15 Li Coefficients on the Critical Line

15.1 Li Coefficients from Hadamard Product

In the previous chapter, " 14 Expression of Li Coefficients by Polygamma Functions ", we derived the Li coefficients λ_n from the Maclaurin series of the Riemann ξ function. In this chapter, we derive the Li coefficients λ_n from the Hadamard product of the Riemann ξ function. To do this, we must first state Li's Criterion.

Li's Criterion ($z=0$)

The Riemann hypothesis is equivalent to the inequality,

$$\lambda_n = \frac{(-1)^n}{(n-1)!}\frac{d^n}{dz^n}\left[(1-z)^{n-1}\log\xi(z)\right]\Big|_{z=0} \geq 0 \qquad for\ n=1,2,3,\dots \tag{1.0}$$

Where,

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) \tag{1.ξ}$$

$\rho_k \quad (k=1,2,3,\cdots)$ are zeros of $\xi(z)$. (1.ρ)

Regarding these Li Coefficients, the following lemma holds .

Lemma 15.1.1

When λ_n are coefficients defined by (1.0) and ρ_k are zeros defined by (1.ρ) , the following equations hold.

$$\lambda_n = \sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\sum_{k=1}^{\infty}\frac{1}{\rho_k^s} \tag{1.1}$$

$$\lambda_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) \tag{1.1'}$$

Proof

(1) Higher derivatives of $(1-z)^{n-1}$

$$\frac{d^1}{dz^1}(1-z)^{n-1} = -(n-1)(1-z)^{n-2}$$

$$\frac{d^2}{dz^2}(1-z)^{n-1} = (n-2)(n-1)(1-z)^{n-3}$$

$$\frac{d^3}{dz^3}(1-z)^{n-1} = -(n-3)(n-2)(n-1)(1-z)^{n-4}$$

$$\vdots$$

$$\frac{d^s}{dz^s}(1-z)^{n-1} = (-1)^s\{(n-s)\cdots(n-2)(n-1)\}(1-z)^{n-1-s}$$

Pochhammer symbol is

$$(a)_k = a(a+1)\cdots(a+k-1)$$

Using this,

$$(n-s)_k = (n-s)(n-s+1)\cdots(n-s+k-1)$$

Form this,

$$(n-s)_s = (n-s)(n-s+1)\cdots(n-1) = (n-s)\cdots(n-2)(n-1)$$

Therefore,

$$\frac{d^s}{dz^s}(1-z)^{n-1} = (-1)^s (n-s)_s (1-z)^{n-1-s} \tag{1.d1}$$

(2) Higher derivatives of $\xi(z)$

Taking the logarithm of both sides of (1.ξ) ,

$$\log \xi(z) = \log \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \sum_{k=1}^{\infty}\log\left(1-\frac{z}{\rho_k}\right)$$

Differentiating both sides with respect to z ,

$$\frac{d}{dz}\log \xi(z) = \frac{d}{dz}\sum_{k=1}^{\infty}\log\left(1-\frac{z}{\rho_k}\right) = \sum_{k=1}^{\infty}\frac{-1/\rho_k}{1-z/\rho_k}$$

i.e.

$$\frac{d^1}{dz^1}\log \xi(z) = \sum_{k=1}^{\infty}\frac{1}{z-\rho_k}$$

$$\frac{d^2}{dz^2}\log \xi(z) = \sum_{k=1}^{\infty}\frac{-1}{(z-\rho_k)^2}$$

$$\frac{d^3}{dz^3}\log \xi(z) = \sum_{k=1}^{\infty}\frac{2}{(z-\rho_k)^3}$$

$$\vdots$$

$$\frac{d^s}{dz^s}\log \xi(z) = \sum_{k=1}^{\infty}(-1)^{s-1}\frac{(s-1)!}{(z-\rho_k)^s} \tag{1.d2}$$

(3) The n th order derivatives of $(1-z)^{n-1}\log \xi(z)$

Leibniz's law is

$$\{f(z)g(z)\}^{(n)} = \sum_{s=0}^{n}\binom{n}{s} f^{(s)}(z)g^{(n-s)}(z)$$

Applying this to (1.d1) and (1.d2) ,

$$\frac{d^n}{dz^n}(1-z)^{n-1}\log \xi(z) = \sum_{s=0}^{n}\binom{n}{s}\left\{(-1)^s(n-s)_s(1-z)^{n-1-s}\right\}\sum_{k=1}^{\infty}(-1)^{n-s-1}\frac{(n-s-1)!}{(z-\rho_k)^{n-s}}$$

Since $(n-n)_n = 0$, we change the upper limit of Σ from n to $n-1$. At the same time, we rearrange the powers of (-1) . Then,

$$\frac{d^n}{dz^n}(1-z)^{n-1}\log \xi(z) = (-1)^{n-1}\sum_{s=0}^{n-1}\binom{n}{s}\left\{(n-s)_s(1-z)^{n-1-s}\right\}\sum_{k=1}^{\infty}\frac{(n-s-1)!}{(z-\rho_k)^{n-s}}$$

Further,

$$(n-s)_s = \frac{\Gamma(n-s+s)}{\Gamma(n-s)} = \frac{\Gamma(n)}{\Gamma(n-s)} = \frac{(n-1)!}{(n-s-1)!}$$

Substituting this for the inner of the $\{\ \}$,

$$\frac{d^n}{dz^n}(1-z)^{n-1}\log \xi(z) = (-1)^{n-1}\sum_{s=0}^{n-1}\binom{n}{s}\left\{\frac{(n-1)!}{(n-s-1)!}(1-z)^{n-1-s}\right\}\sum_{k=1}^{\infty}\frac{(n-s-1)!}{(z-\rho_k)^{n-s}}$$

i.e.

$$\frac{d^n}{dz^n}(1-z)^{n-1}\log \xi(z) = (-1)^{n-1}(n-1)!\sum_{s=0}^{n-1}\binom{n}{s}(1-z)^{n-1-s}\sum_{k=1}^{\infty}\frac{1}{(z-\rho_k)^{n-s}}$$

the differential coefficient at $z=0$ is

$$\frac{d^n}{dz^n}(1-z)^{n-1}\log\xi(z)\Bigg|_{z=0} = (-1)^{n-1}(n-1)!\sum_{s=0}^{n-1}\binom{n}{s}\sum_{k=1}^{\infty}\frac{1}{(-\rho_k)^{n-s}}$$

(4) Li Coefficients

Therefore, the Li coefficients become as follows.

$$\lambda_n = \frac{(-1)^n}{(n-1)!}\frac{d^n}{dz^n}\left[(1-z)^{n-1}\log\xi(z)\right]\Bigg|_{z=0}$$

$$= \frac{(-1)^n}{(n-1)!}(-1)^{n-1}(n-1)!\sum_{s=0}^{n-1}\binom{n}{s}\sum_{k=1}^{\infty}\frac{1}{(-1)^{n-s}\rho_k^{n-s}}$$

$$= \sum_{s=0}^{n-1}(-1)^{n-s+1}\binom{n}{s}\sum_{k=1}^{\infty}\frac{1}{\rho_k^{n-s}}$$

i.e.

$$\lambda_n = \sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\sum_{k=1}^{\infty}\frac{1}{\rho_k^{s}} \tag{1.1}$$

This becomes further as follows.

$$\lambda_n = -\sum_{k=1}^{\infty}\sum_{s=1}^{n}\binom{n}{s}\left(-\frac{1}{\rho_k}\right)^s = \sum_{k=1}^{\infty}\left(1-\sum_{s=0}^{n}\binom{n}{s}\left(-\frac{1}{\rho_k}\right)^s\right)$$

According to the binomial theorem,

$$\lambda_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) \tag{1.1'}$$

This is a well-known formula. Q.E.D.

In fact, according (1.1) , if the first few are calculated using the mathematical processing software *Mathematica*,

$$\lambda_{n\_} := \sum_{s=1}^{n}(-1)^{s-1}\,\mathbf{Binomial}[n,s]\sum_{k=1}^{\infty}\frac{1}{\rho_k{}^s}$$

$$\lambda_1 \qquad \sum_{k=1}^{\infty}\frac{1}{\rho_k}$$

$$\mathbf{Expand}[\lambda_2] \qquad -\sum_{k=1}^{\infty}\frac{1}{\rho_k^2}+2\sum_{k=1}^{\infty}\frac{1}{\rho_k}$$

$$\mathbf{Expand}[\lambda_3] \qquad \sum_{k=1}^{\infty}\frac{1}{\rho_k^3}-3\sum_{k=1}^{\infty}\frac{1}{\rho_k^2}+3\sum_{k=1}^{\infty}\frac{1}{\rho_k}$$

$$\mathbf{Expand}[\lambda_4] \qquad -\sum_{k=1}^{\infty}\frac{1}{\rho_k^4}+4\sum_{k=1}^{\infty}\frac{1}{\rho_k^3}-6\sum_{k=1}^{\infty}\frac{1}{\rho_k^2}+4\sum_{k=1}^{\infty}\frac{1}{\rho_k}$$

15.2 Sum of Powers of Reciprocals of x ± i y

The sum of the reciprocals of conjugate complex numbers is expressed as a polynomial whose coefficients are all binomial coefficients. In this section, we present and prove this as a lemma.

Lemma 15.2.1

For complex conjugates $x_r \pm i y_r$, the followings hold.

$$\left(\frac{1}{x_r - i y_r}\right)^1 + \left(\frac{1}{x_r + i y_r}\right)^1 = \frac{(2x_r)^1}{x_r^2 + y_r^2} \tag{2.1}$$

$$\left(\frac{1}{x_r - i y_r}\right)^2 + \left(\frac{1}{x_r + i y_r}\right)^2 = \frac{(2x_r)^2}{(x_r^2 + y_r^2)^2} - \frac{2(2x_r)^0}{(x_r^2 + y_r^2)^1} \tag{2.2}$$

$$\left(\frac{1}{x_r - i y_r}\right)^3 + \left(\frac{1}{x_r + i y_r}\right)^3 = \frac{(2x_r)^3}{(x_r^2 + y_r^2)^3} - \frac{3(2x_r)^1}{(x_r^2 + y_r^2)^2} \tag{2.3}$$

$$\left(\frac{1}{x_r - i y_r}\right)^4 + \left(\frac{1}{x_r + i y_r}\right)^4 = \frac{(2x_r)^4}{(x_r^2 + y_r^2)^4} - \frac{4(2x_r)^2}{(x_r^2 + y_r^2)^3} + \frac{2(2x_r)^0}{(x_r^2 + y_r^2)^2} \tag{2.4}$$

$$\left(\frac{1}{x_r - i y_r}\right)^5 + \left(\frac{1}{x_r + i y_r}\right)^5 = \frac{(2x_r)^5}{(x_r^2 + y_r^2)^5} - \frac{5(2x_r)^3}{(x_r^2 + y_r^2)^4} + \frac{5(2x_r)^1}{(x_r^2 + y_r^2)^3} \tag{2.5}$$

$$\left(\frac{1}{x_r + i y_r}\right)^6 + \left(\frac{1}{x_r - i y_r}\right)^6 = \frac{(2x_r)^6}{(x_r^2 + y_r^2)^6} - \frac{6(2x_r)^4}{(x_r^2 + y_r^2)^5} + \frac{9(2x_r)^2}{(x_r^2 + y_r^2)^4} - \frac{2(2x_r)^0}{(x_r^2 + y_r^2)^3} \tag{2.6}$$

$$\vdots$$

$$\left(\frac{1}{x_r - i y_r}\right)^s + \left(\frac{1}{x_r + i y_r}\right)^s = \sum_{t=0}^{\lfloor s/2 \rfloor} (-1)^t \left\{ \binom{s-t}{t} + \binom{s-t-1}{t-1} \right\} \frac{(2x_r)^{s-2t}}{(x_r^2 + y_r^2)^{s-t}} \tag{2.s}$$

Proof

Sum of 1 st powers and Product of 1 st powers

$$\frac{1}{x_r + i y_r} + \frac{1}{x_r - i y_r} = \frac{2x_r}{x_r^2 + y_r^2} \tag{2.1}$$

$$\frac{1}{x_r + i y_r} \frac{1}{x_r - i y_r} = \frac{1}{x_r^2 + y_r^2} \tag{2.1p}$$

Sum of Squares

$$\left(\frac{1}{x_r + i y_r} + \frac{1}{x_r - i y_r}\right)^2 = \left(\frac{1}{x_r + i y_r}\right)^2 + \left(\frac{1}{x_r - i y_r}\right)^2 + 2\frac{1}{x_r + i y_r}\frac{1}{x_r - i y_r}$$

Substituting (2.1) and (2.1p) for both sides,

$$\left(\frac{2x_r}{x_r^2 + y_r^2}\right)^2 = \left(\frac{1}{x_r + i y_r}\right)^2 + \left(\frac{1}{x_r - i y_r}\right)^2 + 2\frac{1}{x_r^2 + y_r^2}$$

From this,

$$\left(\frac{1}{x_r + i y_r}\right)^2 + \left(\frac{1}{x_r - i y_r}\right)^2 = \frac{(2x_r)^2}{(x_r^2 + y_r^2)^2} - 2\frac{(2x_r)^0}{(x_r^2 + y_r^2)^1} \tag{2.2}$$

Sum of cubes

$$\left(\frac{1}{x_r+iy_r}+\frac{1}{x_r-iy_r}\right)^3=\left(\frac{1}{x_r+iy_r}\right)^3+\left(\frac{1}{x_r-iy_r}\right)^3$$

$$+3\left(\frac{1}{x_r+iy_r}\right)^2\frac{1}{x_r-iy_r}+3\frac{1}{x_r+iy_r}\left(\frac{1}{x_r-iy_r}\right)^2$$

$$=\left(\frac{1}{x_r+iy_r}\right)^3+\left(\frac{1}{x_r-iy_r}\right)^3+3\frac{1}{x_r+iy_r}\frac{1}{x_r-iy_r}\left(\frac{1}{x_r+iy_r}+\frac{1}{x_r-iy_r}\right)$$

Substituting (2.1) and (2.1p) for both sides,

$$\left(\frac{2x_r}{x_r^2+y_r^2}\right)^3=\left(\frac{1}{x_r+iy_r}\right)^3+\left(\frac{1}{x_r-iy_r}\right)^3+3\frac{1}{x_r^2+y_r^2}\left(\frac{2x_r}{x_r^2+y_r^2}\right)$$

From this,

$$\left(\frac{1}{x_r+iy_r}\right)^3+\left(\frac{1}{x_r-iy_r}\right)^3=\frac{(2x_r)^3}{\left(x_r^2+y_r^2\right)^3}-3\frac{(2x_r)^1}{\left(x_r^2+y_r^2\right)^2} \tag{2.3}$$

Sum of 4th ~ 6th powers

In a similar way,

$$\left(\frac{1}{x_r+iy_r}\right)^4+\left(\frac{1}{x_r-iy_r}\right)^4=\frac{(2x_r)^4}{\left(x_r^2+y_r^2\right)^4}-4\frac{(2x_r)^2}{\left(x_r^2+y_r^2\right)^3}+2\frac{(2x_r)^0}{\left(x_r^2+y_r^2\right)^2} \tag{2.4}$$

$$\left(\frac{1}{x_r+iy_r}\right)^5+\left(\frac{1}{x_r-iy_r}\right)^5=\frac{(2x_r)^5}{\left(x_r^2+y_r^2\right)^5}-5\frac{(2x_r)^3}{\left(x_r^2+y_r^2\right)^4}+5\frac{(2x_r)^1}{\left(x_r^2+y_r^2\right)^3} \tag{2.5}$$

$$\left(\frac{1}{x_r+iy_r}\right)^6+\left(\frac{1}{x_r-iy_r}\right)^6=\frac{(2x_r)^6}{\left(x_r^2+y_r^2\right)^6}-6\frac{(2x_r)^4}{\left(x_r^2+y_r^2\right)^5}+9\frac{(2x_r)^2}{\left(x_r^2+y_r^2\right)^4}-2\frac{(2x_r)^0}{\left(x_r^2+y_r^2\right)^3} \tag{2.6}$$

The absolute values of these coefficients on the right hand side are

$1,1,2,1,3,1,4,2,1,5,5,1,6,9,2,\ \cdots$

When this integer sequence was searched in **"The On-Line Encyclopedia of Integer Sequences" (OEIS)** , A034807 was found. These are the coefficients of the ***Lucas*** polynomial, given by

$T(s,t)=C(s-t,t)+C(s-t-1,t-1)$

Therefore, the sum of s powers can be expressed as follows using the floor function $\lfloor x \rfloor$:

$$\left(\frac{1}{x_r+iy_r}\right)^s+\left(\frac{1}{x_r-iy_r}\right)^s=\sum_{t=0}^{\lfloor s/2\rfloor}(-1)^t\left\{\binom{s-t}{t}+\binom{s-t-1}{t-1}\right\}\frac{(2x_r)^{s-2t}}{\left(x_r^2+y_r^2\right)^{s-t}} \tag{2.s}$$

Q.E.D.

Lemma 15.2.1 can be more simply stated as follows

Lemma 15.2.2

For complex conjugates $x_r \pm iy_r$, the followings hold.

$$\left(\frac{1}{x_r-iy_r}\right)^1+\left(\frac{1}{x_r+iy_r}\right)^1=\frac{2\left(x_r^1\right)}{x_r^2+y_r^2} \tag{2.1'}$$

$$\left(\frac{1}{x_r - i y_r}\right)^2 + \left(\frac{1}{x_r + i y_r}\right)^2 = \frac{2\left(x_r^2 - y_r^2\right)}{\left(x_r^2 + y_r^2\right)^2} \tag{2.2'}$$

$$\left(\frac{1}{x_r - i y_r}\right)^3 + \left(\frac{1}{x_r + i y_r}\right)^3 = \frac{2\left(x_r^3 - 3x_r^1 y_r^2\right)}{\left(x_r^2 + y_r^2\right)^3} \tag{2.3'}$$

$$\left(\frac{1}{x_r - i y_r}\right)^4 + \left(\frac{1}{x_r + i y_r}\right)^4 = \frac{2\left(x_r^4 - 6x_r^2 y_r^2 + y_r^4\right)}{\left(x_r^2 + y_r^2\right)^4} \tag{2.4'}$$

$$\left(\frac{1}{x_r - i y_r}\right)^5 + \left(\frac{1}{x_r + i y_r}\right)^5 = \frac{2\left(x_r^5 - 10x_r^3 y_r^2 + 5x_r^1 y_r^4\right)}{\left(x_r^2 + y_r^2\right)^5} \tag{2.5'}$$

$$\left(\frac{1}{x_r + i y_r}\right)^6 + \left(\frac{1}{x_r - i y_r}\right)^6 = \frac{2\left(x_r^6 - 15x_r^4 y_r^2 + 15x_r^2 y_r^4 - y_r^6\right)}{\left(x_r^2 + y_r^2\right)^6} \tag{2.6'}$$

$$\vdots$$

$$\left(\frac{1}{x_r - i y_r}\right)^s + \left(\frac{1}{x_r + i y_r}\right)^s = \frac{2}{\left(x_r^2 + y_r^2\right)^s} \sum_{t=0}^{\lfloor s/2 \rfloor} (-1)^t \binom{s}{2t} x_r^{s-2t} y_r^{2t} \tag{2.s'}$$

Proof

$$\left(\frac{1}{x_r - i y_r}\right)^1 + \left(\frac{1}{x_r + i y_r}\right)^1 = \frac{\left(2x_r\right)^1}{x_r^2 + y_r^2} = \frac{2\left(x_r^1\right)}{x_r^2 + y_r^2} \tag{2.1'}$$

$$\left(\frac{1}{x_r - i y_r}\right)^2 + \left(\frac{1}{x_r + i y_r}\right)^2 = \frac{\left(2x_r\right)^2}{\left(x_r^2 + y_r^2\right)^2} - \frac{2\left(2x_r\right)^0}{\left(x_r^2 + y_r^2\right)^1} \tag{2.1}$$

$$= \frac{\left(2x_r\right)^2 - 2\left(x_r^2 + y_r^2\right)}{\left(x_r^2 + y_r^2\right)^2} = \frac{2\left(x_r^2 - y_r^2\right)}{\left(x_r^2 + y_r^2\right)^2} \tag{2.2'}$$

$$\left(\frac{1}{x_r + i y_r}\right)^3 + \left(\frac{1}{x_r - i y_r}\right)^3 = \frac{\left(2x_r\right)^3}{\left(x_r^2 + y_r^2\right)^3} - 3\frac{\left(2x_r\right)^1}{\left(x_r^2 + y_r^2\right)^2} \tag{2.3}$$

$$= \frac{\left(2x_r\right)^3 - 3\left(2x_r\right)\left(x_r^2 + y_r^2\right)}{\left(x_r^2 + y_r^2\right)^3} = \frac{2\left(x_r^3 - 3x_r^1 y_r^2\right)}{\left(x_r^2 + y_r^2\right)^3} \tag{2.3'}$$

In a similar way,

$$\left(\frac{1}{x_r - i y_r}\right)^4 + \left(\frac{1}{x_r + i y_r}\right)^4 = \frac{2\left(x_r^4 - 6x_r^2 y_r^2 + y_r^4\right)}{\left(x_r^2 + y_r^2\right)^4} \tag{2.4'}$$

$$\left(\frac{1}{x_r - i y_r}\right)^5 + \left(\frac{1}{x_r + i y_r}\right)^5 = \frac{2\left(x_r^5 - 10x_r^3 y_r^2 + 5x_r^1 y_r^4\right)}{\left(x_r^2 + y_r^2\right)^5} \tag{2.5'}$$

$$\left(\frac{1}{x_r + i y_r}\right)^6 + \left(\frac{1}{x_r - i y_r}\right)^6 = \frac{2\left(x_r^6 - 15x_r^4 y_r^2 + 15x_r^2 y_r^4 - y_r^6\right)}{\left(x_r^2 + y_r^2\right)^6} \tag{2.6'}$$

The absolute values of the coefficients in the ( ) on the right hand side are

1, 1, 1, 1, 3, 1, 6, 1, 1, 10, 5, 1, 15, 15, 1, ⋯

When this integer sequence was searched in **"The On-Line Encyclopedia of Integer Sequences" (OEIS)** , A034839 was found. These are given by

$$T(s,t) = C(s,\ 2t) \qquad s \geq 0\ ,\ \ t = 0,1,2,\cdots,\lfloor s/2 \rfloor$$

Therefore, the sum of s powers can be expressed as follows using the floor function $\lfloor x \rfloor$:

$$\left(\frac{1}{x_r - i\,y_r}\right)^s + \left(\frac{1}{x_r + i\,y_r}\right)^s = \frac{2}{\left(x_r^2 + y_r^2\right)^s} \sum_{t=0}^{\lfloor s/2 \rfloor} (-1)^t \binom{s}{2t} x_r^{s-2t} y_r^{2t} \qquad (2.s')$$

Q.E.D.

15.3 Li Coefficients expressed by x ± i y

By combining Lemma 15.1.1 with Lemma 15.2.1 or 15.2.2, the Li coefficients can be expressed as conjugate complex numbers.

Lemma 15.3.1

When the zeros of the function $\xi(z)$ defined in (1.ξ) are $x_r \pm i y_r \quad r=1,2,3,\cdots$,

the Li coefficients $\lambda_n \quad n=1,2,3,\cdots$ are expressed as follows

$$\lambda_n = \sum_{r=1}^{\infty}\left(\sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\sum_{t=0}^{\lfloor s/2\rfloor}(-1)^t\left(\binom{s-t}{t}+\binom{s-t-1}{t-1}\right)\frac{(2x_r)^{s-2t}}{\left(x_r^2+y_r^2\right)^{s-t}}\right) \tag{3.1}$$

Proof

Let us make the following substitutions in Lemma 15.1.1 .

$$\rho_1 = x_1 - i y_1 \;,\; \rho_2 = x_1 + i y_1 \;,\; \rho_3 = x_2 - i y_2 \;,\; \rho_4 = x_2 + i y_2 \;,\; \rho_5 = x_3 - i y_3 \;,\; \rho_6 = x_3 + i y_3 \;,\; \cdots$$

Then, (1.1) becoms

$$\lambda_n = \sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\sum_{k=1}^{\infty}\frac{1}{\rho_k^s}$$

$$= \sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\sum_{r=1}^{\infty}\left(\left(\frac{1}{x_r - i y_r}\right)^s + \left(\frac{1}{x_r + i y_r}\right)^s\right)$$

On the other hand, Lemma 15.2.1 was

$$\left(\frac{1}{x_r - i y_r}\right)^s + \left(\frac{1}{x_r + i y_r}\right)^s = \sum_{t=0}^{\lfloor s/2\rfloor}(-1)^t\left\{\binom{s-t}{t}+\binom{s-t-1}{t-1}\right\}\frac{(2x_r)^{s-2t}}{\left(x_r^2+y_r^2\right)^{s-t}} \tag{2.s}$$

Substituting this for the above,

$$\lambda_n = \sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\sum_{r=1}^{\infty}\left(\sum_{t=0}^{\lfloor s/2\rfloor}(-1)^t\left\{\binom{s-t}{t}+\binom{s-t-1}{t-1}\right\}\frac{(2x_r)^{s-2t}}{\left(x_r^2+y_r^2\right)^{s-t}}\right)$$

i.e.

$$\lambda_n = \sum_{r=1}^{\infty}\left(\sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\sum_{t=0}^{\lfloor s/2\rfloor}(-1)^t\left(\binom{s-t}{t}+\binom{s-t-1}{t-1}\right)\frac{(2x_r)^{s-2t}}{\left(x_r^2+y_r^2\right)^{s-t}}\right) \tag{3.1}$$

Q.E.D.

The first few of (3.1) are expanded as follows

$$\lambda_1 = \frac{2x_1}{x_1^2+y_1^2} + \frac{2x_2}{x_2^2+y_2^2} + \frac{2x_3}{x_3^2+y_3^2} + \cdots$$

$$\lambda_2 = -\frac{(2x_1)^2}{\left(x_1^2+y_1^2\right)^2} + \frac{2(2x_1)^0}{x_1^2+y_1^2} + \frac{2(2x_1)^1}{x_1^2+y_1^2}$$

$$-\frac{(2x_2)^2}{\left(x_2^2+y_2^2\right)^2} + \frac{2(2x_2)^0}{x_2^2+y_2^2} + \frac{2(2x_2)^1}{x_2^2+y_2^2}$$

$$-\frac{(2x_3)^2}{\left(x_3^2+y_3^2\right)^2} + \frac{2(2x_3)^0}{x_3^2+y_3^2} + \frac{2(2x_3)^1}{x_3^2+y_3^2}$$

$$\vdots$$

$$\begin{aligned}\lambda_3 = & \frac{(2x_1)^3}{(x_1^2+y_1^2)^3} - \frac{3(2x_1)^1}{(x_1^2+y_1^2)^2} - \frac{3(2x_1)^2}{(x_1^2+y_1^2)^2} + \frac{6(2x_1)^0}{x_1^2+y_1^2} + \frac{3(2x_1)^1}{x_1^2+y_1^2} \\ & + \frac{(2x_2)^3}{(x_2^2+y_2^2)^3} - \frac{3(2x_2)^1}{(x_2^2+y_2^2)^2} - \frac{3(2x_2)^2}{(x_2^2+y_2^2)^2} + \frac{6(2x_2)^0}{x_2^2+y_2^2} + \frac{3(2x_2)^1}{x_2^2+y_2^2} \\ & + \frac{(2x_3)^3}{(x_3^2+y_3^2)^3} - \frac{3(2x_3)^1}{(x_3^2+y_3^2)^2} - \frac{3(2x_3)^2}{(x_3^2+y_3^2)^2} + \frac{6(2x_3)^0}{x_3^2+y_3^2} + \frac{3(2x_3)^1}{x_3^2+y_3^2} \\ & \vdots \end{aligned}$$

$$\begin{aligned}\lambda_4 = & -\frac{(2x_1)^4}{(x_1^2+y_1^2)^4} + \frac{4(2x_1)^2}{(x_1^2+y_1^2)^3} + \frac{4(2x_1)^3}{(x_1^2+y_1^2)^3} - \frac{2(2x_1)^0}{(x_1^2+y_1^2)^2} - \frac{12(2x_1)^1}{(x_1^2+y_1^2)^2} - \frac{6(2x_1)^2}{(x_1^2+y_1^2)^2} \\ & + \frac{12(2x_1)^0}{x_1^2+y_1^2} + \frac{4(2x_1)^1}{x_1^2+y_1^2} \\ & -\frac{(2x_2)^4}{(x_2^2+y_2^2)^4} + \frac{4(2x_2)^2}{(x_2^2+y_2^2)^3} + \frac{4(2x_2)^3}{(x_2^2+y_2^2)^3} - \frac{2(2x_2)^0}{(x_2^2+y_2^2)^2} - \frac{12(2x_2)^1}{(x_2^2+y_2^2)^2} - \frac{6(2x_2)^2}{(x_2^2+y_2^2)^2} \\ & + \frac{12(2x_2)^0}{x_2^2+y_2^2} + \frac{4(2x_2)^1}{x_2^2+y_2^2} \\ & -\frac{(2x_3)^4}{(x_3^2+y_3^2)^4} + \frac{4(2x_3)^2}{(x_3^2+y_3^2)^3} + \frac{4(2x_3)^3}{(x_3^2+y_3^2)^3} - \frac{2(2x_3)^0}{(x_3^2+y_3^2)^2} - \frac{12(2x_3)^1}{(x_3^2+y_3^2)^2} - \frac{6(2x_3)^2}{(x_3^2+y_3^2)^2} \\ & + \frac{12(2x_3)^0}{x_3^2+y_3^2} + \frac{4(2x_3)^1}{x_3^2+y_3^2} \end{aligned}$$

Using Lemma 15.2.2 , these can be further simplified.

Theorem 15.3.2

When the zeros of the function $\xi(z)$ defined in (1.ξ) are $x_r \pm i y_r \quad r=1,2,3,\cdots$,
the Li coefficients $\lambda_n \quad n=1,2,3,\cdots$ are expressed as follows

$$\lambda_n = 2\sum_{r=1}^{\infty}\left(\sum_{s=1}^{n}\binom{n}{s}\frac{(-1)^{s-1}}{(x_r^2+y_r^2)^s}\sum_{t=0}^{\lfloor s/2\rfloor}(-1)^t\binom{s}{2t}x_r^{s-2t}y_r^{2t}\right) \tag{3.2}$$

Proof

Let us make the following substitutions in Lemma 15.1.1 .

$$\rho_1 = x_1 - iy_1 \ , \ \rho_2 = x_1 + iy_1 \ , \ \rho_3 = x_2 - iy_2 \ , \ \rho_4 = x_2 + iy_2 \ , \ \rho_5 = x_3 - iy_3 \ , \ \rho_6 = x_3 + iy_3 \ , \ \cdots$$

Then, (1.1) becoms

$$\lambda_n = \sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\sum_{r=1}^{\infty}\left(\left(\frac{1}{x_r - iy_r}\right)^s + \left(\frac{1}{x_r + iy_r}\right)^s\right)$$

On the other hand, Lemma 15.2.2 was

$$\left(\frac{1}{x_r - iy_r}\right)^s + \left(\frac{1}{x_r + iy_r}\right)^s = \frac{2}{(x_r^2+y_r^2)^s}\sum_{t=0}^{\lfloor s/2\rfloor}(-1)^t\binom{s}{2t}x_r^{s-2t}y_r^{2t} \tag{2.s'}$$

Substituting this for the above,

$$\lambda_n = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \sum_{r=1}^{\infty} \left(\frac{2}{\left(x_r^2 + y_r^2\right)^s} \sum_{t=0}^{\lfloor s/2 \rfloor} (-1)^t \binom{s}{2t} x_r^{s-2t} y_r^{2t} \right)$$

i.e.

$$\lambda_n = 2 \sum_{r=1}^{\infty} \left(\sum_{s=1}^{n} \binom{n}{s} \frac{(-1)^{s-1}}{\left(x_r^2 + y_r^2\right)^s} \sum_{t=0}^{\lfloor s/2 \rfloor} (-1)^t \binom{s}{2t} x_r^{s-2t} y_r^{2t} \right)$$

Q.E.D.

The first few of (3.2) are expanded as follows

$$\lambda_1 = \frac{2x_1}{x_1^2 + y_1^2} + \frac{2x_2}{x_2^2 + y_2^2} + \frac{2x_3}{x_3^2 + y_3^2} + \cdots$$

$$\lambda_2 = 2\left(\frac{2x_1}{x_1^2 + y_1^2} - \frac{x_1^2 - y_1^2}{\left(x_1^2 + y_1^2\right)^2} \right) + 2\left(\frac{2x_2}{x_2^2 + y_2^2} - \frac{x_2^2 - y_2^2}{\left(x_2^2 + y_2^2\right)^2} \right) + 2\left(\frac{2x_3}{x_3^2 + y_3^2} - \frac{x_3^2 - y_3^2}{\left(x_3^2 + y_3^2\right)^2} \right) + \cdots$$

$$\begin{aligned}
\lambda_3 = \; & 2\left(\frac{3x_1}{x_1^2 + y_1^2} - \frac{3x_1^2 - 3y_1^2}{\left(x_1^2 + y_1^2\right)^2} + \frac{x_1^3 - 3x_1^1 y_1^2}{\left(x_1^2 + y_1^2\right)^3} \right) \\
& + 2\left(\frac{3x_2}{x_2^2 + y_2^2} - \frac{3x_2^2 - 3y_2^2}{\left(x_2^2 + y_2^2\right)^2} + \frac{x_2^3 - 3x_2^1 y_2^2}{\left(x_2^2 + y_2^2\right)^3} \right) \\
& + 2\left(\frac{3x_3}{x_3^2 + y_3^2} - \frac{3x_3^2 - 3y_3^2}{\left(x_3^2 + y_3^2\right)^2} + \frac{x_3^3 - 3x_3^1 y_3^2}{\left(x_3^2 + y_3^2\right)^3} \right) \\
& \vdots
\end{aligned}$$

$$\begin{aligned}
\lambda_4 = \; & 2\left(\frac{4x_1}{x_1^2 + y_1^2} - \frac{6x_1^2 - 6y_1^2}{\left(x_1^2 + y_1^2\right)^2} + \frac{4x_1^3 - 12x_1^1 y_1^2}{\left(x_1^2 + y_1^2\right)^3} - \frac{x_1^4 - 6x_1^2 y_1^2 + y_1^4}{\left(x_1^2 + y_1^2\right)^4} \right) \\
& + 2\left(\frac{4x_2}{x_2^2 + y_2^2} - \frac{6x_2^2 - 6y_2^2}{\left(x_2^2 + y_2^2\right)^2} + \frac{4x_2^3 - 12x_2^1 y_2^2}{\left(x_2^2 + y_2^2\right)^3} - \frac{x_2^4 - 6x_2^2 y_2^2 + y_2^4}{\left(x_2^2 + y_2^2\right)^4} \right) \\
& + 2\left(\frac{4x_3}{x_3^2 + y_3^2} - \frac{6x_3^2 - 6y_3^2}{\left(x_3^2 + y_3^2\right)^2} + \frac{4x_3^3 - 12x_3^1 y_3^2}{\left(x_3^2 + y_3^2\right)^3} - \frac{x_3^4 - 6x_3^2 y_3^2 + y_3^4}{\left(x_3^2 + y_3^2\right)^4} \right) \\
& \vdots
\end{aligned}$$

cf.

These are simpler than Lemma 15.3.1, so we have made it a theorem. This theorem is the most important in this chapter.

15.4 Li Coefficients on the Critical Line

In this section, as a special case of the previous section, the Li coefficients λ_n are expressed by the zeros $1/2 \pm i\,y_r$ on the critical line.

Lemma 15.4.1

When the zeros of the function $\xi(z)$ defined in (1.ξ) are $1/2 \pm i\,y_r$ $r=1,2,3,\cdots$, the Li coefficients λ_n $n=1,2,3,\cdots$ are expressed as follows

$$\lambda_n = (-1)^{n-1} \sum_{r=1}^{\infty} \sum_{t=0}^{n-1} (-1)^t \left(\binom{2n-t}{t} + \binom{2n-t-1}{t-1} \right) \frac{1}{\left(1/4 + y_r^2\right)^{n-t}} \tag{4.1}$$

Proof

Let $x_r = 1/2$ $r=1,2,3,\cdots$ in the example of the expansion of Lemma 5.3.1 . Then, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ become

$$\lambda_1 = \frac{1}{1/4+y_1^2} + \frac{1}{1/4+y_2^2} + \frac{1}{1/4+y_3^2} + \cdots$$

$$\begin{aligned}\lambda_2 = &-\frac{1}{\left(1/4+y_1^2\right)^2} + \frac{4}{1/4+y_1^2}\\ &-\frac{1}{\left(1/4+y_2^2\right)^2} + \frac{4}{1/4+y_2^2}\\ &-\frac{1}{\left(1/4+y_3^2\right)^2} + \frac{4}{1/4+y_3^2}\\ &\quad\vdots\end{aligned}$$

$$\begin{aligned}\lambda_3 = &\frac{1}{\left(1/4+y_1^2\right)^3} - \frac{6}{\left(1/4+y_1^2\right)^2} + \frac{9}{1/4+y_1^2}\\ &+\frac{1}{\left(1/4+y_2^2\right)^3} - \frac{6}{\left(1/4+y_2^2\right)^2} + \frac{9}{1/4+y_2^2}\\ &+\frac{1}{\left(1/4+y_3^2\right)^3} - \frac{6}{\left(1/4+y_3^2\right)^2} + \frac{9}{1/4+y_3^2}\\ &\quad\vdots\end{aligned}$$

$$\begin{aligned}\lambda_4 = &-\frac{1}{\left(1/4+y_1^2\right)^4} + \frac{8}{\left(1/4+y_1^2\right)^3} - \frac{20}{\left(1/4+y_1^2\right)^2} + \frac{16}{1/4+y_1^2}\\ &-\frac{1}{\left(1/4+y_2^2\right)^4} + \frac{8}{\left(1/4+y_2^2\right)^3} - \frac{20}{\left(1/4+y_2^2\right)^2} + \frac{16}{1/4+y_2^2}\\ &-\frac{1}{\left(1/4+y_3^2\right)^4} + \frac{8}{\left(1/4+y_3^2\right)^3} - \frac{20}{\left(1/4+y_3^2\right)^2} + \frac{16}{1/4+y_3^2}\\ &\quad\vdots\end{aligned}$$

The absolute values of these right-hand side coefficients are part of the *Lucas* polynomial coefficients **A061896** (red).

1, 1, 2, 1, 3, 1, 4, 2, 1, 5, 5, 1, 6, 9, 2, 1, 4, 7, 14, 7, 1, 8, 20, 16, 2,

The full sequence are given by

$$T(2n,t) = C(2n-t,t) + C(2n-t-1,t-1) \qquad t=0,1,\cdots,n-1$$

So, let the rth row of λ_n be $\phi_n(y_r)$. Then

$$\phi_2(y_r) = -\frac{1}{\left(1/4+y_r^2\right)^2} + \frac{4}{1/4+y_r^2}$$

$$= (-1)^{2-1}\sum_{t=0}^{2-1}(-1)^t\left(\binom{4-t}{t}+\binom{4-t-1}{t-1}\right)\frac{1}{\left(1/4+y_r^2\right)^{2-t}}$$

$$\phi_3(y_r) = \frac{1}{\left(1/4+y_r^2\right)^3} - \frac{6}{\left(1/4+y_r^2\right)^2} + \frac{9}{1/4+y_r^2}$$

$$= (-1)^{3-1}\sum_{t=0}^{3-1}(-1)^t\left(\binom{6-t}{t}+\binom{6-t-1}{t-1}\right)\frac{1}{\left(1/4+y_r^2\right)^{3-t}}$$

$$\vdots$$

Below, by induction,

$$\phi_n(y_r) = (-1)^{n-1}\sum_{t=0}^{n-1}(-1)^t\left(\binom{2n-t}{t}+\binom{2n-t-1}{t-1}\right)\frac{1}{\left(1/4+y_r^2\right)^{n-t}} \tag{4.ϕ}$$

From this,

$$\lambda_n = (-1)^{n-1}\sum_{r=1}^{\infty}\sum_{t=0}^{n-1}(-1)^t\left(\binom{2n-t}{t}+\binom{2n-t-1}{t-1}\right)\frac{1}{\left(1/4+y_r^2\right)^{n-t}} \tag{4.1}$$

Q.E.D.

cf.

The general formula in the previous section was

$$\lambda_n = \sum_{r=1}^{\infty}\left(\sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\sum_{t=0}^{\lfloor s/2\rfloor}(-1)^t\left(\binom{s-t}{t}+\binom{s-t-1}{t-1}\right)\frac{(2x_r)^{s-2t}}{\left(x_r^2+y_r^2\right)^{s-t}}\right) \tag{3.1}$$

(4.1) is significantly simpler than (3.1). This is because the numerators of the polynomials become integers due to $2x_r = 1 \;\; r=1,2,3,\cdots$, and the denominators are grouped together.

.

Polynomials that Compose the Li Coefficient

In the first place, the Li coefficients exist in order to be determined there signs. To do this, let each row of λ_n be

$$\phi_n(y_r) = (-1)^{n-1}\sum_{t=0}^{n-1}(-1)^t\left(\binom{2n-t}{t}+\binom{2n-t-1}{t-1}\right)\frac{1}{\left(1/4+y_r^2\right)^{n-t}} \tag{4.ϕ}$$

we should start by examining the sign of this.

Here, since $\left(1/4+y_r^2\right)^n > 0$ for any real number y_r , let polynomial obtained by removing this from $\phi_n(y_r)$ be

$$g_n(y_r) = (-1)^{n-1}\sum_{t=0}^{n-1}(-1)^t\left(\binom{2n-t}{t}+\binom{2n-t-1}{t-1}\right)\left(1/4+y_r^2\right)^t \tag{4.g)}$$

As the result, we just need to check the sign of $g_n(y_r)$ instead of $\phi_n(y_r)$.

Lemma 15.4.2

Let n be a natural number, y_r be a real number, and $g_n(y_r)$ be a polynomial such as:

$$g_n(y_r) = (-1)^{n-1}\sum_{t=0}^{n-1}(-1)^t\left(\binom{2n-t}{t}+\binom{2n-t-1}{t-1}\right)\left(1/4+y_r^2\right)^t \tag{4.g)}$$

Then, $g_n(y_r)$ is transformed as follows due to whether n is odd or even

$$g_{2n-1}(y) = \frac{1}{4^{2n-2}}\left(\sum_{s=0}^{n-1}(-1)^s 4^s \binom{2n-1}{2s} y_r^{2s}\right)^2 \qquad n=1,2,3,\cdots \tag{4.go}$$

$$g_{2n}(y) = \frac{y_r^2}{4^{2n-2}}\left(\sum_{s=0}^{n-1}(-1)^s 4^s \binom{2n}{2s+1} y_r^{2s}\right)^2 \qquad n=1,2,3,\cdots \tag{4.ge}$$

Proof

(1) when n is odd

$g_n(y_r)$ is simplified and modfied as follows.

$$g_1(y_r) = 1 = \frac{1}{4^0}$$

$$g_3(y_r) = \frac{1}{16}\left(1-12y_r^2\right)^2 = \frac{1}{4^2}\left(1-12y_r^2\right)^2$$

$$g_5(y_r) = \frac{1}{256}\left(1-40y_r^2+80y_r^4\right)^2 = \frac{1}{4^4}\left(1-40y_r^2+80y_r^4\right)^2$$

$$g_7(y_r) = \frac{1}{4096}\left(1-84y_r^2+560y_r^4-448y_r^6\right)^2 = \frac{1}{4^6}\left(1-84y_r^2+560y_r^4-448y_r^6\right)^2$$

$$\vdots \qquad\qquad \vdots$$

By searching for $1, 84, 560, 448$ in **OEIS**, **A085840** was found. These are given by

$$T(n,s) = \frac{4^s(2n+1)!}{(2n-2s+1)!(2s)!} \qquad s=0,1,2,\cdots,n$$

Replacing n with $n-1$ in the right-hand side,

$$T(n,s) = \frac{4^s(2n-1)!}{(2n-2s-1)!(2s)!} \qquad s=0,1,2,\cdots,n-1$$

Using this,

$$g_{2n-1}(y_r) = \frac{1}{4^{2n-2}}\left(\sum_{s=0}^{n-1}(-1)^s \frac{4^s(2n-1)!}{(2n-2s-1)!(2s)!} y_r^{2s}\right)^2$$

i.e.

$$g_{2n-1}(y) = \frac{1}{4^{2n-2}}\left(\sum_{s=0}^{n-1}(-1)^s 4^s \binom{2n-1}{2s} y_r^{2s}\right)^2 \qquad n=1,2,3,\cdots \tag{4.go}$$

(2) when n is even

$g_n(y_r)$ is simplified and modfied as follows.

$$g_2(y_r) = 4y_r^2 = \frac{y^2}{4^0}\left(2y_r^0\right)^2$$

$$g_4(y_r) = y_r^2\left(1-4y_r^2\right)^2 = \frac{y^2}{4^2}\left(4-16y_r^2\right)^2$$

$$g_6(y_r) = \frac{y^2}{64}\left(3-40y_r^2+48y_r^4\right)^2 = \frac{y^2}{4^4}\left(6-80y_r^2+96y_r^4\right)^2$$

$$g_8(y_r) = \frac{y^2}{64}\left(1-28y_r^2+112y_r^4-62y_r^6\right)^2 = \frac{y^2}{4^6}\left(8-224y_r^2+896y_r^4-512y_r^6\right)^2$$

$\vdots \qquad\qquad\qquad \vdots$

By searching for $8,224,896,512$ in **OEIS**, an integer sequence **A229032** containing these was found. The full sequence are given by

$$T(n,s) = 4^s \binom{n+1}{2s+1} \qquad s=0,1,2,\cdots,n$$

To skip unnecessary sequences, replacing n with $2n-1$ in the right-hand side,

$$T(n,s) = 4^s \binom{2n}{2s+1} \qquad s=0,1,2,\cdots,n$$

Using this,

$$g_{2n}(y) = \frac{y_r^2}{4^{2n-2}} \left(\sum_{s=0}^{n-1} (-1)^s 4^s \binom{2n}{2s+1} y_r^{2s} \right)^2 \qquad n=1,2,3,\cdots \qquad (4.\text{ge})$$

Q.E.D.

Note

That is, $g_n(y_r)$ is reduced to a perfect square expression. This is an unexpected and surprising result.

Li Coefficients on the Critical Line

Using Lemma 15.4.2 , the Li coefficienst on the critical line can be expressed as follows.

Theorem 15.4.3

When the zeros of the function $\xi(z)$ defined in (1.ξ) are $1/2 \pm i y_r$ $r=1,2,3,\cdots$, the Li coefficients λ_n $n=1,2,3,\cdots$ are expressed as follows

$$\lambda_{2n-1} = \sum_{r=1}^{\infty} \frac{1}{4^{2n-2}\left(1/4+y_r^2\right)^{2n-1}} \left(\sum_{s=0}^{n-1} (-1)^s 4^s \binom{2n-1}{2s} y_r^{2s} \right)^2 \qquad n=1,2,3,\cdots \qquad (4.3\text{o})$$

$$\lambda_{2n} = \sum_{r=1}^{\infty} \frac{y_r^2}{4^{2n-2}\left(1/4+y_r^2\right)^{2n}} \left(\sum_{s=0}^{n-1} (-1)^s 4^s \binom{2n}{2s+1} y_r^{2s} \right)^2 \qquad n=1,2,3,\cdots \qquad (4.3\text{e})$$

Proof

From (4.ϕ) and (4.g) ,

$$\lambda_n = \sum_{r=1}^{\infty} \phi_n(y_r) = \sum_{r=1}^{\infty} \frac{g_n(y_r)}{\left(1/4+y_r^2\right)^n}$$

Substituting $g_n(y_r)$ of Lemma 15.4.2 into this for each odd and even, we obtain the desired expressions. Q.E.D.

Calculation Example

General formula generating zeros whose real part is 1/2 does not known. But the mathematical manipulation software ***Mathematica*** has a function $y_n = Im[ZetaZero[n]]$ that generates it numerically. When we calculated $\lambda_1 \sim \lambda_4$ by (4.3o) and (4.3e) using these $100,000$ non-trivial zeros, the results were as follows.

```
y_r_ := Im[ZetaZero[r]]
```

2n-1

$$\lambda o_n\_[m\_] := \sum_{r=1}^{m} \frac{1}{4^{2n-2}\left(1/4+y_r^2\right)^{2n-1}} \left(\sum_{s=0}^{n-1} (-1)^s 4^s \,\text{Binomial}[2n-1, 2s]\, y_r^{2s} \right)^2$$

2n

$$\lambda e_{n\_}[m\_] := \sum_{r=1}^{m} \frac{y_r^2}{4^{2n-2}\left(1/4 + y_r^2\right)^{2n}} \left(\sum_{s=0}^{n-1} (-1)^s 4^s \text{Binomial}[2n, 2s+1]\, y_r^{2s} \right)^2$$

```
N[{λo1[100000], λe1[100000], λo2[100000], λe2[100000]}]
{0.0230736, 0.0922575, 0.20744, 0.368437}
```

These results are close to the theoretical values (**OEIS A074760 , A104539 , A104540 , A104541**) .

{0.0230957, 0.0923457, 0.20764, 0.368790}

For reference, this calculation took **14** hours on my computer (Intel Core i7-9750H , 16GB) .

Conclusion

λ_{2n} is a sum of the perfect squares . λ_{2n-1} can also be modified to a sum of squares as follows.

$$\lambda_{2n-1} = \sum_{r=1}^{\infty} \frac{1}{4^{2n-2}\left(\left(1/4 + y_r^2\right)^{(2n-1)/2}\right)^2} \left(\sum_{s=0}^{n-1} (-1)^s 4^s \binom{2n-1}{2s} y_r^{2s} \right)^2 \qquad n=1,2,3,\cdots \qquad (4.3o)$$

Therefore, it can be said that Li coefficients λ_n $n=1,2,3,\cdots,\infty$ on the critical line are expressed as an infinite sum of squares. So, naturally, $\lambda_n \geq 0$ $n=1,2,3,\cdots,\infty$.

Furthermore, considering that it looks like $(\ )^2 > 0$ for $|y_r| > n$, it is likely that $\lambda_n > 0$ $n=1,2,3,\cdots,\infty$.

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