13 Probability that the Riemann Hypothesis is false

Abstract

(1) The completed Riemann Zeta function $\xi(z)$ is expanded to Maclaurin series, and the values of coefficients A_r $r=1,2,3,\cdots$ are obtain.

(2) Present the Vieta's formula for the completed Riemann Zeta function $\xi(z)$.

(3) If the Riemann Hypothesis holds, then the semi-multiple series consisting of zeros on the critical line are equal to the polynomial of A_r .

(4) If the Riemann Hypothesis holds, then the power series consisting of zeros on the critical line are equal to the polynomial of A_r .

(5) Using (1) and (4), we calculate the probability that the Riemann Hypothesis is true, and obtain the probability that it is false.

13.1 Maclaurin Series of $\xi(z)$

The Maclaurin series of the completed Riemann Zeta function $\xi(z)$ is given in Theorem 9.1.3 of " **09 Maclaurin Series of Completed Riemann Zeta** ". This is rewritten in a slightly different form as follows.

Theorem 13.1.1

Let the completed Riemann zeta function $\xi(z)$ and the Maclaurin series be as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \sum_{r=0}^{\infty} A_r z^r \tag{1.0}$$

Then, these coefficients A_r $r=0,1,2,3,\cdots$ are given by

$$A_r = \sum_{s=0}^{r}\sum_{t=0}^{s} \frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{(-1)^{s-t}g_{s-t}(3/2)}{2^{s-t}(s-t)!}h_t \tag{1.a}$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \cdots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$h_r = \begin{cases} 1 & r=0 \\ -\dfrac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

Example

The first four of (1.a) are

$$A_0 = \frac{\log^0\pi}{2^0 0!}\frac{(-1)^0 g_0(3/2)}{2^0 0!}h_0 = 1$$

$$A_1 = \frac{\log^1\pi}{2^1 1!} - \frac{g_1(3/2)}{2^1 1!} - \frac{\gamma_0}{0!}$$

$$A_2 = \frac{\log^2\pi}{2^2 2!} + \frac{g_2(3/2)}{2^2 2!} - \frac{\gamma_1}{1!} - \frac{\log^1\pi}{2^1 1!}\frac{g_1(3/2)}{2^1 1!} + \frac{g_1(3/2)}{2^1 1!}\frac{\gamma_0}{0!} - \frac{\log^1\pi}{2^1 1!}\frac{\gamma_0}{0!}$$

$$A_3 = \frac{\log^3\pi}{2^3 3!} - \frac{g_3(3/2)}{2^3 3!} - \frac{\gamma_2}{2!} - \frac{\log^2\pi}{2^2 2!}\frac{g_1(3/2)}{2^1 1!} - \frac{\log^2\pi}{2^2 2!}\frac{\gamma_0}{0!} - \frac{g_2(3/2)}{2^2 2!}\frac{\gamma_0}{0!}$$
$$+ \frac{\log^1\pi}{2^1 1!}\frac{g_2(3/2)}{2^2 2!} - \frac{\log^1\pi}{2^1 1!}\frac{\gamma_1}{1!} + \frac{g_1(3/2)}{2^1 1!}\frac{\gamma_1}{1!} + \frac{\log^1\pi}{2^1 1!}\frac{g_1(3/2)}{2^1 1!}\frac{\gamma_0}{0!}$$

The contents of g_r are polygamma functions, so further expanded, it is as follows

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Clear[A, g, h, γ, ψ]
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$$A_{r\_} := \sum_{s=0}^{r}\sum_{t=0}^{s} \frac{\text{Log}[\pi]^{r-s}}{2^{r-s}(r-s)!}\frac{(-1)^{s-t}g_{s-t}[3/2]}{2^{s-t}(s-t)!}h_t$$

$$g_{r\_}\left[\frac{3}{2}\right] := \text{If}\left[r == 0, 1, \sum_{k=1}^{r} \text{BellY}\left[r, k, \text{Tbl}\psi\left[r, \frac{3}{2}\right]\right]\right]$$

$$h_{r\_} := \text{If}\left[r == 0, 1, -\frac{\gamma_{r-1}}{(r-1)!}\right]$$

$$\text{Tbl}\psi[r\_, z\_] := \text{Table}[\psi_k[z], \{k, 0, r-1\}]$$

$$A_0 \quad 1$$

$$A_1 \quad \frac{\text{Log}[\pi]}{2} - \gamma_0 - \frac{1}{2}\psi_0\left[\frac{3}{2}\right]$$

$$A_2 \quad \frac{\text{Log}[\pi]^2}{8} - \frac{1}{2}\text{Log}[\pi]\,\gamma_0 - \gamma_1 - \frac{1}{4}\text{Log}[\pi]\,\psi_0\left[\frac{3}{2}\right] + \frac{1}{2}\gamma_0\,\psi_0\left[\frac{3}{2}\right] + \frac{1}{8}\left(\psi_0\left[\frac{3}{2}\right]^2 + \psi_1\left[\frac{3}{2}\right]\right)$$

$$A_3 \quad \frac{\text{Log}[\pi]^3}{48} - \frac{1}{8}\text{Log}[\pi]^2\,\gamma_0 - \frac{1}{2}\text{Log}[\pi]\,\gamma_1 - \frac{\gamma_2}{2} - \frac{1}{16}\text{Log}[\pi]^2\,\psi_0\left[\frac{3}{2}\right] + \frac{1}{4}\text{Log}[\pi]\,\gamma_0\,\psi_0\left[\frac{3}{2}\right] + \frac{1}{2}\gamma_1\,\psi_0\left[\frac{3}{2}\right] + \frac{1}{16}\text{Log}[\pi]\left(\psi_0\left[\frac{3}{2}\right]^2 + \psi_1\left[\frac{3}{2}\right]\right) - \frac{1}{8}\gamma_0\left(\psi_0\left[\frac{3}{2}\right]^2 + \psi_1\left[\frac{3}{2}\right]\right) + \frac{1}{48}\left(-\psi_0\left[\frac{3}{2}\right]^3 - 3\,\psi_0\left[\frac{3}{2}\right]\psi_1\left[\frac{3}{2}\right] - \psi_2\left[\frac{3}{2}\right]\right)$$

Finally, these values are as follows.

$$\text{Clear}[A, g, h, \gamma, \psi]$$

$$A_{r\_} := \sum_{s=0}^{r}\sum_{t=0}^{s} \frac{\text{Log}[\pi]^{r-s}}{2^{r-s}\,(r-s)!}\,\frac{(-1)^{s-t}\,g_{s-t}[3/2]}{2^{s-t}\,(s-t)!}\,h_t$$

$$g_{r\_}\left[\frac{3}{2}\right] := \text{If}\left[r == 0, 1, \sum_{k=1}^{r} \text{BellY}\left[r, k, \text{Tbl}\psi\left[r, \frac{3}{2}\right]\right]\right]$$

$$h_{r\_} := \text{If}\left[r == 0, 1, -\frac{\gamma_{r-1}}{(r-1)!}\right]$$

$$\text{Tbl}\psi[r\_, z\_] := \text{Table}[\text{PolyGamma}[k, z], \{k, 0, r-1\}]$$

$$\gamma_{s\_} := \text{StieltjesGamma}[s]$$

$$\text{SetPrecision}[\{A_1, A_2, A_3, A_4, A_5\}, 14]$$

$$\{-0.0230957089661, 0.0233438645342, -0.0004979838499, 0.0002531817303, -5.0502548\times10^{-6}\}$$

$$\text{SetPrecision}[A_{16}, 100]$$

$$4.484340507245494493012998364541513049822570640735984986589156619586973608357503 75\times10^{-19}$$

13.2 Vieta's Formula in ξ(z)

13.2.1 Infinite-degree Equations and Vieta's formula

Reprinting Formula 3.2.1 from " **03 Vieta's formula in Infinite-degree Equation** " (Infinite-degree Equation) , it is as follows:

Formula 3.2.1 (Vieta's Formulas) (Reprint)

Assume that the function $f(z)$ on the complex plane has zeros $z_1, z_2, z_3, z_4, \cdots$ and is completely factored as follows.

$$f(z) = \left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\left(1-\frac{z}{z_3}\right)\left(1-\frac{z}{z_4}\right)\cdots$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \tag{2.0}$$

Where,

$$a_1 = -\sum_{r_1=1}^{\infty} \frac{1}{z_{r_1}}$$

$$a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{1}{z_{r_1} z_{r_2}}$$

$$a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{1}{z_{r_1} z_{r_2} z_{r_3}}$$

$$a_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} \frac{1}{z_{r_1} z_{r_2} z_{r_3} z_{r_3}}$$

$$\vdots$$

$$a_n = (-1)^n \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{z_{r_1} z_{r_2} \cdots z_{r_n}}$$

Using this formula, the following theorem can be proven.

Theorem 13.2.1 (Infinite-degree Equation with Conjugate Complex Roots)

Assume that the function $f(z)$ on the complex plane has zeros $z_k = x_k \pm i y_k$, $y_k \ne 0$ $(k=1,2,3,\cdots)$ and is completely factored as follows.

$$f(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right) = \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)$$

Then, $f(z)$ is expanded to a power series as follows.

$$f(z) = 1+a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \tag{2.1}$$

Where,

$$a_1 = -\sum_{r_1=1}^{\infty} \frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2} \qquad {}_1C_1 = 1$$

$$a_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^2 x_{r_1} x_{r_2}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{2^0}{x_{r_1}^2+y_{r_1}^2} \qquad {}_2C_2 = 1 \ , \ {}_1C_0 = 1$$

$$a_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{2^3 x_{r_1} x_{r_2} x_{r_3}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^1\left(x_{r_1}+x_{r_2}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} \qquad {}_3C_3 = 1 \ \ {}_2C_1 = 2$$

$$a_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} \frac{2^4 x_{r_1} x_{r_2} x_{r_3} x_{r_4}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)} \qquad {}_4C_4 = 1$$

$$+ \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{2^2\left(x_{r_1}x_{r_2}+x_{r_1}x_{r_3}+x_{r_2}x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} \qquad {}_3C_2 = 3$$

$$+ \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} \qquad {}_2C_0 = 1$$

$$a_5 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\sum_{r_5=r_4+1}^{\infty}\frac{2^5 x_{r_1}x_{r_2}x_{r_3}x_{r_4}x_{r_5}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)\left(x_{r_5}^2+y_{r_5}^2\right)} \qquad {}_5C_5 = 1$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{2^3\left(x_{r_1}x_{r_2}x_{r_3}+x_{r_1}x_{r_2}x_{r_4}+x_{r_1}x_{r_3}x_{r_4}+x_{r_2}x_{r_3}x_{r_4}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)} \qquad {}_4C_3 = 4$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}+x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} \qquad {}_3C_1 = 3$$

⋮

$$a_{2n} = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty}\frac{2^{2n} x_{r_1}x_{r_2}\cdots x_{r_{2n}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)} \qquad {}_{2n}C_{2n} = 1$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-2}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)} \qquad {}_{2n-1}C_{2n-2} = 2n-1$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-2}=r_{2n-3}+1}^{\infty}\frac{2^{2n-4}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-4}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+\cdots+x_{r_3}x_{r_4}\cdots x_{r_{2n-2}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-2}}^2+y_{r_{2n-2}}^2\right)} \qquad {}_{2n-2}C_{2n-4}$$

⋮

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-n}=r_{2n-n-1}+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-n}}^2+y_{r_{2n-n}}^2\right)} \qquad {}_{2n-n}C_{2n-2n} = 1$$

$$a_{2n+1} = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n+1}=r_{2n}+1}^{\infty}\frac{2^{2n+1} x_{r_1}x_{r_2}\cdots x_{r_{2n+1}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n+1}}^2+y_{r_{2n+1}}^2\right)} \qquad {}_{2n+1}C_{2n+1} = 1$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty}\frac{2^{2n-1}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+x_{r_1}x_{r_2}\cdots x_{r_{2n}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)} \qquad {}_{2n}C_{2n-3} = 2n$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-3}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+\cdots+x_{r_3}x_{r_4}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)} \qquad {}_{2n-1}C_{2n-3}$$

⋮

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n+1-n}=r_{2n-n}+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}+\cdots+x_{r_{2n+1-n}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n+1-n}}^2+y_{r_{2n+1-n}}^2\right)} \qquad {}_{2n+1-n}C_{2n+1-2n} = n+1$$

In addition, the binomial coefficient on the right hand side is the number of terms in the numerator of each semi-multiple series.

Proof

Let the roots of (2.1) are $z_k = x_k \pm i y_k$, $y_k \neq 0$ $(k=1,2,3,\cdots)$. From Formula 3.2.1 ,

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right) = \prod_{r=1}^{\infty}\left(1-\frac{z}{x_r - i y_r}\right)\left(1-\frac{z}{x_r + i y_r}\right) = 0$$

i.e.

$$\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right) = 0 \tag{2.p}$$

For simplicity, we make the following substitution.

$$\frac{2x_r}{x_r^2+y_r^2} = X_r \quad , \quad \frac{1}{x_r^2+y_r^2} = I_r$$

Then, (2.p) becomes

$$\prod_{r=1}^{\infty}\left(1-X_r z + I_r z^2\right) = \left(1-X_1 z + I_1 z^2\right)\left(1-X_2 z + I_2 z^2\right)\left(1-X_3 z + I_3 z^2\right)\cdots \tag{2.p'}$$

If (2.1) and (2.p') are compared and the coefficient of (2.1) is calculated, it is as follows.

$$a_1 = -X_1 - X_2 - X_3 - \cdots = -\sum_{r_1=1}^{\infty} X_{r_1}$$

$$a_2 = X_1(X_2+X_3+X_4+\cdots) + X_2(X_3+X_4+X_5+\cdots) + X_3(X_4+X_5+X_6+\cdots) + \cdots + I_1+I_2+I_3+\cdots$$

$$= \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} X_{r_1}X_{r_2} + \sum_{r_1=1}^{\infty} I_{r_1}$$

$$a_3 = -X_1X_2(X_3+X_4+X_5+\cdots) - X_1X_3(X_4+X_5+X_6+\cdots) - X_1X_4(X_5+X_6+X_4+\cdots) - \cdots$$

$$-X_2X_3(X_4+X_5+X_6+\cdots) - X_2X_4(X_5+X_6+X_4+\cdots) - X_2X_5(X_6+X_7+X_8+\cdots) - \cdots$$

$$\vdots$$

$$-X_1(I_2+I_3+I_4+\cdots) - X_2(I_3+I_4+I_5+\cdots) - X_3(I_4+I_5+I_6+\cdots) - \cdots$$

$$-I_1(X_2+X_3+X_4+\cdots) - I_2(X_3+X_4+X_5+\cdots) - I_3(X_4+X_5+X_6+\cdots) - \cdots$$

$$= -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} X_{r_1}X_{r_2}X_{r_3} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\left(X_{r_1}I_{r_2} + I_{r_1}X_{r_2}\right)$$

$$a_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} X_{r_1}X_{r_2}X_{r_3}X_{r_4}$$

$$+X_1X_2(I_3+I_4+I_5+\cdots) + X_1X_3(I_4+I_5+I_6+\cdots) + X_1X_4(I_5+I_6+I_4+\cdots) + \cdots$$

$$+X_2X_3(I_4+I_5+I_6+\cdots) + X_2X_4(I_5+I_6+I_4+\cdots) + X_2X_5(I_6+I_7+I_8+\cdots) + \cdots$$

$$\vdots$$

$$+X_1I_2(X_3+X_4+X_5+\cdots) + X_1I_3(X_4+X_5+X_6+\cdots) + X_1I_4(X_5+X_6+X_4+\cdots) + \cdots$$

$$+X_2I_3(X_4+X_5+X_6+\cdots) + X_2I_4(X_5+X_6+X_4+\cdots) + X_2I_5(X_6+X_7+X_8+\cdots) + \cdots$$

$$\vdots$$

$$+I_1X_2(X_3+X_4+X_5+\cdots) + I_1X_3(X_4+X_5+X_6+\cdots) + I_1X_4(X_5+X_6+X_4+\cdots) + \cdots$$

$$+I_2X_3(X_4+X_5+X_6+\cdots) + I_2X_4(X_5+X_6+X_4+\cdots) + I_2X_5(X_6+X_7+X_8+\cdots) + \cdots$$

$$\vdots$$

$$+I_1(I_2+I_3+I_4+\cdots) + I_2(I_3+I_4+I_5+\cdots) + I_3(I_4+I_5+I_6+\cdots) + \cdots$$

$$= \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} X_{r_1}X_{r_2}X_{r_3}X_{r_4}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\left(X_{r_1}X_{r_2}I_{r_3} + X_{r_1}I_{r_2}X_{r_3} + I_{r_1}X_{r_2}X_{r_3}\right) + \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} I_{r_1}I_{r_2}$$

Returning to the original symbol, we obtain $a_1 \sim a_4$. And we obtain a_{2n-1}, a_{2n} by induction. Q.E.D.

13.2.2 Relationship between zeros and coefficients of ξ(z)

Theorem 13.2.1 can be applied to the completed Riemann zeta function $\xi(z)$, yielding the following theorem.

Theorem 13.2.2

Let the completed Riemann zeta function $\xi(z)$ and the Maclaurin series be as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \sum_{r=0}^{\infty} B_r z^r \tag{2.2}$$

Then the following expressions hold for non-trivial zeros $z_k = x_k \pm i y_k$, $y_k \neq 0$ $k=1,2,3,\cdots$ of $\zeta(z)$.

$$B_1 = -\sum_{r_1=1}^{\infty}\frac{2^1 x_{r_1}}{x_{r_1}^2+y_{r_1}^2} \qquad {}_1C_1 = 1$$

$$B_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^2 x_{r_1}x_{r_2}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty}\frac{2^0}{x_{r_1}^2+y_{r_1}^2} \qquad {}_2C_2 = 1,\ {}_1C_0 = 1$$

$$B_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^3 x_{r_1}x_{r_2}x_{r_3}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} \qquad {}_3C_3 = 1,\ {}_2C_1 = 2$$

$$B_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{2^4 x_{r_1}x_{r_2}x_{r_3}x_{r_4}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)} \qquad {}_4C_4 = 1$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^2\left(x_{r_1}x_{r_2}+x_{r_1}x_{r_3}+x_{r_2}x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} \qquad {}_3C_2 = 3$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} \qquad {}_2C_0 = 1$$

$$B_5 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\sum_{r_5=r_4+1}^{\infty}\frac{2^5 x_{r_1}x_{r_2}x_{r_3}x_{r_4}x_{r_5}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)\left(x_{r_5}^2+y_{r_5}^2\right)} \qquad {}_5C_5 = 1$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\frac{2^3\left(x_{r_1}x_{r_2}x_{r_3}+x_{r_1}x_{r_2}x_{r_4}+x_{r_1}x_{r_3}x_{r_4}+x_{r_2}x_{r_3}x_{r_4}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)} \qquad {}_4C_3 = 4$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}+x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} \qquad {}_3C_1 = 3$$

$\vdots$

$$B_{2n} = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty}\frac{2^{2n} x_{r_1}x_{r_2}\cdots x_{r_{2n}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)} \qquad {}_{2n}C_{2n} = 1$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-2}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)} \qquad {}_{2n-1}C_{2n-2} = 2n-1$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-2}=r_{2n-3}+1}^{\infty}\frac{2^{2n-4}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-4}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+\cdots+x_{r_3}x_{r_4}\cdots x_{r_{2n-2}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-2}}^2+y_{r_{2n-2}}^2\right)} \qquad {}_{2n-2}C_{2n-4}$$

$\vdots$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-n}=r_{2n-n-1}+1}^{\infty}\frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-n}}^2+y_{r_{2n-n}}^2\right)} \qquad {}_{2n-n}C_{2n-2n} = 1$$

$$B_{2n+1} = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n+1}=r_{2n}+1}^{\infty}\frac{2^{2n+1} x_{r_1}x_{r_2}\cdots x_{r_{2n+1}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n+1}}^2+y_{r_{2n+1}}^2\right)} \qquad {}_{2n+1}C_{2n+1} = 1$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty}\frac{2^{2n-1}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+x_{r_1}x_{r_2}\cdots x_{r_{2n}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)} \qquad {}_{2n}C_{2n-3} = 2n$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty}\frac{2^{2n-3}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+\cdots+x_{r_3}x_{r_4}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)} \qquad {}_{2n-1}C_{2n-3}$$

$\vdots$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n+1-n}=r_{2n-n}+1}^{\infty}\frac{2^1\left(x_{r_1}+x_{r_2}+\cdots+x_{r_{2n+1-n}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n+1-n}}^2+y_{r_{2n+1-n}}^2\right)} \qquad {}_{2n+1-n}C_{2n+1-2n} = n+1$$

In addition, the binomial coefficient on the right hand side is the number of terms in the numerator of each semi-multiple series.

Proof

According to Theorem 8.3.1 in " **08 Factorization of Completed Riemann Zeta** ", when the Riemann Zeta Function is $\zeta(z)$ and the non-trivial zeros are $z_n = x_n \pm i y_n \quad n=1,2,3,\cdots$, The completed Riemann Zeta function $\xi(z)$ can be factorized as follows.

$$\xi(z) = \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)$$

Therefore Theorem 13.2.1 is applicable, the desired expressions hold. Q.E.D.

Note

This theorem holds true regardless of whether the Riemann hypothesis is true or not.

13.2.3 Case where the Riemann hypothesis is true

In this case, the only zeros are $1/2 \pm y_{r_t}$. So $B_1, B_2, B_3, \cdots$ in Theorem 13.2.2 become as follows.

$$B_1 = -\sum_{r_1=1}^{\infty} \frac{{}_1C_1}{1/4+y_{r_1}^2} \qquad {}_1C_1 = 1$$

$$B_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{{}_2C_2}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{{}_1C_0}{1/4+y_{r_1}^2} \qquad {}_2C_2 = 1,\ {}_1C_0 = 1$$

$$B_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{{}_3C_3}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{{}_2C_1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} \qquad {}_3C_3 = 1,\ {}_2C_1 = 2$$

$\vdots$

13.2.4 Case where the Riemann hypothesis is false

In this case, in addition to $1/2 \pm y_{r_t}$, there are also $1/2 \pm \alpha_{r_t} \pm \beta_{r_t}$ $\left(0 < \alpha_{r_t} < 1/2\right)$ as zeros.

So $B_1, B_2, B_3, \cdots$ in Theorem 13.2.2 become as follows.

$$B_1 = -\sum_{r_1=1}^{\infty} \frac{{}_1C_1}{1/4+y_{r_1}^2} - \sum_{r_1=1} \frac{2^1\left(1/2+\alpha_{r_1}\right)}{\left(1/2+\alpha_{r_1}\right)^2+\beta_{r_1}^2} - \sum_{r_1=1} \frac{2^1\left(1/2-\alpha_{r_1}\right)}{\left(1/2-\alpha_{r_1}\right)^2+\beta_{r_1}^2}$$

$$\begin{aligned} B_2 = & \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{{}_2C_2}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{{}_1C_0}{1/4+y_{r_1}^2} \\ & + \sum_{r_1=1}\sum_{r_2=r_1+1} \frac{2^2\left(1/2+\alpha_{r_1}\right)\left(1/2+\alpha_{r_2}\right)}{\left\{\left(1/2+\alpha_{r_1}\right)^2+\beta_{r_1}^2\right\}\left\{\left(1/2+\alpha_{r_2}\right)^2+\beta_{r_2}^2\right\}} + \sum_{r_1=1} \frac{2^0}{\left(1/2+\alpha_{r_1}\right)^2+\beta_{r1}^2} \\ & + \sum_{r_1=1}\sum_{r_2=r_1+1} \frac{2^2\left(1/2-\alpha_{r_1}\right)\left(1/2-\alpha_{r_2}\right)}{\left\{\left(1/2-\alpha_{r_1}\right)^2+\beta_{r1}^2\right\}\left\{\left(1/2-\alpha_{r_2}\right)^2+\beta_{r_2}^2\right\}} + \sum_{r_1=1} \frac{2^0}{\left(1/2-\alpha_{r_1}\right)^2+\beta_{r1}^2} \end{aligned}$$

$$\begin{aligned} B_3 = & -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{{}_3C_3}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{{}_2C_1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} \\ & - \sum_{r_1=1}\sum_{r_2=r_1+1}\sum_{r_3=r_2+1} \frac{2^3\left(1/2+\alpha_{r_1}\right)\left(1/2+\alpha_{r_2}\right)\left(1/2+\alpha_{r_3}\right)}{\left\{\left(1/2+\alpha_{r_1}\right)^2+\beta_{r_1}^2\right\}\left\{\left(1/2+\alpha_{r_2}\right)^2+\beta_{r_2}^2\right\}\left\{\left(1/2+\alpha_{r_3}\right)^2+\beta_{r_3}^2\right\}} \\ & - \sum_{r_1=1}\sum_{r_2=r_1+1}\sum_{r_3=r_2+1} \frac{2^3\left(1/2-\alpha_{r_1}\right)\left(1/2-\alpha_{r_2}\right)\left(1/2-\alpha_{r_3}\right)}{\left\{\left(1/2-\alpha_{r_1}\right)^2+\beta_{r1}^2\right\}\left\{\left(1/2-\alpha_{r_2}\right)^2+\beta_{r_2}^2\right\}\left\{\left(1/2-\alpha_{r_3}\right)^2+\beta_{r_3}^2\right\}} \\ & - \sum_{r_1=1}\sum_{r_2=r_1+1} \frac{2^1\left\{\left(1/2+\alpha_{r_1}\right)+\left(1/2+\alpha_{r_2}\right)\right\}}{\left\{\left(1/2+\alpha_{r_1}\right)^2+\beta_{r_1}^2\right\}\left\{\left(1/2+\alpha_{r_2}\right)^2+\beta_{r_2}^2\right\}} - \sum_{r_1=1}\sum_{r_2=r_1+1} \frac{2^1\left\{\left(1/2-\alpha_{r_1}\right)+\left(1/2-\alpha_{r_2}\right)\right\}}{\left\{\left(1/2-\alpha_{r_1}\right)^2+\beta_{r_1}^2\right\}\left\{\left(1/2-\alpha_{r_2}\right)^2+\beta_{r_2}^2\right\}} \end{aligned}$$

$\vdots$

Note

(1) It is known that there are an infinite number of zeros on the critical line, but the number of zeros off the critical line is unknown.

(2) B_r $r=0,1,2,\cdots$ are the sum of 2 semi-multiple series, one with zeros on the critical line and the other with zeros off the critical line.

(3) Since $0 < \alpha_{r_t} < 1/2$, semi-multiple series with zeros outside the critical line cannot cancel each other out to 0.

13.3 Proposition equivalent to the Riemann Hypothesis - 1

As seen in the previous section **13.2.3** , if the Riemann hypothesis holds, then the following equivalent lemma holds.

Lemma 13.3.1

Let the completed Riemann zeta function $\xi(z)$ and the Maclaurin series be as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \sum_{r=0}^{\infty} B_r z^r \tag{2.2}$$

Then the following expressions hold for non-trivial zeros $z_k = 1/2 \pm iy_k$, $y_k \neq 0$ $k=1,2,3,\cdots$ of $\zeta(z)$.

$$B_1 = -\sum_{r_1=1}^{\infty} \frac{{}_1C_1}{1/4+y_{r_1}^2} \qquad {}_1C_1 = 1$$

$$B_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{{}_2C_2}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{{}_1C_0}{1/4+y_{r_1}^2} \qquad {}_2C_2 = 1\,,\ {}_1C_0 = 1$$

$$B_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{{}_3C_3}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{{}_2C_1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} \qquad {}_3C_3 = 1,\ {}_2C_1 = 2$$

$$B_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} \frac{{}_4C_4}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)\left(1/4+y_{r_4}^2\right)} \qquad {}_4C_4 = 1$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{{}_3C_2}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)} \qquad {}_3C_2 = 3$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{{}_2C_0}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} \qquad {}_2C_0 = 1$$

$$B_5 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty}\sum_{r_5=r_4+1}^{\infty} \frac{{}_5C_5}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)\left(1/4+y_{r_4}^2\right)\left(1/4+y_{r_5}^2\right)} \qquad {}_5C_5 = 1$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} \frac{{}_4C_3}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)\left(1/4+y_{r_4}^2\right)} \qquad {}_4C_3 = 4$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{{}_3C_1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)} \qquad {}_3C_1 = 3$$

$\vdots$

$$B_{2n} = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty} \frac{{}_{2n}C_{2n}}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\cdots\left(1/4+y_{r_{2n}}^2\right)} \qquad {}_{2n}C_{2n} = 1$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty} \frac{{}_{2n-1}C_{2n-2}}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\cdots\left(1/4+y_{r_{2n-1}}^2\right)} \qquad {}_{2n-1}C_{2n-2} = 2n-1$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-2}=r_{2n-3}+1}^{\infty} \frac{{}_{2n-2}C_{2n-4}}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\cdots\left(1/4+y_{r_{2n-2}}^2\right)} \qquad {}_{2n-2}C_{2n-4}$$

$\vdots$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} \frac{{}_nC_0}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\cdots\left(1/4+y_{r_n}^2\right)} \qquad {}_{2n-n}C_{2n-2n} = 1$$

$$B_{2n+1} = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n+1}=r_{2n}+1}^{\infty} \frac{{}_{2n+1}C_{2n+1}}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\cdots\left(1/4+y_{r_{2n+1}}^2\right)} \qquad {}_{2n+1}C_{2n+1} = 1$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty} \frac{{}_{2n}C_{2n-3}}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\cdots\left(1/4+y_{r_{2n}}^2\right)} \qquad {}_{2n}C_{2n-3} = 2n$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty} \frac{{}_{2n-1}C_{2n-3}}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\cdots\left(1/4+y_{r_{2n-1}}^2\right)} \qquad {}_{2n-1}C_{2n-3}$$

$\vdots$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n+1-n}=r_{2n-n}+1}^{\infty}\frac{{}_{n+1}C_1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\cdots\left(1/4+y_{r_{2n+1-n}}^2\right)} \qquad {}_{2n+1-n}C_{2n+1-2n}=n+1$$

However, this lemma is complicated. If we assume the Riemann hypothesis, a better proposition can be presented.

Proposition 13.3.2

Let n be a natural number, A_n be the constant obtained in Theorem 13.1.1, and y_{r_t} be a zero on the critical line of the Riemann Zeta function $\zeta(z)$. Then the following expression hollds.

$$H_n=\sum_{k=0}^{n}\frac{(-1)^n}{n}\binom{n-1+k}{n-1}(n-k)A_{n-k} \tag{3.2}$$

Where,

$$H_1=\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r_1}^2}$$

$$H_2=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)}$$

$$H_3=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)}$$

$$\vdots$$

$$H_n=\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty}\frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)\cdots\left(1/4+y_{r_n}^2\right)}$$

Proof

In Lemma 13.3.1 , the binomial coefficients can be placed before $\Sigma,\Sigma\Sigma,\cdots$. So, if we abbreviate the semi-multiple series according to the provisos of the proposition, each equation in Lemma 13.3.1 can be written as follows.

$$B_1=-H_1$$
$$B_2=\ H_2+{}_1C_0H_1$$
$$B_3=-H_3-{}_2C_1H_2$$
$$B_4=\ H_4+{}_3C_2H_3+{}_2C_0H_2$$
$$B_5=-H_5-{}_4C_3H_4-{}_3C_1H_3$$
$$B_6=\ H_6+{}_5C_4H_5+{}_4C_2H_4+{}_3C_0H_3$$
$$B_7=-H_7-{}_6C_5H_6-{}_5C_3H_5-{}_4C_1H_4$$
$$\vdots$$

Replacing the binomial coefficients with numerical values and swapping B_r and H,

$$H_1=-B_1$$
$$H_2=\ B_2-H_1$$
$$H_3=-B_3-2H_2$$
$$H_4=\ B_4-3H_3-H_2$$
$$H_5=-B_5-4H_4-3H_3$$
$$H_6=\ B_6-5H_5-6H_4-H_3$$
$$H_7=-B_7-6H_6-10H_5-4H_4$$
$$\vdots$$

Substituting H_r from the top in order using the recursive function of *Mathematica*,

$$H_1=-B_1$$
$$H_2=\ B_1+B_2$$
$$H_3=-2\left(B_1+B_2\right)-B_3$$

$$H_4 = 5(B_1+B_2) + 3B_3 + B_4$$
$$H_5 = -14(B_1+B_2) - 9B_3 - 4B_4 - B_5$$
$$H_6 = 42(B_1+B_2) + 28B_3 + 14B_4 + 5B_5 + B_6$$
$$H_7 = -132(B_1+B_2) - 90B_3 - 48B_4 - 20B_5 - 6B_6 - B_7$$
$$\vdots$$

Here, according "The On-Line Encyclopedia of Integer Sequences", these coefficients are the constituent sequence of the Catalan triangle (OEIS A009766) and are given by

$$T(n,m) = {}_{n+m}C_n\,(n-m+1)/(n+1) \qquad 0 \le m \le n$$

Using this, above formulas can be expressed in general form as follows:

$$H_n = \sum_{k=0}^{n} \frac{(-1)^n}{n}\binom{n-1+k}{n-1}(n-k)B_{n-k} \qquad n=1,2,3,\cdots$$

Finally, since the Maclaurin series of the completed Riemann Zeta function $\xi(z)$ is unique, $B_r = A_r$ $r=1,2,3,\cdots$. So, replacing B_r with A_r , we obtain the desired expression. Q.E.D.

Example

The first few of (3.2) are,

$$\sum_{r_1=1}^{\infty} \frac{1}{1/4+y_{r_1}^2} = -A_1 = 0.0230957089\cdots$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} = A_1 + A_2 = 0.000248155568\cdots$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)} = -2(A_1+A_2) - A_3 = 1.672713713\times10^{-6}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} \frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)\left(1/4+y_{r_4}^2\right)} = 5(A_1+A_2) + 3A_3 + A_4$$
$$= 8.021073428\times10^{-9}$$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\cdots\sum_{r_5=r_4+1}^{\infty} \frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)\cdots\left(1/4+y_{r_5}^2\right)} = -14(A_1+A_2) - 9A_3 - 4A_4 - A_5$$
$$= 2.936055872\times10^{-11}$$

Semi-multiple Series and Theoretical Values

The left-hand sides of these expressions are semi-multiple series consisting of zeros on the critical line, and the right-hand sides are theoretical values consisting of $\log\pi$, Stieltjes constants and the polygamma functions. To verify the validity of the Riemann hypothesis, we can take several zeros on the critical line, calculate the value of the semimultiple series, and compare it with the theoretical value.

The theoretical value can be calculated in an instant. However, calculating the semi-multiple series H_r is not easy. When the calculation of the half multiple series is truncated at m , the amount of calculations for H_r becomes ${}_mC_r$. For example, when the calculation is truncated at $m=100$, the amount of calculations for H_8 becomes ${}_{100}C_8 = 186,087,894,300$. This is not a quantity that can be calculated on a laptop computer. We have to think of another way.

13.4 Proposition equivalent to the Riemann Hypothesis - 2

As mentioned at the end of the previous section, calculating semi-multiple series is not realistic. So what I came up with was to transfer the calculation of semi-multiple series to the calculation of power series. For example, in the case of a half double series, the following equation holds:

$$\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^2 = \left(\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r_1}^2}\right)^2 - 2\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)}$$

Here, obtained in the previous section

$$\sum_{r_1=1}^{\infty}\frac{1}{1/4+y_{r_1}^2} = -A_1 \qquad , \qquad \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)} = A_1 + A_2$$

are substituted for the rhighr hand side,

$$\sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^2 = A_1^2 - 2\left(A_1 + A_2\right)$$

Thus, the calculation of a half-double series is transferred to the calculation of a square series. The latter converges much faster than the former. This example is 2 nd order, but the higher the deree the faster the convergence. Finding such a general formula is the purpose of this section.

Theorem 5.2.2 in " **05 Power Series and Semi Multiple Series** " (**Infinite degree Equation**) was as follows:

Theorem 5.2.2 (Reprint)

When n is a natural number s.t. $n \ge 2$, the following holds for a convergent series.

$$\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^n = \sum_{r_1=1}^{\infty} a_r^n + 2\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^{n-2} H_2$$
$$+ \sum_{s=0}^{n-3}\left(\sum_{r_1=1}^{\infty} a_{r_1}\right)^s \left(\sum_{t=2}^{n-s-1}(-1)^t\left(\sum_{r_1=1}^{\infty} a_{r_1}^{n-s-t}\right)H_t + (-1)^{n-s}(n-s)H_{n-s}\right) \tag{2.2$_n$}$$

Where,

$$H_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} a_{r_1}a_{r_2}$$

$$H_3 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} a_{r_1}a_{r_2}a_{r_3}$$

$$\vdots$$

$$H_n = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} a_{r_1}a_{r_2}a_{r_3}\cdots a_{r_n}$$

When $n \le 2$, the 3 rd term of (2.2$_n$) is ignored.

From this theorem, we obtain the following lemma.

Lemma 13.4.1

When n is a natural number s.t. $n \ge 2$ and y_{r_t} $t=1,2,3,\cdots$ are non-zeoro real numbers, the following holds for a convergent series

$$G_1^n = G_n + 2G_1^{n-2}H_2 + \sum_{s=0}^{n-3} G_1^s\left(\sum_{t=2}^{n-s-1}(-1)^t G_1^{n-s-t}H_t + (-1)^{n-s}(n-s)H_{n-s}\right) \tag{4.1}$$

Where,

$$G_n = \sum_{r_1=1}^{\infty}\left(\frac{1}{1/4+y_{r_1}^2}\right)^n \qquad n=1,2,3,\cdots$$

$$H_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)}$$

$$H_3 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right)\left(1/4+y_{r_3}^2\right)}$$

$$\vdots$$

$$H_n = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \cdots \sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right) \cdots \left(1/4+y_{r_n}^2\right)}$$

When $n \le 2$, the 3 rd term of (4.1) is ignored.

Proof

In Theorem 5.2.2 , let

$$a_{r_n} = \frac{1}{1/4+y_{r_n}^2} \quad , \quad G_n = \sum_{r_1=1}^{\infty} \left(\frac{1}{1/4+y_{r_1}^2} \right)^n \qquad n=1,2,3,\cdots$$

Then,

$$G_1 = \sum_{r_1=1}^{\infty} \frac{1}{1/4+y_{r_1}^2} = H_1$$

Therefore, (2.2n) and the belows are expressed as follows.

$$G_1^n = G_n + 2G_1^{n-2}H_2 + \sum_{s=0}^{n-3} G_1^s \left(\sum_{t=2}^{n-s-1} (-1)^t G_1^{n-s-t} H_t + (-1)^{n-s}(n-s)H_{n-s} \right)$$

Where,

$$H_n = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \cdots \sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{\left(1/4+y_{r_1}^2\right)\left(1/4+y_{r_2}^2\right) \cdots \left(1/4+y_{r_n}^2\right)} \qquad n=2,3,4,\cdots$$

Q.E.D.

Example

The first few of (4.1) are,

$$G_1^2 = G_2 + 2H_2$$

$$G_1^3 = G_3 + 3G_1 H_2 - 3H_3$$

$$G_1^4 = G_4 + 3G_1^2 H_2 + G_2 H_2 - 4G_1 H_3 + 4H_4$$

$$G_1^5 = G_5 + 3G_1^3 H_2 + G_1 G_2 H_2 + G_3 H_2 - 4G_1^2 H_3 - G_2 H_3 + 5G_1 H_4 - 5H_5$$

Problem with Lemma 13.4.1

For given y_{r_t} $t=1,2,3,\cdots$, these equations can be checked, but the computation speed of H_n slows exponentially as n increases. The best way to solve this problem is to eliminate H_n. In general, we do not expect such luck. But if y_{r_t} are zeros on the critical line of the Riemann Zeta function $\zeta(z)$, we can replace H_n with a constant using Proposition 13.3.2 in the previous section. Thus we obtain the following proposition, which is equivalent to the Riemann hypothesis.

Proposition 13.4.2

When n is a natural number s.t. $n \ge 2$ and A_n is the constant obtained in Theorem 13.1.1 , the following expression holds.

$$G_n = G_1^n - 2G_1^{n-1}H_2 - \sum_{s=0}^{n-3} G_1^s \left(\sum_{t=2}^{n-s-1} (-1)^t G_{n-s-t} H_t + (-1)^{n-s}(n-s)H_{n-s} \right) \tag{4.2}$$

Where,

$$G_n = \sum_{r_1=1}^{\infty} \left(\frac{1}{1/4+y_{r_1}^2} \right)^n \qquad n=1,2,3,\cdots$$

$$H_n = \sum_{k=0}^{n} \frac{(-1)^n}{n} \binom{n-1+k}{n-1} (n-k)A_{n-k} \qquad n=2,3,4,\cdots$$

$$G_1 = -A_1 \quad (=H_1)$$

When $n \le 2$, the 3 rd term of (4.2) is ignored.

Proof

In the formula of Lemma 13.4.1 , exchange G_1^n and G_n. And replace H_n with the formula of Proposition 13.3.2 in the previous section.

At this time, replace $H_1 = -A_1$ with $G_1 = -A_1$. Q.E.D.

Example

If this proposition is computed as a recursive formula using *Mathematica*, the first few of (4.2) are as follows.

```
Clear[G, H, A]
```

$$G_{n\_} := G_1{}^n - \sum_{s=0}^{n-3} G_1{}^s \left(\sum_{t=2}^{n-1-s} (-1)^t G_{n-s-t} H_t + (-1)^{n-s} (n-s) H_{n-s} \right) - 2 G_1{}^{n-2} H_2$$

$$H_{n\_} := \sum_{k=0}^{n} \frac{(-1)^n}{n} \text{Binomial}[n-1+k, n-1] (n-k) A_{n-k}$$

$$G_1 := -A_1$$

Expand[G_2] $-2 A_1 + A_1^2 - 2 A_2$

Expand[G_3] $-6 A_1 + 3 A_1^2 - A_1^3 - 6 A_2 + 3 A_1 A_2 - 3 A_3$

Expand[G_4] $-20 A_1 + 10 A_1^2 - 4 A_1^3 + A_1^4 - 20 A_2 + 12 A_1 A_2 - 4 A_1^2 A_2 + 2 A_2^2 - 12 A_3 + 4 A_1 A_3 - 4 A_4$

Expand[G_5] $-70 A_1 + 35 A_1^2 - 15 A_1^3 + 5 A_1^4 - A_1^5 - 70 A_2 + 45 A_1 A_2 - 20 A_1^2 A_2 + 5 A_1^3 A_2 +$
$10 A_2^2 - 5 A_1 A_2^2 - 45 A_3 + 20 A_1 A_3 - 5 A_1^2 A_3 + 5 A_2 A_3 - 20 A_4 + 5 A_1 A_4 - 5 A_5$

Expand[G_6] $-252 A_1 + 126 A_1^2 - 56 A_1^3 + 21 A_1^4 - 6 A_1^5 + A_1^6 - 252 A_2 + 168 A_1 A_2 - 84 A_1^2 A_2 + 30 A_1^3 A_2 -$
$6 A_1^4 A_2 + 42 A_2^2 - 30 A_1 A_2^2 + 9 A_1^2 A_2^2 - 2 A_2^3 - 168 A_3 + 84 A_1 A_3 - 30 A_1^2 A_3 + 6 A_1^3 A_3 +$
$30 A_2 A_3 - 12 A_1 A_2 A_3 + 3 A_3^2 - 84 A_4 + 30 A_1 A_4 - 6 A_1^2 A_4 + 6 A_2 A_4 - 30 A_5 + 6 A_1 A_5 - 6 A_6$

Expand[G_{16}] $-155\,117\,520 A_1 + 77\,558\,760 A_1^2 - 37\,442\,160 A_1^3 + 17\,383\,860 A_1^4 - 7\,726\,160 A_1^5 + 3\,268\,760 A_1^6 -$
$1\,307\,504 A_1^7 + 490\,314 A_1^8 - 170\,544 A_1^9 + 54\,264 A_1^{10} - 15\,504 A_1^{11} + 3876 A_1^{12} - 816 A_1^{13} + 136 A_1^{14} -$
$16 A_1^{15} + A_1^{16} - 155\,117\,520 A_2 + 112\,326\,480 A_1 A_2 - 69\,535\,440 A_1^2 A_2 + 38\,630\,800 A_1^3 A_2 -$
$19\,612\,560 A_1^4 A_2 + 9\,152\,528 A_1^5 A_2 - 3\,922\,512 A_1^6 A_2 + 1\,534\,896 A_1^7 A_2 - 542\,640 A_1^8 A_2 +$
$170\,544 A_1^9 A_2 - 46\,512 A_1^{10} A_2 + 10\,608 A_1^{11} A_2 - 1904 A_1^{12} A_2 + 240 A_1^{13} A_2 - 16 A_1^{14} A_2 + 34\,767\,720 A_2^2 -$

⋮

The middle parts were omitted

⋮

$3808 A_1 A_2 A_{11} + 720 A_1^2 A_2 A_{11} - 64 A_1^3 A_2 A_{11} - 240 A_2^2 A_{11} + 48 A_1 A_2^2 A_{11} + 1904 A_3 A_{11} - 480 A_1 A_3 A_{11} +$
$48 A_1^2 A_3 A_{11} - 32 A_2 A_3 A_{11} + 240 A_4 A_{11} - 32 A_1 A_4 A_{11} + 16 A_5 A_{11} - 46\,512 A_{12} + 10\,608 A_1 A_{12} -$
$1904 A_1^2 A_{12} + 240 A_1^3 A_{12} - 16 A_1^4 A_{12} + 1904 A_2 A_{12} - 480 A_1 A_2 A_{12} + 48 A_1^2 A_2 A_{12} - 16 A_2^2 A_{12} +$
$240 A_3 A_{12} - 32 A_1 A_3 A_{12} + 16 A_4 A_{12} - 10\,608 A_{13} + 1904 A_1 A_{13} - 240 A_1^2 A_{13} + 16 A_1^3 A_{13} + 240 A_2 A_{13} -$
$32 A_1 A_2 A_{13} + 16 A_3 A_{13} - 1904 A_{14} + 240 A_1 A_{14} - 16 A_1^2 A_{14} + 16 A_2 A_{14} - 240 A_{15} + 16 A_1 A_{15} - 16 A_{16}$

Note

The last G_{16} is a long list of 3.3 pages, but it took about 2 seconds to output.

13.5 Probability that the Riemann Hypothesis is false

According **Theorem 13.1.1** and **Proposition 13.4.2** , we calculate this probability using *Mathematica* . The tools are as follows.

```
Tblψ[r_, z_] := Table[PolyGamma[k, z], {k, 0, r - 1}]
```

$\gamma_{s\_}$:= StieltjesGamma[s]

$$g_{r\_}\left[\frac{3}{2}\right] := \mathrm{If}\left[r == 0,\ 1,\ \sum_{k=1}^{r} \mathrm{BellY}\left[r,\ k,\ \mathrm{Tbl}\psi\left[r,\ \frac{3}{2}\right]\right]\right]$$

$$h_{r\_} := \mathrm{If}\left[r == 0,\ 1,\ -\frac{\gamma_{r-1}}{(r-1)\,!}\right]$$

$$A_{r\_} := \sum_{s=0}^{r}\sum_{t=0}^{s} \frac{\mathrm{Log}[\pi]^{r-s}}{2^{r-s}\ (r-s)\,!}\ \frac{(-1)^{s-t}\ g_{s-t}[3/2]}{2^{s-t}\ (s-t)\,!}\ h_t$$

$y_{r\_}$:= Im[ZetaZero[r]]

The first 5 lines are a tool to find the theoretical value A_r according to **Theorem 13.1.1** . The last y_r is the zero point on the critical line. Using these, the values of A_r and y_r are obtained quickly. Hereafter, calculations are performed for each degree according to the equation of **Proposition 13.4.2** .

5 th degree

$$\mathrm{f5}[m\_] := \sum_{r=1}^{m}\left(\frac{1}{1/4 + y_r^{\,2}}\right)^5$$

$$\mathrm{g5} := -70 A_1 + 35 A_1^2 - 15 A_1^3 + 5 A_1^4 - A_1^5 - 70 A_2 + 45 A_1 A_2 - 20 A_1^2 A_2 + 5 A_1^3 A_2 + 10 A_2^2 - 5 A_1 A_2^2 - 45 A_3 + 20 A_1 A_3 - 5 A_1^2 A_3 + 5 A_2 A_3 - 20 A_4 + 5 A_1 A_4 - 5 A_5$$

$f5(m)$ is the sum of the 5 th power, and $g5$ is the righ side of Proposition 13.4.2 . In fact, this is a copy and paste of the example from the previous section. If we take 30,000 zeros on the critical line and calculate them, it is as follows.

```
SetPrecision[f5[30000], 50]
```

$3.1938918608673242323122528742640502953125718597 44 \times 10^{-12}$

```
SetPrecision[g5, 50]
```

$3.19389186086732423231225287429394519 \times 10^{-12}$

The 29 digits on both sides are the same. So, dividing the left side by the right side,

```
3.19389186086732423231225287426405029531257185974384170520685133391`49.15040414601467*^-12 / 3.1938918608673242323122528742939451943793138898`35.60665719160085*^-12
```

0.999999999999999999999999999990639978

That is, the sum of the 5 th powers of 30,000 zeros on the critical line is 0.99999999999999999999999999999 (29 Nines) of the theoretical value. On the other hand, calculating the complementary event.

```
1 - %
```

9.36002×10^{-30}

This includes residual zeros on the critical line and possible zeros off the critical line. Therefore, the probability that the complementary event contains zeros outside the critical line is less than 10^{-29} . i.e. Probability that the Riemann Hypothesis is false is less than 10^{-29} .

6 th degree

In a similar way, calculating for 30,000 zeros on the critical line , it is as follows.

$$\mathrm{f6}[m\_] := \sum_{r=1}^{m}\left(\frac{1}{1/4 + y_r^{\,2}}\right)^6$$

$$\mathrm{g6} := -252 A_1 + 126 A_1^2 - 56 A_1^3 + 21 A_1^4 - 6 A_1^5 + A_1^6 - 252 A_2 + 168 A_1 A_2 - 84 A_1^2 A_2 + 30 A_1^3 A_2 - 6 A_1^4 A_2 + 42 A_2^2 - 30 A_1 A_2^2 + 9 A_1^2 A_2^2 - 2 A_2^3 - 168 A_3 + 84 A_1 A_3 - 30 A_1^2 A_3 + 6 A_1^3 A_3 + 30 A_2 A_3 - 12 A_1 A_2 A_3 + 3 A_3^2 - 84 A_4 + 30 A_1 A_4 - 6 A_1^2 A_4 + 6 A_2 A_4 - 30 A_5 + 6 A_1 A_5 - 6 A_6$$

```
SetPrecision[f6[30000], 60]
```

$1.57589008816605893377996928745089763437436414944590953960 06 \times 10^{-14}$

```
SetPrecision[g6, 60]
```

$1.5758900881660589337799692874508976380528 92 \times 10^{-14}$

```
1.5758900881660589337799692874508976343743641494459095396005637869169155469583604299`\
  59.07122205853604*^-14 /
 1.575890088166058933779969287450897638052891561323806834559201321`42.71887062740259*\
  ^-14
```

$0.999999999999999999999999999999999999976657462$

```
1 - %
```

2.334254×10^{-36}

That is, the sum of the 6 th powers of $30{,}000$ zeros on the critical line is 35 nines ($35\mathrm{N}$) of the theoretical value. So, the probability that the Riemann Hypothesis is false is less than 10^{-35}. The precision of the calculation is 6 digits higher than that of the 5 th degree. This is due to the faster convergence speed of the 6 th degree.

16 th degree

Finally, jump and perform this calculation. The number of zeros on the critical line is $30{,}000$, the same as in the previous two examples.

$$g16 := -155117520 A_1 + 77558760 A_1^2 - 37442160 A_1^3 + 17383860 A_1^4 - 7726160 A_1^5 + 3268760 A_1^6 -$$
$$1307504 A_1^7 + 490314 A_1^8 - 170544 A_1^9 + 54264 A_1^{10} - 15504 A_1^{11} + 3876 A_1^{12} - 816 A_1^{13} + 136 A_1^{14} -$$
$$16 A_1^{15} + A_1^{16} - 155117520 A_2 + 112326480 A_1 A_2 - 69535440 A_1^2 A_2 + 38630800 A_1^3 A_2 -$$
$$19612560 A_1^4 A_2 + 9152528 A_1^5 A_2 - 3922512 A_1^6 A_2 + 1534896 A_1^7 A_2 - 542640 A_1^8 A_2 +$$
$$170544 A_1^9 A_2 - 46512 A_1^{10} A_2 + 10608 A_1^{11} A_2 - 1904 A_1^{12} A_2 + 240 A_1^{13} A_2 - 16 A_1^{14} A_2 + 34767720 A_2^2 -$$
$$38630800 A_1 A_2^2 + 29418840 A_1^2 A_2^2 - 18305056 A_1^3 A_2^2 + 9806280 A_1^4 A_2^2 - 4604688 A_1^5 A_2^2 +$$
$$1899240 A_1^6 A_2^2 - 682176 A_1^7 A_2^2 + 209304 A_1^8 A_2^2 - 53040 A_1^9 A_2^2 + 10472 A_1^{10} A_2^2 - 1440 A_1^{11} A_2^2 +$$
$$104 A_1^{12} A_2^2 - 6537520 A_2^3 + 9152528 A_1 A_2^3 - 7845024 A_1^2 A_2^3 + 5116320 A_1^3 A_2^3 - 2713200 A_1^4 A_2^3 +$$
$$1193808 A_1^5 A_2^3 - 434112 A_1^6 A_2^3 + 127296 A_1^7 A_2^3 - 28560 A_1^8 A_2^3 + 4400 A_1^9 A_2^3 - 352 A_1^{10} A_2^3 + 980628 A_2^4 -$$
$$1534896 A_1 A_2^4 + 1356600 A_1^2 A_2^4 - 852720 A_1^3 A_2^4 + 406980 A_1^4 A_2^4 - 148512 A_1^5 A_2^4 + 39984 A_1^6 A_2^4 -$$
$$7200 A_1^7 A_2^4 + 660 A_1^8 A_2^4 - 108528 A_2^5 + 170544 A_1 A_2^5 - 139536 A_1^2 A_2^5 + 74256 A_1^3 A_2^5 - 26656 A_1^4 A_2^5 +$$
$$6048 A_1^5 A_2^5 - 672 A_1^6 A_2^5 + 7752 A_2^6 - 10608 A_1 A_2^6 + 6664 A_1^2 A_2^6 - 2240 A_1^3 A_2^6 + 336 A_1^4 A_2^6 - 272 A_2^7 +$$
$$240 A_1 A_2^7 - 64 A_1^2 A_2^7 + 2 A_2^8 - 112326480 A_3 + 69535440 A_1 A_3 - 38630800 A_1^2 A_3 + 19612560 A_1^3 A_3 -$$
$$9152528 A_1^4 A_3 + 3922512 A_1^5 A_3 - 1534896 A_1^6 A_3 + 542640 A_1^7 A_3 - 170544 A_1^8 A_3 + 46512 A_1^9 A_3 -$$
$$10608 A_1^{10} A_3 + 1904 A_1^{11} A_3 - 240 A_1^{12} A_3 + 16 A_1^{13} A_3 + 38630800 A_2 A_3 - 39225120 A_1 A_2 A_3 +$$
$$27457584 A_1^2 A_2 A_3 - 15690048 A_1^3 A_2 A_3 + 7674480 A_1^4 A_2 A_3 - 3255840 A_1^5 A_2 A_3 + 1193808 A_1^6 A_2 A_3 -$$
$$372096 A_1^7 A_2 A_3 + 95472 A_1^8 A_2 A_3 - 19040 A_1^9 A_2 A_3 + 2640 A_1^{10} A_2 A_3 - 192 A_1^{11} A_2 A_3 - 9152528 A_2^2 A_3 +$$
$$11767536 A_1 A_2^2 A_3 - 9209376 A_1^2 A_2^2 A_3 + 5426400 A_1^3 A_2^2 A_3 - 2558160 A_1^4 A_2^2 A_3 + 976752 A_1^5 A_2^2 A_3 -$$
$$297024 A_1^6 A_2^2 A_3 + 68544 A_1^7 A_2^2 A_3 - 10800 A_1^8 A_2^2 A_3 + 880 A_1^9 A_2^2 A_3 + 1534896 A_2^3 A_3 -$$
$$2170560 A_1 A_2^3 A_3 + 1705440 A_1^2 A_2^3 A_3 - 930240 A_1^3 A_2^3 A_3 + 371280 A_1^4 A_2^3 A_3 - 106624 A_1^5 A_2^3 A_3 +$$
$$20160 A_1^6 A_2^3 A_3 - 1920 A_1^7 A_2^3 A_3 - 170544 A_2^4 A_3 + 232560 A_1 A_2^4 A_3 - 159120 A_1^2 A_2^4 A_3 +$$
$$66640 A_1^3 A_2^4 A_3 - 16800 A_1^4 A_2^4 A_3 + 2016 A_1^5 A_2^4 A_3 + 10608 A_2^5 A_3 - 11424 A_1 A_2^5 A_3 + 5040 A_1^2 A_2^5 A_3 -$$
$$896 A_1^3 A_2^5 A_3 - 240 A_2^6 A_3 + 112 A_1 A_2^6 A_3 + 9806280 A_3^2 - 9152528 A_1 A_3^2 + 5883768 A_1^2 A_3^2 -$$
$$3069792 A_1^3 A_3^2 + 1356600 A_1^4 A_3^2 - 511632 A_1^5 A_3^2 + 162792 A_1^6 A_3^2 - 42432 A_1^7 A_3^2 + 8568 A_1^8 A_3^2 -$$
$$1200 A_1^9 A_3^2 + 88 A_1^{10} A_3^2 - 3922512 A_2 A_3^2 + 4604688 A_1 A_2 A_3^2 - 3255840 A_1^2 A_2 A_3^2 + 1705440 A_1^3 A_2 A_3^2 -$$
$$697680 A_1^4 A_2 A_3^2 + 222768 A_1^5 A_2 A_3^2 - 53312 A_1^6 A_2 A_3^2 + 8640 A_1^7 A_2 A_3^2 - 720 A_1^8 A_2 A_3^2 + 813960 A_2^2 A_3^2 -$$
$$1023264 A_1 A_2^2 A_3^2 + 697680 A_1^2 A_2^2 A_3^2 - 318240 A_1^3 A_2^2 A_3^2 + 99960 A_1^4 A_2^2 A_3^2 - 20160 A_1^5 A_2^2 A_3^2 +$$
$$2016 A_1^6 A_2^2 A_3^2 - 93024 A_2^3 A_3^2 + 106080 A_1 A_2^3 A_3^2 - 57120 A_1^2 A_2^3 A_3^2 + 16800 A_1^3 A_2^3 A_3^2 - 2240 A_1^4 A_2^3 A_3^2 +$$
$$4760 A_2^4 A_3^2 - 3600 A_1 A_2^4 A_3^2 + 840 A_1^2 A_2^4 A_3^2 - 48 A_2^5 A_3^2 - 511632 A_3^3 + 542640 A_1 A_3^3 - 341088 A_1^2 A_3^3 +$$
$$155040 A_1^3 A_3^3 - 53040 A_1^4 A_3^3 + 13328 A_1^5 A_3^3 - 2240 A_1^6 A_3^3 + 192 A_1^7 A_3^3 + 170544 A_2 A_3^3 - 186048 A_1 A_2 A_3^3 +$$

$106\,080\,A_1^2 A_2 A_3^3 - 38\,080\,A_1^3 A_2 A_3^3 + 8400\,A_1^4 A_2 A_3^3 - 896\,A_1^5 A_2 A_3^3 - 21\,216\,A_2^2 A_3^3 + 19\,040\,A_1 A_2^2 A_3^3 -$
$7200\,A_1^2 A_2^2 A_3^3 + 1120\,A_1^3 A_2^2 A_3^3 + 800\,A_2^3 A_3^3 - 320\,A_1 A_2^3 A_3^3 + 11\,628\,A_3^4 - 10\,608\,A_1 A_3^4 + 4760\,A_1^2 A_3^4 -$
$1200\,A_1^3 A_3^4 + 140\,A_1^4 A_3^4 - 1904\,A_2 A_3^4 + 1200\,A_1 A_2 A_3^4 - 240\,A_1^2 A_2 A_3^4 + 40\,A_2^2 A_3^4 - 48\,A_3^5 + 16\,A_1 A_3^5 -$
$69\,535\,440\,A_4 + 38\,630\,800\,A_1 A_4 - 19\,612\,560\,A_1^2 A_4 + 9\,152\,528\,A_1^3 A_4 - 3\,922\,512\,A_1^4 A_4 + 1\,534\,896\,A_1^5 A_4 -$
$542\,640\,A_1^6 A_4 + 170\,544\,A_1^7 A_4 - 46\,512\,A_1^8 A_4 + 10\,608\,A_1^9 A_4 - 1904\,A_1^{10} A_4 + 240\,A_1^{11} A_4 - 16\,A_1^{12} A_4 +$
$19\,612\,560\,A_2 A_4 - 18\,305\,056\,A_1 A_2 A_4 + 11\,767\,536\,A_1^2 A_2 A_4 - 6\,139\,584\,A_1^3 A_2 A_4 + 2\,713\,200\,A_1^4 A_2 A_4 -$
$1\,023\,264\,A_1^5 A_2 A_4 + 325\,584\,A_1^6 A_2 A_4 - 84\,864\,A_1^7 A_2 A_4 + 17\,136\,A_1^8 A_2 A_4 - 2400\,A_1^9 A_2 A_4 +$
$176\,A_1^{10} A_2 A_4 - 3\,922\,512\,A_2^2 A_4 + 4\,604\,688\,A_1 A_2^2 A_4 - 3\,255\,840\,A_1^2 A_2^2 A_4 + 1\,705\,440\,A_1^3 A_2^2 A_4 -$
$697\,680\,A_1^4 A_2^2 A_4 + 222\,768\,A_1^5 A_2^2 A_4 - 53\,312\,A_1^6 A_2^2 A_4 + 8640\,A_1^7 A_2^2 A_4 - 720\,A_1^8 A_2^2 A_4 + 542\,640\,A_2^3 A_4 -$
$682\,176\,A_1 A_2^3 A_4 + 465\,120\,A_1^2 A_2^3 A_4 - 212\,160\,A_1^3 A_2^3 A_4 + 66\,640\,A_1^4 A_2^3 A_4 - 13\,440\,A_1^5 A_2^3 A_4 +$
$1344\,A_1^6 A_2^3 A_4 - 46\,512\,A_2^4 A_4 + 53\,040\,A_1 A_2^4 A_4 - 28\,560\,A_1^2 A_2^4 A_4 + 8400\,A_1^3 A_2^4 A_4 - 1120\,A_1^4 A_2^4 A_4 +$
$1904\,A_2^5 A_4 - 1440\,A_1 A_2^5 A_4 + 336\,A_1^2 A_2^5 A_4 - 16\,A_2^6 A_4 + 9\,152\,528\,A_3 A_4 - 7\,845\,024\,A_1 A_3 A_4 +$
$4\,604\,688\,A_1^2 A_3 A_4 - 2\,170\,560\,A_1^3 A_3 A_4 + 852\,720\,A_1^4 A_3 A_4 - 279\,072\,A_1^5 A_3 A_4 + 74\,256\,A_1^6 A_3 A_4 -$
$15\,232\,A_1^7 A_3 A_4 + 2160\,A_1^8 A_3 A_4 - 160\,A_1^9 A_3 A_4 - 3\,069\,792\,A_2 A_3 A_4 + 3\,255\,840\,A_1 A_2 A_3 A_4 -$
$2\,046\,528\,A_1^2 A_2 A_3 A_4 + 930\,240\,A_1^3 A_2 A_3 A_4 - 318\,240\,A_1^4 A_2 A_3 A_4 + 79\,968\,A_1^5 A_2 A_3 A_4 -$
$13\,440\,A_1^6 A_2 A_3 A_4 + 1152\,A_1^7 A_2 A_3 A_4 + 511\,632\,A_2^2 A_3 A_4 - 558\,144\,A_1 A_2^2 A_3 A_4 + 318\,240\,A_1^2 A_2^2 A_3 A_4 -$
$114\,240\,A_1^3 A_2^2 A_3 A_4 + 25\,200\,A_1^4 A_2^2 A_3 A_4 - 2688\,A_1^5 A_2^2 A_3 A_4 - 42\,432\,A_2^3 A_3 A_4 + 38\,080\,A_1 A_2^3 A_3 A_4 -$
$14\,400\,A_1^2 A_2^3 A_3 A_4 + 2240\,A_1^3 A_2^3 A_3 A_4 + 1200\,A_2^4 A_3 A_4 - 480\,A_1 A_2^4 A_3 A_4 - 542\,640\,A_3^2 A_4 +$
$511\,632\,A_1 A_3^2 A_4 - 279\,072\,A_1^2 A_3^2 A_4 + 106\,080\,A_1^3 A_3^2 A_4 - 28\,560\,A_1^4 A_3^2 A_4 + 5040\,A_1^5 A_3^2 A_4 -$
$448\,A_1^6 A_3^2 A_4 + 139\,536\,A_2 A_3^2 A_4 - 127\,296\,A_1 A_2 A_3^2 A_4 + 57\,120\,A_1^2 A_2 A_3^2 A_4 - 14\,400\,A_1^3 A_2 A_3^2 A_4 +$
$1680\,A_1^4 A_2 A_3^2 A_4 - 11\,424\,A_2^2 A_3^2 A_4 + 7200\,A_1 A_2^2 A_3^2 A_4 - 1440\,A_1^2 A_2^2 A_3^2 A_4 + 160\,A_2^3 A_3^2 A_4 + 10\,608\,A_3^3 A_4 -$
$7616\,A_1 A_3^3 A_4 + 2400\,A_1^2 A_3^3 A_4 - 320\,A_1^3 A_3^3 A_4 - 960\,A_2 A_3^3 A_4 + 320\,A_1 A_2 A_3^3 A_4 - 16\,A_3^4 A_4 + 1\,961\,256\,A_4^2 -$
$1\,534\,896\,A_1 A_4^2 + 813\,960\,A_1^2 A_4^2 - 341\,088\,A_1^3 A_4^2 + 116\,280\,A_1^4 A_4^2 - 31\,824\,A_1^5 A_4^2 + 6664\,A_1^6 A_4^2 -$
$960\,A_1^7 A_4^2 + 72\,A_1^8 A_4^2 - 542\,640\,A_2 A_4^2 + 511\,632\,A_1 A_2 A_4^2 - 279\,072\,A_1^2 A_2 A_4^2 + 106\,080\,A_1^3 A_2 A_4^2 -$
$28\,560\,A_1^4 A_2 A_4^2 + 5040\,A_1^5 A_2 A_4^2 - 448\,A_1^6 A_2 A_4^2 + 69\,768\,A_2^2 A_4^2 - 63\,648\,A_1 A_2^2 A_4^2 + 28\,560\,A_1^2 A_2^2 A_4^2 -$
$7200\,A_1^3 A_2^2 A_4^2 + 840\,A_1^4 A_2^2 A_4^2 - 3808\,A_2^3 A_4^2 + 2400\,A_1 A_2^3 A_4^2 - 480\,A_1^2 A_2^3 A_4^2 + 40\,A_2^4 A_4^2 - 170\,544\,A_3 A_4^2 +$
$139\,536\,A_1 A_3 A_4^2 - 63\,648\,A_1^2 A_3 A_4^2 + 19\,040\,A_1^3 A_3 A_4^2 - 3600\,A_1^4 A_3 A_4^2 + 336\,A_1^5 A_3 A_4^2 + 31\,824\,A_2 A_3 A_4^2 -$
$22\,848\,A_1 A_2 A_3 A_4^2 + 7200\,A_1^2 A_2 A_3 A_4^2 - 960\,A_1^3 A_2 A_3 A_4^2 - 1440\,A_2^2 A_3 A_4^2 + 480\,A_1 A_2^2 A_3 A_4^2 +$
$2856\,A_3^2 A_4^2 - 1440\,A_1 A_3^2 A_4^2 + 240\,A_1^2 A_3^2 A_4^2 - 96\,A_2 A_3^2 A_4^2 - 15\,504\,A_4^3 + 10\,608\,A_1 A_4^3 - 3808\,A_1^2 A_4^3 +$
$800\,A_1^3 A_4^3 - 80\,A_1^4 A_4^3 + 1904\,A_2 A_4^3 - 960\,A_1 A_2 A_4^3 + 160\,A_1^2 A_2 A_4^3 - 32\,A_2^2 A_4^3 + 240\,A_3 A_4^3 - 64\,A_1 A_3 A_4^3 +$
$4\,A_4^4 - 38\,630\,800\,A_5 + 19\,612\,560\,A_1 A_5 - 9\,152\,528\,A_1^2 A_5 + 3\,922\,512\,A_1^3 A_5 - 1\,534\,896\,A_1^4 A_5 +$
$542\,640\,A_1^5 A_5 - 170\,544\,A_1^6 A_5 + 46\,512\,A_1^7 A_5 - 10\,608\,A_1^8 A_5 + 1904\,A_1^9 A_5 - 240\,A_1^{10} A_5 + 16\,A_1^{11} A_5 +$
$9\,152\,528\,A_2 A_5 - 7\,845\,024\,A_1 A_2 A_5 + 4\,604\,688\,A_1^2 A_2 A_5 - 2\,170\,560\,A_1^3 A_2 A_5 + 852\,720\,A_1^4 A_2 A_5 -$
$279\,072\,A_1^5 A_2 A_5 + 74\,256\,A_1^6 A_2 A_5 - 15\,232\,A_1^7 A_2 A_5 + 2160\,A_1^8 A_2 A_5 - 160\,A_1^9 A_2 A_5 - 1\,534\,896\,A_2^2 A_5 +$
$1\,627\,920\,A_1 A_2^2 A_5 - 1\,023\,264\,A_1^2 A_2^2 A_5 + 465\,120\,A_1^3 A_2^2 A_5 - 159\,120\,A_1^4 A_2^2 A_5 + 39\,984\,A_1^5 A_2^2 A_5 -$
$6720\,A_1^6 A_2^2 A_5 + 576\,A_1^7 A_2^2 A_5 + 170\,544\,A_2^3 A_5 - 186\,048\,A_1 A_2^3 A_5 + 106\,080\,A_1^2 A_2^3 A_5 - 38\,080\,A_1^3 A_2^3 A_5 +$
$8400\,A_1^4 A_2^3 A_5 - 896\,A_1^5 A_2^3 A_5 - 10\,608\,A_2^4 A_5 + 9520\,A_1 A_2^4 A_5 - 3600\,A_1^2 A_2^4 A_5 + 560\,A_1^3 A_2^4 A_5 +$
$240\,A_2^5 A_5 - 96\,A_1 A_2^5 A_5 + 3\,922\,512\,A_3 A_5 - 3\,069\,792\,A_1 A_3 A_5 + 1\,627\,920\,A_1^2 A_3 A_5 - 682\,176\,A_1^3 A_3 A_5 +$
$232\,560\,A_1^4 A_3 A_5 - 63\,648\,A_1^5 A_3 A_5 + 13\,328\,A_1^6 A_3 A_5 - 1920\,A_1^7 A_3 A_5 + 144\,A_1^8 A_3 A_5 - 1\,085\,280\,A_2 A_3 A_5 +$
$1\,023\,264\,A_1 A_2 A_3 A_5 - 558\,144\,A_1^2 A_2 A_3 A_5 + 212\,160\,A_1^3 A_2 A_3 A_5 - 57\,120\,A_1^4 A_2 A_3 A_5 + 10\,080\,A_1^5 A_2 A_3 A_5 -$
$896\,A_1^6 A_2 A_3 A_5 + 139\,536\,A_2^2 A_3 A_5 - 127\,296\,A_1 A_2^2 A_3 A_5 + 57\,120\,A_1^2 A_2^2 A_3 A_5 - 14\,400\,A_1^3 A_2^2 A_3 A_5 +$
$1680\,A_1^4 A_2^2 A_3 A_5 - 7616\,A_2^3 A_3 A_5 + 4800\,A_1 A_2^3 A_3 A_5 - 960\,A_1^2 A_2^3 A_3 A_5 + 80\,A_2^4 A_3 A_5 - 170\,544\,A_3^2 A_5 +$
$139\,536\,A_1 A_3^2 A_5 - 63\,648\,A_1^2 A_3^2 A_5 + 19\,040\,A_1^3 A_3^2 A_5 - 3600\,A_1^4 A_3^2 A_5 + 336\,A_1^5 A_3^2 A_5 + 31\,824\,A_2 A_3^2 A_5 -$
$22\,848\,A_1 A_2 A_3^2 A_5 + 7200\,A_1^2 A_2 A_3^2 A_5 - 960\,A_1^3 A_2 A_3^2 A_5 - 1440\,A_2^2 A_3^2 A_5 + 480\,A_1 A_2^2 A_3^2 A_5 + 1904\,A_3^3 A_5 -$
$960\,A_1 A_3^3 A_5 + 160\,A_1^2 A_3^3 A_5 - 64\,A_2 A_3^3 A_5 + 1\,534\,896\,A_4 A_5 - 1\,085\,280\,A_1 A_4 A_5 + 511\,632\,A_1^2 A_4 A_5 -$
$186\,048\,A_1^3 A_4 A_5 + 53\,040\,A_1^4 A_4 A_5 - 11\,424\,A_1^5 A_4 A_5 + 1680\,A_1^6 A_4 A_5 - 128\,A_1^7 A_4 A_5 - 341\,088\,A_2 A_4 A_5 +$
$279\,072\,A_1 A_2 A_4 A_5 - 127\,296\,A_1^2 A_2 A_4 A_5 + 38\,080\,A_1^3 A_2 A_4 A_5 - 7200\,A_1^4 A_2 A_4 A_5 + 672\,A_1^5 A_2 A_4 A_5 +$

$$31\,824\,A_2^2 A_4 A_5 - 22\,848\,A_1 A_2^2 A_4 A_5 + 7200\,A_1^2 A_2^2 A_4 A_5 - 960\,A_1^3 A_2^2 A_4 A_5 - 960\,A_2^3 A_4 A_5 +$$
$$320\,A_1 A_2^3 A_4 A_5 - 93\,024\,A_3 A_4 A_5 + 63\,648\,A_1 A_3 A_4 A_5 - 22\,848\,A_1^2 A_3 A_4 A_5 + 4800\,A_1^3 A_3 A_4 A_5 -$$
$$480\,A_1^4 A_3 A_4 A_5 + 11\,424\,A_2 A_3 A_4 A_5 - 5760\,A_1 A_2 A_3 A_4 A_5 + 960\,A_1^2 A_2 A_3 A_4 A_5 - 192\,A_2^2 A_3 A_4 A_5 +$$
$$720\,A_3^2 A_4 A_5 - 192\,A_1 A_3^2 A_4 A_5 - 10\,608\,A_4^2 A_5 + 5712\,A_1 A_4^2 A_5 - 1440\,A_1^2 A_4^2 A_5 + 160\,A_1^3 A_4^2 A_5 +$$
$$720\,A_2 A_4^2 A_5 - 192\,A_1 A_2 A_4^2 A_5 + 48\,A_3 A_4^2 A_5 + 271\,320\,A_5^2 - 170\,544\,A_1 A_5^2 + 69\,768\,A_1^2 A_5^2 - 21\,216\,A_1^3 A_5^2 +$$
$$4760\,A_1^4 A_5^2 - 720\,A_1^5 A_5^2 + 56\,A_1^6 A_5^2 - 46\,512\,A_2 A_5^2 + 31\,824\,A_1 A_2 A_5^2 - 11\,424\,A_1^2 A_2 A_5^2 + 2400\,A_1^3 A_2 A_5^2 -$$
$$240\,A_1^4 A_2 A_5^2 + 2856\,A_2^2 A_5^2 - 1440\,A_1 A_2^2 A_5^2 + 240\,A_1^2 A_2^2 A_5^2 - 32\,A_2^3 A_5^2 - 10\,608\,A_3 A_5^2 + 5712\,A_1 A_3 A_5^2 -$$
$$1440\,A_1^2 A_3 A_5^2 + 160\,A_1^3 A_3 A_5^2 + 720\,A_2 A_3 A_5^2 - 192\,A_1 A_2 A_3 A_5^2 + 24\,A_3^2 A_5^2 - 1904\,A_4 A_5^2 + 720\,A_1 A_4 A_5^2 -$$
$$96\,A_1^2 A_4 A_5^2 + 48\,A_2 A_4 A_5^2 - 80\,A_5^3 + 16\,A_1 A_5^3 - 19\,612\,560\,A_6 + 9\,152\,528\,A_1 A_6 - 3\,922\,512\,A_1^2 A_6 +$$
$$1\,534\,896\,A_1^3 A_6 - 542\,640\,A_1^4 A_6 + 170\,544\,A_1^5 A_6 - 46\,512\,A_1^6 A_6 + 10\,608\,A_1^7 A_6 - 1904\,A_1^8 A_6 +$$
$$240\,A_1^9 A_6 - 16\,A_1^{10} A_6 + 3\,922\,512\,A_2 A_6 - 3\,069\,792\,A_1 A_2 A_6 + 1\,627\,920\,A_1^2 A_2 A_6 - 682\,176\,A_1^3 A_2 A_6 +$$
$$232\,560\,A_1^4 A_2 A_6 - 63\,648\,A_1^5 A_2 A_6 + 13\,328\,A_1^6 A_2 A_6 - 1920\,A_1^7 A_2 A_6 + 144\,A_1^8 A_2 A_6 - 542\,640\,A_2^2 A_6 +$$
$$511\,632\,A_1 A_2^2 A_6 - 279\,072\,A_1^2 A_2^2 A_6 + 106\,080\,A_1^3 A_2^2 A_6 - 28\,560\,A_1^4 A_2^2 A_6 + 5040\,A_1^5 A_2^2 A_6 -$$
$$448\,A_1^6 A_2^2 A_6 + 46\,512\,A_2^3 A_6 - 42\,432\,A_1 A_2^3 A_6 + 19\,040\,A_1^2 A_2^3 A_6 - 4800\,A_1^3 A_2^3 A_6 + 560\,A_1^4 A_2^3 A_6 -$$
$$1904\,A_2^4 A_6 + 1200\,A_1 A_2^4 A_6 - 240\,A_1^2 A_2^4 A_6 + 16\,A_2^5 A_6 + 1\,534\,896\,A_3 A_6 - 1\,085\,280\,A_1 A_3 A_6 +$$
$$511\,632\,A_1^2 A_3 A_6 - 186\,048\,A_1^3 A_3 A_6 + 53\,040\,A_1^4 A_3 A_6 - 11\,424\,A_1^5 A_3 A_6 + 1680\,A_1^6 A_3 A_6 -$$
$$128\,A_1^7 A_3 A_6 - 341\,088\,A_2 A_3 A_6 + 279\,072\,A_1 A_2 A_3 A_6 - 127\,296\,A_1^2 A_2 A_3 A_6 + 38\,080\,A_1^3 A_2 A_3 A_6 -$$
$$7200\,A_1^4 A_2 A_3 A_6 + 672\,A_1^5 A_2 A_3 A_6 + 31\,824\,A_2^2 A_3 A_6 - 22\,848\,A_1 A_2^2 A_3 A_6 + 7200\,A_1^2 A_2^2 A_3 A_6 -$$
$$960\,A_1^3 A_2^2 A_3 A_6 - 960\,A_2^3 A_3 A_6 + 320\,A_1 A_2^3 A_3 A_6 - 46\,512\,A_3^2 A_6 + 31\,824\,A_1 A_3^2 A_6 - 11\,424\,A_1^2 A_3^2 A_6 +$$
$$2400\,A_1^3 A_3^2 A_6 - 240\,A_1^4 A_3^2 A_6 + 5712\,A_2 A_3^2 A_6 - 2880\,A_1 A_2 A_3^2 A_6 + 480\,A_1^2 A_2 A_3^2 A_6 - 96\,A_2^2 A_3^2 A_6 +$$
$$240\,A_3^3 A_6 - 64\,A_1 A_3^3 A_6 + 542\,640\,A_4 A_6 - 341\,088\,A_1 A_4 A_6 + 139\,536\,A_1^2 A_4 A_6 - 42\,432\,A_1^3 A_4 A_6 +$$
$$9520\,A_1^4 A_4 A_6 - 1440\,A_1^5 A_4 A_6 + 112\,A_1^6 A_4 A_6 - 93\,024\,A_2 A_4 A_6 + 63\,648\,A_1 A_2 A_4 A_6 - 22\,848\,A_1^2 A_2 A_4 A_6 +$$
$$4800\,A_1^3 A_2 A_4 A_6 - 480\,A_1^4 A_2 A_4 A_6 + 5712\,A_2^2 A_4 A_6 - 2880\,A_1 A_2^2 A_4 A_6 + 480\,A_1^2 A_2^2 A_4 A_6 - 64\,A_2^3 A_4 A_6 -$$
$$21\,216\,A_3 A_4 A_6 + 11\,424\,A_1 A_3 A_4 A_6 - 2880\,A_1^2 A_3 A_4 A_6 + 320\,A_1^3 A_3 A_4 A_6 + 1440\,A_2 A_3 A_4 A_6 -$$
$$384\,A_1 A_2 A_3 A_4 A_6 + 48\,A_3^2 A_4 A_6 - 1904\,A_4^2 A_6 + 720\,A_1 A_4^2 A_6 - 96\,A_1^2 A_4^2 A_6 + 48\,A_2 A_4^2 A_6 + 170\,544\,A_5 A_6 -$$
$$93\,024\,A_1 A_5 A_6 + 31\,824\,A_1^2 A_5 A_6 - 7616\,A_1^3 A_5 A_6 + 1200\,A_1^4 A_5 A_6 - 96\,A_1^5 A_5 A_6 - 21\,216\,A_2 A_5 A_6 +$$
$$11\,424\,A_1 A_2 A_5 A_6 - 2880\,A_1^2 A_2 A_5 A_6 + 320\,A_1^3 A_2 A_5 A_6 + 720\,A_2^2 A_5 A_6 - 192\,A_1 A_2^2 A_5 A_6 -$$
$$3808\,A_3 A_5 A_6 + 1440\,A_1 A_3 A_5 A_6 - 192\,A_1^2 A_3 A_5 A_6 + 96\,A_2 A_3 A_5 A_6 - 480\,A_4 A_5 A_6 + 96\,A_1 A_4 A_5 A_6 -$$
$$16\,A_5^2 A_6 + 23\,256\,A_6^2 - 10\,608\,A_1 A_6^2 + 2856\,A_1^2 A_6^2 - 480\,A_1^3 A_6^2 + 40\,A_1^4 A_6^2 - 1904\,A_2 A_6^2 + 720\,A_1 A_2 A_6^2 -$$
$$96\,A_1^2 A_2 A_6^2 + 24\,A_2^2 A_6^2 - 240\,A_3 A_6^2 + 48\,A_1 A_3 A_6^2 - 16\,A_4 A_6^2 - 9\,152\,528\,A_7 + 3\,922\,512\,A_1 A_7 -$$
$$1\,534\,896\,A_1^2 A_7 + 542\,640\,A_1^3 A_7 - 170\,544\,A_1^4 A_7 + 46\,512\,A_1^5 A_7 - 10\,608\,A_1^6 A_7 + 1904\,A_1^7 A_7 -$$
$$240\,A_1^8 A_7 + 16\,A_1^9 A_7 + 1\,534\,896\,A_2 A_7 - 1\,085\,280\,A_1 A_2 A_7 + 511\,632\,A_1^2 A_2 A_7 - 186\,048\,A_1^3 A_2 A_7 +$$
$$53\,040\,A_1^4 A_2 A_7 - 11\,424\,A_1^5 A_2 A_7 + 1680\,A_1^6 A_2 A_7 - 128\,A_1^7 A_2 A_7 - 170\,544\,A_2^2 A_7 + 139\,536\,A_1 A_2^2 A_7 -$$
$$63\,648\,A_1^2 A_2^2 A_7 + 19\,040\,A_1^3 A_2^2 A_7 - 3600\,A_1^4 A_2^2 A_7 + 336\,A_1^5 A_2^2 A_7 + 10\,608\,A_2^3 A_7 - 7616\,A_1 A_2^3 A_7 +$$
$$2400\,A_1^2 A_2^3 A_7 - 320\,A_1^3 A_2^3 A_7 - 240\,A_2^4 A_7 + 80\,A_1 A_2^4 A_7 + 542\,640\,A_3 A_7 - 341\,088\,A_1 A_3 A_7 +$$
$$139\,536\,A_1^2 A_3 A_7 - 42\,432\,A_1^3 A_3 A_7 + 9520\,A_1^4 A_3 A_7 - 1440\,A_1^5 A_3 A_7 + 112\,A_1^6 A_3 A_7 - 93\,024\,A_2 A_3 A_7 +$$
$$63\,648\,A_1 A_2 A_3 A_7 - 22\,848\,A_1^2 A_2 A_3 A_7 + 4800\,A_1^3 A_2 A_3 A_7 - 480\,A_1^4 A_2 A_3 A_7 + 5712\,A_2^2 A_3 A_7 -$$
$$2880\,A_1 A_2^2 A_3 A_7 + 480\,A_1^2 A_2^2 A_3 A_7 - 64\,A_2^3 A_3 A_7 - 10\,608\,A_3^2 A_7 + 5712\,A_1 A_3^2 A_7 - 1440\,A_1^2 A_3^2 A_7 +$$
$$160\,A_1^3 A_3^2 A_7 + 720\,A_2 A_3^2 A_7 - 192\,A_1 A_2 A_3^2 A_7 + 16\,A_3^3 A_7 + 170\,544\,A_4 A_7 - 93\,024\,A_1 A_4 A_7 +$$
$$31\,824\,A_1^2 A_4 A_7 - 7616\,A_1^3 A_4 A_7 + 1200\,A_1^4 A_4 A_7 - 96\,A_1^5 A_4 A_7 - 21\,216\,A_2 A_4 A_7 + 11\,424\,A_1 A_2 A_4 A_7 -$$
$$2880\,A_1^2 A_2 A_4 A_7 + 320\,A_1^3 A_2 A_4 A_7 + 720\,A_2^2 A_4 A_7 - 192\,A_1 A_2^2 A_4 A_7 - 3808\,A_3 A_4 A_7 + 1440\,A_1 A_3 A_4 A_7 -$$
$$192\,A_1^2 A_3 A_4 A_7 + 96\,A_2 A_3 A_4 A_7 - 240\,A_4^2 A_7 + 48\,A_1 A_4^2 A_7 + 46\,512\,A_5 A_7 - 21\,216\,A_1 A_5 A_7 +$$
$$5712\,A_1^2 A_5 A_7 - 960\,A_1^3 A_5 A_7 + 80\,A_1^4 A_5 A_7 - 3808\,A_2 A_5 A_7 + 1440\,A_1 A_2 A_5 A_7 - 192\,A_1^2 A_2 A_5 A_7 +$$
$$48\,A_2^2 A_5 A_7 - 480\,A_3 A_5 A_7 + 96\,A_1 A_3 A_5 A_7 - 32\,A_4 A_5 A_7 + 10\,608\,A_6 A_7 - 3808\,A_1 A_6 A_7 + 720\,A_1^2 A_6 A_7 -$$
$$64\,A_1^3 A_6 A_7 - 480\,A_2 A_6 A_7 + 96\,A_1 A_2 A_6 A_7 - 32\,A_3 A_6 A_7 + 952\,A_7^2 - 240\,A_1 A_7^2 + 24\,A_1^2 A_7^2 - 16\,A_2 A_7^2 -$$
$$3\,922\,512\,A_8 + 1\,534\,896\,A_1 A_8 - 542\,640\,A_1^2 A_8 + 170\,544\,A_1^3 A_8 - 46\,512\,A_1^4 A_8 + 10\,608\,A_1^5 A_8 -$$

$$1904 A_1^6 A_8 + 240 A_1^7 A_8 - 16 A_1^8 A_8 + 542\,640 A_2 A_8 - 341\,088 A_1 A_2 A_8 + 139\,536 A_1^2 A_2 A_8 -$$
$$42\,432 A_1^3 A_2 A_8 + 9520 A_1^4 A_2 A_8 - 1440 A_1^5 A_2 A_8 + 112 A_1^6 A_2 A_8 - 46\,512 A_2^2 A_8 + 31\,824 A_1 A_2^2 A_8 -$$
$$11\,424 A_1^2 A_2^2 A_8 + 2400 A_1^3 A_2^2 A_8 - 240 A_1^4 A_2^2 A_8 + 1904 A_2^3 A_8 - 960 A_1 A_2^3 A_8 + 160 A_1^2 A_2^3 A_8 -$$
$$16 A_2^4 A_8 + 170\,544 A_3 A_8 - 93\,024 A_1 A_3 A_8 + 31\,824 A_1^2 A_3 A_8 - 7616 A_1^3 A_3 A_8 + 1200 A_1^4 A_3 A_8 -$$
$$96 A_1^5 A_3 A_8 - 21\,216 A_2 A_3 A_8 + 11\,424 A_1 A_2 A_3 A_8 - 2880 A_1^2 A_2 A_3 A_8 + 320 A_1^3 A_2 A_3 A_8 + 720 A_2^2 A_3 A_8 -$$
$$192 A_1 A_2^2 A_3 A_8 - 1904 A_3^2 A_8 + 720 A_1 A_3^2 A_8 - 96 A_1^2 A_3^2 A_8 + 48 A_2 A_3^2 A_8 + 46\,512 A_4 A_8 - 21\,216 A_1 A_4 A_8 +$$
$$5712 A_1^2 A_4 A_8 - 960 A_1^3 A_4 A_8 + 80 A_1^4 A_4 A_8 - 3808 A_2 A_4 A_8 + 1440 A_1 A_2 A_4 A_8 - 192 A_1^2 A_2 A_4 A_8 +$$
$$48 A_2^2 A_4 A_8 - 480 A_3 A_4 A_8 + 96 A_1 A_3 A_4 A_8 - 16 A_4^2 A_8 + 10\,608 A_5 A_8 - 3808 A_1 A_5 A_8 + 720 A_1^2 A_5 A_8 -$$
$$64 A_1^3 A_5 A_8 - 480 A_2 A_5 A_8 + 96 A_1 A_2 A_5 A_8 - 32 A_3 A_5 A_8 + 1904 A_6 A_8 - 480 A_1 A_6 A_8 + 48 A_1^2 A_6 A_8 -$$
$$32 A_2 A_6 A_8 + 240 A_7 A_8 - 32 A_1 A_7 A_8 + 8 A_8^2 - 1\,534\,896 A_9 + 542\,640 A_1 A_9 - 170\,544 A_1^2 A_9 +$$
$$46\,512 A_1^3 A_9 - 10\,608 A_1^4 A_9 + 1904 A_1^5 A_9 - 240 A_1^6 A_9 + 16 A_1^7 A_9 + 170\,544 A_2 A_9 - 93\,024 A_1 A_2 A_9 +$$
$$31\,824 A_1^2 A_2 A_9 - 7616 A_1^3 A_2 A_9 + 1200 A_1^4 A_2 A_9 - 96 A_1^5 A_2 A_9 - 10\,608 A_2^2 A_9 + 5712 A_1 A_2^2 A_9 -$$
$$1440 A_1^2 A_2^2 A_9 + 160 A_1^3 A_2^2 A_9 + 240 A_2^3 A_9 - 64 A_1 A_2^3 A_9 + 46\,512 A_3 A_9 - 21\,216 A_1 A_3 A_9 +$$
$$5712 A_1^2 A_3 A_9 - 960 A_1^3 A_3 A_9 + 80 A_1^4 A_3 A_9 - 3808 A_2 A_3 A_9 + 1440 A_1 A_2 A_3 A_9 - 192 A_1^2 A_2 A_3 A_9 +$$
$$48 A_2^2 A_3 A_9 - 240 A_3^2 A_9 + 48 A_1 A_3^2 A_9 + 10\,608 A_4 A_9 - 3808 A_1 A_4 A_9 + 720 A_1^2 A_4 A_9 - 64 A_1^3 A_4 A_9 -$$
$$480 A_2 A_4 A_9 + 96 A_1 A_2 A_4 A_9 - 32 A_3 A_4 A_9 + 1904 A_5 A_9 - 480 A_1 A_5 A_9 + 48 A_1^2 A_5 A_9 - 32 A_2 A_5 A_9 +$$
$$240 A_6 A_9 - 32 A_1 A_6 A_9 + 16 A_7 A_9 - 542\,640 A_{10} + 170\,544 A_1 A_{10} - 46\,512 A_1^2 A_{10} + 10\,608 A_1^3 A_{10} -$$
$$1904 A_1^4 A_{10} + 240 A_1^5 A_{10} - 16 A_1^6 A_{10} + 46\,512 A_2 A_{10} - 21\,216 A_1 A_2 A_{10} + 5712 A_1^2 A_2 A_{10} -$$
$$960 A_1^3 A_2 A_{10} + 80 A_1^4 A_2 A_{10} - 1904 A_2^2 A_{10} + 720 A_1 A_2^2 A_{10} - 96 A_1^2 A_2^2 A_{10} + 16 A_2^3 A_{10} + 10\,608 A_3 A_{10} -$$
$$3808 A_1 A_3 A_{10} + 720 A_1^2 A_3 A_{10} - 64 A_1^3 A_3 A_{10} - 480 A_2 A_3 A_{10} + 96 A_1 A_2 A_3 A_{10} - 16 A_3^2 A_{10} +$$
$$1904 A_4 A_{10} - 480 A_1 A_4 A_{10} + 48 A_1^2 A_4 A_{10} - 32 A_2 A_4 A_{10} + 240 A_5 A_{10} - 32 A_1 A_5 A_{10} + 16 A_6 A_{10} -$$
$$170\,544 A_{11} + 46\,512 A_1 A_{11} - 10\,608 A_1^2 A_{11} + 1904 A_1^3 A_{11} - 240 A_1^4 A_{11} + 16 A_1^5 A_{11} + 10\,608 A_2 A_{11} -$$
$$3808 A_1 A_2 A_{11} + 720 A_1^2 A_2 A_{11} - 64 A_1^3 A_2 A_{11} - 240 A_2^2 A_{11} + 48 A_1 A_2^2 A_{11} + 1904 A_3 A_{11} - 480 A_1 A_3 A_{11} +$$
$$48 A_1^2 A_3 A_{11} - 32 A_2 A_3 A_{11} + 240 A_4 A_{11} - 32 A_1 A_4 A_{11} + 16 A_5 A_{11} - 46\,512 A_{12} + 10\,608 A_1 A_{12} -$$
$$1904 A_1^2 A_{12} + 240 A_1^3 A_{12} - 16 A_1^4 A_{12} + 1904 A_2 A_{12} - 480 A_1 A_2 A_{12} + 48 A_1^2 A_2 A_{12} - 16 A_2^2 A_{12} +$$
$$240 A_3 A_{12} - 32 A_1 A_3 A_{12} + 16 A_4 A_{12} - 10\,608 A_{13} + 1904 A_1 A_{13} - 240 A_1^2 A_{13} + 16 A_1^3 A_{13} + 240 A_2 A_{13} -$$
$$32 A_1 A_2 A_{13} + 16 A_3 A_{13} - 1904 A_{14} + 240 A_1 A_{14} - 16 A_1^2 A_{14} + 16 A_2 A_{14} - 240 A_{15} + 16 A_1 A_{15} - 16 A_{16}$$

```
SetPrecision[f16[30000], 105]
```

1.52095375313428790416691125096776570270017381985973087702928876918849076318838048230\
44671528194807135528 × 10^{-37}

```
SetPrecision[g16, 150]
```

1.52095375313428790416691125096776570270017381985973087702928876918849076318838048230\
44671528194807135606 × 10^{-37}

```
1.52095375313428790416691125096776570270017381985973087702928876918849076318838048230\
  446715281948071355277615215142778`103.64525271013926*^-37 /
 1.52095375313428790416691125096776570270017381985973087702928876918849076318838048230\
  4467152819480713560634755589948988`103.84156841793063*^-37
```

0.99999999999999999999999999999999999999999999999999999999999999999999999999999999999\
999999999999999999483

```
1 - %
```

5.2 × 10^{-102}

That is, the sum of the 16 th powers of 30,000 zeros on the critical line is 101 nines (101N) of the theoretical value. Therefore, the probability that the Riemann Hypothesis is false is less than 10^{-101}.

For reference, on my computer (Intel Core i7 - 9750H , 16GB), it took 3.5 hours to calculate $f_{16}(30000)$ and 2 seconds to calculate g_{16}.

The precision of the calculation appears to increase by 6.3 digits as the degree increases by one. That is, the degree of exponent and the precision of the calculation are roughly proportional. If so, may be 10^{-1000} at the 160 th degree and 10^{-10000} at the 1600 th degree. Such calculations are possible using the formulas presented in this chapter. Therefore, the probability that the Riemann Hypothesis is false is very close to zero.

2025.01.29

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Alien's Mathematics