08 Power Series of Completed Dirichlet Beta

When $\zeta(z,a)$ is the Hurwitz zeta function and $\Gamma(z)$ is the gamma function, the Dirichlet beta function $\beta(z)$ and the Completed Dirichlet Beta function $\omega(z)$ are expressed by the following equations, respectively.

$$\beta(z) = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + \cdots = \frac{1}{4^z}\left\{\zeta\left(z,\frac{1}{4}\right) - \zeta\left(z,\frac{3}{4}\right)\right\} \tag{0.0}$$

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z) \tag{0.1}$$

Observing the two equations, we can see that 2^z and 4^z in $\beta(z)$ overlap. That is, if we simply expand (0.1) to a series, it will contain unnecessary series. This has a significant impact on the speed of the calculation. Therefore, we process (01) as follows to remove 4^z in advance.

Lemma 8.0

When $\zeta(z,a)$ is the Hurwitz zeta function and $\Gamma(z)$ is the gamma function, $\omega(z)$ is expressed as follows:

$$\omega(z) = 4\left(\frac{1}{2\sqrt{\pi}}\right)^{1+z}\left\{\zeta\left(z,\frac{1}{4}\right) - \zeta\left(z,\frac{3}{4}\right)\right\}\Gamma\left(\frac{1+z}{2}\right) \tag{0.1'}$$

Proof

Substituting (0.0) for (0.1) ,

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\frac{1}{4^z}\left\{\zeta\left(z,\frac{1}{4}\right) - \zeta\left(z,\frac{3}{4}\right)\right\}$$

Here,

$$\left(\frac{2}{\sqrt{\pi}}\right)^{1+z} = \frac{4^{1+z}}{4^{1+z}}\left(\frac{2}{\sqrt{\pi}}\right)^{1+z} = 4^{1+z}\left(\frac{1}{2\sqrt{\pi}}\right)^{1+z}$$

Substituting this for the right side,

$$\omega(z) = 4^{1+z}\left(\frac{1}{2\sqrt{\pi}}\right)^{1+z}\frac{1}{4^z}\left\{\zeta\left(z,\frac{1}{4}\right) - \zeta\left(z,\frac{3}{4}\right)\right\}\Gamma\left(\frac{1+z}{2}\right)$$

i.e.

$$\omega(z) = 4\left(\frac{1}{2\sqrt{\pi}}\right)^{1+z}\left\{\zeta\left(z,\frac{1}{4}\right) - \zeta\left(z,\frac{3}{4}\right)\right\}\Gamma\left(\frac{1+z}{2}\right)$$

Q.E.D.

8.1 Taylor series of $\omega(z)$ around 1

We expand each of the 3 functions that consist of (0.1') into a Taylor series, and let their Cauchy product be the Taylor series of $\omega(z)$.

Lemma 8.1.1

The following holds on the whole complex plane:

$$\left(\frac{1}{2\sqrt{\pi}}\right)^{1+z} = \frac{1}{4\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\log 2 + \frac{1}{2}\log\pi\right)^r (z-1)^r \tag{9.1}$$

Proof

$$\frac{1}{2\sqrt{\pi}} = e^{\log\frac{1}{2\sqrt{\pi}}} = e^{-\log 2 - \frac{1}{2}\log\pi}$$

Then,

$$f(z) = \left(\frac{1}{2\sqrt{\pi}}\right)^{1+z} = e^{-\left(\log 2+\frac{1}{2}\log\pi\right)z - \left(\log 2+\frac{1}{2}\log\pi\right)}$$

According to Formula 9.2.2 in my paper " **09 Higher Derivative** " (**SuperCalculus**) ,

$$\left(e^{ax+b}\right)^{(n)} = \left(\frac{1}{a}\right)^{-n} e^{ax+b}$$

Applying this to the above equation,

$$f^{(r)}(z) = (-1)^r\left(\log 2+\frac{1}{2}\log\pi\right)^r e^{-\left(\log 2+\frac{1}{2}\log\pi\right)z - \left(\log 2+\frac{1}{2}\log\pi\right)}$$

When $z=1$,

$$f^{(r)}(1) = (-1)^r\left(\log 2+\frac{1}{2}\log\pi\right)^r e^{-(\log 4+\log\pi)} = \frac{(-1)^r}{4\pi}\left(\log 2+\frac{1}{2}\log\pi\right)^r$$

Therefore,

$$f(z) = \sum_{r=0}^{\infty}\frac{f^{(r)}(1)}{r!}(z-1)^r = \frac{1}{4\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\log 2+\frac{1}{2}\log\pi\right)^r(z-1)^r$$

Q.E.D.

Lemma 8.1.2

The following holds on the whole complex plane:

$$\zeta\left(z,\frac{1}{4}\right) - \zeta\left(z,\frac{3}{4}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left\{\gamma_r\left(\frac{1}{4}\right) - \gamma_r\left(\frac{3}{4}\right)\right\}(z-1)^r \qquad (9.2)$$

Where, $\gamma_r(a)$ is Generalized Stieltjes constant defined by the following expression.

$$\gamma_r(a) = \lim_{m\to\infty}\left\{\sum_{k=0}^{m}\frac{\log^r(k+a)}{k+a} - \frac{\log^{r+1}(m+a)}{r+1}\right\} \qquad r=0,1,2,\cdots \quad a\ne 0,-1,-2,\cdots$$

Proof

The Hurwitz zeta function $\zeta(z,a)$ can be expanded to a Laurent series using the generalized Stieltjes constant $\gamma_r(a)$ as follows:

$$\zeta(z,a) = \frac{1}{z-1} + \sum_{r=0}^{\infty}(-1)^r\gamma_r(a)\frac{(z-1)^r}{r!}$$

From this,

$$\zeta\left(z,\frac{1}{4}\right) = \frac{1}{z-1} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\left(\frac{1}{4}\right)\frac{(z-1)^r}{r!}$$

$$\zeta\left(z,\frac{3}{4}\right) = \frac{1}{z-1} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\left(\frac{3}{4}\right)\frac{(z-1)^r}{r!}$$

Calculating the difference between the two, we obtain the desired expression.

Q.E.D.

Lemma 8.1.3

When $\Gamma(z)$ is Gamma function, $\psi_n(z)$ is Polygamma function, and $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomial , then, the following holds within a circle of radius 2 centered at $z=1$.

$$\Gamma\left(\frac{1+z}{2}\right) = \sum_{r=0}^{\infty} \frac{g_r(1)}{2^r r!}(z-1)^r \tag{9.3}$$

Where,

$$g_r(1) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) & r=1,2,3,\cdots \end{cases}$$

Proof

According to Formula 12.1.1 in my paper "**12 Series Expansion of Gamma Function & the Reciprocal** " (**Alacarte**) , when $\Gamma(z)$ is Gamma function, $\psi_n(z)$ is Polygamma function, and $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomial ,

$$\Gamma(z) = \Gamma(a) + \sum_{n=1}^{\infty} \frac{c_n(a)}{n!}(z-a)^n \qquad a \ne 0,-1,-2,-3,\cdots$$

Where,

$$c_n(a) = \Gamma(a) \sum_{k=1}^{n} B_{n,k}\left(\psi_0(a), \psi_1(a), \ldots, \psi_{n-1}(a)\right) \qquad n=1,2,3,\cdots$$

Using this, $\Gamma\{(1+z)/2\}$ can be expanded to Taylor series as follows.

$$\Gamma\left(\frac{1+z}{2}\right) = \Gamma(a) + \sum_{r=1}^{\infty} \frac{\Gamma^{(r)}(a)}{r!}\left(\frac{1+z}{2}-a\right)^r$$

$$\Gamma^{(r)}(a) = \Gamma(a) \sum_{k=1}^{r} B_{r,k}\left(\psi_0(a), \psi_1(a), \ldots, \psi_{r-1}(a)\right) \qquad r=1,2,3,\cdots$$

Putting $a=1$,

$$\Gamma\left(\frac{1+z}{2}\right) = \Gamma(1) + \sum_{r=1}^{\infty} \frac{\Gamma^{(r)}(1)}{r!}\left(\frac{z-1}{2}\right)^r \tag{9.4}$$

$$\Gamma^{(r)}(1) = \Gamma(1) \sum_{k=1}^{r} B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) \qquad r=1,2,3,\cdots$$

Here, let $g_r(1)$ be

$$g_r(1) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0(1), \psi_1(1), \ldots, \psi_{r-1}(1)\right) & r=1,2,3,\cdots \end{cases}$$

Then

$$\Gamma^{(r)}(1) = \Gamma(1) g_r(1) \qquad r=0,1,2,\cdots$$

So, substituting this for (9.4) ,

$$\Gamma\left(\frac{1+z}{2}\right) = \Gamma(1) + \Gamma(1) \sum_{r=1}^{\infty} \frac{g_r(1)}{r!}\left(\frac{z-1}{2}\right)^r$$

i.e.

$$\Gamma\left(\frac{1+z}{2}\right) = \sum_{r=0}^{\infty} \frac{g_r(1)}{2^r r!}(z-1)^r \qquad \{\because \Gamma(1)=1\}$$

Q.E.D.

Combining all of the above, we obtain the following theorem.

Theorem 8.1.4 (Taylor series of $\omega(z)$ around 1)

Let the Completed Dirichlet Beta Function $\omega(z)$ and its Taylor series be as follows.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1+z}{2}\right)\beta(z) = \sum_{r=0}^{\infty} B_r (z-1)^r \tag{1.1}$$

Then, these coefficients $B_r \quad r=0,1,2,3,\cdots$ are given by,

$$B_r = \frac{1}{\pi}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^{r-s}}{(r-s)!}\left(\log 2+\frac{1}{2}\log\pi\right)^{r-s}\frac{(-1)^{s-t}}{(s-t)!}\left\{\gamma_{s-t}\left(\frac{1}{4}\right)-\gamma_{s-t}\left(\frac{3}{4}\right)\right\}\frac{g_t(1)}{2^t t!} \tag{1.2}$$

$$g_r(1) = \begin{cases} 1 & r=0 \\ \displaystyle\sum_{k=1}^{r} B_{r,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{r-1}(1)\right) & r=1,2,3,\cdots \end{cases}$$

Where, $\psi_n(z)$ is the Polygamma function, $B_{n,k}(f_1, f_2, \dots)$ is the Bell polynomial and $\gamma_r(a)$ is Generalized Stieltjes constant defined by the following.

$$\gamma_r(a) = \lim_{m\to\infty}\left\{\sum_{k=0}^{m}\frac{\log^r(k+a)}{k+a} - \frac{\log^{r+1}(m+a)}{r+1}\right\} \qquad \begin{array}{l} r=0,1,2,\cdots \\ a\ne 0,-1,-2,\cdots \end{array}$$

Proof

From Lemma 8.0 ,

$$\omega(z) = 4\left(\frac{1}{2\sqrt{\pi}}\right)^{1+z}\left\{\zeta\left(z,\frac{1}{4}\right)-\zeta\left(z,\frac{3}{4}\right)\right\}\Gamma\left(\frac{1+z}{2}\right) \tag{0.1'}$$

From Lemma 8.1.1 , Lemma 8.1.2 and Lemma 8.1.3 ,

$$\left(\frac{1}{2\sqrt{\pi}}\right)^{1+z} = \frac{1}{4\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\log 2+\frac{1}{2}\log\pi\right)^r (z-1)^r$$

$$\zeta\left(z,\frac{1}{4}\right)-\zeta\left(z,\frac{3}{4}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left\{\gamma_r\left(\frac{1}{4}\right)-\gamma_r\left(\frac{3}{4}\right)\right\}(z-1)^r$$

$$\Gamma\left(\frac{1+z}{2}\right) = \sum_{r=0}^{\infty}\frac{g_r(1)}{2^r r!}(z-1)^r$$

Where,

$$g_r(1) = \begin{cases} 1 & r=0 \\ \displaystyle\sum_{k=1}^{r} B_{r,k}\left(\psi_0(1), \psi_1(1), \dots, \psi_{r-1}(1)\right) & r=1,2,3,\cdots \end{cases}$$

Substituting the three lemma formulas for the right-hand side of (0.1') ,

$$\omega(z) = 4\frac{1}{4\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\log 2+\frac{1}{2}\log\pi\right)^r (z-1)^r$$
$$\times\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left\{\gamma_r\left(\frac{1}{4}\right)-\gamma_r\left(\frac{3}{4}\right)\right\}(z-1)^r \times \sum_{r=0}^{\infty}\frac{g_r(1)}{2^r r!}(z-1)^r$$

According to Formula 1.1.2 in my paper " **01 Power of Infinite Series** " (**Infinite-degree Equation**) ,

$$\left(\sum_{r=0}^{\infty}a_r z^r\right)\left(\sum_{r=0}^{\infty}b_r z^r\right)\left(\sum_{r=0}^{\infty}c_r z^r\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}a_{r-s}b_{s-t}c_t z^r$$

therefore, let

$$a_r = \frac{(-1)^r}{r!}\left(\log 2 + \frac{1}{2}\log \pi \right)^r \quad , \quad b_r = \frac{(-1)^r}{r\,!}\left\{ \gamma_r\!\left(\frac{1}{4} \right) - \gamma_r\!\left(\frac{3}{4} \right) \right\} \quad , \quad c_r = \frac{g_r(1)}{2^r\, r\,!}$$

Then,

$$\omega(z) = \frac{1}{\pi}\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^{r-s}}{(r-s)!}\left(\log 2 + \frac{1}{2}\log \pi \right)^{r-s} \frac{(-1)^{s-t}}{(s-t)\,!}\left\{ \gamma_{s-t}\!\left(\frac{1}{4} \right) - \gamma_{s-t}\!\left(\frac{3}{4} \right) \right\} \times \frac{g_t(1)}{2^t\, t\,!}(z-1)^r$$

So, putting

$$B_r = \frac{1}{\pi}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^{r-s}}{(r-s)!}\left(\log 2 + \frac{1}{2}\log \pi \right)^{r-s} \frac{(-1)^{s-t}}{(s-t)\,!}\left\{ \gamma_{s-t}\!\left(\frac{1}{4} \right) - \gamma_{s-t}\!\left(\frac{3}{4} \right) \right\} \frac{g_t(1)}{2^t\, t\,!} \tag{1.2}$$

we obtain the right hand side of (1.1) .

Q.E.D.

Examples

In (1.2) , B_r is expressed by the generalized Stieltjes constant $\gamma_r(a)$ and the intermediate constant $g(1)$, but since $g(1)$ is a polynomial of the Polygamma $\psi_r(1)$, B_r is ultimately expressed by $\gamma_r(a)$ and $\psi_r(1)$ .. Using the mathematical processing software *Mathematica*, the first few are as follows.

$$\mathbf{B}_{r\_} := \frac{1}{\pi}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^{r-s}}{(r-s)\,!}\left(\text{Log}[2] + \frac{1}{2}\text{Log}[\pi] \right)^{r-s} \frac{(-1)^{s-t}}{(s-t)\,!}\left(\gamma_{s-t}\left[\frac{1}{4}\right] - \gamma_{s-t}\left[\frac{3}{4}\right] \right) \frac{g_t[1]}{2^t\, t\,!}$$

$$g_{r\_}[1] := \text{If}\left[r == 0,\ 1,\ \sum_{k=1}^{r} \text{BellY}[r,\ k,\ \text{Tbl}\psi[r,\ 1]] \right]$$

$$\text{Tbl}\psi[r\_,\ z\_] := \text{Table}[\psi_k[z],\ \{k,\ 0,\ r-1\}]$$

$$\mathbf{B}_0 \quad \frac{1}{\pi}\left(\gamma_0\left[\frac{1}{4}\right] - \gamma_0\left[\frac{3}{4}\right] \right) \qquad (=1)$$

$$\mathbf{B}_1 \quad \frac{1}{\pi}\left(-\left(\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)\left(\gamma_0\left[\frac{1}{4}\right] - \gamma_0\left[\frac{3}{4}\right] \right) \right) - \gamma_1\left[\frac{1}{4}\right] + \gamma_1\left[\frac{3}{4}\right] + \frac{1}{2}\left(\gamma_0\left[\frac{1}{4}\right] - \gamma_0\left[\frac{3}{4}\right] \right)\psi_0[1] \right)$$

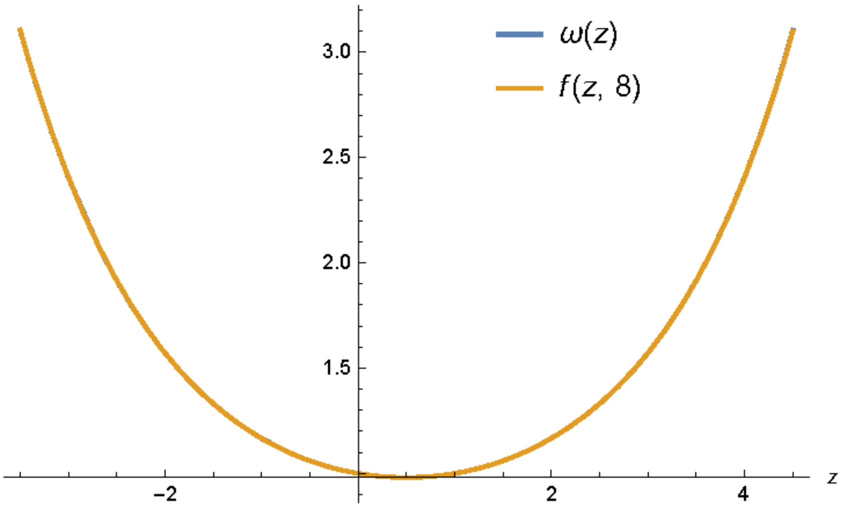
$$\begin{aligned}\mathbf{B}_2 \quad \frac{1}{\pi}\Bigg(& \frac{1}{2}\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)^2 \left(\gamma_0\left[\frac{1}{4}\right] - \gamma_0\left[\frac{3}{4}\right] \right) + \left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)\left(\gamma_1\left[\frac{1}{4}\right] - \gamma_1\left[\frac{3}{4}\right] \right) \\ & + \frac{1}{2}\left(\gamma_2\left[\frac{1}{4}\right] - \gamma_2\left[\frac{3}{4}\right] \right) - \frac{1}{2}\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)\left(\gamma_0\left[\frac{1}{4}\right] - \gamma_0\left[\frac{3}{4}\right] \right)\psi_0[1] \\ & - \frac{1}{2}\left(\gamma_1\left[\frac{1}{4}\right] - \gamma_1\left[\frac{3}{4}\right] \right)\psi_0[1] + \frac{1}{8}\left(\gamma_0\left[\frac{1}{4}\right] - \gamma_0\left[\frac{3}{4}\right] \right)\left(\psi_0[1]^2 + \psi_1[1] \right) \Bigg)\end{aligned}$$

$$\begin{aligned}\mathbf{B}_3 \quad \frac{1}{\pi}\Bigg(& -\frac{1}{6}\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)^3 \left(\gamma_0\left[\frac{1}{4}\right] - \gamma_0\left[\frac{3}{4}\right] \right) \\ & - \frac{1}{2}\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)^2 \left(\gamma_1\left[\frac{1}{4}\right] - \gamma_1\left[\frac{3}{4}\right] \right) - \frac{1}{2}\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)\left(\gamma_2\left[\frac{1}{4}\right] - \gamma_2\left[\frac{3}{4}\right] \right) \\ & + \frac{1}{6}\left(-\gamma_3\left[\frac{1}{4}\right] + \gamma_3\left[\frac{3}{4}\right] \right) + \frac{1}{4}\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)^2 \left(\gamma_0\left[\frac{1}{4}\right] - \gamma_0\left[\frac{3}{4}\right] \right)\psi_0[1] \\ & + \frac{1}{2}\left(\text{Log}[2] + \frac{\text{Log}[\pi]}{2} \right)\left(\gamma_1\left[\frac{1}{4}\right] - \gamma_1\left[\frac{3}{4}\right] \right)\psi_0[1] + \frac{1}{4}\left(\gamma_2\left[\frac{1}{4}\right] - \gamma_2\left[\frac{3}{4}\right] \right)\psi_0[1]\end{aligned}$$

$$-\frac{1}{8}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\left(\psi_0[1]^2+\psi_1[1]\right)$$
$$-\frac{1}{8}\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)\left(\psi_0[1]^2+\psi_1[1]\right)$$
$$+\frac{1}{48}\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\left(\psi_0[1]^3+3\,\psi_0[1]\,\psi_1[1]+\psi_2[1]\right)\Bigg)$$

Figure

Both sides of (1.1) are drawn as follows. The left side (function) is blue and the right side (series) is orange. The right side is calculated up to $(z-1)^8$, but both sides overlap and the left side (blue) is not visible.



The singular points of the Gamma function (9.3) are cancelled out with the trivial zeros of the Dirichlet beta function (0.0). As the result, the convergence radius of this series is ∞.

8.2 Maclaurin series of $\omega(z)$

Maclaurin series of $\omega(z)$ is surprisingly easy to obtain from the Taylor series around 1 using functional equations.

Theorem 8.2.1 (Maclaurin series of $\omega(z)$)

Let the Completed Dirichlet Beta Function $\omega(z)$ and its Maclaurin series be as follows.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1+z}{2}\right)\beta(z) = \sum_{r=0}^{\infty} A_r z^r \tag{2.1}$$

Then, these coefficients $A_r \quad r=0,1,2,3,\cdots$ are given by,

$$A_r = \frac{1}{\pi}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^{-s}}{(r-s)!}\left(\log 2+\frac{1}{2}\log\pi\right)^{r-s}\frac{(-1)^{s-t}}{(s-t)!}\left\{\gamma_{s-t}\left(\frac{1}{4}\right)-\gamma_{s-t}\left(\frac{3}{4}\right)\right\}\frac{g_t(1)}{2^t t!} \tag{2.2}$$

$$g_r(1) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0(1), \psi_1(1), \cdots, \psi_{r-1}(1)\right) & r=1,2,3,\cdots \end{cases}$$

Where, $\psi_n(z)$ is the Polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is the Bell polynomial and $\gamma_r(a)$ is Generalized Stieltjes constant defined by the following.

$$\gamma_r(a) = \lim_{m\to\infty}\left\{\sum_{k=0}^{m}\frac{\log^r(k+a)}{k+a} - \frac{\log^{r+1}(m+a)}{r+1}\right\} \qquad \begin{array}{l} r=0,1,2,\cdots \\ a\ne 0,-1,-2,\cdots \end{array}$$

Proof

From Theorem 8.1.4 ,

$$B_r = \frac{1}{\pi}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^{r-s}}{(r-s)!}\left(\log 2+\frac{1}{2}\log\pi\right)^{r-s}\frac{(-1)^{s-t}}{(s-t)!}\left\{\gamma_{s-t}\left(\frac{1}{4}\right)-\gamma_{s-t}\left(\frac{3}{4}\right)\right\}\frac{g_t(1)}{2^t t!} \tag{1.2}$$

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1+z}{2}\right)\beta(z) = \sum_{r=0}^{\infty} B_r (z-1)^r \tag{1.1}$$

Replacing z with $1-z$ in (1.1),

$$\omega(1-z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+1-z} \Gamma\left(\frac{1+1-z}{2}\right)\beta(1-z) = \sum_{r=0}^{\infty} B_r (-z)^r \tag{9.5}$$

According to Formula 4.1.1 in my paper " **04 Completed Dirichlet Beta** " ,

$$\left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1}{2}+\frac{z}{2}\right)\beta(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{2-z} \Gamma\left(\frac{1}{2}+\frac{1-z}{2}\right)\beta(1-z)$$

So, the middle sides (functions) of (1.1) and (9.5) are equal, and the following functional equation holds on the whole complex plane .

$$\omega(z) = \omega(1-z)$$

Therefore, the right-hand sides (series) of (1.1) and (9.5) become as follows.

$$\sum_{r=0}^{\infty} B_r (z-1)^r = \sum_{r=0}^{\infty} B_r (-z)^r = \sum_{r=0}^{\infty} (-1)^r B_r z^r = \sum_{r=0}^{\infty} A_r z^r$$

That is $A_r = (-1)^r B_r \quad r=0,1,2,\cdots$.

So, if we multiply (1.2) by $(-1)^r$, the first sign on the right hand side becomes,

$$(-1)^r(-1)^{r-s} = (-1)^{-s}$$

Thus, we obtain (2.2) .

Q.E.D.

Examples

Using the mathematical processing software *Mathematica*, the first few of A_r are as follows.
Compared to B_r in the previous section, we can see that the signs of the odd terms are reversed.

$$A_{r\_} := \frac{1}{\pi}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^{-s}}{(r-s)!}\left(\text{Log}[2]+\frac{1}{2}\text{Log}[\pi]\right)^{r-s}\frac{(-1)^{s-t}}{(s-t)!}\left(\gamma_{s-t}\left[\frac{1}{4}\right]-\gamma_{s-t}\left[\frac{3}{4}\right]\right)\frac{g_t[1]}{2^t\, t!}$$

$$g_{r\_}[1] := \text{If}\left[r == 0, 1, \sum_{k=1}^{r}\text{BellY}[r, k, \text{Tbl}\psi[r, 1]]\right]$$

$$\text{Tbl}\psi[r\_, z\_] := \text{Table}[\psi_k[z], \{k, 0, r-1\}]$$

$$A_0 \quad \frac{1}{\pi}\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)$$

$$A_1 \quad \frac{1}{\pi}\left(\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)+\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]-\frac{1}{2}\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\psi_0[1]\right)$$

$$A_2 \quad \frac{1}{\pi}\left(\frac{1}{2}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)^2\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)+\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)\right.$$
$$+\frac{1}{2}\left(\gamma_2\left[\frac{1}{4}\right]-\gamma_2\left[\frac{3}{4}\right]\right)-\frac{1}{2}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\psi_0[1]$$
$$\left.-\frac{1}{2}\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)\psi_0[1]+\frac{1}{8}\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\left(\psi_0[1]^2+\psi_1[1]\right)\right)$$

$$A_3 \quad \frac{1}{\pi}\left(\frac{1}{6}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)^3\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\right.$$
$$+\frac{1}{2}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)^2\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)+\frac{1}{2}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_2\left[\frac{1}{4}\right]-\gamma_2\left[\frac{3}{4}\right]\right)$$
$$+\frac{1}{6}\left(\gamma_3\left[\frac{1}{4}\right]-\gamma_3\left[\frac{3}{4}\right]\right)-\frac{1}{4}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)^2\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\psi_0[1]$$
$$-\frac{1}{2}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)\psi_0[1]-\frac{1}{4}\left(\gamma_2\left[\frac{1}{4}\right]-\gamma_2\left[\frac{3}{4}\right]\right)\psi_0[1]$$
$$+\frac{1}{8}\left(\text{Log}[2]+\frac{\text{Log}[\pi]}{2}\right)\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\left(\psi_0[1]^2+\psi_1[1]\right)$$
$$+\frac{1}{8}\left(\gamma_1\left[\frac{1}{4}\right]-\gamma_1\left[\frac{3}{4}\right]\right)\left(\psi_0[1]^2+\psi_1[1]\right)$$
$$\left.-\frac{1}{48}\left(\gamma_0\left[\frac{1}{4}\right]-\gamma_0\left[\frac{3}{4}\right]\right)\left(\psi_0[1]^3+3\,\psi_0[1]\,\psi_1[1]+\psi_2[1]\right)\right)$$

Confirmation Calculation

Left hande side

$$\omega[z\_] := \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\text{Gamma}\left[\frac{1+z}{2}\right]\text{DirichletBeta}[z]$$

N[Series[ω[z], {z, 0, 8}]]

$$1. - 0.077784\,(z+0.) + 0.0803502\,(z+0.)^2 - 0.00518246\,(z+0.)^3 + 0.00271685\,(z+0.)^4 - 0.000152607\,(z+0.)^5 + 0.000053991\,(z+0.)^6 - 2.71502\times10^{-6}\,(z+0.)^7 + 7.27982\times10^{-7}\,(z+0.)^8 + O[z+0.]^9$$

Right hande side

```
Unprotect[Power]; Power[0, 0] = 1;
```

$$f[z\_, m\_] := \sum_{r=0}^{m} A_r z^r$$

$$A_{r\_} := \frac{1}{\pi} \sum_{s=0}^{r} \sum_{t=0}^{s} \frac{(-1)^{-s}}{(r-s)!} \left(\text{Log}[2] + \frac{1}{2} \text{Log}[\pi]\right)^{r-s} \frac{(-1)^{s-t}}{(s-t)!} \left(\gamma_{s-t}\left[\frac{1}{4}\right] - \gamma_{s-t}\left[\frac{3}{4}\right]\right) \frac{g_t[1]}{2^t\, t!}$$

$$g_{r\_}[1] := \text{If}\left[r == 0, 1, \sum_{k=1}^{r} \text{BellY}[r, k, \text{Tbl}\psi[r, 1]]\right]$$

$$\text{Tbl}\psi[r\_, z\_] := \text{Table}[\text{PolyGamma}[k, z], \{k, 0, r-1\}]$$

$$\gamma_{s\_}[a\_] := \text{StieltjesGamma}[s, a]$$

```
N[f[z, 8]]
```

$$1. - 0.077784\, z + 0.0803502\, z^2 - 0.00518246\, z^3 + 0.00271685\, z^4 - 0.000152607\, z^5 + 0.000053991\, z^6 - 2.71502 \times 10^{-6}\, z^7 + 7.27982 \times 10^{-7}\, z^8$$

Note

Based on the Maclaurin series, $\omega(z)$ is as follows:

$$\omega(z) = \sum_{r=0}^{\infty} A_r z^r = \sum_{r=0}^{\infty} A_r (1-z)^r = \sum_{r=0}^{\infty} (-z)^r A_r (z-1)^r = \sum_{r=0}^{\infty} B_r (z-1)^r$$

That is, the Maclaurin series of $\omega(z)$ with z replaced by $1-z$ is the Taylor series of $\omega(z)$ around 1. And the coefficients of both are the same for even terms, and differ only in sign for odd terms. It is an interesting property. What is even more interesting is that the central points $0, 1$ of the expansion of both series are left and right edge points of the critical strip.

These are derived from functional equations. Therefore, these relations also hold for the completed Riemann zeta function $\xi(z)$.

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Alien's Mathematics