a \neq 0, -1, -2, \cdots \end{array}$$

Using the mathematical processing software **Mathematica**, the first few of A_r are,

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Tblψ[r_, z_] := Table[ψₖ[z], {k, 0, r - 1}]
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$A_0 \quad \dfrac{1}{\pi}\left(\gamma_0\left[\dfrac{1}{4}\right] - \gamma_0\left[\dfrac{3}{4}\right]\right) \quad (=1)$

$A_1 \quad \dfrac{1}{\pi}\left(\left(\mathbf{Log}[2] + \dfrac{\mathbf{Log}[\pi]}{2}\right)\left(\gamma_0\left[\dfrac{1}{4}\right] - \gamma_0\left[\dfrac{3}{4}\right]\right) + \gamma_1\left[\dfrac{1}{4}\right] - \gamma_1\left[\dfrac{3}{4}\right] - \dfrac{1}{2}\left(\gamma_0\left[\dfrac{1}{4}\right] - \gamma_0\left[\dfrac{3}{4}\right]\right)\psi_0[1]\right)$

$A_2 \quad \dfrac{1}{\pi}\left(\dfrac{1}{2}\left(\mathbf{Log}[2] + \dfrac{\mathbf{Log}[\pi]}{2}\right)^2\left(\gamma_0\left[\dfrac{1}{4}\right] - \gamma_0\left[\dfrac{3}{4}\right]\right) + \left(\mathbf{Log}[2] + \dfrac{\mathbf{Log}[\pi]}{2}\right)\left(\gamma_1\left[\dfrac{1}{4}\right] - \gamma_1\left[\dfrac{3}{4}\right]\right)\right.$

$\qquad \quad + \dfrac{1}{2}\left(\gamma_2\left[\dfrac{1}{4}\right] - \gamma_2\left[\dfrac{3}{4}\right]\right) - \dfrac{1}{2}\left(\mathbf{Log}[2] + \dfrac{\mathbf{Log}[\pi]}{2}\right)\left(\gamma_0\left[\dfrac{1}{4}\right] - \gamma_0\left[\dfrac{3}{4}\right]\right)\psi_0[1]$

$\qquad \quad \left. - \dfrac{1}{2}\left(\gamma_1\left[\dfrac{1}{4}\right] - \gamma_1\left[\dfrac{3}{4}\right]\right)\psi_0[1] + \dfrac{1}{8}\left(\gamma_0\left[\dfrac{1}{4}\right] - \gamma_0\left[\dfrac{3}{4}\right]\right)\left(\psi_0[1]^2 + \psi_1[1]\right)\right)$

$\vdots$

```
Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]
γₛ_[a_] := StieltjesGamma[s, a]

SetPrecision[{A₁, A₂, A₃, A₄}, 14]
{-0.0777839899618, 0.080350229317, -0.005182462271, 0.002716852531}
```

Combining Lemma 9.1.1 $\sim$ Lemma 9.1.3 and Theorem 8.2.1 , we obtain the following theorem.

Theorem 9.1.4

If ρ_{r_1} $r_1 = 1, 2, 3, \cdots$ are nontrivial zeros of the Dirichlet Beta Function $\beta(z)$, the Li Coefficients λ_n $n = 1, 2, 3, \cdots$ are expressed as follows.

$$\lambda_n = -nA_1 + \sum_{k=2}^{n} (-1)^{k-1}\binom{n}{k}\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^k} \qquad (\text{When } n = 1, \text{ the 2 rd term is ignored.})$$

$$\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}} = -A_1$$

$$\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^k} = (-A_1)^k - 2(-A_1)^{k-2}A_2 - \sum_{s=0}^{k-3}(-A_1)^s\left\{\sum_{t=2}^{k-s-1}\left(\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^{k-s-t}}\right)A_t + (k-s)A_{k-s}\right\}$$

$$(\text{When } k \leq 2, \text{ the 3 nd term is ignored.})$$

$$A_r = \frac{1}{\pi}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^{-s}}{(r-s)!}\left(\log 2 + \frac{1}{2}\log \pi\right)^{r-s}\frac{(-1)^{s-t}}{(s-t)!}\left\{\gamma_{s-t}\left(\frac{1}{4}\right) - \gamma_{s-t}\left(\frac{3}{4}\right)\right\}\frac{g_t(1)}{2^t t!}$$

$$g_r(1) = \begin{cases} 1 & r = 0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{r-1}(1)\right) & r = 1, 2, 3, \cdots \end{cases}$$

Where, $\psi_n(z)$ is the Polygamma function, $B_{n,k}(f_1, f_2, \dots)$ is the Bell polynomial and $\gamma_r(a)$ is Generalized Stieltjes constant.

Proof

From Lemma 9.1.1 and Lemma 9.1.2

$$\lambda_n = n\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}} + \sum_{k=2}^{n}(-1)^{k-1}\binom{n}{k}\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^{k}} \tag{1.1n}$$

$$\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^{n}} = \left(\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}}\right)^{n} - 2\left(\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}}\right)^{n-2}H_2$$

$$- \sum_{s=0}^{n-3}\left(\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}}\right)^{s}\left\{\sum_{t=2}^{n-s-1}(-1)^{t}\left(\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^{n-s-t}}\right)H_t + (-1)^{n-s}(n-s)H_{n-s}\right\} \tag{1.2n}$$

$$H_n = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{1}{\rho_{r_1}\rho_{r_2}\rho_{r_3}\cdots\rho_{r_n}} \tag{1.Hn}$$

Next, from Lemma 9.1.3 ,

$$\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}} = -A_1 \tag{1.3$_1$}$$

Substituting this for (1.1n) and (1.2n) ,

$$\lambda_n = -nA_1 + \sum_{k=2}^{n}(-1)^{k-1}\binom{n}{k}\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^{k}}$$

$$\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^{n}} = \left(-A_1\right)^{n} - 2\left(-A_1\right)^{n-2}H_2$$

$$- \sum_{s=0}^{n-3}\left(-A_1\right)^{s}\left\{\sum_{t=2}^{n-s-1}(-1)^{t}\left(\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^{n-s-t}}\right)H_t + (-1)^{n-s}(n-s)H_{n-s}\right\}$$

From (1.Hn) and Lemm 9.1.3 (1.3n) ,

$$H_n = (-1)^{n}A_n \qquad n = 2, 3, 4, \cdots$$

Substituting these for the above

$$\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^{k}} = \left(-A_1\right)^{k} - 2\left(-A_1\right)^{k-2}A_2 - \sum_{s=0}^{k-3}\left(-A_1\right)^{s}\left\{\sum_{t=2}^{k-s-1}\left(\sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^{k-s-t}}\right)A_t + (k-s)A_{k-s}\right\}$$

A_r , $g_r(1)$ and provisos follow from Theorem 8.2.1 .

<div align="right">Q.E.D.</div>

The statement in Theorem 9.1.4 is easy to read but not suitable for calculations, so we replace the sum of powers with

$$G_1 = -A_1 \quad , \quad G_k = \sum_{r_1=1}^{\infty}\frac{1}{\rho_{r_1}^{k}} \quad k = 2, 3, 4, \cdots$$

This leads to the following theorem, which is suitable for recursive calculations.

Theorem 9.1.5

The Li Coefficients λ_n $n = 1, 2, 3, \cdots$ can be calculated as follows.

$$\lambda_n = -nA_1 + \sum_{k=2}^{n}(-1)^{k-1}\binom{n}{k}G_k \qquad (\text{When } n = 1 \text{, the 2 rd term is ignored.})$$

$$G_1 = -A_1$$

$$G_k = \left(-A_1\right)^{k} - 2\left(-A_1\right)^{k-2}A_2 - \sum_{s=0}^{k-3}\left(-A_1\right)^{s}\left\{\sum_{t=2}^{k-s-1}G_{k-s-t}A_t + (k-s)A_{k-s}\right\}$$

<div align="right">(When $k \leq 2$, the 3 nd term is ignored.)</div>

$$A_r = \frac{1}{\pi}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^{-s}}{(r-s)!}\left(log\,2 + \frac{1}{2}log\,\pi\right)^{r-s}\frac{(-1)^{s-t}}{(s-t)!}\left\{\gamma_{s-t}\left(\frac{1}{4}\right) - \gamma_{s-t}\left(\frac{3}{4}\right)\right\}\frac{g_t(1)}{2^{t}t!}$$

$$g_r(1) = \begin{cases} 1 & r = 0 \\ \sum_{k=1}^{r} B_{r,k}\big(\psi_0(1),\,\psi_1(1),\,\dots,\,\psi_{r-1}(1)\big) & r = 1,2,3,\cdots \end{cases}$$

Where, $\psi_n(z)$ is the Polygamma function, $B_{n,k}(f_1,f_2,\dots)$ is the Bell polynomial and $\gamma_r(a)$ is Generalized Stieltjes constant.

In fact, when this thorem is executed using the mathematical processing software ***Mathematica***, it is as follows.

Li's coefficient (symbolic) λ_n

```
Unprotect[Power];  Power[0, 0] = 1;
```

$$\lambda_{n\_} := -n\,A_1 + \sum_{k=2}^{n} (-1)^{k-1}\,\text{Binomial}[n, k]\,G_k$$

$$G_1 := -A_1$$

$$G_{k\_} := (-A_1)^k - 2\,(-A_1)^{k-2}\,A_2 - \sum_{s=0}^{k-3} (-A_1)^s \left(\sum_{t=2}^{k-s-1} G_{k-s-t}\,A_t + (k-s)\,A_{k-s} \right)$$

$$A_{r\_} := \frac{1}{\pi} \sum_{s=0}^{r} \sum_{t=0}^{s} \frac{(-1)^{-s}}{(r-s)!} \left(\text{Log}[2] + \frac{1}{2}\,\text{Log}[\pi] \right)^{r-s} \frac{(-1)^{s-t}}{(s-t)!} \left(\gamma_{s-t}\!\left[\frac{1}{4}\right] - \gamma_{s-t}\!\left[\frac{3}{4}\right] \right) \frac{g_t[1]}{2^t\,t!}$$

$$g_{r\_}[1] := \text{If}\!\left[r = 0,\ 1,\ \sum_{k=1}^{r} \text{BellY}[r, k, \text{Tbl}\psi[r, 1]] \right]$$

$$\text{Tbl}\psi[r\_, z\_] := \text{Table}[\psi_k[z], \{k, 0, r-1\}]$$

$$\lambda_1 \quad -\frac{1}{\pi} \left(\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right) \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) + \gamma_1\!\left[\frac{1}{4}\right] - \gamma_1\!\left[\frac{3}{4}\right] - \frac{1}{2} \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) \psi_0[1] \right)$$

$$\lambda_2 \quad -\frac{2}{\pi} \left(\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right) \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) + \gamma_1\!\left[\frac{1}{4}\right] - \gamma_1\!\left[\frac{3}{4}\right] - \frac{1}{2} \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) \psi_0[1] \right)$$

$$-\frac{1}{\pi^2} \left(\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right) \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) + \gamma_1\!\left[\frac{1}{4}\right] - \gamma_1\!\left[\frac{3}{4}\right] - \frac{1}{2} \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) \psi_0[1] \right)^2$$

$$+\frac{2}{\pi} \left(\frac{1}{2} \left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)^2 \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) + \left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right) \left(\gamma_1\!\left[\frac{1}{4}\right] - \gamma_1\!\left[\frac{3}{4}\right] \right) \right.$$

$$+ \frac{1}{2} \left(\gamma_2\!\left[\frac{1}{4}\right] - \gamma_2\!\left[\frac{3}{4}\right] \right) - \frac{1}{2} \left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right) \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) \psi_0[1]$$

$$\left. - \frac{1}{2} \left(\gamma_1\!\left[\frac{1}{4}\right] - \gamma_1\!\left[\frac{3}{4}\right] \right) \psi_0[1] + \frac{1}{8} \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) \left(\psi_0[1]^2 + \psi_1[1] \right) \right)$$

$$\lambda_3 \quad -\frac{3}{\pi} \left(\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right) \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) + \gamma_1\!\left[\frac{1}{4}\right] - \gamma_1\!\left[\frac{3}{4}\right] - \frac{1}{2} \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) \psi_0[1] \right)$$

$$-\frac{1}{\pi^3} \left(\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right) \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) + \gamma_1\!\left[\frac{1}{4}\right] - \gamma_1\!\left[\frac{3}{4}\right] - \frac{1}{2} \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) \psi_0[1] \right)^3$$

$$+\frac{3}{\pi^2} \left(\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right) \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) + \gamma_1\!\left[\frac{1}{4}\right] - \gamma_1\!\left[\frac{3}{4}\right] - \frac{1}{2} \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) \psi_0[1] \right)$$

$$\times \left(\frac{1}{2} \left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)^2 \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) + \left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right) \left(\gamma_1\!\left[\frac{1}{4}\right] - \gamma_1\!\left[\frac{3}{4}\right] \right) \right.$$

$$+ \frac{1}{2} \left(\gamma_2\!\left[\frac{1}{4}\right] - \gamma_2\!\left[\frac{3}{4}\right] \right) - \frac{1}{2} \left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right) \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) \psi_0[1]$$

$$\left. - \frac{1}{2} \left(\gamma_1\!\left[\frac{1}{4}\right] - \gamma_1\!\left[\frac{3}{4}\right] \right) \psi_0[1] + \frac{1}{8} \left(\gamma_0\!\left[\frac{1}{4}\right] - \gamma_0\!\left[\frac{3}{4}\right] \right) \left(\psi_0[1]^2 + \psi_1[1] \right) \right)$$

$$-3\left(\frac{1}{\pi^2}\left(\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)+\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]-\frac{1}{2}\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\psi_0[1]\right)^2\right.$$

$$-\frac{2}{\pi}\left(\frac{1}{2}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)^2\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)+\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)\right.$$

$$+\frac{1}{2}\left(\gamma_2\left[\frac{1}{4}\right]-\gamma_2\left[\frac{3}{4}\right]\right)-\frac{1}{2}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\psi_0[1]$$

$$\left.\left.-\frac{1}{2}\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)\psi_0[1]+\frac{1}{8}\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\left(\psi_0[1]^2+\psi_1[1]\right)\right)\right)$$

$$-\frac{3}{\pi}\left(\frac{1}{6}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)^3\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)+\frac{1}{2}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)^2\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)\right.$$

$$+\frac{1}{2}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_2\left[\frac{1}{4}\right]-\gamma_2\left[\frac{3}{4}\right]\right)+\frac{1}{6}\left(\gamma_3\left[\frac{1}{4}\right]-\gamma_3\left[\frac{3}{4}\right]\right)$$

$$-\frac{1}{4}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)^2\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\psi_0[1]$$

$$-\frac{1}{2}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)\psi_0[1]-\frac{1}{4}\left(\gamma_2\left[\frac{1}{4}\right]-\gamma_2\left[\frac{3}{4}\right]\right)\psi_0[1]$$

$$+\frac{1}{8}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\left(\psi_0[1]^2+\psi_1[1]\right)+\frac{1}{8}\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)$$

$$\left.\times\left(\psi_0[1]^2+\psi_1[1]\right)-\frac{1}{48}\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\left(\psi_0[1]^3+3\psi_0[1]\,\psi_1[1]+\psi_2[1]\right)\right)$$

Given the values of $\psi_n(1)$ and $\gamma_n(a)$, the Li Coefficients are calculated as follows.

Li's coefficient (numerical) λ_n

```
Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]
γ_s_[a_] := StieltjesGamma[s, a]
```

λ_1	0.777839899618		λ_{11}	8.39856156
λ_2	0.310218089464		λ_{12}	9.78922816
λ_3	0.69457042132		λ_{13}	11.23628437
λ_4	1.22635973042		λ_{14}	12.7283053
λ_5	1.8994612040		λ_{15}	14.2545345
λ_6	2.7062506951		λ_{16}	15.8051035
λ_7	3.637783553		λ_{17}	17.3712243
λ_8	4.684003358		λ_{18}	18.945349
λ_9	5.833975075		λ_{19}	20.521296
λ_{10}	7.07613652		λ_{20}	22.094333
			λ_{21}	23.661221

Of these, λ_1 is equal to the sum of the reciprocals of the zeros of the Dirichlet Beta Function. (**OEIS A360807**)

cf.

The Li Coefficients for the Completed Riemann Zeta, e.g., λ_3 is

$$\lambda_3 = -\frac{3}{2}\log\pi + \frac{3}{2}\psi_0\left(\frac{3}{2}\right) + \frac{3}{4}\psi_1\left(\frac{3}{2}\right) + \frac{1}{16}\psi_2\left(\frac{3}{2}\right)$$

$$+ 3\gamma_0 - 3\gamma_0^2 + \gamma_0^3 - 6\gamma_1 + 3\gamma_0\gamma_1 + \frac{3}{2}\gamma_2$$

It is separated into three groups: $log\,\pi$, $\psi_n(3/2)$ and γ_n . The same is true for the other λ_n . Therefore, the recursive calculation only needed to be performed for γ_n .

On the other hand, the Li coefficients for the Completed Dirichlet Beta cannot be grouped as seen above. So there is no other way than to calculate $log\,\pi$, $\psi_n(1)$, and $\gamma_n(a)$ together recursively as above.

9.2 Li Coefficients expressed by x ± i y

In the previous section, the Li Coefficient obtained from the Hadamard product of the Completed Dirichlet Beta Function was shown as follows:

$$\lambda_n = \sum_{\rho} \left(1 - \left(1 - \frac{1}{\rho} \right)^n \right) \qquad \text{where } \rho \text{ runs over all the zeros.}$$

In this section, we express the Li Coefficients by the conjugate zeros $x_r \pm i\, y_r$ of the Completed Dirichlet Beta Function.

Lemma 9.2.1

For complex conjugates $x_r \pm i\, y_r$, the followings hold.

$$\left(\frac{1}{x_r - i\, y_r} \right)^1 + \left(\frac{1}{x_r + i\, y_r} \right)^1 = \frac{(2x_r)^1}{x_r^2 + y_r^2} \tag{9.1}$$

$$\left(\frac{1}{x_r - i\, y_r} \right)^2 + \left(\frac{1}{x_r + i\, y_r} \right)^2 = \frac{(2x_r)^2}{\left(x_r^2 + y_r^2\right)^2} - \frac{2(2x_r)^0}{\left(x_r^2 + y_r^2\right)^1} \tag{9.2}$$

$$\left(\frac{1}{x_r - i\, y_r} \right)^3 + \left(\frac{1}{x_r + i\, y_r} \right)^3 = \frac{(2x_r)^3}{\left(x_r^2 + y_r^2\right)^3} - \frac{3(2x_r)^1}{\left(x_r^2 + y_r^2\right)^2} \tag{9.3}$$

$$\left(\frac{1}{x_r - i\, y_r} \right)^4 + \left(\frac{1}{x_r + i\, y_r} \right)^4 = \frac{(2x_r)^4}{\left(x_r^2 + y_r^2\right)^4} - \frac{4(2x_r)^2}{\left(x_r^2 + y_r^2\right)^3} + \frac{2(2x_r)^0}{\left(x_r^2 + y_r^2\right)^2} \tag{9.4}$$

$$\left(\frac{1}{x_r - i\, y_r} \right)^5 + \left(\frac{1}{x_r + i\, y_r} \right)^5 = \frac{(2x_r)^5}{\left(x_r^2 + y_r^2\right)^5} - \frac{5(2x_r)^3}{\left(x_r^2 + y_r^2\right)^4} + \frac{5(2x_r)^1}{\left(x_r^2 + y_r^2\right)^3} \tag{9.5}$$

$$\left(\frac{1}{x_r - i\, y_r} \right)^6 + \left(\frac{1}{x_r + i\, y_r} \right)^6 = \frac{(2x_r)^6}{\left(x_r^2 + y_r^2\right)^6} - \frac{6(2x_r)^4}{\left(x_r^2 + y_r^2\right)^5} + \frac{9(2x_r)^2}{\left(x_r^2 + y_r^2\right)^4} - \frac{2(2x_r)^0}{\left(x_r^2 + y_r^2\right)^3} \tag{9.6}$$

$$\vdots$$

$$\left(\frac{1}{x_r - i\, y_r} \right)^s + \left(\frac{1}{x_r + i\, y_r} \right)^s = \sum_{t=0}^{\lfloor s/2 \rfloor} (-1)^t \left\{ \binom{s-t}{t} + \binom{s-t-1}{t-1} \right\} \frac{(2x_r)^{s-2t}}{\left(x_r^2 + y_r^2\right)^{s-t}} \tag{9.s}$$

Proof
Sum of 1 st powers and Product of 1 st powers

$$\frac{1}{x_r + i\, y_r} + \frac{1}{x_r - i\, y_r} = \frac{2x_r}{x_r^2 + y_r^2} \tag{9.1}$$

$$\frac{1}{x_r + i\, y_r} \frac{1}{x_r - i\, y_r} = \frac{1}{x_r^2 + y_r^2} \tag{9.1p}$$

Sum of Squares

$$\left(\frac{1}{x_r + i\, y_r} + \frac{1}{x_r - i\, y_r} \right)^2 = \left(\frac{1}{x_r + i\, y_r} \right)^2 + \left(\frac{1}{x_r - i\, y_r} \right)^2 + 2 \frac{1}{x_r + i\, y_r} \frac{1}{x_r - i\, y_r}$$

Substituting (9.1) and (9.1p) for both sides,

$$\left(\frac{2x_r}{x_r^2 + y_r^2} \right)^2 = \left(\frac{1}{x_r + i\, y_r} \right)^2 + \left(\frac{1}{x_r - i\, y_r} \right)^2 + 2 \frac{1}{x_r^2 + y_r^2}$$

From this,

$$\left(\frac{1}{x_r + i\, y_r} \right)^2 + \left(\frac{1}{x_r - i\, y_r} \right)^2 = \frac{(2x_r)^2}{\left(x_r^2 + y_r^2\right)^2} - 2 \frac{(2x_r)^0}{\left(x_r^2 + y_r^2\right)^1} \tag{9.2}$$

Sum of cubes

$$\left(\frac{1}{x_r+i\,y_r}+\frac{1}{x_r-i\,y_r}\right)^3 = \left(\frac{1}{x_r+i\,y_r}\right)^3 + \left(\frac{1}{x_r-i\,y_r}\right)^3$$

$$+\ 3\left(\frac{1}{x_r+i\,y_r}\right)^2\frac{1}{x_r-i\,y_r}\ +\ 3\frac{1}{x_r+i\,y_r}\left(\frac{1}{x_r-i\,y_r}\right)^2$$

$$=\left(\frac{1}{x_r+i\,y_r}\right)^3 + \left(\frac{1}{x_r-i\,y_r}\right)^3 +\ 3\frac{1}{x_r+i\,y_r}\frac{1}{x_r-i\,y_r}\left(\frac{1}{x_r+i\,y_r}+\frac{1}{x_r-i\,y_r}\right)$$

Substituting (9.1) and (9.1p) for both sides,

$$\left(\frac{2x_r}{x_r^2+y_r^2}\right)^3=\left(\frac{1}{x_r+i\,y_r}\right)^3+\left(\frac{1}{x_r-i\,y_r}\right)^3+\ 3\frac{1}{x_r^2+y_r^2}\left(\frac{2x_r}{x_r^2+y_r^2}\right)$$

From this,

$$\left(\frac{1}{x_r+i\,y_r}\right)^3+\left(\frac{1}{x_r-i\,y_r}\right)^3=\frac{\left(2x_r\right)^3}{\left(x_r^2+y_r^2\right)^3}\ -\ 3\frac{\left(2x_r\right)^1}{\left(x_r^2+y_r^2\right)^2} \tag{9.3}$$

Sum of 4th ~ 6th powers

In a similar way,

$$\left(\frac{1}{x_r+i\,y_r}\right)^4+\left(\frac{1}{x_r-i\,y_r}\right)^4=\frac{\left(2x_r\right)^4}{\left(x_r^2+y_r^2\right)^4}\ -\ 4\frac{\left(2x_r\right)^2}{\left(x_r^2+y_r^2\right)^3}\ +\ 2\frac{\left(2x_r\right)^0}{\left(x_r^2+y_r^2\right)^2} \tag{9.4}$$

$$\left(\frac{1}{x_r+i\,y_r}\right)^5+\left(\frac{1}{x_r-i\,y_r}\right)^5=\frac{\left(2x_r\right)^5}{\left(x_r^2+y_r^2\right)^5}\ -\ 5\frac{\left(2x_r\right)^3}{\left(x_r^2+y_r^2\right)^4}\ +\ 5\frac{\left(2x_r\right)^1}{\left(x_r^2+y_r^2\right)^3} \tag{9.5}$$

$$\left(\frac{1}{x_r+i\,y_r}\right)^6+\left(\frac{1}{x_r-i\,y_r}\right)^6=\frac{\left(2x_r\right)^6}{\left(x_r^2+y_r^2\right)^6}\ -\ 6\frac{\left(2x_r\right)^4}{\left(x_r^2+y_r^2\right)^5}\ +\ 9\frac{\left(2x_r\right)^2}{\left(x_r^2+y_r^2\right)^4}\ -\ 2\frac{\left(2x_r\right)^0}{\left(x_r^2+y_r^2\right)^3}$$

$$\tag{9.6}$$

The absolute values of these coefficients on the right hand side are

$$1, 1, 2, 1, 3, 1, 4, 2, 1, 5, 5, 1, 6, 9, 2,\ \cdots$$

When this integer sequence was searched in **"The On-Line Encyclopedia of Integer Sequences" (OEIS)** , $\mathrm{A034807}$ was found. These are the coefficients of the *Lucas* polynomial, given by

$$T(s,t)\ =\ C(s-t,\,t)\ +\ C(s-t-1,\,t-1)$$

Therefore, the sum of s powers can be expressed as follows using the floor function $\lfloor x\rfloor$:

$$\left(\frac{1}{x_r+i\,y_r}\right)^s+\left(\frac{1}{x_r-i\,y_r}\right)^s=\sum_{t=0}^{\lfloor s/2\rfloor}(-1)^t\left\{\binom{s-t}{t}+\binom{s-t-1}{t-1}\right\}\frac{\left(2x_r\right)^{s-2t}}{\left(x_r^2+y_r^2\right)^{s-t}} \tag{9.s}$$

Q.E.D.

Using Lemma 9.1.1 in the previous section and this Lemma 9.2.1, we obtain the following theorem.

Theorem 9.2.2

When the zeros of the Dirichlet Beta Function $\omega(z)$ are $x_r\pm i\,y_r\ \ r=1, 2, 3, \cdots$,

the Li Coefficients $\lambda_n\ \ n=1, 2, 3, \cdots$ are expressed as follows

$$\lambda_n=\sum_{r=1}^{\infty}\left(\sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\sum_{t=0}^{\lfloor s/2\rfloor}(-1)^t\left(\binom{s-t}{t}+\binom{s-t-1}{t-1}\right)\frac{\left(2x_r\right)^{s-2t}}{\left(x_r^2+y_r^2\right)^{s-t}}\right) \tag{2.2}$$

Proof

From Lemma 9.1.1 ,

$$\lambda_n = n \sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}} + \sum_{k=2}^{n} (-1)^{k-1} \binom{n}{k} \sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}^k} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \sum_{r_1=1}^{\infty} \frac{1}{\rho_{r_1}^k}$$

Replacing k with s and r_1 with k ,

$$\lambda_n = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \sum_{k=1}^{\infty} \frac{1}{\rho_k^s} \qquad (1.1n)$$

Here, let

$$\rho_1 = x_1 - i\,y_1 \ , \ \ \rho_2 = x_1 + i\,y_1 \ , \ \ \rho_3 = x_2 - i\,y_2 \ , \ \ \rho_4 = x_2 + i\,y_2 \ , \ \ \rho_5 = x_3 - i\,y_3 \ , \ \ \rho_6 = x_3 + i\,y_3 \ , \ \ \cdots$$

Then, (1.1n) becoms

$$\lambda_n = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \sum_{k=1}^{\infty} \frac{1}{\rho_k^s}$$

$$= \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \sum_{r=1}^{\infty} \left(\left(\frac{1}{x_r - i\,y_r} \right)^s + \left(\frac{1}{x_r + i\,y_r} \right)^s \right)$$

On the other hand, Lemma 9.2.1 was

$$\left(\frac{1}{x_r - i\,y_r} \right)^s + \left(\frac{1}{x_r + i\,y_r} \right)^s = \sum_{t=0}^{\lfloor s/2 \rfloor} (-1)^t \left\{ \binom{s-t}{t} + \binom{s-t-1}{t-1} \right\} \frac{(2x_r)^{s-2t}}{\left(x_r^2 + y_r^2 \right)^{s-t}} \qquad (9.s)$$

Substituting this for the above,

$$\lambda_n = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \sum_{r=1}^{\infty} \left(\sum_{t=0}^{\lfloor s/2 \rfloor} (-1)^t \left\{ \binom{s-t}{t} + \binom{s-t-1}{t-1} \right\} \frac{(2x_r)^{s-2t}}{\left(x_r^2 + y_r^2 \right)^{s-t}} \right)$$

i.e.

$$\lambda_n = \sum_{r=1}^{\infty} \left(\sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \sum_{t=0}^{\lfloor s/2 \rfloor} (-1)^t \left(\binom{s-t}{t} + \binom{s-t-1}{t-1} \right) \frac{(2x_r)^{s-2t}}{\left(x_r^2 + y_r^2 \right)^{s-t}} \right) \qquad (2.2)$$

Q.E.D.

The first few of (2.2) are expanded as follows

$$\lambda_1 = \frac{2x_1}{x_1^2 + y_1^2} + \frac{2x_2}{x_2^2 + y_2^2} + \frac{2x_3}{x_3^2 + y_3^2} + \cdots$$

$$\lambda_2 = -\frac{(2x_1)^2}{\left(x_1^2 + y_1^2 \right)^2} + \frac{2(2x_1)^0}{x_1^2 + y_1^2} + \frac{2(2x_1)^1}{x_1^2 + y_1^2}$$

$$- \frac{(2x_2)^2}{\left(x_2^2 + y_2^2 \right)^2} + \frac{2(2x_2)^0}{x_2^2 + y_2^2} + \frac{2(2x_2)^1}{x_2^2 + y_2^2}$$

$$- \frac{(2x_3)^2}{\left(x_3^2 + y_3^2 \right)^2} + \frac{2(2x_3)^0}{x_3^2 + y_3^2} + \frac{2(2x_3)^1}{x_3^2 + y_3^2}$$

$$\vdots$$

$$\lambda_3 = \frac{(2x_1)^3}{\left(x_1^2 + y_1^2 \right)^3} - \frac{3(2x_1)^1}{\left(x_1^2 + y_1^2 \right)^2} - \frac{3(2x_1)^2}{\left(x_1^2 + y_1^2 \right)^2} + \frac{6(2x_1)^0}{x_1^2 + y_1^2} + \frac{3(2x_1)^1}{x_1^2 + y_1^2}$$

$$+ \frac{(2x_2)^3}{\left(x_2^2 + y_2^2 \right)^3} - \frac{3(2x_2)^1}{\left(x_2^2 + y_2^2 \right)^2} - \frac{3(2x_2)^2}{\left(x_2^2 + y_2^2 \right)^2} + \frac{6(2x_2)^0}{x_2^2 + y_2^2} + \frac{3(2x_2)^1}{x_2^2 + y_2^2}$$

$$+ \frac{(2x_3)^3}{(x_3^2 + y_3^2)^3} - \frac{3(2x_3)^1}{(x_3^2 + y_3^2)^2} - \frac{3(2x_3)^2}{(x_3^2 + y_3^2)^2} + \frac{6(2x_3)^0}{x_3^2 + y_3^2} + \frac{3(2x_3)^1}{x_3^2 + y_3^2}$$

$$\vdots$$

$$\lambda_4 = - \frac{(2x_1)^4}{(x_1^2 + y_1^2)^4} + \frac{4(2x_1)^2}{(x_1^2 + y_1^2)^3} + \frac{4(2x_1)^3}{(x_1^2 + y_1^2)^3} - \frac{2(2x_1)^0}{(x_1^2 + y_1^2)^2} - \frac{12(2x_1)^1}{(x_1^2 + y_1^2)^2} - \frac{6(2x_1)^2}{(x_1^2 + y_1^2)^2}$$

$$+ \frac{12(2x_1)^0}{x_1^2 + y_1^2} + \frac{4(2x_1)^1}{x_1^2 + y_1^2}$$

$$- \frac{(2x_2)^4}{(x_2^2 + y_2^2)^4} + \frac{4(2x_2)^2}{(x_2^2 + y_2^2)^3} + \frac{4(2x_2)^3}{(x_2^2 + y_2^2)^3} - \frac{2(2x_2)^0}{(x_2^2 + y_2^2)^2} - \frac{12(2x_2)^1}{(x_2^2 + y_2^2)^2} - \frac{6(2x_2)^2}{(x_2^2 + y_2^2)^2}$$

$$+ \frac{12(2x_2)^0}{x_2^2 + y_2^2} + \frac{4(2x_2)^1}{x_2^2 + y_2^2}$$

$$- \frac{(2x_3)^4}{(x_3^2 + y_3^2)^4} + \frac{4(2x_3)^2}{(x_3^2 + y_3^2)^3} + \frac{4(2x_3)^3}{(x_3^2 + y_3^2)^3} - \frac{2(2x_3)^0}{(x_3^2 + y_3^2)^2} - \frac{12(2x_3)^1}{(x_3^2 + y_3^2)^2} - \frac{6(2x_3)^2}{(x_3^2 + y_3^2)^2}$$

$$+ \frac{12(2x_3)^0}{x_3^2 + y_3^2} + \frac{4(2x_3)^1}{x_3^2 + y_3^2}$$

$$\vdots$$

9.3 Li Coefficients on the Critical Line

In this section, as a special case of the previous section, the Li Coefficients λ_n are expressed by the zeros $1/2 \pm i\, y_r$ on the critical line.

Lemma 9.3.1

When the zeros of the Dirichlet Beta Function $\omega(z)$ are $1/2 \pm i\, y_r$ $r = 1, 2, 3, \cdots$,

the Li Coefficients λ_n $n = 1, 2, 3, \cdots$ are expressed as follows

$$\lambda_n = (-1)^{n-1} \sum_{r=1}^{\infty} \sum_{t=0}^{n-1} (-1)^t \left(\binom{2n-t}{t} + \binom{2n-t-1}{t-1} \right) \frac{1}{\left(1/4 + y_r^2 \right)^{n-t}} \tag{3.1}$$

Proof

Let $x_r = 1/2$ $r = 1, 2, 3, \cdots$ in the example of the expansion of Theorem 9.2.2 . Then, λ_1 , λ_2 , λ_3 , λ_4 become

$$\lambda_1 = \frac{1}{1/4 + y_1^2} + \frac{1}{1/4 + y_2^2} + \frac{1}{1/4 + y_3^2} + \cdots$$

$$\lambda_2 = -\frac{1}{\left(1/4 + y_1^2 \right)^2} + \frac{4}{1/4 + y_1^2}$$

$$-\frac{1}{\left(1/4 + y_2^2 \right)^2} + \frac{4}{1/4 + y_2^2}$$

$$-\frac{1}{\left(1/4 + y_3^2 \right)^2} + \frac{4}{1/4 + y_3^2}$$

$$\vdots$$

$$\lambda_3 = \frac{1}{\left(1/4 + y_1^2 \right)^3} - \frac{6}{\left(1/4 + y_1^2 \right)^2} + \frac{9}{1/4 + y_1^2}$$

$$+ \frac{1}{\left(1/4 + y_2^2 \right)^3} - \frac{6}{\left(1/4 + y_2^2 \right)^2} + \frac{9}{1/4 + y_2^2}$$

$$+ \frac{1}{\left(1/4 + y_3^2 \right)^3} - \frac{6}{\left(1/4 + y_3^2 \right)^2} + \frac{9}{1/4 + y_3^2}$$

$$\vdots$$

$$\lambda_4 = -\frac{1}{\left(1/4 + y_1^2 \right)^4} + \frac{8}{\left(1/4 + y_1^2 \right)^3} - \frac{20}{\left(1/4 + y_1^2 \right)^2} + \frac{16}{1/4 + y_1^2}$$

$$-\frac{1}{\left(1/4 + y_2^2 \right)^4} + \frac{8}{\left(1/4 + y_2^2 \right)^3} - \frac{20}{\left(1/4 + y_2^2 \right)^2} + \frac{16}{1/4 + y_2^2}$$

$$-\frac{1}{\left(1/4 + y_3^2 \right)^4} + \frac{8}{\left(1/4 + y_3^2 \right)^3} - \frac{20}{\left(1/4 + y_3^2 \right)^2} + \frac{16}{1/4 + y_3^2}$$

$$\vdots$$

The absolute values of these right-hand side coefficients are part of the *Lucas* polynomial coefficients **A061896** (red).

1, 1, 2, 1, 3, 1, 4, 2, 1, 5, 5, 1, 6, 9, 2, 1, 4, 7, 14, 7, 1, 8, 20, 16, 2,

The sequence of only the red characters can be obtained by the following.

$$T(2n, t) = C(2n-t, t) + C(2n-t-1, t-1) \qquad t = 0, 1, \cdots, n-1$$

So, let the rth row of λ_n be $\phi_n(y_r)$. Then

$$\phi_2(y_r) = -\frac{1}{\left(1/4 + y_r^2\right)^2} + \frac{4}{1/4 + y_r^2}$$

$$= (-1)^{2-1} \sum_{t=0}^{2-1} (-1)^t \left(\binom{4-t}{t} + \binom{4-t-1}{t-1} \right) \frac{1}{\left(1/4 + y_r^2\right)^{2-t}}$$

$$\phi_3(y_r) = \frac{1}{\left(1/4 + y_r^2\right)^3} - \frac{6}{\left(1/4 + y_r^2\right)^2} + \frac{9}{1/4 + y_r^2}$$

$$= (-1)^{3-1} \sum_{t=0}^{3-1} (-1)^t \left(\binom{6-t}{t} + \binom{6-t-1}{t-1} \right) \frac{1}{\left(1/4 + y_r^2\right)^{3-t}}$$

$$\vdots$$

Here after, by induction,

$$\phi_n(y_r) = (-1)^{n-1} \sum_{t=0}^{n-1} (-1)^t \left(\binom{2n-t}{t} + \binom{2n-t-1}{t-1} \right) \frac{1}{\left(1/4 + y_r^2\right)^{n-t}} \tag{3.ϕ}$$

From this,

$$\lambda_n = (-1)^{n-1} \sum_{r=1}^{\infty} \sum_{t=0}^{n-1} (-1)^t \left(\binom{2n-t}{t} + \binom{2n-t-1}{t-1} \right) \frac{1}{\left(1/4 + y_r^2\right)^{n-t}} \tag{3.1}$$

Q.E.D.

cf.

The general formula in the previous section was

$$\lambda_n = \sum_{r=1}^{\infty} \left(\sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \sum_{t=0}^{\lfloor s/2 \rfloor} (-1)^t \left(\binom{s-t}{t} + \binom{s-t-1}{t-1} \right) \frac{\left(2x_r\right)^{s-2t}}{\left(x_r^2 + y_r^2\right)^{s-t}} \right) \tag{2.2}$$

(3.1) is significantly simpler than (2.2). This is because the numerators of the polynomials become integers due to $2x_r = 1 \quad r=1, 2, 3, \cdots$, and the denominators are grouped together.

.

Polynomials that Compose the Li Coefficient

In the first place, the Li Coefficients exist in order to be determined there signs. To do so, let each row of λ_n be

$$\phi_n(y_r) = (-1)^{n-1} \sum_{t=0}^{n-1} (-1)^t \left(\binom{2n-t}{t} + \binom{2n-t-1}{t-1} \right) \frac{1}{\left(1/4 + y_r^2\right)^{n-t}} \tag{9.ϕ}$$

we should start by examining the sign of this.

Here, since $\left(1/4 + y_r^2\right)^n > 0$ for any real number y_r, let polynomial obtained by removing this from $\phi_n(y_r)$ be

$$g_n(y_r) = (-1)^{n-1} \sum_{t=0}^{n-1} (-1)^t \left(\binom{2n-t}{t} + \binom{2n-t-1}{t-1} \right) \left(1/4 + y_r^2\right)^t \tag{9.g)}$$

As the result, we just need to check the sign of $g_n(y_r)$ instead of $\phi_n(y_r)$.

Lemma 9.3.2

Let n be a natural number, y_r be a real number, and $g_n(y_r)$ be a polynomial such as:

$$g_n(y_r) = (-1)^{n-1} \sum_{t=0}^{n-1} (-1)^t \left(\binom{2n-t}{t} + \binom{2n-t-1}{t-1} \right) \left(1/4 + y_r^2\right)^t \tag{9.g)}$$

Then, $g_n(y_r)$ is transformed as follows due to whether n is odd or even

$$g_{2n-1}(y) = \frac{1}{4^{2n-2}} \left(\sum_{s=0}^{n-1} (-1)^s 4^s \binom{2n-1}{2s} y_r^{2s} \right)^2 \qquad n=1, 2, 3, \cdots \tag{9.go}$$

$$g_{2n}(y) = \frac{y_r^2}{4^{2n-2}} \left(\sum_{s=0}^{n-1} (-1)^s \, 4^s \binom{2n}{2s+1} y_r^{2s} \right)^2 \qquad n=1, 2, 3, \cdots \qquad\qquad (9.\text{ge})$$

Proof

(1) when n is odd

$g_n(y_r)$ is simplified and modfied as follows.

$$g_1(y_r) = 1 \qquad\qquad\qquad\qquad\qquad = \frac{1}{4^0}$$

$$g_3(y_r) = \frac{1}{16} \left(1 - 12y_r^2 \right)^2 \qquad\qquad = \frac{1}{4^2} \left(1 - 12y_r^2 \right)^2$$

$$g_5(y_r) = \frac{1}{256} \left(1 - 40y_r^2 + 80y_r^4 \right)^2 \qquad = \frac{1}{4^4} \left(1 - 40y_r^2 + 80y_r^4 \right)^2$$

$$g_7(y_r) = \frac{1}{4096} \left(1 - 84y_r^2 + 560y_r^4 - 448y_r^6 \right)^2 = \frac{1}{4^6} \left(1 - 84y_r^2 + 560y_r^4 - 448y_r^6 \right)^2$$

$$\vdots \qquad\qquad\qquad\qquad\qquad \vdots$$

By searching for $1, 84, 560, 448$ in **OEIS, A085840** was found. These are given by

$$T(n, s) = \frac{4^s (2n+1)!}{(2n-2s+1)!(2s)!} \qquad s = 0, 1, 2, \cdots, n$$

Replacing n with $n-1$ in the right-hand side,

$$T(n, s) = \frac{4^s (2n-1)!}{(2n-2s-1)!(2s)!} \qquad s = 0, 1, 2, \cdots, n-1$$

Using this,

$$g_{2n-1}(y_r) = \frac{1}{4^{2n-2}} \left(\sum_{s=0}^{n-1} (-1)^s \frac{4^s (2n-1)!}{(2n-2s-1)!(2s)!} y_r^{2s} \right)^2$$

i.e.

$$g_{2n-1}(y) = \frac{1}{4^{2n-2}} \left(\sum_{s=0}^{n-1} (-1)^s \, 4^s \binom{2n-1}{2s} y_r^{2s} \right)^2 \qquad n=1, 2, 3, \cdots \qquad\qquad (9.\text{go})$$

(2) when n is even

$g_n(y_r)$ is simplified and modfied as follows.

$$g_2(y_r) = 4y_r^2 \qquad\qquad\qquad\qquad\qquad = \frac{y_r^2}{4^0} \left(2y_r^0 \right)^2$$

$$g_4(y_r) = y_r^2 \left(1 - 4y_r^2 \right)^2 \qquad\qquad\qquad = \frac{y_r^2}{4^2} \left(4 - 16y_r^2 \right)^2$$

$$g_6(y_r) = \frac{y_r^2}{64} \left(3 - 40y_r^2 + 48y_r^4 \right)^2 \qquad = \frac{y_r^2}{4^4} \left(6 - 80y_r^2 + 96y_r^4 \right)^2$$

$$g_8(y_r) = \frac{y_r^2}{64} \left(1 - 28y_r^2 + 112y_r^4 - 62y_r^6 \right)^2 = \frac{y_r^2}{4^6} \left(8 - 224y_r^2 + 896y_r^4 - 512y_r^6 \right)^2$$

$$\vdots \qquad\qquad\qquad\qquad\qquad \vdots$$

By searching for $8, 224, 896, 512$ in **OEIS**, an integer sequence **A229032** containing these was found.

The full sequence are given by

$$T(n, s) = 4^s \binom{n+1}{2s+1} \qquad s = 0, 1, 2, \cdots, n$$

To skip unnecessary sequences, replacing n with $2n-1$ in the right-hand side,

$$T(n,s) \;=\; 4^s \begin{pmatrix} 2n \\ 2s+1 \end{pmatrix} \qquad s=0,1,2,\cdots,n$$

Using this,

$$g_{2n}(y) \;=\; \frac{y_r^{\,2}}{4^{\,2n-2}} \left(\sum_{s=0}^{n-1} (-1)^s \, 4^s \begin{pmatrix} 2n \\ 2s+1 \end{pmatrix} y_r^{\,2s} \right)^2 \qquad n=1,2,3,\cdots \qquad (9.\text{ge})$$

Q.E.D.

Note

That is, $g_n(y_r)$ is reduced to a perfect square expression. This is an unexpected and surprising result.

Li Coefficients on the Critical Line

Using Lemma 9.3.2 , the Li Coefficients on the Critical Line can be expressed as follows.

Theorem 9.3.3

When the zeros of the Dirichlet Beta Function $\omega(z)$ are $1/2 \pm i\, y_r \quad r=1,2,3,\cdots$,

the Li Coefficients $\lambda_n \quad n=1,2,3,\cdots$ are expressed as follows

$$\lambda_{2n-1} \;=\; \sum_{r=1}^{\infty} \frac{1}{4^{\,2n-2}\left(\left(1/4+y_r^{\,2}\right)^{(2n-1)/2}\right)^2} \left(\sum_{s=0}^{n-1} (-1)^s \, 4^s \begin{pmatrix} 2n-1 \\ 2s \end{pmatrix} y_r^{\,2s} \right)^2 \qquad n=1,2,3,\cdots \qquad (3.3\text{o})$$

$$\lambda_{2n} \;=\; \sum_{r=1}^{\infty} \frac{y_r^{\,2}}{4^{\,2n-2}\left(1/4+y_r^{\,2}\right)^{2n}} \left(\sum_{s=0}^{n-1} (-1)^s \, 4^s \begin{pmatrix} 2n \\ 2s+1 \end{pmatrix} y_r^{\,2s} \right)^2 \qquad n=1,2,3,\cdots \qquad (3.3\text{e})$$

Proof

From (3.1) , (9.ϕ) and (9.g) ,

$$\lambda_n \;=\; \sum_{r=1}^{\infty} \phi_n(y_r) \;=\; \sum_{r=1}^{\infty} \frac{g_n(y_r)}{\left(1/4+y_r^{\,2}\right)^n}$$

Substituting $g_n(y_r)$ of Lemma 9.3.2 into this for each odd and even,

$$\lambda_{2n-1} \;=\; \sum_{r=1}^{\infty} \frac{1}{4^{\,2n-2}\left(1/4+y_r^{\,2}\right)^{2n-1}} \left(\sum_{s=0}^{n-1} (-1)^s \, 4^s \begin{pmatrix} 2n-1 \\ 2s \end{pmatrix} y_r^{\,2s} \right)^2$$

$$=\; \sum_{r=1}^{\infty} \frac{1}{4^{\,2n-2}\left(\left(1/4+y_r^{\,2}\right)^{(2n-1)/2}\right)^2} \left(\sum_{s=0}^{n-1} (-1)^s \, 4^s \begin{pmatrix} 2n-1 \\ 2s \end{pmatrix} y_r^{\,2s} \right)^2$$

$$\lambda_{2n} \;=\; \sum_{r=1}^{\infty} \frac{y_r^{\,2}}{4^{\,2n-2}\left(1/4+y_r^{\,2}\right)^{2n}} \left(\sum_{s=0}^{n-1} (-1)^s \, 4^s \begin{pmatrix} 2n \\ 2s+1 \end{pmatrix} y_r^{\,2s} \right)^2$$

Q.E.D.

Note

Naturally, $\lambda_n \geq 0 \quad n=1,2,3,\cdots,\infty$. That is, the Li Coefficients on the critical line satisfy the Li's Criterion.

Calculation Example

The formula for generating the zeros of the Dirichlet Beta Function with real part $1/2$ is unknown, but the first 10000 are provided by **Tomas oliveira e Silva** (http://sweet.ua.pt/tos/zeta.html (004-001)) .

So, we calculate $\lambda_1 \sim \lambda_4$ using these according to (3.3o) and (3.3e) , it is as follows.

Loading Zeros of Dirichlet Beta Function

```
SetDirectory[NotebookDirectory[]];
y := ReadList["BetaZeros.prn", Number]
```

Li coefficients on the Critical Line

2n-1

$$\lambda o_n\_[m\_] := \sum_{r=1}^{m} \frac{1}{4^{2n-2} \left(\left(1/4 + y[[r]]^2 \right)^{(2n-1)/2} \right)^2} \left(\sum_{s=0}^{n-1} (-1)^s 4^s \text{Binomial}[2n-1, 2s] y[[r]]^{2s} \right)^2$$

2n

$$\lambda e_n\_[m\_] := \sum_{r=1}^{m} \frac{y[[r]]^2}{4^{2n-2} \left(1/4 + y[[r]]^2 \right)^{2n}} \left(\sum_{s=0}^{n-1} (-1)^s 4^s \text{Binomial}[2n, 2s+1] y[[r]]^{2s} \right)^2$$

```
{λo₁[10 000], λe₁[10 000], λo₂[10 000], λe₂[10 000]}
{0.0776004, 0.309484, 0.692918, 1.22342}
```

Comparing these with the calculation results in Section 1 , we can see that they match to 2 significant digits.

2026.02.09

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Alien's Mathematics