Proof of the Riemann Hypothesis by Vieta's Formulas

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Abstract

(1) Riemann xi functkion $\xi(z)$ and $\xi(1-z)$ are factored into the Hadamard products as follows.

$$\xi(z) = \Pi(1-z/\rho) \quad , \quad \xi(1-z) = \Pi(1-(1-z)/\rho)$$

(2) $\xi(z) = \xi(1-z)$ holds on the whole complex plane..

(3) $\Pi(1-z/\rho) = \Pi(1-(1-z)/\rho)$ holds on the whole complex plane if and only if all $Re(\rho)$ are $1/2$.

(4) Thus, $\Pi(1-z/\rho) = \xi(z) = \xi(1-z) = \Pi(1-(1-z)/\rho)$. Then, the Riemann hypothesis holds.

1. Functions studied in this paper

In this paper, we study the Riemann zeta function and the Riemann xi function, which are defined as follows.

(0.0) $$\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots$$

(0.1) $$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Hereafter, we will simply refer to these as the "zeta function" and the "xi function" .

In addition, it is known that these zeros are equivalent in the critical strip ($0 < Re(z) < 1$) .

2. Notation for zeros of ζ(z) and ξ(z)

The zeros ρ of the zeta function $\zeta(z)$ and the xi function $\xi(z)$ are usually written as follows.

$$\sum_{\rho} \frac{1}{\rho} \quad , \quad \prod_{\rho}\left(1-\frac{z}{\rho}\right)$$ where ρ runs over all the zeros.

However, this notation is conceptual and vague, and cannot be used for practical calculations.

(1) Complex number notation

In 1914, Hardy and Littlewood proved that there are an infinite number of zeros on the critical line. This means that there are an infinite number of zeros in the critical strip. So, in this paper, we use the following notation.

$$\sum_{k=1}^{\infty} \frac{1}{\rho_k} \quad , \quad \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(2) Real and imaginary parts notation

However, even with the notation (1), it is difficult to examine the real and imaginary parts of the zeros ρ_k in detail. So, considering that the xi function has conjugate zeros, we replace ρ_k $k=1,2,3,\cdots$ as follows.

$$\rho_{2r-1} = x_r - iy_r \ , \ \rho_{2r} = x_r + iy_r \quad r=1,2,3,\cdots \qquad (y_r > 0)$$

Using this, the example (1) can be rewritten as follows.

$$\sum_{r=1}^{\infty}\left(\frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r}\right) \quad , \quad \prod_{r=1}^{\infty}\left(1-\frac{z}{x_r - iy_r}\right)\left(1-\frac{z}{x_r + iy_r}\right)$$

3. Proof of the Riemann Hypothesis

Theorem 3.1 (Hadamard product)

The xi functions $\xi(z)$ and $\xi(1-z)$ are factorized by the zeros ρ_k $k=1,2,3,\cdots$ respectively as follows.

(0.1) $$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(0.2) $$\xi(1-z) = -z(1-z)\pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Proof

In 1893, ***Hadamard*** proved the following theorem.

$$\zeta(z) = \frac{(2\pi/e)^z}{2(z-1)}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}}$$

The ***Weierstrass*** expression for the gamma function is

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}} = \frac{e^{-\gamma z/2}}{\Gamma(1+z/2)}$$

Substituting this for the right hand side of the above equation,

$$\zeta(z) = \frac{(2\pi/e)^z}{2(z-1)}\frac{e^{-\gamma z/2}}{\Gamma(1+z/2)}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

$$= \frac{e^{z\log 2\pi}e^{-z}}{z-1}\frac{e^{-\gamma z/2}}{2\Gamma(1+z/2)}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

$$= \frac{1}{(z-1)z\Gamma(z/2)}e^{\left(\log 2\pi-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

From this,

$$-z(1-z)\Gamma\left(\frac{z}{2}\right)\zeta(z) = e^{\left(\log 2\pi-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

Multiplying both sides by $\pi^{-z/2} = e^{-(z\log\pi)/2}$,

$$-z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = e^{\left(\log 2\pi-\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

i.e.

(9.1) $$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)\cdot e^{\sum_{k=1}^{\infty}\frac{z}{\rho_k}}$$

Here, let

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right) = \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)$$

$$\sum_{k=1}^{\infty}\frac{z}{\rho_k} = \sum_{r=1}^{\infty}\left(\frac{z}{x_r-iy_r}+\frac{z}{x_r+iy_r}\right) = \sum_{r=1}^{\infty}\frac{2x_r z}{x_r^2+y_r^2}$$

Then, (9.1) becomes

(9.1') $$\xi(z) = e^{\left(\log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}\right)z} \prod_{r=1}^{\infty}\left(1 - \frac{2x_r z}{x_r^2 + y_r^2} + \frac{z^2}{x_r^2 + y_r^2}\right) \cdot e^{\sum_{r=1}^{\infty}\frac{2x_r z}{x_r^2 + y_r^2}}$$

Furthermore, if we express $x_n + iy_n \quad n=1,2,3,\cdots$ whose real parts are $1/2$ as $1/2 \pm iy_r \quad r=1,2,3,\cdots$ and those whose real parts are not $1/2$ as $1/2 \pm \alpha_s \pm i\beta_s \quad (0 < \alpha_s < 1/2) \quad s=1,2,3,\cdots$, then (9.1') becomes

(9.1") $$\xi(z) = e^{\left(\log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}\right)z} \prod_{r=1}^{\infty}\left(1 - \frac{z}{1/4 + y_r^2} + \frac{z^2}{1/4 + y_r^2}\right) \cdot e^{\sum_{r=1}^{\infty}\frac{z}{1/4 + y_r^2}}$$

$$\times \prod_{s=1}\left\{1 - \frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2 + \beta_s^2} + \frac{z^2}{(1/2-\alpha_s)^2 + \beta_s^2}\right\} \cdot e^{\sum_{s=1}\frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2 + \beta_s^2}}$$

$$\times \prod_{s=1}\left\{1 - \frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2 + \beta_s^2} + \frac{z^2}{(1/2+\alpha_s)^2 + \beta_s^2}\right\} \cdot e^{\sum_{s=1}\frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2 + \beta_s^2}}$$

Substituting $z=1$ for both sides of (9.1') and (9.1") ,

(9.2') $$\xi(1) = 1 = e^{\log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}} \prod_{r=1}^{\infty}\left(1 - \frac{2x_r - 1}{x_r^2 + y_r^2}\right) \cdot e^{\sum_{r=1}^{\infty}\frac{2x}{x_r^2 + y_r^2}}$$

(9.2") $$\xi(1) = 1 = e^{\log 2 + \frac{\log\pi}{2} - 1 - \frac{\gamma}{2}} \times \prod_{s=1}\left\{1 + \frac{2\alpha_s}{(1/2-\alpha_s)^2 + \beta_s^2}\right\}\left\{1 - \frac{2\alpha_s}{(1/2+\alpha_s)^2 + \beta_s^2}\right\}$$

$$\times e^{\sum_{r=1}^{\infty}\frac{1}{1/4 + y_r^2} + \sum_{s=1}\left\{\frac{1-2\alpha_s}{(1/2-\alpha_s)^2 + \beta_s^2} + \frac{1+2\alpha_s}{(1/2+\alpha_s)^2 + \beta_s^2}\right\}}$$

From these,

(9.3) $$\prod_{r=1}^{\infty}\left(1 - \frac{2x_r - 1}{x_r^2 + y_r^2}\right) = \prod_{s=1}\left\{1 + \frac{2\alpha_s}{(1/2-\alpha_s)^2 + \beta_s^2}\right\}\left\{1 - \frac{2\alpha_s}{(1/2+\alpha_s)^2 + \beta_s^2}\right\}$$

(9.4) $$e^{\frac{2x_r}{x_r^2 + y_r^2}} = e^{\sum_{r=1}^{\infty}\frac{1}{1/4 + y_r^2} + \sum_{s=1}\left\{\frac{1-2\alpha_s}{(1/2-\alpha_s)^2 + \beta_s^2} + \frac{1+2\alpha_s}{(1/2+\alpha_s)^2 + \beta_s^2}\right\}}$$

Here, conveniently,

$$\left\{1 + \frac{2\alpha_s}{(1/2-\alpha_s)^2 + \beta_s^2}\right\}\left\{1 - \frac{2\alpha_s}{(1/2+\alpha_s)^2 + \beta_s^2}\right\}$$

$$= 1 + \frac{2\alpha_s}{(1/2-\alpha_s)^2 + \beta_s^2} - \frac{2\alpha_s}{(1/2+\alpha_s)^2 + \beta_s^2} - \frac{2\alpha_s}{(1/2-\alpha_s)^2 + \beta_s^2}\,\frac{2\alpha_s}{(1/2+\alpha_s)^2 + \beta_s^2}$$

$$= 1 + \frac{2\alpha_s}{(1/2-\alpha_s)^2 + \beta_s^2}\,\frac{2\alpha_s}{(1/2+\alpha_s)^2 + \beta_s^2} - \frac{2\alpha_s}{(1/2-\alpha_s)^2 + \beta_s^2}\,\frac{2\alpha_s}{(1/2+\alpha_s)^2 + \beta_s^2}$$

$$= 1$$

Then, (9.3) , (9.4) becom

(9.3') $$\prod_{n=1}^{\infty}\left(1 - \frac{2x_n - 1}{x_n^2 + y_n^2}\right) = 1 \qquad \left\{\text{i.e. } \prod_{r=1}^{\infty}\left(1 - \frac{1}{x_r - iy_r}\right)\left(1 - \frac{1}{x_r + iy_r}\right) = \prod_{k=1}^{\infty}\left(1 - \frac{1}{\rho_k}\right) = 1\right\}$$

(9.4') $$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2}=\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}$$

Substituting (9.3') for (9.2') ,

$$\xi(1)=1=e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}\cdot e^{\sum_{r=1}^{\infty}\frac{2x}{x_r^2+y_r^2}}=e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}e^{\sum_{k=1}^{\infty}\frac{1}{\rho_k}}$$

From this,

(9.5) $$\sum_{k=1}^{\infty}\frac{1}{\rho_k}=1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}=0.0230957\cdots$$

Therefore,

$$e^{\sum_{k=1}^{\infty}\frac{z}{\rho_k}}=e^{\left(1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}\right)z}$$

Substituting this for the right side of (9.1) ,

$$\xi(z)=e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)\cdot e^{\left(1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}\right)z}$$

i.e.

(0.1) $$\xi(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

Since both sides are holomorphic on the whole complex plane, replacing z with $1-z$

(0.2) $$\xi(1-z)=\prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Q.E.D.

Note

When $\xi(z)=1+A_1z^1+A_2z^2+A_3z^4+\cdots$, according to Vieta's formulas, the following holds.

(9.5) $$\sum_{k=1}^{\infty}\frac{1}{\rho_k}=-A_1=-\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)=-(-0.0230957\cdots)$$

Theorem 3.2 (Functional Equation)

Let xi function be

(0.1) $$\xi(z)=-z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Then, the following expression holds on the whole complex plane .

(0.3) $$\xi(z)=\xi(1-z)$$

Proof

According to ***Riemann***, the following holds except for two points on the complex plane .

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)=\pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)\qquad Re(z)\neq 0,1$$

$$z(1-z)=(1-z)\{1-(1-z)\}$$

Substituting these for the right side of (0.1) ,

$$\xi(z)=-(1-z)\{1-(1-z)\}\pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)=\xi(1-z)$$

Since $\xi(z)$ is holomorphic on the whole complex plane, (0.3) also holds on the whole complex plane.

Q.E.D.

Theorem 3.3 (Hadamard product Equality)

Let the xi functions $\xi(z)$ and $\xi(1-z)$ are factorized by the zeros ρ_k $k=1,2,3,\cdots$ respectively as follows.

(0.1') $$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(0.2') $$\xi(1-z) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Then, if and only if $Re(\rho_k)=1/2$ $k=1,2,3,\cdots$, the following equation holds on the whole complex plane.

(0.4) $$\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Proof

Express the zeros ρ_k as follows, with separate real and imaginary parts:

$$\rho_{2r-1}=x_r-iy_r \ , \ \rho_{2r}=x_r+iy_r \quad r=1,2,3,\cdots \qquad (y_r > 0)$$

Then (0.4) becoms

(0.4') $$\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right) = \prod_{r=1}^{\infty}\left(1-\frac{1-z}{x_r-iy_r}\right)\left(1-\frac{1-z}{x_r+iy_r}\right)$$

I. Sufficiency

If $x_r = 1/2$ $r=1,2,3,\cdots$, (0.4') becomes

(0.4") $$\prod_{r=1}^{\infty}\left(1-\frac{z}{1/2-iy_r}\right)\left(1-\frac{z}{1/2+iy_r}\right) = \prod_{r=1}^{\infty}\left(1-\frac{1-z}{1/2-iy_r}\right)\left(1-\frac{1-z}{1/2+iy_r}\right)$$

Let us find the roots of the pair on both sides. then,

Left side $1-\dfrac{z}{1/2-iy_r}=0$, $1-\dfrac{z}{1/2+iy_r}=0$ $\qquad r=1,2,3,\cdots$

Right side $1-\dfrac{1-z}{1/2-iy_r}=0$, $1-\dfrac{1-z}{1/2+iy_r}=0$ $\qquad r=1,2,3,\cdots$

From these,

Left side $z = 1/2-iy_r$, $z = 1/2+iy_r$ $\qquad r=1,2,3,\cdots$

Right side $z = 1/2+iy_r$, $z = 1/2-iy_r$ $\qquad r=1,2,3,\cdots$

The zeros on the left and right sides are crosswise coincident. That is, (0.4") holds identically. This is equivalent to (0.4) holds identically.

As the result, from Theorem 3.1 and Theorem 3.2 , the following is completed on the whole complex plane.

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \xi(z) = \xi(1-z) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

And, the following equation holds from Vieta's formula (9.5) .

(9.5') $$\sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right) = 1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2} = 0.0230957\cdots$$

II. Necessity

Now, suppose that in addition to the zeros on the critical line, there are also zeros off the critical line within the critical strip. It is known that a set of such zeros should consist of the following four.

$$1/2 - \alpha_s \pm i\beta_s \ , \ 1/2 + \alpha_s \pm i\beta_s \quad \left(0 < \alpha_s < 1/2 \ , \ \beta_s > 0 \right)$$

Then, according to Vieta's formula (9.5) , the following equation has to hold.

$$\sum_{r=1}^{\infty} \left(\frac{1}{1/2 - iy_r} + \frac{1}{1/2 + iy_r} \right) + \sum_{s=1} \left(\frac{1}{1/2 - \alpha_s - i\beta_s} + \frac{1}{1/2 - \alpha_s + i\beta_s} \right)$$
$$+ \sum_{s=1} \left(\frac{1}{1/2 + \alpha_s - i\beta_s} + \frac{1}{1/2 + \alpha_s + i\beta_s} \right) = 1 + \frac{\gamma}{2} - \log 2 - \frac{\log \pi}{2}$$

i.e.

(9.5")
$$\sum_{r=1}^{\infty} \left(\frac{1}{1/2 - iy_r} + \frac{1}{1/2 + iy_r} \right) + \sum_{s=1} \left\{ \frac{1 - 2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2} + \frac{1 + 2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2} \right\}$$
$$= 1 + \frac{\gamma}{2} - \log 2 - \frac{\log \pi}{2} = 0.0230957\cdots$$

However, since $0 < 1 - 2\alpha_s < 1 + 2\alpha_s < 2$ for $0 < \alpha_s < 1/2$,

$$\sum_{s=1} \left\{ \frac{1 - 2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2} + \frac{1 + 2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2} \right\} > 0$$

Therefore, (9.5") contradicts (9.5') . Thus, there should be no zeros outside the critical line within the critical strip. As the result, (0.4) is holds only if $Re(\rho_k) = 1/2 \quad k = 1, 2, 3, \cdots$.

III. Thus, $Re(\rho_k) = 1/2 \quad k = 1, 2, 3, \cdots$ is a necessary and sufficient condition for (0.4) .

Q.E.D.

Based on the above three theorems, if and only if $Re(\rho_k) = 1/2 \quad k = 1, 2, 3, \cdots$, the following **the four step reasoning** is completed on the whole complex plane.

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right) = \xi(z) = \xi(1 - z) = \prod_{k=1}^{\infty} \left(1 - \frac{1 - z}{\rho_k} \right)$$

Thus, the Riemann hypothesis is established as a theorem.

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2026.01.08 Supplemented Proof of Theorem 3.1.

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