Proof of the Riemann Hypothesis by Factorization

Abstract

(1) Riemann xi functkion $\xi(z)$ and $\xi(1-z)$ are factored into the Hadamard products as follows.

$$\xi(z) = \Pi(1-z/\rho) \quad , \quad \xi(1-z) = \Pi(1-(1-z)/\rho)$$

(2) $\xi(z) = \xi(1-z)$ holds on the whole complex plane..

(3) Despite (1) and (2) , $\Pi(1-z/\rho) \neq \Pi(1-(1-z)/\rho)$ in general.

(4) $\Pi(1-z/\rho) = \Pi(1-(1-z)/\rho)$ holds on the whole complex plane if and only if all $Re(\rho)$ are $1/2$.

Thus, only if all $Re(\rho)$ are $1/2$, the four-step logic $\Pi(1-z/\rho) = \xi(z) = \xi(1-z) = \Pi(1-(1-z)/\rho)$ complete on the whole complex plane, and the Riemann Hypothesis holds.

Introduction

Functions studied in this paper

In this paper, we study the Riemann zeta function and the Riemann xi function, which are defined as follows.

(0.0) $$\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots$$

(0.1) $$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Hereafter, we will simply refer to these as the "zeta function" and the "xi function" .

In addition, it is known that these zeros are equivalent in the critical strip ($0 < Re(z) < 1$) .

Notation for zeros of ζ(z) and ξ(z)

The zeros ρ of the zeta function $\zeta(z)$ and the xi function $\xi(z)$ are usually written as follows.

$$\sum_{\rho} \frac{1}{\rho} \quad , \quad \prod_{\rho}\left(1-\frac{z}{\rho}\right) \qquad \text{where } \rho \text{ runs over all the zeros.}$$

However, this notation is conceptual and vague, and cannot be used for practical calculations.

(1) Complex number notation

In 1914, Hardy and Littlewood proved that there are an infinite number of zeros on the critical line. This means that there are an infinite number of zeros in the critical strip. So, in this paper, we use the following notation.

$$\sum_{k=1}^{\infty} \frac{1}{\rho_k} \quad , \quad \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(2) Real and imaginary parts notation

However, even with the notation (1), it is difficult to examine the real and imaginary parts of the zeros ρ_k in detail. So, considering that the xi function has conjugate zeros, we replace ρ_k $k=1,2,3,\cdots$ as follows.

$$\rho_{2r-1} = x_r - iy_r \ , \ \rho_{2r} = x_r + iy_r \qquad r=1,2,3,\cdots \qquad (y_r > 0)$$

Using this, the example (1) can be rewritten as follows.

$$\sum_{r=1}^{\infty}\left(\frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r}\right) \quad , \quad \prod_{r=1}^{\infty}\left(1-\frac{z}{x_r - iy_r}\right)\left(1-\frac{z}{x_r + iy_r}\right)$$

1 Two Hadamard Products

Theorem 1.1 (Hadamard product)

Let the Completed Rie,mann Zeta Function $\xi(z)$ be as follws.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) \tag{0.1}$$

Then $\xi(z)$ and $\xi(1-z)$ are factorized by their zeros ρ_k $k=1,2,3,\cdots$ respectively as follows:

$$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) \tag{1.1}$$

$$\xi(1-z) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right) \tag{1.2}$$

Proof

In 1893, ***Hadamard*** proved the following theorem.

$$\zeta(z) = \frac{(2\pi/e)^z}{2(z-1)}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}}$$

The ***Weierstrass*** expression for the gamma function is

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}} = \frac{e^{-\gamma z/2}}{\Gamma(1+z/2)}$$

Substituting this for the right hand side of the above equation,

$$\begin{aligned}\zeta(z) &= \frac{(2\pi/e)^z}{2(z-1)}\frac{e^{-\gamma z/2}}{\Gamma(1+z/2)}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}} \\ &= \frac{e^{z\log 2\pi}e^{-z}}{z-1}\frac{e^{-\gamma z/2}}{2\Gamma(1+z/2)}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}} \\ &= \frac{1}{(z-1)z\Gamma(z/2)}e^{\left(\log 2\pi-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}\end{aligned}$$

From this,

$$-z(1-z)\Gamma\left(\frac{z}{2}\right)\zeta(z) = e^{\left(\log 2\pi-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

Multiplying both sides by $\pi^{-z/2} = e^{-(z\log\pi)/2}$,

$$-z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = e^{\left(\log 2\pi-\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

i.e.

$$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\sum_{k=1}^{\infty}\frac{z}{\rho_k}} \tag{9.1}$$

Here, let

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)=\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right)=\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)$$

$$\sum_{k=1}^{\infty}\frac{z}{\rho_k}=\sum_{r=1}^{\infty}\left(\frac{z}{x_r-iy_r}+\frac{z}{x_r+iy_r}\right)=\sum_{r=1}^{\infty}\frac{2x_r z}{x_r^2+y_r^2}$$

Then, (9.1) becomes

(9.1') $$\xi(z)=e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)\cdot e^{\sum_{r=1}^{\infty}\frac{2x_r z}{x_r^2+y_r^2}}$$

Furthermore, if we express x_n+iy_n $n=1,2,3,\cdots$ whose real parts are $1/2$ as $1/2\pm iy_r$ $r=1,2,3,\cdots$ and those whose real parts are not $1/2$ as $1/2\pm\alpha_s\pm i\beta_s$ $(0<\alpha_s<1/2)$ $s=1,2,3,\cdots$, then (9.1') becomes

(9.1") $$\xi(z)=e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)\cdot e^{\sum_{r=1}^{\infty}\frac{z}{1/4+y_r^2}}$$

$$\times\prod_{s=1}\left\{1-\frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2+\beta_s^2}+\frac{z^2}{(1/2-\alpha_s)^2+\beta_s^2}\right\}\cdot e^{\sum_{s=1}\frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2+\beta_s^2}}$$

$$\times\prod_{s=1}\left\{1-\frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{z^2}{(1/2+\alpha_s)^2+\beta_s^2}\right\}\cdot e^{\sum_{s=1}\frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2+\beta_s^2}}$$

Substituting $z=1$ for both sides of (9.1') and (9.1") ,

(9.2') $$\xi(1)=1=e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}\prod_{r=1}^{\infty}\left(1-\frac{2x_r-1}{x_r^2+y_r^2}\right)\cdot e^{\sum_{r=1}^{\infty}\frac{2x}{x_r^2+y_r^2}}$$

(9.2") $$\xi(1)=1=e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}\times\prod_{s=1}\left\{1+\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}\left\{1-\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}$$

$$\times e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}+\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}}$$

From these,

(9.3) $$\prod_{r=1}^{\infty}\left(1-\frac{2x_r-1}{x_r^2+y_r^2}\right)=\prod_{s=1}\left\{1+\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}\left\{1-\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}$$

(9.4) $$e^{\frac{2x_r}{x_r^2+y_r^2}}=e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}+\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}}$$

Here, conveniently,

$$\left\{1+\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}\left\{1-\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}$$

$$=1+\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}-\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}-\frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}$$

$$= 1 + \frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2} - \frac{2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\frac{2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}$$

$$= 1$$

Then, (9.3) , (9.4) becom

(9.3') $$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right) = 1 \qquad \left\{\text{i.e. } \prod_{r=1}^{\infty}\left(1-\frac{1}{x_r-iy_r}\right)\left(1-\frac{1}{x_r+iy_r}\right) = \prod_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right) = 1\right\}$$

(9.4') $$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = \sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} + \sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2} + \frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}$$

Substituting (9.3') for (9.2') ,

$$\xi(1) = 1 = e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}} \cdot e^{\sum_{r=1}^{\infty}\frac{2x}{x_r^2+y_r^2}} = e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}} e^{\sum_{k=1}^{\infty}\frac{1}{\rho_k}}$$

Since $1 = e^0$,

(9.5) $$\sum_{k=1}^{\infty}\frac{1}{\rho_k} = 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2} = 0.0230957\cdots$$

Therefore,

$$e^{\sum_{k=1}^{\infty}\frac{z}{\rho_k}} = e^{\left(1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}\right)z}$$

Substituting this for the right side of (9.1) ,

$$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)\cdot e^{\left(1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}\right)z}$$

i.e.

(0.1) $$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

Since both sides are holomorphic on the whole complex plane, replacing z with $1-z$

(0.2) $$\xi(1-z) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Q.E.D.

Note

When $\xi(z) = 1+A_1z^1+A_2z^2+A_3z^4+\cdots$, according to Vieta's formulas, the following holds.

(9.5) $$\sum_{k=1}^{\infty}\frac{1}{\rho_k} = -A_1 = -\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right) = -(-0.0230957\cdots)$$

2 Functional Equation

Theorem 2.1 (Functional Equation)

Let xi function be

(0.1) $$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Then, the following expression holds on the whole complex plane .

(2.1) $$\xi(z) = \xi(1-z)$$

Proof

According to ***Riemann***, the following holds except for two points on the complex plane .

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \qquad Re(z) \neq 0,1$$

$$z(1-z) = (1-z)\{1-(1-z)\}$$

Substituting these for the right side of (0.1) ,

$$\xi(z) = -(1-z)\{1-(1-z)\}\pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) = \xi(1-z)$$

Since $\xi(z)$ is holomorphic on the whole complex plane, the fnctional equation holds on the whole complex plane.

Q.E.D.

3 Inequality of two Hadamard Products

From Theorem 1.1 , $\xi(z)$ and $\xi(1-z)$ are factorized by their zeros ρ_k $k=1,2,3,\cdots$ respectively as follows:

(1.1) $$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(1.2) $$\xi(1-z) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

And, from Theorem 2.1 , the following equation holds on the whole complex plane.

(2.1) $$\xi(z) = \xi(1-z)$$

The question here is whether the following equation holds on the whole complex plane.

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Putting $\rho_{2r-1}=x_r-iy_r$, $\rho_{2r}=x_r+iy_r$ $r=1,2,3,\cdots$,

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right) = \prod_{r=1}^{\infty}\left(1-\frac{1-z}{x_r-iy_r}\right)\left(1-\frac{1-z}{x_r+iy_r}\right)$$

Expanding the ( )( ) on both sides,

$$\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right) = \prod_{r=1}^{\infty}\left(1-\frac{2x_r-1}{x_r^2+y_r^2}-\frac{2(1-x_r)z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)$$

Observing this, both sides are different series generally. That is, this equality does not hold on the whole complex plane generally.

Remark

The above means that the following four-step logic does not hold unconditionally.

$$\Pi(1-z/\rho)=\xi(z) \quad , \quad \xi(z) = \xi(1-z) \quad , \quad \xi(1-z)=\Pi(1-(1-z)/\rho)$$

$$\Downarrow$$

$$\Pi(1-z/\rho)=\Pi(1-(1-z)/\rho)$$

4 Equality Conditions for two Hadamard Products

In this chapter, we will find the necessary and sufficient condition for $\Pi(1-z/\rho)=\Pi(1-(1-z)/\rho)$, thereby proving that the Riemann hypothesis holds as a theorem.

Theorem 4.1 (Functional Equation for Hadamard Products)

Let the xi Function $\xi(z)$ and $\xi(1-z)$ are factorized by their zeros ρ_k $k=1,2,3,\cdots$ as follows.

(1.1) $$\xi(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(1.2) $$\xi(1-z)=\prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Then, if and only if $Re(\rho_k)=1/2$ $k=1,2,3,\cdots$, The following equation holds on the whole complex plane.

(4.1) $$\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)=\prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Proof

Express the zeros ρ_k as follows, with separate real and imaginary parts:

$$\rho_{2r-1}=x_r-iy_r\ ,\ \rho_{2r}=x_r+iy_r\quad r=1,2,3,\cdots\qquad (y_r>0)$$

Then (4.1) becomes

(4.1') $$\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right)=\prod_{r=1}^{\infty}\left(1-\frac{1-z}{x_r-iy_r}\right)\left(1-\frac{1-z}{x_r+iy_r}\right)$$

I. Sufficiency

If $x_r=1/2$ $r=1,2,3,\cdots$, (4.1') becoms

(0.4") $$\prod_{r=1}^{\infty}\left(1-\frac{z}{1/2-iy_r}\right)\left(1-\frac{z}{1/2+iy_r}\right)=\prod_{r=1}^{\infty}\left(1-\frac{1-z}{1/2-iy_r}\right)\left(1-\frac{1-z}{1/2+iy_r}\right)$$

Let us find the roots of the pair on both sides. then,

Left side $$1-\frac{z}{1/2-iy_r}=0\ ,\quad 1-\frac{z}{1/2+iy_r}=0\qquad r=1,2,3,\cdots$$

Right side $$1-\frac{1-z}{1/2-iy_r}=0\ ,\quad 1-\frac{1-z}{1/2+iy_r}=0\qquad r=1,2,3,\cdots$$

From these,

Left side $z=1/2-iy_r$, $z=1/2+iy_r$ $r=1,2,3,\cdots$

Right side $z=1/2+iy_r$, $z=1/2-iy_r$ $r=1,2,3,\cdots$

The zeros on the left and right sides are crosswise coincident. This means that the left and right sides are the same.

In fact, (4.1") can be rewritten as follows.

(4.1") $$\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)=\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)$$

This holds clearly on the whole complex plane. So, (4.1) holds on the whole complex plane.

II. Necessity

(1) When neither x_r nor y_r overlap

Let $\rho_{2r-1} = x_r - iy_r$, $\rho_{2r} = x_r + iy_r$ $r=1,2,3,\cdots$. Then

(4.1')
$$\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right) = \prod_{r=1}^{\infty}\left(1-\frac{1-z}{x_r-iy_r}\right)\left(1-\frac{1-z}{x_r+iy_r}\right)$$

Let us find the roots of the pair on both sides. then,

Left side $1-\dfrac{z}{x_r-iy_r}=0$, $1-\dfrac{z}{x_r+iy_r}=0$ $r=1,2,3,\cdots$

Right side $1-\dfrac{1-z}{x_r-iy_r}=0$, $1-\dfrac{1-z}{x_r+iy_r}=0$ $r=1,2,3,\cdots$

From these,

Left side $z=x_r-iy_r$, $z=x_r+iy_r$ $r=1,2,3,\cdots$

Right side $z=1-x_r+iy_r$, $z=1-x_r-iy_r$ $r=1,2,3,\cdots$

Since both sides have to be equal, make equations where the signs and symbols of the imaginary parts match.

Then,

$$x_r-iy_r = 1-x_r-iy_r \qquad r=1,2,3,\cdots$$

$$x_r+iy_r = 1-x_r+iy_r \qquad r=1,2,3,\cdots$$

From these,

$$x_r = 1-x_r \qquad r=1,2,3,\cdots$$

i.e. $x_r = 1/2$ $r=1,2,3,\cdots$

(2) When x_r overlap

For x_r to overlap, at least two sets of conjugate zeros are required.

For ease of viewing, swap the order so that x_1 and x_2 overlap but there are no overlap after x_3 .

Substituting x_2 for x_1, ρ_k $k=1,\cdots,4$ become:

$$\rho_1 = x_1 - iy_1 \ , \ \rho_2 = x_1 + iy_1 \ , \ \rho_3 = x_1 - iy_2 \ , \ \rho_4 = x_1 + iy_2$$

Then,

Left side $\displaystyle\prod_{k=1}^{4}\left(1-\frac{z}{\rho_k}\right) = \left(1-\frac{z}{x_1-iy_1}\right)\left(1-\frac{z}{x_1+iy_1}\right)\left(1-\frac{z}{x_1-iy_2}\right)\left(1-\frac{z}{x_1+iy_2}\right)$

Right side $\displaystyle\prod_{k=1}^{4}\left(1-\frac{1-z}{\rho_k}\right) = \left(1-\frac{1-z}{x_1-iy_1}\right)\left(1-\frac{1-z}{x_1+iy_1}\right)\left(1-\frac{1-z}{x_1-iy_2}\right)\left(1-\frac{1-z}{x_1+iy_2}\right)$

Let us find the roots of the pairs on both sides. then,

Left side $1-\dfrac{z}{x_1-iy_1}=0$, $1-\dfrac{z}{x_1+iy_1}=0$, $1-\dfrac{z}{x_1-iy_2}=0$, $1-\dfrac{z}{x_1+iy_2}=0$

Right side $1-\dfrac{1-z}{x_1-iy_1}=0$, $1-\dfrac{1-z}{x_1+iy_1}=0$, $1-\dfrac{1-z}{x_1-iy_2}=0$, $1-\dfrac{1-z}{x_1+iy_2}=0$

From these,

Left side $z=x_1-iy_1$, $z=x_1+iy_1$, $z=x_1-iy_2$, $z=x_1+iy_2$

Right side $z=1-x_1+iy_1$, $z=1-x_1-iy_1$, $z=1-x_1+iy_2$, $z=1-x_1-iy_2$

Since both sides of each pair have to be equal, make equations where the signs and symbols of the imaginary parts match. Then,

$$x_1 - iy_1 = 1 - x_1 - iy_1 \quad , \quad x_1 - iy_2 = 1 - x_1 - iy_2$$

$$x_1 + iy_1 = 1 - x_1 + iy_1 \quad , \quad x_1 + iy_2 = 1 - x_1 + iy_2$$

From these,

$$x_1 = 1 - x_1$$

i.e. $\quad x_1 = 1/2$

As the result,

Left side $\; z = 1/2 - iy_1 \;,\; z = 1/2 + iy_1 \quad , \quad z = 1/2 - iy_2 \;,\; z = 1/2 + iy_2$

Right side $\; z = 1/2 + iy_1 \;,\; z = 1/2 - iy_1 \quad , \quad z = 1/2 + iy_2 \;,\; z = 1/2 - iy_2$

That is, $z = 1/2 \pm iy_1$, $z = 1/2 \pm iy_2$ are double roots respectively

(3) When y_r overlap

For y_r to overlap, at least two sets of conjugate zeros are required.

For ease of viewing, swap the order so that y_1 and y_2 overlap but there are no overlap after y_3.

Substituting y_2 for y_1, $\rho_k \quad k = 1, \cdots, 4$ become:

$$\rho_1 = x_1 - iy_1 \;,\; \rho_2 = x_1 + iy_1 \;,\; \rho_3 = x_2 - iy_1 \;,\; \rho_4 = x_2 + iy_1$$

Then,

Left side $\displaystyle \prod_{k=1}^{4}\left(1 - \frac{z}{\rho_k}\right) = \left(1 - \frac{z}{x_1 - iy_1}\right)\left(1 - \frac{z}{x_1 + iy_1}\right)\left(1 - \frac{z}{x_2 - iy_1}\right)\left(1 - \frac{z}{x_2 + iy_1}\right)$

Right side $\displaystyle \prod_{k=1}^{4}\left(1 - \frac{1-z}{\rho_k}\right) = \left(1 - \frac{1-z}{x_1 - iy_1}\right)\left(1 - \frac{1-z}{x_1 + iy_1}\right)\left(1 - \frac{1-z}{x_2 - iy_1}\right)\left(1 - \frac{1-z}{x_2 + iy_1}\right)$

If the following calculation gives $x_1 \neq x_2$, then these are probably zeros outside the critical line.

Let us find the roots of the pairs on both sides. then,

Left side $\displaystyle 1 - \frac{z}{x_1 - iy_1} = 0 \;,\; 1 - \frac{z}{x_1 + iy_1} = 0 \quad , \quad 1 - \frac{z}{x_2 - iy_1} = 0 \;,\; 1 - \frac{z}{x_2 + iy_1} = 0$

Right side $\displaystyle 1 - \frac{1-z}{x_1 - iy_1} = 0 \;,\; 1 - \frac{1-z}{x_1 + iy_1} = 0 \quad , \quad 1 - \frac{1-z}{x_2 - iy_1} = 0 \;,\; 1 - \frac{1-z}{x_2 + iy_1} = 0$

From these,

Left side $\; z = x_1 - iy_1 \quad , \quad z = x_1 + iy_1 \quad , \quad z = x_2 - iy_1 \quad , \quad z = x_2 + iy_1$

Right side $\; z = 1 - x_1 + iy_1 \;,\; z = 1 - x_1 - iy_1 \quad , \quad z = 1 - x_2 + iy_1 \;,\; z = 1 - x_2 - iy_1$

Since both sides of each pair have to be equal, make equations where the signs and symbols of the imaginary parts match. Then,

$$x_1 - iy_1 = 1 - x_1 - iy_1 \quad , \quad x_2 - iy_1 = 1 - x_2 - iy_1$$

$$x_1 + iy_1 = 1 - x_1 + iy_1 \quad , \quad x_2 + iy_1 = 1 - x_2 + iy_1$$

From these,

$$x_1 = 1 - x_1 \quad , \quad x_2 = 1 - x_2$$

i.e. $\quad x_1 = x_2 = 1/2$

As the result,

Left side $\; z = 1/2 - iy_1 \;,\; z = 1/2 + iy_1 \quad , \quad z = 1/2 - iy_1 \;,\; z = 1/2 + iy_1$

Right side $\; z = 1/2 + iy_1 \;,\; z = 1/2 - iy_1 \quad , \quad z = 1/2 + iy_1 \;,\; z = 1/2 - iy_1$

That is, $z = 1/2 \pm iy_1$ are quadruple roots. The possibility of zeros outside the critical line disappears.

(4) When both x_r and y_r overlap

For these to overlap, at least two sets of conjugate zeros are required.
For ease of viewing, swap the order so that there are no overlap after x_3, y_3.

Substituting x_2, y_2 for x_1, y_1, ρ_k $k = 1, \cdots, 4$ become:

$$\rho_1 = x_1 - iy_1 \ , \ \rho_2 = x_1 + iy_1 \ , \ \rho_3 = x_1 - iy_1 \ , \ \rho_4 = x_1 + iy_1$$

Using a similar way to (3), $x_1 = 1/2$, and $z = 1/2 \pm iy_1$ are quadruple roots.

As the result, if and only if $Re(\rho_k) = 1/2 \quad k = 1, 2, 3, \cdots$, (4.1) holds on the whole comples plane.

Q.E.D.

Another Proof

Let $\rho_{2r-1} = x_r - iy_r \ , \ \rho_{2r} = x_r + iy_r \quad r = 1, 2, 3, \cdots \qquad (y_r > 0)$. Then (4.1) becomes

(4.1') $$\prod_{r=1}^{\infty}\left(1 - \frac{z}{x_r - iy_r}\right)\left(1 - \frac{z}{x_r + iy_r}\right) = \prod_{r=1}^{\infty}\left(1 - \frac{1-z}{x_r - iy_r}\right)\left(1 - \frac{1-z}{x_r + iy_r}\right)$$

I. Sufficiency

If $x_r = 1/2 \quad r = 1, 2, 3, \cdots$, (4.1') becomes

(0.4") $$\prod_{r=1}^{\infty}\left(1 - \frac{z}{1/2 - iy_r}\right)\left(1 - \frac{z}{1/2 + iy_r}\right) = \prod_{r=1}^{\infty}\left(1 - \frac{1-z}{1/2 - iy_r}\right)\left(1 - \frac{1-z}{1/2 + iy_r}\right)$$

Expanding the ( )( ) on both sides,

(4.1") $$\prod_{r=1}^{\infty}\left(1 - \frac{z}{1/4 + y_r^2} + \frac{z^2}{1/4 + y_r^2}\right) = \prod_{r=1}^{\infty}\left(1 - \frac{z}{1/4 + y_r^2} + \frac{z^2}{1/4 + y_r^2}\right)$$

This holds clearly on the whole complex plane. So, (4.1) holds on the whole complex plane.

The important thing here is that these zeros actually exist. (e.g. $1/2 + i\,14.13472514\cdots$)

For these existing zerors, the following is completed from Theorem 1.1 and Theorem 2.1 .

$$\xi(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{\rho_k}\right) = \prod_{k=1}^{\infty}\left(1 - \frac{1-z}{\rho_k}\right) = \xi(1-z)$$

As the result, the following equation holds from Vieta's formula (9.5) .

(9.5') $$\sum_{r=1}^{\infty}\left(\frac{1}{1/2 - iy_r} + \frac{1}{1/2 + iy_r}\right) = 1 + \frac{\gamma}{2} - \log 2 - \frac{\log \pi}{2} = 0.0230957\cdots$$

II. Necessity

Currently, no nontrivial zeros outside the critical line have been found. But now, we will assume that such zeros exist. It is known that a set of such zeros should consist of the following four.

$$1/2 - \alpha_s \pm i\beta_s \ , \ 1/2 + \alpha_s \pm i\beta_s \quad (0 < \alpha_s < 1/2 \ , \ \beta_s > 0)$$

Then, according to Vieta's formula (9.5) , the following equation has to hold.

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/2 - iy_r} + \frac{1}{1/2 + iy_r}\right) + \sum_{s=1}\left(\frac{1}{1/2 - \alpha_s - i\beta_s} + \frac{1}{1/2 - \alpha_s + i\beta_s}\right)$$
$$+ \sum_{s=1}\left(\frac{1}{1/2 + \alpha_s - i\beta_s} + \frac{1}{1/2 + \alpha_s + i\beta_s}\right) = 1 + \frac{\gamma}{2} - \log 2 - \frac{\log \pi}{2}$$

i.e.

(9.5")
$$\sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right)+\sum_{s=1}\left\{\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}+\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}$$
$$=1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}=0.0230957\cdots$$

However, since $0<1-2\alpha_s<1+2\alpha_s<2$ for $0<\alpha_s<1/2$,

$$\sum_{s=1}\left\{\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}+\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}>0$$

Therefore, (9.5") contradicts (9.5') . Thus, there should be no zeros outside the critical line within the critical strip. As the result, (4.1) is holds only if $Re(\rho_k)=1/2 \quad k=1,2,3,\cdots$

Q.E.D.

Theorem 4.2 (Riemann)

When $\rho_k \quad k=1,2,3,\cdots$ are non-trivial zeros of the Riemann Zeta Function $\zeta(z)$,

$Re(\rho_k)=1/2 \quad k=1,2,3,\cdots$

Proof

As first, the non-trivial zeros ρ_k of the Riemann Zeta Function $\zeta(z)$ are equivalent to the zeros of the Completed Riemann Zeta Function $\xi(z)$.

At these zeros, the functional equation has to hold not only for the functions but also for the Hadamard products. That is, the followings have to hold on the whole complex plane:

$$\xi(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)=\prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)=\xi(1-z)$$

Theorem 4.1 proves that the necessary and sufficient condition for this is $Re(\rho_k)=1/2 \quad k=1,2,3,\cdots$.

Therefore, the Riemann Hypothesis holds as a theorem.

Q.E.D.

Note

The proof method in this paper can also be applied to Dirichlet L- functions. That is, it appears that the Riemann hypothesis also holds for Dirichlet L- functions.

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