3 Generalized Multinomial Theorem

3.1 Binomial Theorem

Theorem 3.1.1

If \( x_1, x_2 \) are real numbers and \( n \) is a positive integer, then

\[
(x_1 + x_2)^n = \sum_{r=0}^{n} \binom{n}{r} x_1^{n-r} x_2^r
\]  

(1.1)

Binomial Coefficients

Binomial Coefficient in (1.1) is a positive number and is described as \( \binom{n}{r} \). Here, \( n \) and \( r \) are both non-negative integer. \( \binom{n}{r} \) is the number of ways of picking \( r \) unordered outcomes from \( n \) possibilities and is calculated as follows.

\[
\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}
\]

Pascal's Triangle

About \( \binom{n}{r} \), what arranged \( n \) in the row and \( r \) in the column is called Pascal's Triangle.

\[
\begin{array}{ccccccc}
0 & & & & & & 1 \\
0 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
2 & 3 & 3 & 1 & & & \\
3 & 4 & 6 & 4 & 1 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Properties of the Binomial Coefficient

Although a lot of properties of the binomial coefficient are known, fundamental (understood from Pascal’s Triangle immediately) some are as follows. Among these, ii is used for step-by-step calculation of \( \binom{n}{r} \).

i \( \binom{n}{r} = \binom{n}{n-r} \)

ii \( \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \)

iii \( \binom{n-1}{r} = \sum_{k=r-1}^{n-1} \binom{k}{r-1} \)

iv \( \sum_{r=0}^{n} \binom{n}{r} = 2^n \)

v \( \sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0 \)
3.2 Generalized Binomial Theorem

3.2.1 Newton's Generalized Binomial Theorem

Theorem 3.2.1

The following formulas hold for a real number $\alpha$.

\[(1+x)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r \quad |x| \leq 1 \quad (|x|=1 \text{ is allowed at } \alpha > 0) \quad (1.1)\]

\[= \sum_{r=0}^{\infty} \binom{\alpha}{\alpha-r} (x^{-1})^r \quad |x| > 1 \quad (1.2)\]

Proof

When $n$ is a natural number, the following expression holds from the binomial theorem.

\[(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r\]

Since $\binom{n}{r} = 0$ for $r > n$, this can be written as follows.

\[(1+x)^n = \sum_{r=0}^{\infty} nCr x^r\]

Extending $n$ to real number $\alpha$,

\[(1+x)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r = \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1) \, r!} x^r \quad (1.1)\]

Here, let

\[a_r = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1) \, r!} x^r \equiv a_r.\]

Then

\[a_{r+1} = \frac{\Gamma(p+1)}{\Gamma(p-r) \, (r+1)!} x^{r+1} = \frac{(p-r) \, x}{r+1}.\]

From this,

\[\lim_{r \to \infty} \left| \frac{a_{r+1}}{a_r} \right| = \lim_{r \to \infty} \left| \frac{(p-r) \, x}{r+1} \right| = \lim_{r \to \infty} \left| \frac{p \, (r+1) x}{1+1/r} \right| = \left| \frac{-x}{1} \right| = |x|\]

According to d'Alembert's ratio test, (1.1) converges absolutely if $|x| < 1$.

Next, if $|x| > 1$ then $|x^{-1}| < 1$. Therefore, from (1.1),

\[\frac{1}{x} = \left(1+\frac{1}{x}\right)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} \left(x^{-1}\right)^r\]

Multiplying by $x^\alpha$ both sides,

\[x^\alpha = x^\alpha \sum_{r=0}^{\infty} \binom{\alpha}{r} x^{-r} = \sum_{r=0}^{\infty} \binom{\alpha}{\alpha-r} x^{\alpha-r} \quad (1.2)\]

The proof in case of $|x| = 1$ is accomplished in the following sub section.
### 3.2.2 General Binomial Coefficient

The coefficient \(_\binom{\alpha}{r}\) in Theorem 3.2.1 is called **General Binomial Coefficient** and is as follows.

\[
\binom{\alpha}{r} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)\Gamma(r+1)} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-r+1)}{r!}
\]  
(2.0)

The first few are as follows.

\[
\begin{align*}
\binom{\alpha}{0} &= 1, \\
\binom{\alpha}{1} &= \frac{\alpha}{1!}, \\
\binom{\alpha}{2} &= \frac{\alpha(\alpha-1)}{2!}, \\
\binom{\alpha}{3} &= \frac{\alpha(\alpha-1)(\alpha-2)}{3!},
\end{align*}
\]

Although properties similar to binomial coefficient also about general binomial coefficient are known, especially an important thing is sum of the general binomial coefficient. We prepare some Lemma, in order to obtain this.

**Lemma 3.2.2**

When \(\alpha\) is not positive integer, binomial series \(\sum_{r=0}^{\infty} a_r \binom{\alpha-1}{r}\) converges or diverges simultaneously with Dirichlet series \(\sum_{r=1}^{\infty} (-1)^r \frac{a_r}{r^\alpha}\).

**Source** 『岩波数学公式Ⅱ』 p132

**Lemma 3.2.3**

\[
\sum_{r=0}^{\infty} \binom{\alpha}{r}
\]
converges absolutely for non-integer \(\alpha > 0\).

**Proof**

\[
\binom{\alpha}{r} = \frac{\alpha}{\alpha-r} \binom{\alpha-1}{r}
\]

Then

\[
\sum_{r=0}^{\infty} \binom{\alpha}{r} = \sum_{r=0}^{\infty} \frac{\alpha}{\alpha-r} \binom{\alpha-1}{r}
\]

Let \(a_r = \frac{\alpha}{\alpha-r}\). Then

\[
\sum_{r=0}^{\infty} a_r \binom{\alpha-1}{r} = \sum_{r=0}^{\infty} \binom{\alpha}{r}
\]

(\(s1\))

\[
\sum_{r=1}^{\infty} (-1)^r \frac{a_r}{r^\alpha} = \sum_{r=1}^{\infty} (-1)^r \frac{\alpha}{r^\alpha(\alpha-r)}
\]

(\(s2\))

Here, let

\[
(-1)^r \frac{\alpha}{r^\alpha(\alpha-r)} \equiv b_r
\]

Then
\[
\frac{b_{r+1}}{b_r} = \frac{(-1)^{r+1} \alpha}{(r+1)^{\alpha}(\alpha-r-1)} = -\frac{r^\alpha(\alpha-r)}{(r+1)^{\alpha}(\alpha-r-1)}
\]

From this
\[
\lim_{r \to \infty} \left| \frac{b_{r+1}}{b_r} \right| = \lim_{r \to \infty} \left| \frac{r^\alpha(\alpha-r)}{(r+1)^{\alpha}(\alpha-r-1)} \right| = \lim_{r \to \infty} \left| \frac{\alpha}{r} \right| \left( \frac{1}{1+\frac{1}{r}} \right) \left( \frac{1}{\alpha} - \frac{1}{\alpha} - 1 \right) = 1
\]

Since the judgment is impossible, we try Raabe's test for convergence.
\[
\lim_{r \to \infty} r \left( \left| \frac{b_r}{b_{r+1}} \right| - 1 \right) = \lim_{r \to \infty} r \left( \left| \frac{(r+1)^{\alpha}(\alpha-r-1)}{r^\alpha(\alpha-r)} \right| - 1 \right)
\]
\[
= \lim_{r \to \infty} \left( \left| \frac{r^{\alpha+1}}{(r+1)^{\alpha+1}} - 1 \right| \right) = \lim_{r \to \infty} \left( \left| \frac{\alpha r^\alpha}{r^{\alpha+1} - 1} \right| \right)
\]
\[
= \lim_{r \to \infty} \left( \left| r \left(1+\frac{1}{r}\right)^\alpha - \left(1+\frac{1}{r}\right) \frac{\alpha}{\alpha-r} - 1 \right| \right)
\]
\[
= \lim_{r \to \infty} \left( \left| 1+\frac{1}{r} \right| \alpha - 1 \right) + \lim_{r \to \infty} \left( \left| 1+\frac{1}{r} - 1 \right| \right) \frac{1}{\alpha} + \frac{1}{1-\alpha/r}
\]

Here,
\[
\left(1+\frac{1}{r}\right)^\alpha = \sum_{s=0}^{\infty} \left( \alpha \right)_s \frac{1}{r^s} = 1 + \frac{\alpha}{1!} \frac{1}{r} + \frac{\alpha(\alpha-1)}{2!} \frac{1}{r^2} + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} \frac{1}{r^3} + ...
\]

Then,
\[
r \left( \left(1+\frac{1}{r}\right)^\alpha - 1 \right) = r \sum_{s=1}^{\infty} \left( \alpha \right)_s \frac{1}{r^s} = \sum_{s=1}^{\infty} \left( \alpha \right)_s \frac{1}{r^{s+1}} = \alpha + \sum_{s=2}^{\infty} \left( \alpha \right)_s \frac{1}{r^{s-1}}
\]

Therefore,
\[
\lim_{r \to \infty} r \left( \left(1+\frac{1}{r}\right)^\alpha - 1 \right) = \alpha + \lim_{r \to \infty} \sum_{s=2}^{\infty} \left( \alpha \right)_s \frac{1}{r^{s-1}} = \alpha
\]

Moreover,
\[
\lim_{r \to \infty} \left(1+\frac{1}{r}\right)^\alpha \frac{1}{1-\alpha/r} = 1
\]

After all,
Thus, if \( \alpha > 0 \), (s2) converges absolutely. Then (s1) also converges absolutely according to Lemma 3.2.2.

**Theorem 3.2.4**

The following expressions hold for arbitrary real number \( \alpha > 0 \).

\[
\sum_{r=0}^{\infty} \binom{\alpha}{r} = 2^\alpha \quad \text{(2.1)}
\]

\[
\sum_{r=0}^{\infty} (-1)^r \binom{\alpha}{r} = 0 \quad \text{(2.2)}
\]

**Proof**

According to Lemma 3.2.3, \( \sum_{r=0}^{\infty} \binom{\alpha}{r} \) converges absolutely for non-integer \( \alpha > 0 \). Therefore, from Theorem 3.2.1 (1.1),

\[
\sum_{r=0}^{\infty} \binom{\alpha}{r} 1^r = (1+1)^\alpha = 2^\alpha \quad \text{(2.1)}
\]

\[
\sum_{r=0}^{\infty} \binom{\alpha}{r} (-1)^r = (1-1)^\alpha = 0 \quad \text{(2.2)}
\]

**Note**

In fact, it is known that (2.1) holds if \( \alpha > -1 \). (Where, it is conditional convergence.)

For example, when \( \alpha = -0.9 \), the right side is \( 2^{-0.9} = 0.53588673 \ldots \) and the left side seems to converge to this value. However, the confirmation is difficult as the convergence is very slow. So, we apply Knopp Transformation to this and accelerate the convergence. It is as follows.

\[
\text{Knopp Transformation: } \sum_{r=0}^{\infty} \binom{\alpha}{r} 1^r = (1+1)^\alpha = 2^\alpha
\]

3.2.3 Generalized Binomial Theorem

Theorem 3.2.1 can be further generalized.

**Theorem 3.2.5**

When \( \alpha \) is a real number, the following expression holds for \( x_1, x_2 \) s.t. \( |x_1| \geq |x_2| \).

\[
(x_1 + x_2)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} x_1^{\alpha-r} x_2^r \quad (x_1 = x_2 \text{ is allowed at } \alpha > 0)
\]

**Proof**

If \( |x_1| \geq |x_2| \) then \( 1 > \frac{|x_2|}{|x_1|} = \frac{x_2}{x_1} \). Therefore, using Theorem 3.2.1 (1.1).
\[(x_1 + x_2)^\alpha = \left\{ x_1 \left( 1 + \frac{x_2}{x_1} \right) \right\}^\alpha = x_1^\alpha \left( 1 + \frac{x_2}{x_1} \right)^\alpha \]
\[= x_1^\alpha \left\{ 1 + \left( \frac{\alpha}{1} \right) \frac{x_2}{x_1} + \left( \frac{\alpha}{2} \right) \left( \frac{x_2}{x_1} \right)^2 + \left( \frac{\alpha}{3} \right) \left( \frac{x_2}{x_1} \right)^3 + \ldots \right\} \]
\[= \binom{\alpha}{0} x_1^\alpha + \binom{\alpha}{1} x_1^{\alpha-1} x_2^1 + \binom{\alpha}{2} x_1^{\alpha-2} x_2^2 + \binom{\alpha}{3} x_1^{\alpha-3} x_2^3 + \ldots \]
\[= \sum_{r=0}^\infty \binom{\alpha}{r} x_1^{\alpha-r} x_2^r \]

**Note**

As is clear from the process of the proof, if \[|x_1| \geq |x_2|\] then (3.1) converges absolutely.

Where, \[|x_1| = |x_2|\] is allowed at \[\alpha > 0\].

This becomes important in Generalized Multinomial Theorem.
3.3 Multinomial Theorem

Theorem 3.3.0
For real numbers $x_1, x_2, \ldots, x_m$ and non-negative integers $n, r_1, r_2, \ldots, r_m$, the followings hold.

$$
(x_1 + x_2 + \cdots + x_m)^n = \sum_{r_1 \geq 0, r_2 \geq 0, \ldots, r_m \geq 0} \frac{n!}{r_1! \cdot r_2! \cdot \cdots \cdot r_m!} x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m} \quad (0.1)
$$

where $\Sigma$ denotes the sum of all combinations of $(r_1, r_2, \ldots, r_m)$ s.t. $r_1 + r_2 + \cdots + r_m = n$.

$$
(x_1 + x_2 + \cdots + x_m)^n = \sum_{r_1 + r_2 + \cdots + r_m = n} \frac{n!}{(n-r_1-\cdots-r_{m-1})! \cdot r_1! \cdot \cdots \cdot r_{m-1}!} x_1^{n-r_1} x_2^{r_2} \cdots x_m^{r_m} \quad (0.2)
$$

where $\Sigma$ denotes the sum of all possible combinations of $(n, r_1, r_2, \ldots, r_{m-1})$.

Since (0.1) is well known, the proof is omitted. In addition, it is also clear (0.2) and (0.1) are synonymous. These are near a definition rather than a theorem.

How to generate multinomial coefficients
Theorem 3.3.0 is not difficult in theory. Difficulty is its proviso. This is to actually generate combinations (m choose n) with repetition. But this is not easy when it becomes more than 3 terms. Since I found out the formulas which generates these without leak, I present it here as a theorem. (1.2) realizes the provis by an iterated series (multiple series) and (1.1) realizes it by a diagonal series (half-multiple series).

Theorem 3.3.1
For real numbers $x_1, x_2, \ldots, x_m$ and a natural number $n$, the following expressions hold.

$$
(x_1 + x_2 + \cdots + x_m)^n = \sum_{r_1=0}^{n} \sum_{r_2=0}^{r_1} \cdots \sum_{r_m=0}^{r_{m-1}} \left( \begin{array}{c} n \\ r_1 \\ \vdots \\ r_{m-1} \end{array} \right) x_1^{n-r_1} x_2^{r_1-r_2} \cdots x_m^{r_{m-1}} \quad (1.1)
$$

$$
= \sum_{r_1=0}^{n} \sum_{r_2=0}^{r_1} \cdots \sum_{r_m=0}^{r_{m-1}} \left( \begin{array}{c} n \\ r_1 + r_2 + \cdots + r_{m-1} \\ \vdots \\ r_m \end{array} \right) x_1^{n-r_1-\cdots-r_{m-1}} x_2^{r_2} \cdots x_m^{r_m} \quad (1.2)
$$

Proof
According to Theorem 3.1.1, the following expressions hold.

$$
(x_1 + x_2 + x_3 + x_4 + \cdots + x_m)^n = \sum_{r_1=0}^{n} n! C_{r_1} x_1^{n-r_1} (x_2 + x_3 + x_4 + \cdots + x_m)^{r_1} \quad (1)
$$

$$
(x_2 + x_3 + x_4 + \cdots + x_m)^{r_1} = \sum_{r_2=0}^{r_1} r_1! C_{r_2} x_2^{r_2} (x_3 + x_4 + \cdots + x_m)^{r_2} \quad (2)
$$

$$
(x_3 + x_4 + \cdots + x_m)^{r_2} = \sum_{r_3=0}^{r_2} r_2! C_{r_3} x_3^{r_3} (x_4 + \cdots + x_m)^{r_3} \quad (3)
$$

$$
: \quad (x_{m-2} + x_{m-1} + x_m)^{r_{m-2}} = \sum_{r_{m-2}=0}^{r_{m-3}} r_{m-2}! C_{r_{m-2}} x_{m-2}^{r_{m-2}} (x_{m-1} + x_m)^{r_{m-2}} \quad (m-2)
$$
\[(x_{m-1} + x_m)^{r_{m-2}} = \sum_{r_{m-1}=0}^{r_{m-2}} r_{m-1} C_{r_{m-1}} x_{m-1}^{r_{m-2}-r_{m-1}} x_m^{r_{m-1}} \quad (m-1)\]

Substituting (2), (3), \(\ldots\), (m-2), (m-1) for (1) one by one, we obtain (1.1).

Next, according to Theorem 3.4.1 (later), when \(|x_1| \geq |x_2 + x_3 + \ldots + x_m|\),

\[
(x_1 + x_2 + \cdots + x_m)^\alpha = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_m=0}^{\infty} \left( r_1 + r_2 + \cdots + r_{m-1} \right)^\alpha \left( r_2 + \cdots + r_{m-1} \right)^{r_{m-2} + r_{m-1}} \ldots \left( r_{m-1} \right)^{r_{m-2} + r_{m-1}} \times x_1^{\alpha - r_1 - \cdots - r_{m-1}} x_2^{r_1} x_3^{r_2} \cdots x_m^{r_{m-1}}
\]

Replacing the real number \(\alpha\) with non-negative integer \(n\),

\[
(x_1 + x_2 + \cdots + x_m)^n = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_m=0}^{\infty} \left( r_1 + r_2 + \cdots + r_{m-1} \right)^n \left( r_2 + \cdots + r_{m-1} \right)^{r_{m-2} + r_{m-1}} \ldots \left( r_{m-1} \right)^{r_{m-2} + r_{m-1}} \times x_1^{n - r_1 - \cdots - r_{m-1}} x_2^{r_1} x_3^{r_2} \cdots x_m^{r_{m-1}}
\]

Since \(\binom{n}{r} = 0\) for \(r > n, r=1, 2, 3, \ldots\), this is a definite multiple series. Therefore, the condition \(|x_1| \geq |x_2 + x_3 + \ldots + x_m|\) is unnecessary. Although this is not bad as it is, replacing \(\infty\) on the \(\Sigma\) with \(n\), we obtain (1.2).

cf. (1.2) results in (0.2). Because,

\[
\binom{n}{r} = \frac{n!}{(n-r_1-\cdots-r_{m-1})! r_1! \cdots r_{m-1}!}
\]

**Example 1:** The expansion of \((x_1 + x_2 + x_3)^4\)

Using (1.1),

\[
(x_1 + x_2 + x_3)^4 = \sum_{r_0=0}^{4} \sum_{s_0=0}^{r_0} \binom{4}{r_0} \binom{r_0}{s_0} x_1^{4-r_0} x_2^{r_0-s_0} x_3^{s_0}
\]

\[
= \left( \begin{array}{c} 4 \\ 0 \end{array} \right) x_1^4 + \left( \begin{array}{c} 4 \\ 1 \end{array} \right) x_1^3 x_2 + \left( \begin{array}{c} 4 \\ 2 \end{array} \right) x_1^2 x_3^2 + \left( \begin{array}{c} 4 \\ 3 \end{array} \right) x_1 x_3^3 + \left( \begin{array}{c} 4 \\ 4 \end{array} \right) x_3^4
\]

\[
+ 4x_1^3 (x_2 + x_3) + 6x_1^2 (x_2^2 + 2x_2 x_3 + x_3^2) + 4x_1 (x_2^3 + 3x_2^2 x_3 + 3x_2 x_3^2 + x_3^3) + x_2^4 + 4x_2^3 x_3 + 6x_2^2 x_3^2 + 4x_2 x_3^3 + x_3^4
\]
Using (1.2),

\[(x_1 + x_2 + x_3)^4 = \sum_{r=0}^{4} \sum_{s=0}^{4} \binom{4}{r+s} \binom{r+s}{s} x_1^{4-r-s} x_2^r x_3^s\]

\[
= \sum_{s=0}^{4} \binom{4}{0+s} \binom{0+s}{s} x_1^{4-s} x_2^s x_3^s + \sum_{s=0}^{4} \binom{4}{1+s} \binom{1+s}{s} x_1^{3-s} x_2 x_3^s + \sum_{s=0}^{4} \binom{4}{2+s} \binom{2+s}{s} x_1^{2-s} x_2^2 x_3^s + \sum_{s=0}^{4} \binom{4}{3+s} \binom{3+s}{s} x_1^{1-s} x_2^3 x_3^s + \sum_{s=0}^{4} \binom{4}{4+s} \binom{4+s}{s} x_1^{0-s} x_2^4 x_3^s
\]

we can see that what totaled this along the diagonal line is equal to the above.

Example 2: The expansion of \((x_1 + x_2 + x_3 + x_4)^3\)

Now, the formulas of the theorem are expanded using mathematical software. (1.1) and (1.2) are expanded and are verified respectively.

\[f[n] := (x_1 + x_2 + x_3 + x_4)^n\]

\[\text{Expand}[f[3]]\]

\[x_1^3 + 3 x_1^2 x_2 + 3 x_1 x_2^2 + x_2^3 + 3 x_1^2 x_3 + 3 x_1 x_2 x_3 + 3 x_1 x_3^2 + 3 x_2 x_3^2 + 3 x_1^2 x_4 + 6 x_1 x_2 x_4 + 3 x_2^2 x_4 + 6 x_1 x_3 x_4 + 6 x_2 x_3 x_4 + 3 x_1 x_3^2 + 3 x_2 x_3^2 + 3 x_1 x_4^2 + 3 x_2 x_4^2 + 3 x_3 x_4^2\]

\[f[n] := \sum_{r=0}^{n} \sum_{s=0}^{r} \text{Binomial}[n, r] \text{Binomial}[r, s] \text{Binomial}[s, t] x_1^{n-r} x_2^{r-s} x_3^{s-t} x_4^t\]

\[\text{Expand}[f[3]] = f[3]\]

True

\[f[s] := \sum_{r=0}^{s} \sum_{s=0}^{r} \text{Binomial}[n, r + s + t] \text{Binomial}[r + s + t, s + t] \times \text{Binomial}[s + t, t] x_1^{n-r-s-t} x_2^{r} x_3^{s} x_4^{t}\]

\[\text{Expand}[f[3]] = f[3]\]

True
Sum of multinomial coefficients

\[ \sum_{r_1=0}^{n} \ldots \sum_{r_{m-2}=0}^{n} \sum_{r_{m-1}=0}^{n} \binom{n}{r_1} \binom{n}{r_2} \ldots \binom{n}{r_{m-2}} \binom{n}{r_{m-1}} = m^n \] (1.1*)

Proof

\[ \sum_{r=0}^{n} \binom{n}{r} = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} 1^r = (1+1)^n = 2^n \]

\[ \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n}{r} \binom{r}{s} = \sum_{r=0}^{n} \binom{n}{r} \sum_{s=0}^{r} \binom{r}{s} = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} 2^r = (1+2)^n = 3^n \]

\[ \sum_{r=0}^{n} \sum_{s=0}^{r} \sum_{t=0}^{s} \binom{n}{r} \binom{r}{s} \binom{s}{t} = \sum_{r=0}^{n} \binom{n}{r} \left( \sum_{s=0}^{r} \sum_{t=0}^{s} \binom{s}{t} \right) \binom{n}{r} = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} 3^r = (1+3)^n = 4^n \]

Hereafter, by induction we obtain the desired expression.

Example: Sum of multinomial coefficients of \((x_1 + x_2 + x_3)^4\)

Let’s calculate sum of multinomial coefficients in Example 1. Then it is as follows.

\[ 1 + (4+4) + (6+12+6) + (4+12+12+4) + (1+4+6+4+1) = 81 = 3^4 \]
3.4 Generalized Multinomial Theorem

Although I do not know whether the theorem like generalized multinomial theorem exists or not, since this is essential for Higher Calculus of Function Product, I present this here.

**Theorem 3.4.1**

The following expressions hold for real numbers \( \alpha \) and \( x_1, x_2, \ldots, x_m \) s.t. \( |x_1| \geq |x_2 + x_3 + \cdots + x_m| \).

\[
\left( x_1 + x_2 + \cdots + x_m \right)^\alpha = \sum_{r_1, r_2, \ldots, r_m = 0}^{\infty} \sum_{r_{m+1} = 0}^{\infty} \frac{\alpha^r}{r!} x_1^{\alpha r_1} x_2^{r_2} \cdots x_m^{r_m} \quad (1.1)
\]

\[
= \sum_{r_1, r_2, \ldots, r_m = 0}^{\infty} \sum_{r_{m+1} = 0}^{\infty} \left( \begin{array}{c} \alpha \\ r_1 \\ \vdots \\ r_m \\ \end{array} \right) x_1^{\alpha r_1} x_2^{r_2} \cdots x_m^{r_m} \quad (1.2)
\]

Where, \( |x_1| = |x_2 + x_3 + \cdots + x_m| \) is allowed at \( \alpha > 0 \).

**Proof**

From [Theorem 3.2.5](#), when \( |x_1| \geq |x_2 + x_3 + \cdots + x_m| \), the following expression holds.

\[
\left( x_1 + x_2 + x_3 + x_4 + \cdots + x_m \right)^\alpha = \sum_{r_1 = 0}^{\infty} \frac{\alpha^r}{r!} x_1^{\alpha r_1} (x_2 + x_3 + x_4 + \cdots + x_m)^{r_1} \quad (1.1)
\]

Here, the right side converges absolutely.

On the other hand, from [Theorem 3.3.1 (1.1)](#), the following expression holds.

\[
\left( x_2 + x_3 + \cdots + x_m \right)^{r_1} = \sum_{r_1 = 0}^{\infty} \sum_{r_2 = 0}^{\infty} \sum_{r_{m+1} = 0}^{\infty} \frac{r_1^{r_2}}{r_2!} x_2^{r_2} x_3^{r_3} \cdots x_m^{r_m} \quad (1.2)
\]

Substituting the latter for the former,

\[
\left( x_1 + x_2 + \cdots + x_m \right)^\alpha = \sum_{r_1 = 0}^{\infty} \sum_{r_2 = 0}^{\infty} \sum_{r_{m+1} = 0}^{\infty} \frac{\alpha^{r_1}}{r_1!} \frac{r_2^{r_1}}{r_2!} \cdots \frac{r_m^{r_1}}{r_m!} x_2^{r_2} x_3^{r_3} \cdots x_m^{r_m} \quad (1.1)
\]

Naturally, the right side also converges absolutely.

Next, let us describe a multiple series and its iterated series as follows respectively.

\[
\sum_{r_1, r_2, \ldots, r_m = 0}^{\infty} a_{r_1, r_2, \ldots, r_m} r_1 r_2 \cdots r_m, \quad \sum_{r_1 = 0}^{\infty} \sum_{r_2 = 0}^{\infty} \sum_{r_{m+1} = 0}^{\infty} a_{r_1, r_2, \ldots, r_m, r_{m+1}} r_1 r_2 \cdots r_m \quad (1.1)
\]

In order to convert the iterated series to its diagonal series, we should just perform the following operations.

(See "02 Multiple Series & Exponential Function".)

- Replace \( r_{m-1} \) with \( r_{m-1}^\alpha \), and replace the 1st \( \infty \) with \( r_{m-1} \) from the right.
- Replace \( r_{m-2} \) with \( r_{m-2}^\alpha r_{m-1} \), and replace the 2nd \( \infty \) with \( r_{m-2} \) from the right.
- Repeat this infinite times.

If so, in order to return the diagonal series to the original iterated series, we should just perform this opposite operation. That is,

- Replace \( r_1 \) with \( r_1^{-1} \), and replace the \( (m - 1) \)th \( \infty \) with \( r_1 \) from the right.
- Replace \( r_2 \) with \( r_1^{-1} r_2 \), and replace \( r_2 \) on the 2nd \( \sum \) with \( \infty \) from the left.
- Repeat this infinite times.
Replace $r_{m-1}$ with $r_{m-1}+r_m$, and replace $r_{m-1}$ on the $(m-1)$th $\sum$ with $\infty$ from the left.

For example,

$$
(x_1 + x_2 + x_3 + x_4)^\alpha = \sum_{r_{1-0}}^{\infty} \alpha \begin{pmatrix} r_1 \cr r_2 \cr r_3 \cr r_4 \end{pmatrix} x_1^{\alpha-r_1} x_2^{r_1-r_2} x_3^{r_2-r_3} x_4^{r_3-r_4}
$$

Thus, performing this operation to (1.1), we obtain the following.

$$
(\sum_{r_{1-0}}^{\infty} \alpha \begin{pmatrix} r_1+2 \cr r_2 \cr r_3 \cr r_4 \end{pmatrix} x_1^{\alpha-r_1-r_2} x_2^{r_1+r_2} x_3^{r_2+r_3} x_4^{r_3-r_4})
$$

Since (1.1) converges absolutely, this rearrangement is allowed.

**Example 1:** The expansion of $(x_1 + x_2 + x_3)^{3.9}$

Using (1.1),

$$
(x_1 + x_2 + x_3)^{3.9} = \sum_{r=0}^{\infty} \sum_{s=0}^{r} \begin{pmatrix} 3.9 \cr r \cr s \end{pmatrix} x_1^{3.9-r} x_2^r x_3^s
$$

$$
= \begin{pmatrix} 3.9 \cr 0 \cr s \end{pmatrix} x_1^{3.9} x_2^0 x_3^s + \begin{pmatrix} 3.9 \cr 1 \cr s \end{pmatrix} x_1^{2.9} x_2^1 x_3^s + \begin{pmatrix} 3.9 \cr 2 \cr s \end{pmatrix} x_1^{1.9} x_2^2 x_3^s + \begin{pmatrix} 3.9 \cr 3 \cr s \end{pmatrix} x_1^{0.9} x_2^3 x_3^s + \begin{pmatrix} 4 \cr 4 \cr s \end{pmatrix} x_1^{0.1} x_2^4 x_3^s + \ldots
$$

$$
= x_1^{3.9} + 3.9 x_1^{2.9} (x_2 + x_3) + 5.655 x_1^{1.9} (x_2^2 + 2x_2 x_3 + x_3^2) + 3.5815 x_1^{0.9} (x_2^3 + 3x_2^2 x_3 + 3x_2 x_3^2 + x_3^3)
$$

$$
+ 0.805838 x_1^{0.1} (x_2^4 + 4x_2^3 x_3 + 6x_2^2 x_3^2 + 4x_2 x_3^3 + x_3^4)
$$

$$
- 0.0161168 x_1^{1.1} (x_2^5 + 5x_2^4 x_3 + 10x_2^3 x_3^2 + 10x_2^3 x_3^2 + 5x_2 x_3^4 + x_3^5)
$$

Using (1.2),

$$
(x_1 + x_2 + x_3)^{3.9} = \sum_{r=0}^{\infty} \sum_{s=0}^{r} \begin{pmatrix} 3.9 \cr r+s \cr s \end{pmatrix} x_1^{3.9-r-s} x_2^r x_3^s
$$
we can see that what totaled this along the diagonal line is equal to the above.

\textbf{Example 2:} The expansion of \((a + b + c + d)^{2.9}\)

\begin{align*}
\text{Clear}[p] ;
p &= 2.9 ; \\
fl[a_-, b_-, c_-, d_] := (a - b - c - d)^p \\
fl[5, -2, 3, 4] &= 794.328 \\
fr[a_-, b_-, c_-, d_] := \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \binom{p}{r} \binom{r}{s} \binom{s}{t} \cdot \frac{a^{-r} b^{-s} c^{-t} d^{-t}}{x} \\
N[fr[5, -2, 3, 4]] &= 794.328 \\
fs[a_-, b_-, c_-, d_] := \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \binom{p}{r + s + t} \binom{r + s + t}{s + t} \cdot \frac{a^{p-r-s-t} b^{r-s} c^{s-t} d^{t}}{x} \\
fs[5, -2, 3, 4] &= 794.328
\end{align*}

Although \(a = b + c + d\) in this numerical example, \((1.1)\) and \((1.2)\) are consistent.

\textbf{Sum of General Multinomial Coefficients}

\[
\sum_{\alpha} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \binom{p}{r+s+t} \binom{r+s+t}{s+t} \cdot \frac{a^{p-r-s-t} b^{r-s} c^{s-t} d^{t}}{x} = m^\alpha
\]  

\text{(1.1')}
As expected, the following expression does not hold.

\[
\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{r_{m-1}} \sum_{r_{m-1}=0}^{r_{m-2}} \left( \begin{array}{c} \alpha \\ r_1 \end{array} \right) \left( \begin{array}{c} r_1 \\ r_2 \end{array} \right) \cdots \left( \begin{array}{c} r_{m-2} \\ r_{m-1} \end{array} \right) = m^\alpha
\]

It is because \( x_1 = x_2 = \cdots = x_m = 1 \) does not satisfy the condition \(|x_1| \geq |x_2 + x_3 + \cdots + x_m|\).

Let its partial sum be

\[
S_n = \sum_{r_1=0}^{r_{m-1}} \sum_{r_{m-1}=0}^{r_{m-2}} \sum_{r_{m-2}=0}^{r_1} \left( \begin{array}{c} \alpha \\ r_1 \end{array} \right) \left( \begin{array}{c} r_1 \\ r_2 \end{array} \right) \cdots \left( \begin{array}{c} r_{m-2} \\ r_{m-1} \end{array} \right)
\]

Then, when \( n \to \infty \), \( S_n \) oscillates and diverges. And \( m^\alpha \) is the median of this oscillating divergent series.

In fact, applying Knopp Transformation to \( S_n \), we can obtain the approximate value of \( m^\alpha \) with high precision. However, it does not become a series but becomes an asymptotic expansion.