05 Power Series and Semi Multiple Series

Abstract

While studying powers of series, I noticed that this can be expressed as a formula consisting of power series and semi-multiple series. This expression is not unique, and as the power increases the expression becomes more diverse. In the midst of all this, I found that these could be expressed in a unified way. we present this in the first three sections.

In the final section, we present a fast calculation method that replaces the time-consuming calculation of semi-multiple series with polynomials consisting of power series.

5.1 Head, Body and Tail

The next lemma shows that the product of a polynomial and a power polynomial can be split into three parts: head, body, and tail.

Lemma 5.1.1

When n, m are natural numbers and a_{r_1} is a real number, the following holds.

$$\left(\sum_{r_1=1}^{m} a_{r_1}\right)\sum_{r_1=0}^{m} a_{r_1}^{n-1} = \sum_{r_1=1}^{m} a_{r_1}^{n} + \sum_{t=2}^{n-1}(-1)^t\left(\sum_{r_1=1}^{m} a_{r_1}^{n-t}\right)H_t(m) + (-1)^n n H_n(m) \qquad (1.1_n)$$

Where,

$$H_2(m) = \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m} a_{r_1}a_{r_2}$$

$$H_3(m) = \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m} a_{r_1}a_{r_2}a_{r_3}$$

$$\vdots$$

$$H_n(m) = \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m}\cdots\sum_{r_n=r_{n-1}+1}^{m} a_{r_1}a_{r_2}a_{r_3}\cdots a_{r_n}$$

When $n \le 2$, the 2 nd term of (1.1n) is ignored.

Proof

Let a power polynomial as follows.

$$G_n(m) = \sum_{r_1=1}^{m} a_{r_1}^{n}$$

Using these symbols, (1.1n) can be written as follows,

$$G_1(m)G_{n-1}(m) = G_n(m) + \sum_{t=2}^{n-1}(-1)^t G_{n-t}(m)\,H_t(m) + (-1)^n n\,H_n(m) \qquad (1.1_n')$$

We can prove (1.1n') instead of (1.1n).

When $n=3$, if $G_1(3)G_2(3)$ is expanded and arranged

$$G_1(3)G_2(3) = \left(a_1^3+a_2^3+a_3^3\right) + a_1^2a_2 + a_1a_2^2 + a_1a_3^2 + a_2^2a_3 + a_1a_3^2 + a_2a_3^2$$

Here,

$$G_1(3)H_2(3) = a_1^2a_2 + a_1a_2^2 + a_1a_3^2 + a_2^2a_3 + a_1a_3^2 + a_2a_3^2 + 3\,a_1a_2a_3$$

So,

$$a_1^2 a_2 + a_1 a_2^2 + a_1 a_3^2 + a_2^2 a_3 + a_1 a_3^2 + a_2 a_3^2 = G_1(3) H_2(3) - 3 H_3(3)$$

Substituting this for the right side of $G_1(3) G_2(3)$,

$$G_1(3) G_2(3) = G_3(3) + G_1(3) H_2(3) - 3 H_3(3) \qquad (1.1_3')$$

Note that this equation holds true for any natural number m .

When n=4 , if $G_1(4) G_3(4)$ is expanded and arranged

$$G_1(4) G_3(4) = \left(a_1^4 + a_2^4 + a_3^4 + a_4^4 \right) + a_1^3 a_2 + a_1 a_2^3 + \cdots + a_3 a_4^3$$

Here,

$$G_2(4) H_2(4) = \left(a_1^3 a_2 + a_1 a_2^3 + \cdots + a_3 a_4^3 \right) + a_1^2 a_2 a_3 + a_1 a_2^2 a_3 + \cdots + a_2 a_3 a_4^2$$

And

$$a_1^2 a_2 a_3 + a_1 a_2^2 a_3 + \cdots + a_2 a_3 a_4^2 = G_1(4) H_3(4) - 4 H_4(4)$$

From these two fexpressions,

$$a_1^3 a_2 + a_1^3 a_3 + \cdots + a_3 a_4^3 = G_2(4) H_2(4) - G_1(4) H_3(4) + 4 H_4(4)$$

Substituting this for the right side of $G_1(4) G_3(4)$,

$$G_1(4) G_3(4) = G_4(4) + G_2(4) H_2(4) - G_1(4) H_3(4) + 4 H_4(4) \qquad (1.1_4')$$

Note that this equation holds true for any natural number m .

When n=5 , if $G_1(5) G_4(5)$ is expanded and arranged

$$G_1(5) G_4(5) = \left(a_1^5 + a_2^5 + a_3^5 + a_4^5 + a_4^5 \right) + a_1^4 a_2 + a_1 a_2^4 + \cdots + a_4 a_5^4$$

Here,

$$G_3(5) H_2(5) = \left(a_1^4 a_2 + a_1 a_2^4 + \cdots + a_4 a_5^4 \right) + a_1^3 a_2 a_3 + a_1 a_2^3 a_3 + \cdots + a_3 a_4 a_5^3$$

And

$$a_1^3 a_2 a_3 + a_1 a_2^3 a_3 + \cdots + a_3 a_4 a_5^3 = G_2(5) H_3(5) - G_1(5) H_4(5) + 5 H_5(5)$$

From these two fexpressions,

$$a_1^4 a_2 + a_1 a_2^4 + \cdots + a_4 a_5^4 = G_3(5) H_2(5) - G_2(5) H_3(5) + G_1(5) H_4(5) - 5 H_5(5)$$

Substituting this for the right side of $G_1(5) G_4(5)$,

$$G_1(5) G_4(5) = G_5(5) + G_3(5) H_2(5) - G_2(5) H_3(5) + G_1(5) H_4(5) - 5 H_5(5) \qquad (1.1_5')$$

Hereafter, by induction, we obtain

$$G_1(m) G_{n-1}(m) = G_n(m) + \sum_{t=2}^{n-1} (-1)^t G_{n-t}(m) H_t(m) + (-1)^n n H_n(m) \qquad (1.1_n')$$

Let us run this in the mathematical processing software *Mathematica*. First, define the functions as follows.

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Clear[G, H, a, r]
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$$G_{n\_}[m\_] := \sum_{r_1=1}^{m} a_{r_1}{}^{n}$$

$$H_2[m\_] := \sum_{r_1=1}^{m} \sum_{r_2=r_1+1}^{m} a_{r_1} a_{r_2}$$

$$H_3[m\_] := \sum_{r_1=1}^{m} \sum_{r_2=r_1+1}^{m} \sum_{r_3=r_2+1}^{m} a_{r_1} a_{r_2} a_{r_3}$$

$$\vdots$$

$$H_9[m\_] := \sum_{r_1=1}^{m} \sum_{r_2=r_1+1}^{m} \sum_{r_3=r_2+1}^{m} \sum_{r_4=r_3+1}^{m} \sum_{r_5=r_4+1}^{m} \sum_{r_6=r_5+1}^{m} \sum_{r_7=r_6+1}^{m} \sum_{r_8=r_7+1}^{m} \sum_{r_9=r_8+1}^{m} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5} a_{r_6} a_{r_7} a_{r_8} a_{r_9}$$

Then, the upper row is input and the lower row is output.

$$G_1[m]\, G_{n-1}[m] == G_n[m] + \sum_{t=2}^{n-1} (-1)^t G_{n-t}[m]\, H_t[m] + (-1)^n n H_n[m]$$

$$\left(\sum_{r_1=1}^{m} a_{r_1}\right) \sum_{r_1=1}^{m} a_{r_1}^{-1+n} == \sum_{r_1=1}^{m} a_{r_1}^{n} + \sum_{t=2}^{-1+n} (-1)^t \left(\sum_{r_1=1}^{m} a_{r_1}^{n-t}\right) H_t[m] + (-1)^n n H_n[m]$$

Thus , (1.1_n) was obtained. Q.E.D.

Example $n=5$

$$\left(\sum_{r_1=1}^{m} a_{r_1}\right) \sum_{r_1=1}^{m} a_{r_1}^{5-1} = \sum_{r_1=1}^{m} a_{r_1}^{5} + \sum_{t=2}^{5-1} (-1)^t \left(\sum_{r_1=1}^{m} a_{r_1}^{5-t}\right) H_t(m) + (-1)^5 5 H_5(m)$$

If we express each Σ as $G_n(m), H_n(m)$ and verify it using *Mathematica*, It is as follows.

Both $m=6$ and $m=4$ are verified, and these equations are confirmed to be true.

When $m=6$

$$G_1[6]\, G_{5-1}[6] == G_5[6] + \sum_{t=2}^{5-1} (-1)^t G_{5-t}[6]\, H_t[6] + (-1)^5 5 H_5[6]$$

$$\begin{aligned}
&(a_1 + a_2 + a_3 + a_4 + a_5 + a_6)\left(a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4 + a_6^4\right) == \\
&\quad a_1^5 + a_2^5 + a_3^5 + a_4^5 + a_5^5 + a_6^5 + (a_1 + a_2 + a_3 + a_4 + a_5 + a_6) \\
&\qquad (a_1 a_2 a_3 a_4 + a_1 a_2 a_3 a_5 + a_1 a_2 a_4 a_5 + a_1 a_3 a_4 a_5 + a_2 a_3 a_4 a_5 + a_1 a_2 a_3 a_6 + a_1 a_2 a_4 a_6 + a_1 a_3 a_4 a_6 + \\
&\qquad\quad a_2 a_3 a_4 a_6 + a_1 a_2 a_5 a_6 + a_1 a_3 a_5 a_6 + a_2 a_3 a_5 a_6 + a_1 a_4 a_5 a_6 + a_2 a_4 a_5 a_6 + a_3 a_4 a_5 a_6) - \\
&\quad 5\,(a_1 a_2 a_3 a_4 a_5 + a_1 a_2 a_3 a_4 a_6 + a_1 a_2 a_3 a_5 a_6 + a_1 a_2 a_4 a_5 a_6 + a_1 a_3 a_4 a_5 a_6 + a_2 a_3 a_4 a_5 a_6) - \\
&\quad (a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4 + a_1 a_2 a_5 + a_1 a_3 a_5 + a_2 a_3 a_5 + a_1 a_4 a_5 + a_2 a_4 a_5 + a_3 a_4 a_5 + \\
&\qquad a_1 a_2 a_6 + a_1 a_3 a_6 + a_2 a_3 a_6 + a_1 a_4 a_6 + a_2 a_4 a_6 + a_3 a_4 a_6 + a_1 a_5 a_6 + a_2 a_5 a_6 + a_3 a_5 a_6 + a_4 a_5 a_6) \\
&\quad \left(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2\right) + (a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4 + a_1 a_5 + a_2 a_5 + \\
&\qquad a_3 a_5 + a_4 a_5 + a_1 a_6 + a_2 a_6 + a_3 a_6 + a_4 a_6 + a_5 a_6)\left(a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 + a_6^3\right)
\end{aligned}$$

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Simplify[%]
True
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When $m=4$

$$G_1[4]\, G_{5-1}[4] == G_5[4] + \sum_{t=2}^{5-1} (-1)^t G_{5-t}[4]\, H_t[4] + (-1)^5 5 H_5[4]$$

$$\begin{aligned}
&(a_1 + a_2 + a_3 + a_4)\left(a_1^4 + a_2^4 + a_3^4 + a_4^4\right) == a_1^5 + a_2^5 + a_3^5 + a_4^5 + \\
&\quad a_1 a_2 a_3 a_4 (a_1 + a_2 + a_3 + a_4) - (a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4)\left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right) + \\
&\quad (a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4)\left(a_1^3 + a_2^3 + a_3^3 + a_4^3\right)
\end{aligned}$$

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Simplify[%]
True
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5.2 Split of Powers of Series (unified notation)

Theorem 5.2.1

When n is natural number s.t. $n \geq 2$, m is natural number, and a_{r1} is real number, the following holds.

$$\left(\sum_{r_1=1}^{m} a_{r_1}\right)^n = \sum_{r_1=1}^{m} a_r^n + 2\left(\sum_{r_1=1}^{m} a_{r_1}\right)^{n-2} H_2(m)$$

$$+\sum_{s=0}^{n-3}\left(\sum_{r_1=1}^{m} a_{r_1}\right)^s\left(\sum_{t=2}^{n-s-1}(-1)^t\left(\sum_{r_1=1}^{m} a_{r_1}^{n-s-t}\right)H_t(m)+(-1)^{n-s}(n-s)H_{n-s}(m)\right) \quad (2.1_n)$$

Where,

$$H_2(m) = \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m} a_{r_1}a_{r_2}$$

$$H_3(m) = \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m} a_{r_1}a_{r_2}a_{r_3}$$

$$\vdots$$

$$H_n(m) = \sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m}\sum_{r_3=r_2+1}^{m}\cdots\sum_{r_n=r_{n-1}+1}^{m} a_{r_1}a_{r_2}a_{r_3}\cdots a_{r_n}$$

When $n \leq 2$, the 2 nd term of (2.1_n) is ignored.

Proof

Let a power polynomial as follows.

$$G_n(m) = \sum_{r_1=1}^{m} a_{r_1}^n$$

Furthermore, abbreviate $G_n(m)$, $H_n(m)$ as G_n , H_n respectively. Then,

$$\left(\sum_{r_1=1}^{m} a_{r_1}\right)^2 = \sum_{r_1=1}^{m} a_{r_1}^2 + 2\sum_{r_1=1}^{m}\sum_{r_2=r_1+1}^{m} a_{r_1}a_{r_2}$$

is represented as follows

$$G_1^2 = G_2 + 2H_2 \quad (2.1_2')$$

Multiplying both sides by G_1 ,

$$G_1^3 = G_1G_{3-1} + 2G_1H_2$$

From Lemma 5.1.1 ,

$$G_1G_{3-1} = G_3 + \sum_{t=2}^{3-1}(-1)^t G_{3-t}H_t + (-1)^3 3H_3$$

Substituting this for the first term on the right side,

$$G_1^3 = G_3 + G_1^0\left(\sum_{t=2}^{3-1}(-1)^t G_{3-t}H_t + (-1)^3 3H_3\right) + 2G_1H_2 \quad (2.1_3')$$

Multiplying both sides by G_1 ,

$$G_1^4 = G_1G_{4-1} + G_1^1\left(\sum_{t=2}^{3-1}(-1)^t G_{3-t}H_t + (-1)^3 3H_3\right) + 2G_1^2H_2$$

From Lemma 5.1.1 ,

$$G_1G_{4-1} = G_4 + \sum_{t=2}^{4-1}(-1)^t G_{4-t}H_t + (-1)^4 4\,H_4$$

Substituting this for the first term on the right side,

$$G_1^4 = G_4 + G_1^0\left(\sum_{t=2}^{4-1}(-1)^t G_{4-t}H_t + (-1)^4 4\,H_4\right) + G_1^1\left(\sum_{t=2}^{3-1}(-1)^t G_{3-t}H_t + (-1)^3 3\,H_3\right) + 2G_1^2H_2 \qquad (2.1_4')$$

Multiplying both sides by G_1 ,

$$G_1^5 = G_1G_{5-1} + G_1^1\left(\sum_{t=2}^{4-1}(-1)^t G_{4-t}H_t + (-1)^4 4\,H_4\right) + G_1^2\left(\sum_{t=2}^{3-1}(-1)^t G_{3-t}H_t + (-1)^3 3\,H_3\right) + 2G_1^3H_2$$

From Lemma 5.1.1 ,

$$G_1^5 = G_1G_{5-1} + G_1^1\left(\sum_{t=2}^{4-1}(-1)^t G_{4-t}H_t + (-1)^4 4\,H_4\right)$$

Substituting this for the first term on the right side,

$$G_1^5 = G_5 + G_1^0\left(\sum_{t=2}^{5-1}(-1)^t G_{5-t}H_t + (-1)^5 5\,H_5\right) + G_1^1\left(\sum_{t=2}^{4-1}(-1)^t G_{4-t}H_t + (-1)^4 4\,H_4\right) + G_1^2\left(\sum_{t=2}^{3-1}(-1)^t G_{3-t}H_t + (-1)^3 3\,H_3\right) + 2G_1^3H_2 \qquad (2.1_5')$$

⋮

The above can be unified as follows:

$$G_1^n = G_n + \sum_{s=0}^{n-3} G_1^s\left(\sum_{t=2}^{n-1-s}(-1)^t G_{n-s-t}H_t + (-1)^{n-s}(n-s)H_{n-s}\right) + 2G_1^{n-2}H_2 \qquad (2.1_n')$$

And, putting (m) back in place,

$$\left(\sum_{r_1=1}^{m} a_{r_1}\right)^n = \sum_{r_1=1}^{m} a_r^n + 2\left(\sum_{r_1=1}^{m} a_{r_1}\right)^{n-2} H_2(m) + \sum_{s=0}^{n-3}\left(\sum_{r_1=1}^{m} a_{r_1}\right)^s\left(\sum_{t=2}^{n-s-1}(-1)^t\left(\sum_{r_1=1}^{m} a_{r_1}^{n-s-t}\right)H_t(m) + (-1)^{n-s}(n-s)H_{n-s}(m)\right) \qquad (2.1_n)$$

(2.1_n) can be obtained by *Mathematica*. If you input the top two lines, the bottom two lines will be output.

$$G_1[m]^n == G_n[m] + \sum_{s=0}^{n-3} G_1[m]^s\left(\sum_{t=2}^{n-1-s}(-1)^t G_{n-s-t}[m]\,H_t[m] + (-1)^{n-s}(n-s)\,H_{n-s}[m]\right) + 2\,G_1[m]^{n-2}H_2[m]$$

$$\left(\sum_{r_1=1}^{m} a_{r_1}\right)^n == \sum_{r_1=1}^{m} a_{r_1}^n + \sum_{s=0}^{-3+n}\left(\sum_{r_1=1}^{m} a_{r_1}\right)^s\left(\sum_{t=2}^{-1+n-s}(-1)^t\left(\sum_{r_1=1}^{m} a_{r_1}^{n-s-t}\right)H_t[m] + (-1)^{n-s}(n-s)\,H_{n-s}[m]\right) + 2\left(\sum_{r_1=1}^{m} a_{r_1}\right)^{-2+n}\sum_{r_1=1}^{m}\sum_{r_2=1+r_1}^{m} a_{r_1}a_{r_2}$$

Example $n=5$ **,** $m=4$

$$\left(\sum_{r_1=1}^{4} a_{r_1}\right)^5 = \sum_{r_1=1}^{4} a_r^5 + 2\left(\sum_{r_1=1}^{4} a_{r_1}\right)^{5-2} H_2(4)$$

$$+\sum_{s=0}^{5-3}\left(\sum_{r_1=1}^{4} a_{r_1}\right)^s \left(\sum_{t=2}^{5-s-1}(-1)^t\left(\sum_{r_1=1}^{4} a_{r_1}^{5-s-t}\right)H_t(4) + (-1)^{5-s}(5-s)H_{5-s}(4)\right)$$

When this is calculated using (2.1n') , it is as follows.

$$G_1[4]^5 == G_5[4] + 2\,G_1[4]^{5-2}\,H_2[4] + \sum_{s=0}^{5-3} G_1[4]^s \left(\sum_{t=2}^{5-1-s} (-1)^t\, G_{5-s-t}[4]\, H_t[4] + (-1)^{5-s}\,(5-s)\, H_{5-s}[4]\right)$$

$$(a_1+a_2+a_3+a_4)^5 == a_1^5+a_2^5+a_3^5+a_4^5+a_1 a_2 a_3 a_4\,(a_1+a_2+a_3+a_4) +$$
$$2\,(a_1+a_2+a_3+a_4)^3\,(a_1 a_2+a_1 a_3+a_2 a_3+a_1 a_4+a_2 a_4+a_3 a_4) -$$
$$(a_1 a_2 a_3+a_1 a_2 a_4+a_1 a_3 a_4+a_2 a_3 a_4)\,(a_1^2+a_2^2+a_3^2+a_4^2) +$$
$$(a_1 a_2+a_1 a_3+a_2 a_3+a_1 a_4+a_2 a_4+a_3 a_4)\,(a_1^3+a_2^3+a_3^3+a_4^3) +$$
$$(a_1+a_2+a_3+a_4)^2\,((a_1+a_2+a_3+a_4)\,(a_1 a_2+a_1 a_3+a_2 a_3+a_1 a_4+a_2 a_4+a_3 a_4) -$$
$$3\,(a_1 a_2 a_3+a_1 a_2 a_4+a_1 a_3 a_4+a_2 a_3 a_4)) +$$
$$(a_1+a_2+a_3+a_4)\,\big(4\,a_1 a_2 a_3 a_4-(a_1+a_2+a_3+a_4)\,(a_1 a_2 a_3+a_1 a_2 a_4+a_1 a_3 a_4+a_2 a_3 a_4) +$$
$$(a_1 a_2+a_1 a_3+a_2 a_3+a_1 a_4+a_2 a_4+a_3 a_4)\,(a_1^2+a_2^2+a_3^2+a_4^2)\big)$$

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Simplify[%]
True
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As a corollary of Theorem 5.2.1 , we obtain the following. However, since this is the original purpose, we will make it a separate theorem.

Theorem 5.2.2

When n is natural number s.t. $n \geq 2$, for a convergent infinite series, the following holds.

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^n = \sum_{r_1=1}^{\infty} a_r^n + 2\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^{n-2} H_2$$

$$+\sum_{s=0}^{n-3}\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^s \left(\sum_{t=2}^{n-s-1}(-1)^t\left(\sum_{r_1=1}^{\infty} a_{r_1}^{n-s-t}\right)H_t + (-1)^{n-s}(n-s)H_{n-s}\right) \qquad (2.2_n)$$

Where,

$$H_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} a_{r_1} a_{r_2}$$

$$H_3 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$

$$\vdots$$

$$H_n = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} a_{r_1} a_{r_2} a_{r_3}\cdots a_{r_n}$$

When $n \leq 2$, the 2 nd term of (2.2_n) is ignored.

Derivation

In $(2.1_n')$, replace $[m]$ with $[\infty]$ using *Mathematica*. Then, the lower two lines are output for the upper two lines.

$$G_1[\infty]^n == G_n[\infty] + \sum_{s=0}^{n-3} G_1[\infty]^s \left(\sum_{t=2}^{n-1-s} (-1)^t G_{n-s-t}[\infty] H_t[\infty] + (-1)^{n-s} (n-s) H_{n-s}[\infty] \right) + 2 G_1[\infty]^{n-2} H_2[\infty]$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^n == \sum_{r_1=1}^{\infty} a_{r_1}^n + \sum_{s=0}^{-3+n} \left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^s \left(\sum_{t=2}^{-1+n-s} (-1)^t \left(\sum_{r_1=1}^{\infty} a_{r_1}^{n-s-t} \right) H_t[\infty] + (-1)^{n-s} (n-s) H_{n-s}[\infty] \right) + 2 \left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^{-2+n} \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2}$$

Thus, we obtain (2.2_n) . Q.E.D.

First expantion

The first expansion of Theorem 5.2.2 for $n=6$ is as follows.

$$\begin{aligned} \left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^6 == \sum_{r_1=1}^{\infty} a_{r_1}^6 &+ \left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^0 \left(\sum_{t=2}^{6-1} (-1)^t H_t \sum_{r_1=1}^{\infty} a_{r_1}^{6-t} + (-1)^6 6 H_6 \right) \\ &+ \left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^1 \left(\sum_{t=2}^{5-1} (-1)^t H_t \sum_{r_1=1}^{\infty} a_{r_1}^{5-t} + (-1)^5 5 H_5 \right) \\ &+ \left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^2 \left(\sum_{t=2}^{4-1} (-1)^t H_t \sum_{r_1=1}^{\infty} a_{r_1}^{4-t} + (-1)^4 4 H_4 \right) \\ &+ \left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^3 \left(\sum_{t=2}^{3-1} (-1)^t H_t \sum_{r_1=1}^{\infty} a_{r_1}^{3-t} + (-1)^3 3 H_3 \right) + 2 \left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^4 H_2 \end{aligned}$$

This is a unified notation that looks neat, but it is still a little difficult to understand unless we expand Σ with the subscript t . Therefore, in the next section, we will do a full expansion.

5.3 Split of Powers of Series (specific notation)

In this section, we fully expand the unified notation in the previous section and provide specific notations.

Formula 5.3.1

For a convergent infinite series, the following holds.

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 = \sum_{r_1=1}^{\infty} a_{r_1}^2 + 2\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2}$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 = \sum_{r_1=1}^{\infty} a_{r_1}^3 + 3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2} - 3\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^4 = \sum_{r_1=1}^{\infty} a_{r_1}^4 + \left(3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 + \sum_{r_1=1}^{\infty} a_{r_1}^2\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2} - 4\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$
$$+ 4\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4}$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^5 = \sum_{r_1=1}^{\infty} a_{r_1}^5 + \left(3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty} a_{r_1}^2 + \sum_{r_1=1}^{\infty} a_{r_1}^3\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2}$$
$$- \left(4\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 + \sum_{r_1=1}^{\infty} a_{r_1}^2\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$
$$+ 5\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4}$$
$$- 5\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty}\sum_{r_5=1+r_4}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5}$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^6 = \sum_{r_1=1}^{\infty} a_{r_1}^6 + \left(3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^4 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2\sum_{r_1=1}^{\infty} a_{r_1}^2 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty} a_{r_1}^3 + \sum_{r_1=1}^{\infty} a_{r_1}^4\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2}$$
$$- \left(4\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty} a_{r_1}^2 + \sum_{r_1=1}^{\infty} a_{r_1}^3\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$
$$+ \left(5\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 + \sum_{r_1=1}^{\infty} a_{r_1}^2\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4}$$
$$- 6\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty}\sum_{r_5=1+r_4}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5}$$
$$+ 6\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty}\sum_{r_5=1+r_4}^{\infty}\sum_{r_6=1+r_5}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5} a_{r_6}$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^7 = \sum_{r_1=1}^{\infty} a_{r_1}^7$$
$$+ \left(3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^5 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3\sum_{r_1=1}^{\infty} a_{r_1}^2 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2\sum_{r_1=1}^{\infty} a_{r_1}^3 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty} a_{r_1}^4 + \sum_{r_1=1}^{\infty} a_{r_1}^5\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2}$$

$$- \left(4\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^4 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 \sum_{r_1=1}^{\infty} a_{r_1}^2 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} a_{r_1}^3 + \sum_{r_1=1}^{\infty} a_{r_1}^4\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$

$$+ \left(5\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} a_{r_1}^2 + \sum_{r_1=1}^{\infty} a_{r_1}^3\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4}$$

$$- \left(6\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 + \sum_{r_1=1}^{\infty} a_{r_1}^2\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} \sum_{r_5=1+r_4}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5}$$

$$+ 7\left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} \sum_{r_5=1+r_4}^{\infty} \sum_{r_6=1+r_5}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5} a_{r_6}$$

$$- 7 \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} \sum_{r_5=1+r_4}^{\infty} \sum_{r_6=1+r_5}^{\infty} \sum_{r_7=1+r_6}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5} a_{r_6} a_{r_7}$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^8 = \sum_{r_1=1}^{\infty} a_{r_1}^8 + \left(3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^6 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^4 \sum_{r_1=1}^{\infty} a_{r_1}^2 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 \sum_{r_1=1}^{\infty} a_{r_1}^3\right.$$

$$\left. + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 \sum_{r_1=1}^{\infty} a_{r_1}^4 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} a_{r_1}^5 + \sum_{r_1=1}^{\infty} a_{r_1}^6\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2}$$

$$- \left(4\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^5 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 \sum_{r_1=1}^{\infty} a_{r_1}^2 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 \sum_{r_1=1}^{\infty} a_{r_1}^3 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} a_{r_1}^4 + \sum_{r_1=1}^{\infty} a_{r_1}^5\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$

$$+ \left(5\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^4 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 \sum_{r_1=1}^{\infty} a_{r_1}^2 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} a_{r_1}^3 + \sum_{r_1=1}^{\infty} a_{r_1}^4\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4}$$

$$- \left(6\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} a_{r_1}^2 + \sum_{r_1=1}^{\infty} a_{r_1}^3\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} \sum_{r_5=1+r_4}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5}$$

$$+ \left(7\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 + \sum_{r_1=1}^{\infty} a_{r_1}^2\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} \sum_{r_5=1+r_4}^{\infty} \sum_{r_6=1+r_5}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5} a_{r_6}$$

$$- 8\left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} \sum_{r_5=1+r_4}^{\infty} \sum_{r_6=1+r_5}^{\infty} \sum_{r_7=1+r_6}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5} a_{r_6} a_{r_7}$$

$$+ 8 \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} \sum_{r_5=1+r_4}^{\infty} \sum_{r_6=1+r_5}^{\infty} \sum_{r_7=1+r_6}^{\infty} \sum_{r_8=1+r_7}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5} a_{r_6} a_{r_7} a_{r_8}$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^9 = \sum_{r_1=1}^{\infty} a_{r_1}^9 + \left(3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^7 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^5 \sum_{r_1=1}^{\infty} a_{r_1}^2 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^4 \sum_{r_1=1}^{\infty} a_{r_1}^3 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 \sum_{r_1=1}^{\infty} a_{r_1}^4\right.$$

$$\left. + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 \sum_{r_1=1}^{\infty} a_{r_1}^5 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} a_{r_1}^6 + \sum_{r_1=1}^{\infty} a_{r_1}^7\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2}$$

$$- \left(4\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^6 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^4 \sum_{r_1=1}^{\infty} a_{r_1}^2 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 \sum_{r_1=1}^{\infty} a_{r_1}^3\right.$$

$$\left. + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 \sum_{r_1=1}^{\infty} a_{r_1}^4 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} a_{r_1}^5 + \sum_{r_1=1}^{\infty} a_{r_1}^6\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$

$$+ \left(5\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^5 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 \sum_{r_1=1}^{\infty} a_{r_1}^2 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 \sum_{r_1=1}^{\infty} a_{r_1}^3\right.$$

$$\left. + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right) \sum_{r_1=1}^{\infty} a_{r_1}^4 + \sum_{r_1=1}^{\infty} a_{r_1}^5\right) \sum_{r_1=1}^{\infty} \sum_{r_2=1+r_1}^{\infty} \sum_{r_3=1+r_2}^{\infty} \sum_{r_4=1+r_3}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4}$$

$$
-\left(6\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^4+\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2\sum_{r_1=1}^{\infty} a_{r_1}^2+\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty} a_{r_1}^3+\sum_{r_1=1}^{\infty} a_{r_1}^4\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty}\sum_{r_5=1+r_4}^{\infty} a_{r_1}a_{r_2}a_{r_3}a_{r_4}a_{r_5}
$$

$$
+\left(7\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3+\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty} a_{r_1}^2+\sum_{r_1=1}^{\infty} a_{r_1}^3\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty}\sum_{r_5=1+r_4}^{\infty}\sum_{r_6=1+r_5}^{\infty} a_{r_1}a_{r_2}a_{r_3}a_{r_4}a_{r_5}a_{r_6}
$$

$$
-\left(8\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2+\sum_{r_1=1}^{\infty} a_{r_1}^2\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty}\sum_{r_5=1+r_4}^{\infty}\sum_{r_6=1+r_5}^{\infty}\sum_{r_7=1+r_6}^{\infty} a_{r_1}a_{r_2}a_{r_3}a_{r_4}a_{r_5}a_{r_6}a_{r_7}
$$

$$
+9\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty}\sum_{r_5=1+r_4}^{\infty}\sum_{r_6=1+r_5}^{\infty}\sum_{r_7=1+r_6}^{\infty}\sum_{r_8=1+r_7}^{\infty} a_{r_1}a_{r_2}a_{r_3}a_{r_4}a_{r_5}a_{r_6}a_{r_7}a_{r_8}
$$

$$
-9\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty}\sum_{r_5=1+r_4}^{\infty}\sum_{r_6=1+r_5}^{\infty}\sum_{r_7=1+r_6}^{\infty}\sum_{r_8=1+r_7}^{\infty}\sum_{r_9=1+r_8}^{\infty} a_{r_1}a_{r_2}a_{r_3}a_{r_4}a_{r_5}a_{r_6}a_{r_7}a_{r_8}a_{r_9}
$$

Derivation

We use *Mathematica* to prevent mistakes. Assume $G_n(m)$, $H_n(m)$ are defined as shown on page 2 .

$n=2$

Substitute $n=2$ for (2.1_n) in Theorem 5.2.1 and remove $[m]$. The 1 st line is the input and the 2 nd line is the output.

$$
G_1{}^2 == G_2+\sum_{s=0}^{2-3} G_1{}^s\left(\sum_{t=2}^{2-1-s}(-1)^t G_{2-s-t} H_t+(-1)^{2-s}(2-s) H_{2-s}\right)+2 G_1{}^{2-2} H_2
$$

$$
G_1^2 == G_2+2 H_2
$$

Attach $[2]$ to this and verify, then

$$
G_1[2]^2 == G_2[2]+2 H_2[2]
$$

$$
(a_1+a_2)^2 == a_1^2+2 a_1 a_2+a_2^2
$$

```
Simplify[%]
True
```

Replacing $[2]$ with $[\infty]$,

$$
G_1[\infty]^2 == G_2[\infty]+2 H_2[\infty]
$$

$$
\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 == \sum_{r_1=1}^{\infty} a_{r_1}^2+2\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2}
$$

$n=3$

Substitute $n=3$ for (2.1_n) and remove $[m]$. Then,

$$
G_1{}^3 == G_3+\sum_{s=0}^{3-3} G_1{}^s\left(\sum_{t=2}^{3-1-s}(-1)^t G_{3-s-t} H_t+(-1)^{3-s}(3-s) H_{3-s}\right)+2 G_1{}^{3-2} H_2
$$

$$
G_1^3 == G_3+3 G_1 H_2-3 H_3
$$

Attach $[3]$ to this and verify, then

$$
G_1[3]^3 == G_3[3]+3 G_1[3] H_2[3]-3 H_3[3]
$$

$$
(a_1+a_2+a_3)^3 == a_1^3+a_2^3-3 a_1 a_2 a_3+a_3^3+3(a_1+a_2+a_3)(a_1 a_2+a_1 a_3+a_2 a_3)
$$

```
Simplify[%]
True
```

Replacing $[3]$ with $[\infty]$,

$$G_1[\infty]^3 == G_3[\infty] + 3\,G_1[\infty]\,H_2[\infty] - 3\,H_3[\infty]$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 == \sum_{r_1=1}^{\infty} a_{r_1}^3 + 3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty} a_{r_1}\,a_{r_2} - 3\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty} a_{r_1}\,a_{r_2}\,a_{r_3}$$

$n=4$

Substitute $n=4$ for (2.1_n) and remove $[m]$. Then,

$$G_1{}^4 == G_4 + \sum_{s=0}^{4-3} G_1{}^s\left(\sum_{t=2}^{4-1-s}(-1)^t\,G_{4-s-t}\,H_t + (-1)^{4-s}\,(4-s)\,H_{4-s}\right) + 2\,G_1{}^{4-2}\,H_2$$

$$G_1^4 == G_4 + 2\,G_1^2\,H_2 + G_2\,H_2 + G_1\,(G_1\,H_2 - 3\,H_3) - G_1\,H_3 + 4\,H_4$$

Expand[%]

$$G_1^4 == G_4 + 3\,G_1^2\,H_2 + G_2\,H_2 - 4\,G_1\,H_3 + 4\,H_4$$

Summarizing with respect to H_2,

$$G_1^4 == G_4 + \left(3\,G_1^2 + G_2\right)H_2 - 4\,G_1\,H_3 + 4\,H_4$$

Attach $[4]$ to this and verify, then

$$G_1[4]^4 == G_4[4] + \left(3\,G_1[4]^2 + G_2[4]\right)H_2[4] - 4\,G_1[4]\,H_3[4] + 4\,H_4[4]$$

$$(a_1 + a_2 + a_3 + a_4)^4 ==$$
$$a_1^4 + a_2^4 + a_3^4 + 4\,a_1\,a_2\,a_3\,a_4 + a_4^4 - 4\,(a_1 + a_2 + a_3 + a_4)\,(a_1\,a_2\,a_3 + a_1\,a_2\,a_4 + a_1\,a_3\,a_4 + a_2\,a_3\,a_4) +$$
$$(a_1\,a_2 + a_1\,a_3 + a_2\,a_3 + a_1\,a_4 + a_2\,a_4 + a_3\,a_4)\left(a_1^2 + a_2^2 + a_3^2 + a_4^2 + 3\,(a_1 + a_2 + a_3 + a_4)^2\right)$$

Simplify[%]

True

Replacing $[4]$ with $[\infty]$,

$$G_1[\infty]^4 == G_4[\infty] + \left(3\,G_1[\infty]^2 + G_2[\infty]\right)H_2[\infty] - 4\,G_1[\infty]\,H_3[\infty] + 4\,H_4[\infty]$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^4 == \sum_{r_1=1}^{\infty} a_{r_1}^4 + \left(3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 + \sum_{r_1=1}^{\infty} a_{r_1}^2\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty} a_{r_1}\,a_{r_2} -$$
$$4\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty} a_{r_1}\,a_{r_2}\,a_{r_3} + 4\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty} a_{r_1}\,a_{r_2}\,a_{r_3}\,a_{r_4}$$

$n=5$

Substitute $n=5$ for (2.1_n) and remove $[m]$. Then,

$$G_1{}^5 == G_5 + \sum_{s=0}^{5-3} G_1{}^s\left(\sum_{t=2}^{5-1-s}(-1)^t\,G_{5-s-t}\,H_t + (-1)^{5-s}\,(5-s)\,H_{5-s}\right) + 2\,G_1{}^{5-2}\,H_2$$

$$G_1^5 == G_5 + 2\,G_1^3\,H_2 + G_3\,H_2 + G_1^2\,(G_1\,H_2 - 3\,H_3) - G_2\,H_3 + G_1\,H_4 + G_1\,(G_2\,H_2 - G_1\,H_3 + 4\,H_4) - 5\,H_5$$

Expand[%]

$$G_1^5 == G_5 + 3\,G_1^3\,H_2 + G_1\,G_2\,H_2 + G_3\,H_2 - 4\,G_1^2\,H_3 - G_2\,H_3 + 5\,G_1\,H_4 - 5\,H_5$$

Summarizing with respect to H_2, H_3,

$$G_1^5 == G_5 + \left(3 G_1^3 + G_1 G_2 + G_3\right) H_2 - \left(4 G_1^2 + G_2\right) H_3 + 5 G_1 H_4 - 5 H_5$$

Attach [5] to this and verify, then

$$G_1[5]^5 == G_5[5] + \left(3 G_1[5]^3 + G_1[5] G_2[5] + G_3[5]\right) H_2[5]$$
$$- \left(4 G_1[5]^2 + G_2[5]\right) H_3[5] + 5 G_1[5] H_4[5] - 5 H_5[5]$$

$$(a_1 + a_2 + a_3 + a_4 + a_5)^5 == a_1^5 + a_2^5 + a_3^5 + a_4^5 - 5 a_1 a_2 a_3 a_4 a_5 + a_5^5 +$$
$$5 (a_1 + a_2 + a_3 + a_4 + a_5) (a_1 a_2 a_3 a_4 + a_1 a_2 a_3 a_5 + a_1 a_2 a_4 a_5 + a_1 a_3 a_4 a_5 + a_2 a_3 a_4 a_5) -$$
$$(a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4 + a_1 a_2 a_5 + a_1 a_3 a_5 + a_2 a_3 a_5 + a_1 a_4 a_5 + a_2 a_4 a_5 + a_3 a_4 a_5)$$
$$\left(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + 4 (a_1 + a_2 + a_3 + a_4 + a_5)^2\right) +$$
$$(a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4 + a_1 a_5 + a_2 a_5 + a_3 a_5 + a_4 a_5)$$
$$\left(a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 + 3 (a_1 + a_2 + a_3 + a_4 + a_5)^3 + (a_1 + a_2 + a_3 + a_4 + a_5) \left(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2\right)\right)$$

```
Simplify[%]
True
```

Replacing [5] with [∞] ,

$$G_1[\infty]^5 == G_5[\infty] + \left(3 G_1[\infty]^3 + G_1[\infty] G_2[\infty] + G_3[\infty]\right) H_2[\infty]$$
$$- \left(4 G_1[\infty]^2 + G_2[\infty]\right) H_3[\infty] + 5 G_1[\infty] H_4[\infty] - 5 H_5[\infty]$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^5 == \sum_{r_1=1}^{\infty} a_{r_1}^5 + \left(3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 + \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty} a_{r_1}^2 + \sum_{r_1=1}^{\infty} a_{r_1}^3\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty} a_{r_1} a_{r_2} -$$
$$\left(4\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2 + \sum_{r_1=1}^{\infty} a_{r_1}^2\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty} a_{r_1} a_{r_2} a_{r_3} +$$
$$5\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} - 5\sum_{r_1=1}^{\infty}\sum_{r_2=1+r_1}^{\infty}\sum_{r_3=1+r_2}^{\infty}\sum_{r_4=1+r_3}^{\infty}\sum_{r_5=1+r_4}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5}$$

n=6～

In a similar way to $n=4,5$, $n=6\sim 9$ are obtained . It is also possible to calculate $n=10\sim$, but this will not be entered on paper, so we have omitted it.

Note

A feature of Formula 5.3.1 is that it is completely splitted into power series and semi-multiple series. This is thought to be the reason why Formula 5.3.1 was derived in a unified manner from Theorem 5.2.1 . Furthermore, Formula 5.3.1 itself could be written in a unified way, but this would seem cumbersome.

Other Exprssions

As mentioned at the beginning of this chapter, Formula 5.3.1 is not the only expression. There are countless other expressions, many of which have incomplete or excessive divisions into power series and semi-multiple series. For example,

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^3 = -2\sum_{r_1=1}^{\infty} a_{r_1}^3 + 3\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)\sum_{r_1=1}^{\infty} a_{r_1}^2 + 6\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^4 = 2\sum_{r_1=1}^{\infty} a_r^4 - \left(\sum_{r_1=1}^{\infty} a_{r_1}^2\right)^2 + 4\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^2\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} a_{r_1} a_{r_2}$$

$$- \left(\sum_{r_1=1}^{\infty} a_{r_1} \right) \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} a_{r_1} a_{r_2} a_{r_3} + 8 \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4}$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^5 = 6 \sum_{r=1}^{\infty} a_r^5 - 15 \left(\sum_{r_1=1}^{\infty} a_{r_1} \right) \sum_{r_1=1}^{\infty} a_{r_1}^4 + 10 \left(\sum_{r_1=1}^{\infty} a_{r_1}^2 \right) \sum_{r_1=1}^{\infty} a_{r_1}^3$$

$$+ 20 \left(\sum_{r_1=1}^{\infty} a_{r_1}^3 \right) \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} a_{r_1} a_{r_2}$$

$$+ 30 \left(\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} a_{r_1} a_{r_2} \right) \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$

$$+ 30 \left(\sum_{r_1=1}^{\infty} a_{r_1} \right) \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4}$$

$$+ 30 \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} \sum_{r_5=r_4+1}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5}$$

$$\left(\sum_{r_1=1}^{\infty} a_{r_1} \right)^6 = 10 \left(\sum_{r_1=1}^{\infty} a_{r_1}^3 \right)^2 + 15 \left(\sum_{r_1=1}^{\infty} a_{r_1}^2 \right) \sum_{r_1=1}^{\infty} a_{r_1}^4 - 24 \left(\sum_{r_1=1}^{\infty} a_{r_1} \right) \sum_{r_1=1}^{\infty} a_{r_1}^5$$

$$+ 30 \left(\sum_{r_1=1}^{\infty} a_{r_1}^4 \right) \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} a_{r_1} a_{r_2} - 90 \left(\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} a_{r_1} a_{r_2} a_{r_3} \right)^2$$

$$+ 60 \left(\sum_{r_1=1}^{\infty} a_{r_1} \right) \left(\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} a_{r_1} a_{r_2} \right) \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} a_{r_1} a_{r_2} a_{r_3}$$

$$- 60 \left(\sum_{r_1=1}^{\infty} a_{r_1}^2 \right) \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4}$$

$$- 120 \left(\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} a_{r_1} a_{r_2} \right) \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4}$$

$$+ 120 \left(\sum_{r_1=1}^{\infty} a_{r_1} \right) \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} \sum_{r_5=r_4+1}^{\infty} a_{r_1} a_{r_2} a_{r_3} a_{r_4} a_{r_5}$$

5.4 Fast Calculation Method for Semi Multiple Series

In this section, we present a fast calculation method that replaces the time-consuming calculation of semi-multiple series with polynomials consisting of power series.

Formula 5.4.1 (Recursion)

When n is natural number s.t. $n \ge 2$, for a convergent infinite series, the following holds.

$$H_2 = \frac{1}{2}\left(G_1^2 - G_2\right) \tag{4.2}$$

$$H_n = \frac{(-1)^n}{n}\left(G_1^n - G_n - \sum_{s=0}^{n-3}\sum_{t=2}^{n-1-s}(-1)^t G_1^s G_{n-s-t} H_t - \sum_{s=1}^{n-3}(-1)^{n-s} G_1^s (n-s) H_{n-s} - 2G_1^{n-2}H_2\right) \quad n \ge 3 \tag{4.n}$$

Where,

$$G_n = \left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^n$$

$$H_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} a_{r_1}a_{r_2}$$

$$H_3 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} a_{r_1}a_{r_2}a_{r_3}$$

$$\vdots$$

$$H_n = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} a_{r_1}a_{r_2}a_{r_3}\cdots a_{r_n}$$

Derivation

Rewriting Theorem 5.2.2 using the symbols G_n, H_n,

When $n = 2$

$$G_1^2 = G_2 + 2H_2 \quad \Longrightarrow \quad H_2 = \left(G_1^2 - G_2\right)/2$$

When $n \ge 3$

$$G_1^n = G_n + \sum_{s=0}^{n-3} G_1^s \left(\sum_{t=2}^{n-1-s}(-1)^t G_{n-s-t}H_t + (-1)^{n-s}(n-s)H_{n-s}\right) + 2G_1^{n-2}H_2 \tag{2.1n'}$$

Splitting this as follows,

$$G_1^n = G_n + \sum_{s=0}^{n-3} G_1^s \sum_{t=2}^{n-1-s}(-1)^t G_{n-s-t}H_t + \sum_{s=0}^{n-3} G_1^s(-1)^{n-s}(n-s)H_{n-s} + 2G_1^{n-2}H_2$$

$$= G_n + \sum_{s=0}^{n-3} G_1^s \sum_{t=2}^{n-1-s}(-1)^t G_{n-s-t}H_t + (-1)^n nH_n + \sum_{s=1}^{n-3} G_1^s(-1)^{n-s}(n-s)H_{n-s} + 2G_1^{n-2}H_2$$

From this,

$$(-1)^n nH_n = G_1^n - G_n - \sum_{s=0}^{n-3} G_1^s \sum_{t=2}^{n-1-s}(-1)^t G_{n-s-t}H_t - \sum_{s=1}^{n-3} G_1^s(-1)^{n-s}(n-s)H_{n-s} - 2G_1^{n-2}H_2$$

Dividing both sides by $(-1)^n n$,

$$H_n = \frac{(-1)^n}{n}\left(G_1^n - G_n - \sum_{s=0}^{n-3}\sum_{t=2}^{n-1-s}(-1)^t G_1^s G_{n-s-t} H_t - \sum_{s=1}^{n-3}(-1)^{n-s} G_1^s (n-s) H_{n-s} - 2G_1^{n-2}H_2\right) \tag{4.n}$$

Q.E.D.

Executing a Recursive Formula

Since these are recursive formulas, executing them using the mathematical processing software ***Mathematica*** the following results are immediately obtained.

```
Clear[G, H]
```

$$H_2 := \frac{1}{2}\left(G_1{}^2 - G_2\right)$$

$$H_{n\_} := \frac{(-1)^n}{n}\left(G_1{}^n - G_n - \sum_{s=0}^{n-3}\sum_{t=2}^{n-1-s}(-1)^t G_1{}^s G_{n-s-t} H_t - \sum_{s=1}^{n-3}(-1)^{n-s} G_1{}^s (n-s) H_{n-s} - 2 G_1{}^{n-2} H_2\right)$$

Simplify[H₂] $\quad \frac{1}{2}\left(G_1^2 - G_2\right)$

Simplify[H₃] $\quad \frac{1}{6}\left(G_1^3 - 3 G_1 G_2 + 2 G_3\right)$

Simplify[H₄] $\quad \frac{1}{24}\left(G_1^4 - 6 G_1^2 G_2 + 3 G_2^2 + 8 G_1 G_3 - 6 G_4\right)$

Simplify[H₅] $\quad \frac{1}{120}\left(G_1^5 - 10 G_1^3 G_2 + 20 G_1^2 G_3 - 20 G_2 G_3 + 15 G_1 \left(G_2^2 - 2 G_4\right) + 24 G_5\right)$

Simplify[H₆] $\quad \frac{1}{720}\left(G_1^6 - 15 G_1^4 G_2 - 15 G_2^3 + 40 G_1^3 G_3 + 45 G_1^2 \left(G_2^2 - 2 G_4\right) + 90 G_2 G_4 - 24 G_1 (5 G_2 G_3 - 6 G_5) + 40 \left(G_3^2 - 3 G_6\right)\right)$

Simplify[H₇] $\quad \frac{1}{5040}\left(G_1^7 - 21 G_1^5 G_2 + 70 G_1^4 G_3 + 105 G_1^3 \left(G_2^2 - 2 G_4\right) - 84 G_1^2 (5 G_2 G_3 - 6 G_5) - 35 G_1 \left(3 G_2^3 - 8 G_3^2 - 18 G_2 G_4 + 24 G_6\right) + 6 \left(35 G_2^2 G_3 - 70 G_3 G_4 - 84 G_2 G_5 + 120 G_7\right)\right)$

Simplify[H₈] $\quad \frac{1}{40320}\left(G_1^8 - 28 G_1^6 G_2 + 112 G_1^5 G_3 + 210 G_1^4 \left(G_2^2 - 2 G_4\right) - 224 G_1^3 (5 G_2 G_3 - 6 G_5) - 140 G_1^2 \left(3 G_2^3 - 8 G_3^2 - 18 G_2 G_4 + 24 G_6\right) + 48 G_1 \left(35 G_2^2 G_3 - 70 G_3 G_4 - 84 G_2 G_5 + 120 G_7\right) + 7 \left(15 G_2^4 - 180 G_2^2 G_4 - 160 G_2 \left(G_3^2 - 3 G_6\right) + 12 \left(15 G_4^2 + 32 G_3 G_5 - 60 G_8\right)\right)\right)$

Fast Calculation Method for Semi Multiple Series

Using these results, the calculation of semi-multiple series can be replaced by the calculation of power series polynomials. Although The computational amount of the half multiple series $H_m(n)$ is as large as ${}_nC_m$, the computational amount of the polynomial of the power series is small as seen above. Therefore, the effect of this substitution is enormous.

Example $a_{r_t} = 1/r_t^2$

The calculation for this example in ***Mathematica*** is as follows:

```
Clear[G, f, g]
```

$$G_{n\_}[m\_] := \sum_{r_1=1}^{m} \left(\frac{1}{r_1^2}\right)^n$$

(1) Semi-Triple Series

$$f3[m\_] := \sum_{r_1=1}^{m} \sum_{r_2=r_1+1}^{m} \sum_{r_3=r_2+1}^{m} \frac{1}{r_1^2 r_2^2 r_3^2}$$

$$g3[m\_] := \frac{1}{6}\left(G_1[m]^3 - 3 G_1[m] G_2[m] + 2 G_3[m]\right)$$

$f3(m)$ is a half triple series, and $g3(m)$] is the above H_3 transcribed with $[m]$ added.

When $m=1000$, the calculation results are as follows. The two are completely consistent.

```
N[f3[1000]]        N[g3[1000]]
0.189941           0.189941
```

The computational amount of $f3(1000)$ is ${}_{1000}C_3 = 166{,}167{,}000$, and the computational amount of $g3(1000)$ is estimated at 3000 . As a result, the calculation time for $f3(1000)$ was 10 minutes, and the calculation time for $g3(1000)$ was less than 1 second. FYI, my computer is Intel Core i7-9750H , 16GB.

(2) Semi-Quintuple Seriesi

$$f5[m\_] := \sum_{r_1=1}^{m} \sum_{r_2=r_1+1}^{m} \sum_{r_3=r_2+1}^{m} \sum_{r_4=r_3+1}^{m} \sum_{r_5=r_4+1}^{m} \frac{1}{r_1^2 r_2^2 r_3^2 r_4^2 r_5^2}$$

$$g5[m\_] := \frac{1}{120}\left(G_1[m]^5 - 10 G_1[m]^3 G_2[m] + 20 G_1[m]^2 G_3[m] - 20 G_2[m] G_3[m] + 15 G_1[m]\left(G_2[m]^2 - 2 G_4[m]\right) + 24 G_5[m]\right)$$

$f5(m)$ is a half quintuple series, and $g5(m)$] is the above H_5 transcribed with $[m]$ added.

When $m=150$, the calculation results are as follows. The two are completely consistent.

```
N[f5[150]]         N[g5[150]]
0.00217652         0.00217652
```

The computational amount of $f5(150)$ is ${}_{150}C_5 = 591{,}600{,}030$, and the computational amount of $g5(150)$ is estimated at $1{,}050\ (=150\times 7)$ As a result, the calculation time for $f5(150)$ was 61 minutes, and the calculation time for $g5(150)$ was less than 1 second.

Furthermore, $g5(5000)$ was calculated as follows. The calculation time was 1 second.

```
N[g5[5000]]     0.00234086
```

$f5(5000)$ cannot be calculated. The computational amount is ${}_{5000}C_5 = 25{,}989{,}619{,}781{,}251{,}000$. Since the calculation amount $591{,}600{,}030$ of $f5(150)$ took about one hour (61 minutes) , the calculation time for $f5(5000)$ becomes as follows:

$$25{,}989{,}619{,}781{,}251{,}000/591{,}600{,}030/24/365 \approx 5{,}015$$

That is, it would take $5{,}000$ years to calculate $f5(5000)$ on my computer.

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2025.02.02 Added Sec.4

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