Proof of Riemann Hypothesis for Dirichlet Beta using Li Coefficients

Abstract

(1) The Li coefficients for the Completed Dirichlet Beta Function $\omega(z)$ can be defined at the left edge of the critical strip. And this can be derived from both $\omega(z)$ and the Hadamard product. If these are written as ${}_0\lambda_n$ and ${}_0\mu_n$ respectively, then ${}_0\lambda_n = {}_0\mu_n$ must be true.

(2) The Li coefficients can also be defined at the right edge of the critical strip. And this can also be derived from both $\omega(z)$ and the Hadamard product. If these are written as ${}_1\lambda_n$ and ${}_1\mu_n$ respectively, then ${}_1\lambda_n = {}_1\mu_n$ must be true.

(3) The Li coefficients from $\omega(z)$ becomes ${}_0\lambda_n = {}_1\lambda_n$ by functional equation and the definition. On the other hand, ${}_0\mu_n$, ${}_1\mu_n$ obtained from the Hadamard product can be expressed by the conjugate zeros $x_r \pm iy_r \quad r=1,2,3,\cdots$. However, according to this, ${}_0\mu_n \neq {}_1\mu_n$ in generally.

(4) A necessary and sufficient condition for ${}_0\mu_n = {}_1\mu_n$ is that the zeros of $\omega(z)$ are on the critical line $x=1/2$. Only in this case, ${}_0\lambda_n = {}_0\mu_n = {}_1\mu_n = {}_1\lambda_n$ is completed. And these Li Coefficients satisfy Li's Criterion. Thus, the Riemann hypothesis for the Dirichlet Beta Function $\beta(z)$ holds as a theorem.

Introduction

Functions studied in this paper

In this paper, we study the Dirichlet Beta Function $\beta(z)$ and the Completed Dirichlet Beta Function $\omega(z)$, which are defined as follows.

$$\beta(z) = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + \cdots = \frac{1}{4^z}\left\{\zeta\left(z,\frac{1}{4}\right) - \zeta\left(z,\frac{3}{4}\right)\right\} \tag{0.0}$$

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z) \tag{0.1}$$

In addition, it is known that these zeros are equivalent in the critical strip ($0 < Re(z) < 1$) .

Notation for zeros of β(z) and ω(z)

In this paper, the zeros ρ of the $\beta(z)$ and the $\omega(z)$ are described as follows:

(1) Complex number notation

$$\sum_{k=1}^{\infty}\frac{1}{\rho_k} \quad , \quad \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(2) Real and imaginary parts notation

Putting $\rho_{2r-1} = x_r - iy_r \;,\; \rho_{2r} = x_r + iy_r \quad r=1,2,3,\cdots \quad (y_r > 0)$,

$$\sum_{r=1}^{\infty}\left(\frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r}\right) \quad , \quad \prod_{r=1}^{\infty}\left(1-\frac{z}{x_r - iy_r}\right)\left(1-\frac{z}{x_r + iy_r}\right)$$

1 Li Coefficients at the Left Edge of the Critical Strip

The Li coefficients in this chapter are defined at the left edge of the critical strip $z=0$.

1.1 Li Coefficients from the ω function oλn

The Li coefficients in this section are derived from the Completed Dirichlet Beta Function $\omega(z)$

Theorem 1.1

Let us define the Li coefficient ${}_{\omega}\lambda_n$ using the following two formulas.

$$ {}_{\omega}\lambda_n = \frac{(-1)^n}{(n-1)!}\left[\frac{d^n}{dz^n}(1-z)^{n-1}\log\omega(z)\right]_{z=0} \tag{1.1d}$$

$$ \omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z) \tag{0.ω}$$

Then, ${}_{\omega}\lambda_n$ is expressed as follows:

$$ {}_{\omega}\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}(-1)^s\left[\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\frac{B_{s,t}\left(\omega^{(1)},\omega^{(2)},\cdots,\omega^{(s)}\right)}{\omega(z)^t}\right]_{z=0} \tag{1.λ}$$

Where, $(a)_k$ is the Pochhammer symbol and $B_{n,k}(f_1, f_2, \ldots)$ is the Bell polynomial.

Proof

(1) Higher-order Derivatives of $(1-z)^{n-1}$

$$\frac{d^1}{dz^1}(1-z)^{n-1} = -(n-1)(1-z)^{n-2}$$

$$\frac{d^2}{dz^2}(1-z)^{n-1} = (n-2)(n-1)(1-z)^{n-3}$$

$$\frac{d^3}{dz^3}(1-z)^{n-1} = -(n-3)(n-2)(n-1)(1-z)^{n-4}$$

$$\vdots$$

$$\frac{d^s}{dz^s}(1-z)^{n-1} = (-1)^s\{(n-s)\cdots(n-2)(n-1)\}(1-z)^{n-1-s}$$

Pochhammer symbol is

$$(a)_k = a(a+1)\cdots(a+k-1)$$

Using this,

$$(n-s)_k = (n-s)(n-s+1)\cdots(n-s+k-1)$$

From this,

$$(n-s)_s = (n-s)(n-s+1)\cdots(n-1) = (n-s)\cdots(n-2)(n-1)$$

Therefore,

$$\frac{d^s}{dz^s}(1-z)^{n-1} = (-1)^s(n-s)_s(1-z)^{n-1-s}$$

Replacing s with $n-s$,

$$\frac{d^{n-s}}{dz^{n-s}}(1-z)^{n-1} = (-1)^{n-s}(s)_{n-s}(1-z)^{s-1} \tag{1.1}$$

(2) Higher-order Derivatives of $\log\omega(z)$

According to **22.2.3** in my paper "**22 Higher Derivative of Composition**", when $B_{s,t}(f_1, f_2, \dots)$ is Bell polynomial

$$\{\log f(x)\}^{(n)} = \sum_{r=1}^{n} (-1)^{r-1}(r-1)!\,B_{n,r}(f_1, f_2, \dots, f_n) f^{-r} \qquad n \ge 1$$

So,

$$\frac{d^s}{dz^s}\log\omega(z) = \sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(s)}\right)}{\omega(z)^t} \qquad s \ge 1 \tag{1.2λ}$$

(3) Higher-order Derivatives of $(1-z)^{n-1}\log\omega(z)$

Leibniz's law is

$$\{f(z)g(z)\}^{(n)} = \sum_{s=0}^{n}\binom{n}{s} f^{(n-s)}(z)g^{(s)}(z)$$

Substitute (1.1) and (1.2λ) for this. Then since $n,s \ge 1$,

$$\frac{d^n}{dz^n}(1-z)^{n-1}\log\omega(z) = \sum_{s=1}^{n}\binom{n}{s}\left\{(-1)^{n-s}(s)_{n-s}(1-z)^{s-1}\right\}$$
$$\times\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(s)}\right)}{\omega(z)^t} \tag{1.3λ}$$

(4) Li Coefficients ${}_{\omega}\lambda_n$

Substituting (1.3λ) for (1.1d),

$${}_{\omega}\lambda_n = \frac{(-1)^n}{(n-1)!}\left[\sum_{s=1}^{n}\binom{n}{s}\left\{(-1)^{n-s}(s)_{n-s}(1-z)^{s-1}\right\}\right.$$
$$\left.\times\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(s)}\right)}{\omega(z)^t}\right]_{z=0}$$

i.e.

$${}_{\omega}\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}(-1)^s\left[\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(s)}\right)}{\omega(z)^t}\right]_{z=0} \tag{1.λ}$$

Q.E.D.

The first few are

$${}_{\omega}\lambda_1 = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\omega'(z)}{\omega(z)}$$

$${}_{\omega}\lambda_2 = \frac{(-1)^2}{1!}\lim_{z\to 0}\left\{-\frac{2\omega'(z)}{\omega(z)} - \frac{(1-z)\omega'(z)^2}{\omega(z)^2} + \frac{(1-z)\omega''(z)}{\omega(z)}\right\}$$

$${}_{\omega}\lambda_3 = \frac{(-1)^3}{2!}\lim_{z\to 0}\left\{\frac{6\omega'(z)}{\omega(z)} + \frac{6(1-z)\omega'(z)^2}{\omega(z)^2} + \frac{2(1-z)^2\omega'(z)^3}{\omega(z)^3}\right.$$
$$\left. - \frac{6(1-z)\omega''(z)}{\omega(z)} - \frac{3(1-z)^2\omega'(z)\xi''(z)}{\omega(z)^2} + \frac{(1-z)^2\omega^{(3)}(z)}{\omega(z)}\right\}$$

$$\vdots$$

Further, when we calculate these using the mathematical processing software *Mathematica*, it is as follows.

```
f[n_, z_] := (1 - z)^(n-1) Log[ω[z]]
ω[z_] := (2/√π)^(1+z) Gamma[(1 + z)/2] DirichletBeta[z]
Unprotect[Power]; Power[0, 0] = 1;
oλ1 := (-1)^1/0! Limit[FullSimplify[∂z f[1, z]], z → 0]
N[oλ1]    0.0777839899617847`
oλ2 := (-1)^2/1! Limit[FullSimplify[∂z ∂z f[2, z]], z → 0]
N[oλ2]    0.31021808946399876`
oλ3 := (-1)^3/2! Limit[FullSimplify[∂z ∂z ∂z f[3, z]], z → 0]
N[oλ3]    0.6945704213216022`
oλ4 := (-1)^4/3! Limit[Simplify[∂z ∂z ∂z ∂z f[4, z]], z → 0]
N[oλ4]    1.2263597304205356`
```

λ_1 is equal to the sum of the reciprocals of the zeros on the critical line of the Dirichlet Beta Function. (OEIS A360807).

1.2 Li Coefficients from the Hadamard product oμn

The Li coefficients in this section are derived from the Hadamard product.

Theorem 1.2

Let us define the Li coefficient ${}_0\mu_n$ using the following two formulas.

$$ {}_0\mu_n = \frac{(-1)^n}{(n-1)!}\left[\frac{d^n}{dz^n}(1-z)^{n-1}\log\omega(z)\right]_{z=0} \tag{1.2d}$$

$$\omega(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) \quad \text{where, } \rho_k \ (k=1,2,3,\cdots) \text{ are zeros of } \omega(z) \tag{0.ρ}$$

Then, ${}_0\mu_n$ is expressed as follows:

$$ {}_0\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) \tag{1.μ}$$

Proof

(1) Higher-order Derivatives of $(1-z)^{n-1}$

This is the same as 1.1 (1) . That is,

$$\frac{d^{n-s}}{dz^{n-s}}(1-z)^{n-1} = (-1)^{n-s}(s)_{n-s}(1-z)^{s-1} \tag{1.1}$$

(2) Higher-order Derivatives of $\log\omega(z)$

Taking the logarithm of both sides of (0.ρ) ,

$$\log\omega(z) = \log\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \sum_{k=1}^{\infty}\log\left(1-\frac{z}{\rho_k}\right)$$

Differentiating both sides with respect to z ,

$$\frac{d}{dz}\log\omega(z) = \frac{d}{dz}\sum_{k=1}^{\infty}\log\left(1-\frac{z}{\rho_k}\right) = \sum_{k=1}^{\infty}\frac{-1/\rho_k}{1-z/\rho_k}$$

i.e.

$$\frac{d^1}{dz^1}\log\omega(z) = -\sum_{k=1}^{\infty}\frac{1}{\rho_k - z}$$

$$\frac{d^2}{dz^2}\log\omega(z) = -\sum_{k=1}^{\infty}\frac{1}{(\rho_k - z)^2}$$

$$\frac{d^3}{dz^3}\log\omega(z) = -\sum_{k=1}^{\infty}\frac{2}{(\rho_k - z)^3}$$

$$\vdots$$

$$\frac{d^s}{dz^s}\log\omega(z) = -\sum_{k=1}^{\infty}\frac{(s-1)!}{(\rho_k - z)^s} \tag{1.2μ}$$

(3) Higher-order Derivative Coefficients ($z=0$) of $(1-z)^{n-1}\log\omega(z)$

Applying the Leibniz's law to (1.1) and (1.2μ) ,

$$\frac{d^n}{dz^n}(1-z)^{n-1}\log\omega(z) = -\sum_{s=0}^{n}\binom{n}{s}(-1)^{n-s}(s)_{n-s}(1-z)^{s-1}\sum_{k=1}^{\infty}\frac{(s-1)!}{(\rho_k - z)^s}$$

Since $(0-1)!$ is not possible, changing the initial value of the first Σ subscript from 0 to 1,

$$\frac{d^n}{dz^n}(1-z)^{n-1}\log\omega(z) = -\sum_{s=1}^{n}\binom{n}{s}(-1)^{n-s}(s)_{n-s}(1-z)^{s-1}\sum_{k=1}^{\infty}\frac{(s-1)!}{(\rho_k - z)^s}$$

Further,

$$(s)_{n-s} = s(s+1)\cdots(n-1) = \frac{(n-1)!}{(s-1)!}$$

Substituting this for the above,

$$\frac{d^n}{dz^n}(1-z)^{n-1}\log\omega(z) = -\sum_{s=1}^{n}\binom{n}{s}(-1)^{n-s}\frac{(n-1)!}{(s-1)!}(1-z)^{s-1}\sum_{k=1}^{\infty}\frac{(s-1)!}{(\rho_k - z)^s}$$

i.e.

$$\frac{d^n}{dz^n}(1-z)^{n-1}\log\omega(z) = -(-1)^n(n-1)!\sum_{k=1}^{\infty}\sum_{s=1}^{n}\binom{n}{s}(1-z)^{s-1}\frac{(-1)^s}{(\rho_k - z)^s}$$

The derivative coefficient of this at $z=0$, the left edge of the critical strip, is

$$\left[\frac{d^n}{dz^n}(1-z)^{n-1}\log\omega(z)\right]_{z=0} = -(-1)^n(n-1)!\sum_{k=1}^{\infty}\sum_{s=1}^{n}\binom{n}{s}\frac{(-1)^s}{\rho_k^s}$$

$$= -(-1)^n(n-1)!\sum_{k=1}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{\rho_k^s} - 1\right)$$

$$= -(-1)^{-n}(n-1)!\sum_{k=1}^{\infty}\left(\left(1-\frac{1}{\rho_k}\right)^n - 1\right)$$

i.e.

$$\left[\frac{d^n}{dz^n}(1-z)^{n-1}\log\omega(z)\right]_{z=0} = (-1)^{-n}(n-1)!\sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) \tag{1.3μ}$$

(4) Li Coefficients ${}_0\mu_n$

Substituting (1.3μ) for (1.2d) ,

$$ {}_0\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) \tag{1.μ} $$

1.3 ${}_0\lambda_1$ and ${}_0\mu_1$

For the Li coefficients ${}_0\lambda_n$, ${}_0\mu_n$, the following lemma holds especially when $n=1$.

Lemma 1.3

Let ρ_k $k=1,2,3,\cdots$ are zeros of the xi function, and let $\rho_{2r-1}=x_r-iy_r$, $\rho_{2r}=x_r+iy_r$ $r=1,2,3,\cdots$.

Then,

$$ \sum_{r=1}^{\infty}\left(\frac{1}{x_r-iy_r}+\frac{1}{x_r+iy_r}\right) = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\omega'(z)}{\omega(z)} = 0.0777839\cdots $$

Proof

In Theorem 1.2, especially when $n=1$,

$$ {}_0\mu_1 = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^1\right) = \sum_{k=1}^{\infty}\frac{1}{\rho_k} $$

Let $\rho_{2r-1}=x_r-iy_r$, $\rho_{2r}=x_r+iy_r$ $r=1,2,3,\cdots$. Then

$$ {}_0\mu_1 = \sum_{r=1}^{\infty}\left(\frac{1}{x_r-iy_r}+\frac{1}{x_r+iy_r}\right) $$

On the other hand, in Theorem 1.1 , especially when $n=1$,

$$ {}_0\lambda_1 = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\omega'(z)}{\omega(z)} $$

Since ${}_0\lambda_1 = {}_0\mu_1$,

$$ \sum_{r=1}^{\infty}\left(\frac{1}{x_r-iy_r}+\frac{1}{x_r+iy_r}\right) = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\omega'(z)}{\omega(z)} = 0.0777839\cdots $$

Q.E.D.

Note

When $\omega(z) = 1+A_1z^1+A_2z^2+A_3z^4+\cdots$, the following holds according to Theorem 8.2.1 in my papaer " **08 Power Series of Completed Dirichlet Beta** ". This is the Vieta's formula .

$$ \begin{aligned} \sum_{k=1}^{\infty}\frac{1}{\rho_k} &= -A_1 \\ &= -\frac{1}{\pi}\left(\left(\log 2+\frac{\log\pi}{2}\right)\left(\gamma_0\left(\frac{1}{4}\right)-\gamma_0\left(\frac{3}{4}\right)\right)+\gamma_1\left(\frac{1}{4}\right)-\gamma_1\left(\frac{3}{4}\right)-\frac{\psi_0(1)}{2}\left(\gamma_0\left(\frac{1}{4}\right)-\gamma_0\left(\frac{3}{4}\right)\right)\right) \\ &= -(-0.0777839\cdots) \end{aligned} $$

2 Li Coefficients at the Right Edge of the Critical Strip

The Li coefficients in this chapter are defined at the right edge of the critical strip $z=1$.

2.1 Li Coefficients from the ω function 1λn

The Li coefficients in this section are derived from the Completed Dirichlet Beta Function $\omega(z)$

Theorem 2.1

Let us define the Li coefficient ${}_{\omega}\lambda_n$ using the following two formulas.

$$ {}_1\lambda_n = \frac{1}{(n-1)!}\left[\frac{d^n}{dz^n} z^{n-1} \log \omega(z)\right]_{z=1} \tag{2.1d} $$

$$ \omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1+z}{2}\right) \beta(z) \tag{0.ω} $$

Then, ${}_1\lambda_n$ is expressed as follows:

$$ {}_1\lambda_n = \frac{1}{(n-1)!} \sum_{s=1}^{n} \binom{n}{s} (s)_{n-s} \left[\sum_{t=1}^{s} (-1)^{t-1}(t-1)! \frac{B_{s,t}\left(\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(s)}\right)}{\xi(z)^t} \right]_{z=1} $$

Where, $(a)_k$ is the Pochhammer symbol and $B_{n,k}(f_1, f_2, \ldots)$ is the Bell polynomial.

Proof

(1) Higher-order Derivatives of z^{n-1}

$$ \frac{d^s}{dz^s} z^{n-1} = \{(n-s) \cdots (n-2)(n-1)\} z^{n-1-s} $$

Using the Pochhammer symbol $(a)_k$,

$$ \frac{d^s}{dz^s} z^{n-1} = (n-s)_s z^{n-1-s} $$

Replacing s with $n-s$,

$$ \frac{d^{n-s}}{dz^{n-s}} z^{n-1} = (s)_{n-s} z^{s-1} \tag{2.1} $$

(2) Higher-order Derivatives of $\log \omega(z)$

This is the same as 1.1 (2) . That is,

$$ \frac{d^s}{dz^s} \log \omega(z) = \sum_{t=1}^{s} (-1)^{t-1}(t-1)! \frac{B_{s,t}\left(\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(s)}\right)}{\omega(z)^t} \qquad s \ge 1 \tag{1.2λ} $$

(3) Higher-order Derivatives of $z^{n-1} \log \omega(z)$

Leibniz's law is

$$ \{f(z)g(z)\}^{(n)} = \sum_{s=0}^{n} \binom{n}{s} f^{(n-s)}(z) g^{(s)}(z) $$

Substitute (2.1) and (1.2λ) for this. Then since $n,s \ge 1$,

$$ \frac{d^n}{dz^n} z^{n-1} \log \omega(z) = \sum_{s=1}^{n} \binom{n}{s} (s)_{n-s} z^{s-1} \sum_{t=1}^{s} (-1)^{t-1}(t-1)! \frac{B_{s,t}\left(\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(s)}\right)}{\omega(z)^t} \tag{2.3λ} $$

(4) Li Coefficients $_1\lambda_n$

Substituting (2.3λ) for (2.1d) ,

$$_1\lambda_n = \frac{1}{(n-1)!}\left[\sum_{s=1}^{n} \binom{n}{s} (s)_{n-s} z^{s-1} \sum_{t=1}^{s} (-1)^{t-1}(t-1)! \frac{B_{s,t}\left(\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(s)}\right)}{\omega(z)^t} \right]_{z=1}$$

i.e.

$$_1\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n} \binom{n}{s} (s)_{n-s} \left[\sum_{t=1}^{s} (-1)^{t-1}(t-1)! \frac{B_{s,t}\left(\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(s)}\right)}{\omega(z)^t} \right]_{z=1} \qquad (2.\lambda)$$

Q.E.D.

The first few are

$$_1\lambda_1 = \frac{1}{0!} \lim_{z\to 1} \frac{\omega'(z)}{\omega(z)}$$

$$_1\lambda_2 = \frac{1}{1!} \lim_{z\to 1} \left\{ \frac{2\omega'(z)}{\omega(z)} - \frac{z\omega'(z)^2}{\omega(z)^2} + \frac{z\omega''(z)}{\omega(z)} \right\}$$

$$_1\lambda_3 = \frac{1}{2!} \lim_{z\to 1} \left\{ \frac{6\omega'(z)}{\omega(z)} - \frac{6z\,\omega'(z)^2}{\omega(z)^2} + \frac{2z^2\omega'(z)^3}{\omega(z)^3} + \frac{6z\,\omega''(z)}{\omega(z)} - \frac{3z^2\omega'(z)\omega''(z)}{\omega(z)^2} + \frac{z^2\omega^{(3)}(z)}{\omega(z)} \right\}$$

$$\vdots$$

Further, when these were calculated using the mathematical processing software *Mathematica*, they were in perfect consistent with the result in 1.1.(4) .

2.2 Li Coefficients from the Hadamard product $_1\mu_n$

The Li coefficients in this section are derived from the Hadamard product.

Theorem 2.2

Let us define the Li coefficient $_1\mu_n$ using the following two formulas.

$$_1\mu_n = \frac{1}{(n-1)!}\left[\frac{d^n}{dz^n} z^{n-1} \log \omega(z) \right]_{z=1} \qquad (2.2d)$$

$$\omega(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) \quad \textit{where, } \rho_k \ (k=1,2,3,\cdots) \textit{ are zeros of } \omega(z) \qquad (0.\rho)$$

Then, $_1\mu_n$ is expressed as follows:

$$_1\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right) \qquad (2.\mu)$$

Proof

(1) Higher-order Derivatives of z^{n-1}

This is the same as 2.1 (1) . That is,

$$\frac{d^{n-s}}{dz^{n-s}} z^{n-1} = (s)_{n-s} z^{s-1} \qquad (2.1)$$

(2) Higher-order Derivatives of $log\ \omega(z)$

This is the same as 1.2 (2). That is,

$$\frac{d^s}{dz^s} log\ \omega(z) = -\sum_{k=1}^{\infty} \frac{(s-1)!}{(\rho_k - z)^s}$$

However, in this section, we use the following formula, where $\rho_k - z$ is replaced with $z - \rho_k$.

$$\frac{d^s}{dz^s} log\ \omega(z) = \sum_{k=1}^{\infty} (-1)^{s-1} \frac{(s-1)!}{(z-\rho_k)^s} \tag{2.2μ}$$

(3) Higher-order Derivative Coefficients ($z=1$) of $z^{n-1} log\ \omega(z)$

Applying the Leibniz's law to (2.1) and (2.2μ),

$$\frac{d^n}{dz^n} z^{n-1} log\ \omega(z) = \sum_{s=0}^{n} \binom{n}{s} (s)_{n-s} z^{s-1} \sum_{k=1}^{\infty} (-1)^{s-1} \frac{(s-1)!}{(z-\rho_k)^s}$$

Since $(0-1)!$ is not possible, changing the initial value of the first Σ subscript from 0 to 1,

$$\frac{d^n}{dz^n} z^{n-1} log\ \omega(z) = \sum_{s=1}^{n} \binom{n}{s} (s)_{n-s} z^{s-1} \sum_{k=1}^{\infty} (-1)^{s-1} \frac{(s-1)!}{(z-\rho_k)^s}$$

Further,

$$(s)_{n-s} = s(s+1)\cdots(n-1) = \frac{(n-1)!}{(s-1)!}$$

Substituting this for the above,

$$\frac{d^n}{dz^n} z^{n-1} log\ \omega(z) = -(n-1)! \sum_{k=1}^{\infty} \sum_{s=1}^{n} \binom{n}{s} z^{s-1} \frac{(-1)^s}{(z-\rho_k)^s}$$

The derivative coefficient of this at $z=1$, the right edge of the critical strip, is

$$\begin{aligned}\left[\frac{d^n}{dz^n} z^{n-1} log\ \omega(z)\right]_{z=1} &= -(n-1)! \sum_{k=1}^{\infty} \sum_{s=1}^{n} \binom{n}{s} \frac{(-1)^s}{(1-\rho_k)^s} \\ &= -(n-1)! \sum_{k=1}^{\infty} \left(\sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{(1-\rho_k)^s} - 1 \right) \\ &= -(n-1)! \sum_{k=1}^{\infty} \left(\left(1 - \frac{1}{1-\rho_k}\right)^n - 1 \right)\end{aligned}$$

i.e.

$$\left[\frac{d^n}{dz^n} z^{n-1} log\ \omega(z)\right]_{z=1} = (n-1)! \sum_{k=1}^{\infty} \left(1 - \left(1 - \frac{1}{1-\rho_k}\right)^n \right) \tag{2.3μ}$$

(4) Li Coefficients ${}_1\mu_n$

Substituting (2.3μ) for (2.2d),

$${}_1\mu_n = \sum_{k=1}^{\infty} \left(1 - \left(1 - \frac{1}{1-\rho_k}\right)^n \right)$$

Now, for any complex number ρ_k, the following holds:

$$\left(1 - \frac{1}{1-\rho_k}\right)^n = \left(\frac{-\rho_k}{1-\rho_k}\right)^n = \left(\frac{1-\rho_k}{-\rho_k}\right)^{-n} = \left(\frac{\rho_k - 1}{\rho_k}\right)^{-n} = \left(1 - \frac{1}{\rho_k}\right)^{-n}$$

Therefore, ${}_1\mu_n$ becomes further as follows.

$$ {}_1\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right) \qquad \left(={}_0\mu_{-n}\right) \tag{2.μ} $$

Q.E.D.

3 Inequality of Left and Right Coefficients

3.1 Li Coefficients from the ω function 0λn & 1λn

Theorem 1.1 and Theorem 2.1 give the Li coefficients ${}_0\lambda_n$, ${}_1\lambda_n$ from the ω functions at the left and right edge of the critical strip. The question here is whether ${}_0\lambda_n$ and ${}_1\lambda_n$ are equal. The conclusion is that they are equal. This is shown below as a theorem.

Theorem 3.1

Let the Li Coefficients ${}_0\lambda_n$, ${}_1\lambda_n$ at both edges of the critical strip be as follows:

$${}_0\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}(-1)^s\left[\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\frac{B_{s,t}\left(\omega^{(1)},\omega^{(2)},\cdots,\omega^{(s)}\right)}{\omega(z)^t}\right]_{z=0} \tag{1.λ}$$

$${}_1\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}\left[\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\frac{B_{s,t}\left(\omega^{(1)},\omega^{(2)},\cdots,\omega^{(s)}\right)}{\omega(z)^t}\right]_{z=1} \tag{2.λ}$$

Where,

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z) \tag{0.ω}$$

Then, the following equation holds.

$${}_0\lambda_n = {}_1\lambda_n \qquad n=1,2,3,\cdots \tag{3.1}$$

Proof

(1.λ) and (2.λ) can be rewritten as follows:

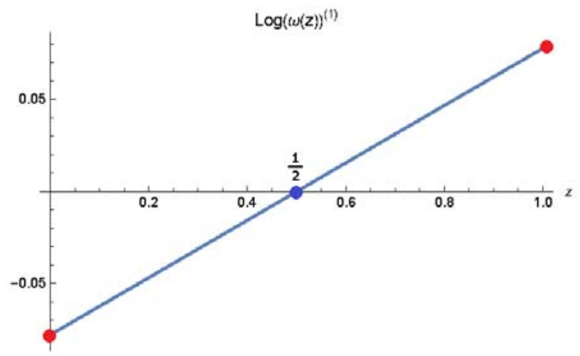
$${}_0\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}(-1)^s\left[\frac{d^s}{dz^s}\log\omega(z)\right]_{z=0}$$

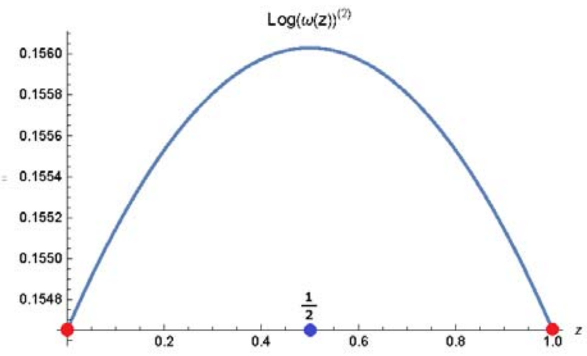
$${}_1\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}\left[\frac{d^s}{dz^s}\log\omega(z)\right]_{z=1}$$

By the functional equation $\omega(z)=\omega(1-z)$, $\omega(z)$ is line symmetry with respect to $z=1/2$. So, $\log\omega(z)$ is also line symmetry with respect to $z=1/2$. And the higher-order derivatives of $\log\omega(z)$ become as follows.

Odd-order derivatives: Point symmetry with respect to $z=1/2$ (example: left figure) .

Even-order derivatives: Line symmetry with respect to $z=1/2$ (example: right figure) .





So, the following equation holds for the higher-order derivative coefficient (red dots) of $\log\omega(z)$ at $s=0$ and $s=1$.

$$(-1)^s\left[\frac{d^s}{dz^s}\log\omega(z)\right]_{s=0} = \left[\frac{d^s}{dz^s}\log\omega(z)\right]_{s=1} \qquad for\ s=1,2,3,\cdots$$

Thus, ${}_0\lambda_n = {}_1\lambda_n$. That is, this holds unconditionally based on the functional equation and the definition.

Q.E.D.

3.2 Li Coefficients from the Hadamard product ${}_0\mu_n$ & ${}_1\mu_n$

Theorem 1.2 and Theorem 2.2 give the Li Coefficients ${}_0\mu_n$, ${}_1\mu_n$ from the Hadamard product at the left and right edge of the critical strip. The question here is whether ${}_0\mu_n$ and ${}_1\mu_n$ are equal.

Lemma 3.2.1

Let ρ_k $k=1,2,3,\cdots$ are zeros of the ω function, and let $\rho_{2r-1}=x_r-iy_r$, $\rho_{2r}=x_r+iy_r$ $r=1,2,3,\cdots$. Then,

(1) Odd order

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n-1}=\sum_{r=1}^{\infty}\frac{2}{\left(x_r^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-(2n-1)}=\frac{2}{\left((1-x_r)^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}$$

(2) Even order

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n}=\sum_{r=1}^{\infty}\frac{2}{\left(x_r^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-2n}=\sum_{r=1}^{\infty}\frac{2}{\left((1-x_r)^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}$$

Proof

Since the imaginary part of ρ_k has $\pm$, assign ρ_k as in the assumption. Then,

$$\left(1-\frac{1}{\rho_{2r-1}}\right)+\left(1-\frac{1}{\rho_{2r}}\right)=\left(1-\frac{1}{x_r-iy_r}\right)+\left(1-\frac{1}{x_r+iy_r}\right)$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-1}+\left(1-\frac{1}{\rho_{2r}}\right)^{-1}=\left(1-\frac{1}{x_r-iy_r}\right)^{-1}+\left(1-\frac{1}{x_r+iy_r}\right)^{-1}$$

Making the denominator on the right hand side real,

$$\left(1-\frac{1}{\rho_{2r-1}}\right)+\left(1-\frac{1}{\rho_{2r}}\right)=\left(\frac{x_r^2+y_r^2-x_r}{x_r^2+y_r^2}-i\frac{y_r}{x_r^2+y_r^2}\right)+\left(\frac{x_r^2+y_r^2-x_r}{x_r^2+y_r^2}+i\frac{y_r}{x_r^2+y_r^2}\right)$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-1}+\left(1-\frac{1}{\rho_{2r}}\right)^{-1}=\left(\frac{x_r^2+y_r^2-x_r}{(1-x_r)^2+y_r^2}-i\frac{y_r}{(1-x_r)^2+y_r^2}\right)$$
$$+\left(\frac{x_r^2+y_r^2-x_r}{(1-x_r)^2+y_r^2}+i\frac{y_r}{(1-x_r)^2+y_r^2}\right)$$

These are complicated, so we will abbreviate them as follows.

$$\frac{x_r^2+y_r^2-x_r}{x_r^2+y_r^2}=A_r\ ,\quad \frac{y_r}{x_r^2+y_r^2}=B_r\ ,\quad \frac{x_r^2+y_r^2-x_r}{(1-x_r)^2+y_r^2}=C_r\ ,\quad \frac{y_r}{(1-x_r)^2+y_r^2}=D_r$$

Then,

When $n=1$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)+\left(1-\frac{1}{\rho_{2r}}\right)=\left(A_r-iB_r\right)+\left(A_r+iB_r\right)=2A_r$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-1}+\left(1-\frac{1}{\rho_{2r}}\right)^{-1}=\left(C_r-iD_r\right)+\left(C_r+iD_r\right)=2C_r$$

When $n=2$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^2+\left(1-\frac{1}{\rho_{2r}}\right)^2=(A_r-iB_r)^2+(A_r+iB_r)^2=2\left(A_r^2-B_r^2\right)$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-2}+\left(1-\frac{1}{\rho_{2r}}\right)^{-2}=(C_r-iD_r)^2+(C_r+iD_r)^2=2\left(C_r^2-D_r^2\right)$$

When $n=3$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^3+\left(1-\frac{1}{\rho_{2r}}\right)^3=(A_r-iB_r)^3+(A_r+iB_r)^3=2\left(A_r^3-3A_rB_r^2\right)$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-3}+\left(1-\frac{1}{\rho_{2r}}\right)^{-3}=(C_r-iD_r)^3+(C_r+iD_r)^3=2\left(C_r^3-3C_rD_r^2\right)$$

When $n=4$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^4+\left(1-\frac{1}{\rho_{2r}}\right)^4=(A_r-iB_r)^4+(A_r+iB_r)^4=2\left(A_r^4-6A_r^2B_r^2+B_r^4\right)$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-4}+\left(1-\frac{1}{\rho_{2r}}\right)^{-4}=(C_r-iD_r)^4+(C_r+iD_r)^4=2\left(C_r^4-6C_r^2D_r^2+D_r^4\right)$$

The absolute values of the coefficient in 2( ) on the right sides are

$1,\ 1,1,\ 1,3,\ 1,6,1,\ 1,10,5,\ 1,15,15,1,\ \cdots$

This integer sequence matches **OEIS A098158** and is given by the following formula

$$T(n,k)=Binomial(n,2k)\ ,\quad for\ \ n\ge 0\ \ \&\ \ k=0,1,2,\cdots$$

Using this,

Odd order

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{2n-1}+\left(1-\frac{1}{\rho_{2r}}\right)^{2n-1}=2\sum_{s=0}^{n-1}(-1)^{n-1-s}\binom{2n-1}{2(n-s-1)}A_r^{2s+1}B_r^{2n-2s-2}$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-(2n-1)}+\left(1-\frac{1}{\rho_{2r}}\right)^{-(2n-1)}=2\sum_{s=0}^{n-1}(-1)^{n-1-s}\binom{2n-1}{2(n-s-1)}C_r^{2s+1}D_r^{2n-2s-2}$$

Even order

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{2n}+\left(1-\frac{1}{\rho_{2r}}\right)^{2n}=2\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}A_r^{2s}B_r^{2n-2s}$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-2n}+\left(1-\frac{1}{\rho_{2r}}\right)^{-2n}=2\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}C_r^{2s}D_r^{2n-2s}$$

Here,

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^n=\sum_{r=1}^{\infty}\left\{\left(1-\frac{1}{\rho_{2r-1}}\right)^n+\left(1-\frac{1}{\rho_{2r}}\right)^n\right\}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-n}=\sum_{r=1}^{\infty}\left\{\left(1-\frac{1}{\rho_{2r-1}}\right)^{-n}+\left(1-\frac{1}{\rho_{2r}}\right)^{-n}\right\}$$

So, changing A_r, B_r, C_r, D_r back into their original symbols,

Odd order

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n-1}=2\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(\frac{x_r^2+y_r^2-x_r}{x_r^2+y_r^2}\right)^{2s+1}\left(\frac{y_r}{x_r^2+y_r^2}\right)^{2n-2s-2}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-(2n-1)}=2\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}$$

$$\times\left(\frac{x_r^2+y_r^2-x_r}{(1-x_r)^2+y_r^2}\right)^{2s+1}\left(\frac{y_r}{(1-x_r)^2+y_r^2}\right)^{2n-2s-2}$$

i.e.

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n-1} = \sum_{r=1}^{\infty}\frac{2}{\left(x_r^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-(2n-1)} = \sum_{r=1}^{\infty}\frac{2}{\left((1-x_r)^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}$$

Even order

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n} = 2\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(\frac{x_r^2+y_r^2-x_r}{x_r^2+y_r^2}\right)^{2s}\left(\frac{y_r}{x_r^2+y_r^2}\right)^{2n-2s}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-2n} = 2\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(\frac{x_r^2+y_r^2-x_r}{(1-x_r)^2+y_r^2}\right)^{2s}\left(\frac{y_r}{(1-x_r)^2+y_r^2}\right)^{2n-2s}$$

i.e.

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n} = \sum_{r=1}^{\infty}\frac{2}{\left(x_r^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-2n} = \sum_{r=1}^{\infty}\frac{2}{\left((1-x_r)^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}$$

Q.E.D.

Using this lemma, the Li Coefficients ${}_0\mu_n$, ${}_1\mu_n$ can be obtained as follows:

Theorem 3.2.2

Let the Li coefficients ${}_0\mu_n$, ${}_1\mu_n$ at both edges of the critical strip be as follows:

$${}_0\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{n}\right) \qquad (1.\mu)$$

$${}_1\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right) \qquad \left(={}_0\mu_{-n}\right) \qquad (2.\mu)$$

Where,

$$\omega(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) \qquad \rho_k\ \ (k=1,2,3,\cdots)\ \textit{are zeros of}\ \omega(z) \qquad (0.\rho)$$

Then, when $\rho_{2r-1}=x_r-iy_r$, $\rho_{2r}=x_r+iy_r$ $r=1,2,3,\cdots$, the Li coefficients ${}_0\mu_n$, ${}_1\mu_n$ can be rewritten as follow.

(1) Odd order

$${}_0\mu_{2n-1} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(x_r^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}\right\}$$

$${}_1\mu_{2n-1} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left((1-x_r)^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}\right\}$$

(2) Even order

$${}_0\mu_{2n} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(x_r^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}\right\}$$

$${}_1\mu_{2n} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left((1-x_r)^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}\right\}$$

Proof

Li Coefficients are

$$_0\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) \quad , \quad {}_1\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right)$$

Since,

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^n = \sum_{r=1}^{\infty}\left\{\left(1-\frac{1}{\rho_{2r-1}}\right)^n + \left(1-\frac{1}{\rho_{2r}}\right)^n\right\}$$

So,

$$_0\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) = \sum_{r=1}^{\infty}\left\{2-\left(1-\frac{1}{\rho_{2r-1}}\right)^n - \left(1-\frac{1}{\rho_{2r}}\right)^n\right\}$$

$$_1\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right) = \sum_{r=1}^{\infty}\left\{2-\left(1-\frac{1}{\rho_{2r-1}}\right)^{-n} - \left(1-\frac{1}{\rho_{2r}}\right)^{-n}\right\}$$

Here, using Lemma 3.2.1 ,

(1) Odd order

$$_0\mu_{2n-1} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(x_r^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}\right\}$$

$$_1\mu_{2n-1} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left((1-x_r)^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}\right\}$$

(2) Even order

$$_0\mu_{2n} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(x_r^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}\right\}$$

$$_1\mu_{2n} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left((1-x_r)^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}\right\}$$

Remark

As seen in 1.2 and 2.2 , the zeros $\rho_k \;\; k=1,2,3,\cdots$ of $\omega(z)$ are values that satisfy the following equations.

$$_0\mu_n = {}_0\lambda_n \quad , \quad {}_1\mu_n = {}_1\lambda_n \qquad n=1,2,3,\cdots$$

And, since ${}_0\lambda_n = {}_1\lambda_n \quad n=1,2,3,\cdots$ from Theorem 3.1, ${}_0\mu_n = {}_1\mu_n \quad n=1,2,3,\cdots$ must be true.

However, observing Theorem 3.2.2, if $x_r^2 \neq (1-x_r)^2 \quad r=1,2,3,\cdots$ then in general

$$_0\mu_n \neq {}_1\mu_n \qquad n=1,2,3,\cdots$$

This means that the following four-step logic does not hold in general.

$$_0\mu_n = {}_0\lambda_n \; , \; {}_1\mu_n = {}_1\lambda_n \; , \; {}_0\lambda_n = {}_1\lambda_n \implies {}_0\mu_n = {}_1\mu_n$$

4 Condition for Equality of Left and Right Li Coefficients

In this chapter, we will find the necessary and sufficient condition for ${}_0\mu_n = {}_1\mu_n \quad n=1,2,3,\cdots$, thereby proving that the Riemann hypothesis for the Dirichlet Beta Function $\beta(z)$ holds as a theorem.

Theorem 4.1

Let the Li coefficients ${}_0\mu_n$, ${}_1\mu_n$ at both edges of the critical strip be as follows:

$$ {}_0\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) \tag{1.μ} $$

$$ {}_1\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right) \qquad \left(={}_0\mu_{-n}\right) \tag{2.μ} $$

Where,

$$ \omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) \qquad \rho_k \ (k=1,2,3,\cdots) \text{ are zeros of } \omega(z) \tag{4.0} $$

Then, the necessary and sufficient condition for ${}_0\mu_n = {}_1\mu_n \quad n=1,2,3,\cdots$ is $Re(\rho_k) = 1/2 \quad k=1,2,3,\cdots$.

Proof

First, the nontrivial zeros of the Dirichlet Beta function $\beta(z)$ are same as the zeros of the Completed Dirichlet Beta function $\omega(z)$ Further, $\rho_{2r-1} = 1/2 - iy_r$, $\rho_{2r} = 1/2 + iy_r \quad r=1,2,3,\cdots$.

I. Sufficiency

If $Re(\rho_k) = x_r = 1/2 \quad r=1,2,3,\cdots$, from Theorem 3.2.2 ,

$$ {}_0\mu_{2n-1} = {}_1\mu_{2n-1} $$

$$ = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(y_r^2+1/4\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(y_r^2-1/4\right)^{2s+1}y_r^{2n-2s-2}\right\} \tag{4.1} $$

$$ {}_0\mu_{2n} = {}_1\mu_{2n} $$

$$ = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(y_r^2+1/4\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(y_r^2-1/4\right)^{2s}y_r^{2n-2s}\right\} \tag{4.2} $$

That is, ${}_0\mu_n = {}_1\mu_n \quad n=1,2,3,\cdots$.

As the result, ${}_0\lambda_n = {}_0\mu_n$, ${}_1\lambda_n = {}_1\mu_n$ and Theorem 3.1 together complete the following.

$$ {}_0\lambda_n = {}_0\mu_n = {}_1\mu_n = {}_1\lambda_n \qquad n=1,2,3,\cdots \tag{4.3} $$

Especially when $n=1$, from Lemma 1.3 ,

$$ \sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right) = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\omega'(z)}{\omega(z)} = 0.0777839\cdots \tag{4.3$_1$} $$

II. Necessity

Now, suppose that in addition to the zeros on the critical line, there are also zeros off the critical line.
It is known that a set of such zeros should consist of the following four.

$$ 1/2-\alpha_s \pm i\beta_s \ , \ 1/2+\alpha_s \pm i\beta_s \qquad (0<\alpha_s<1/2 \ , \ \beta_s>0) $$

Then, from Lemma 1.3 , the following equation has to hold .

$$ \sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right) + \sum_{s=1}\left(\frac{1}{1/2-\alpha_s-i\beta_s}+\frac{1}{1/2-\alpha_s+i\beta_s}\right) $$

$$ + \sum_{s=1}\left(\frac{1}{1/2+\alpha_s-i\beta_s}+\frac{1}{1/2+\alpha_s+i\beta_s}\right) = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\omega'(z)}{\omega(z)} $$

i.e.

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right)+\sum_{s=1}\left\{\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}+\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}=\frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\omega'(z)}{\omega(z)}$$

$$=0.0777839\cdots \qquad (4.3_2)$$

However, since $0<1-2\alpha_s<1+2\alpha_s<2$ for $0<\alpha_s<1/2$,

$$\sum_{s=1}\left\{\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}+\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}>0$$

Therefore, (4.3_2) contradicts (4.3_1). Thus, there should be no zeros outside the critical line within the critical strip.

As the result, ${}_0\mu_n={}_1\mu_n \quad n=1,2,3,\cdots$ holds only if $Re(\rho_k)=1/2 \quad k=1,2,3,\cdots$.

Q.E.D.

Theorem 4.2 (Riemann)

When the nontrivial zeros of the Dirichlet Beta Function $\beta(z)$ are $\rho_k \quad k=1,2,3,\cdots$, $Re(\rho_k)=1/2 \quad k=1,2,3,\cdots$.

Proof

Where the Riemann Hypothesis holds, it must naturally be the case that ${}_0\lambda_n={}_0\mu_n={}_1\mu_n={}_1\lambda_n \quad n=1,2,3,\cdots$. .

The necessary and sufficient condition for this is $Re(\rho_k)=1/2 \quad k=1,2,3,\cdots$, as proved in Theorem 4.1.

Furthermore, all of these Li Coefficients are non-negative and satisfy Li's Criterion. (See **Appendix** .)

Thus, the Riemann Hypothesis for the Dirichlet Beta Function $\beta(z)$ holds as a theorem.

Q.E.D.

Appendix

According to the Proof of Theorem 4.1 , if $Re(\rho_k) = x_r = 1/2 \quad r=1,2,3,\cdots$ then

$$ {}_0\mu_{2n-1} = {}_1\mu_{2n-1} $$

$$ = 2\sum_{r=1}^{\infty}\left\{1 - \frac{1}{\left(y_r^2+1/4\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(y_r^2-1/4\right)^{2s+1}y_r^{2n-2s-2}\right\} \quad (4.1) $$

$$ {}_0\mu_{2n} = {}_1\mu_{2n} $$

$$ = 2\sum_{r=1}^{\infty}\left\{1 - \frac{1}{\left(y_r^2+1/4\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(y_r^2-1/4\right)^{2s}y_r^{2n-2s}\right\} \quad (4.2) $$

Calculation Example

The formula for generating the zeros of the Dirichlet Beta Function with real part $1/2$ is unknown, but the first 10000 are provided by ***Tomas oliveira e Silva*** (http://sweet.ua.pt/tos/zeta.html (004-001)) .

So, we calculate $\mu_1 \sim \mu_4$ using these according to (4.1) and (4.2) , it is as follows.

Loading Zeros of Dirichlet Beta Function

```
SetDirectory[NotebookDirectory[]];
y := ReadList["BetaZeros.prn", Number]
```

2n-1

$$ o\mu_{n\_}[m\_] := 2\sum_{r=1}^{m}\left(1 - \frac{1}{\left(y[[r]]^2+1/4\right)^{2n-1}} \times \sum_{s=0}^{n-1}(-1)^{n-s-1}\,\text{Binomial}[2n-1,\,2(n-s-1)]\,\left(y[[r]]^2-1/4\right)^{2s+1}y[[r]]^{2n-2s-2}\right) $$

2n

$$ e\mu_{n\_}[m\_] := 2\sum_{r=1}^{m}\left(1 - \frac{1}{\left(y[[r]]^2+1/4\right)^{2n}} \times \sum_{s=0}^{n}(-1)^{n-s}\,\text{Binomial}[2n,\,2(n-s)]\,\left(y[[r]]^2-1/4\right)^{2s}y[[r]]^{2n-2s}\right) $$

$$ N[\{o\mu_1[10000],\, e\mu_1[10000],\, o\mu_2[10000],\, e\mu_2[10000]\}] $$

{0.0776004, 0.309484, 0.692918, 1.22342}

Comparing these with the calculation results in 1.1 , we can see that they match to 2 significant digits.

cf.

According to Theorem 9.3.3 in my paper " **09 Li coefficients for the Completed Dirichlet Beta** ",

when $Re(\rho_k) = 1/2 \quad k=1,2,3,\cdots$

$$ {}_0\mu_{2n-1} = \sum_{r=1}^{\infty}\frac{1}{4^{2n-2}\left(\left(1/4+y_r^2\right)^{(2n-1)/2}\right)^2}\left(\sum_{s=0}^{n-1}(-1)^s 4^s\binom{2n-1}{2s}y_r^{2s}\right)^2 \quad n=1,2,3,\cdots $$

$$ {}_0\mu_{2n} = \sum_{r=1}^{\infty}\frac{y^2}{4^{2n-2}\left(1/4+y_r^2\right)^{2n}}\left(\sum_{s=0}^{n-1}(-1)^s 4^s\binom{2n}{2s+1}y_r^{2s}\right)^2 \quad n=1,2,3,\cdots $$

Since these are sums of squares of real numbers, they satisfy Li's Criterion. In fact, these are equivalent to (4.1) and (4.2) .

For example,

$$ {}_{0}\mu_3 = 2\sum_{r=1}^{\infty}\left\{1 - \frac{-3y_r^2\left(y_r^2 - 1/4\right) + \left(y_r^2 - 1/4\right)^2}{\left(y_r^2 + 1/4\right)^3}\right\} = \sum_{r=1}^{\infty}\frac{1 - 24y_r^2 + 144y_r^4}{4^2\left(y_r^2 + 1/4\right)^3} = \sum_{r=1}^{\infty}\frac{\left(1 - 12y_r^2\right)^2}{4^2\left(y_r^2 + 1/4\right)^3} $$

$$ {}_{0}\mu_4 = 2\sum_{r=1}^{\infty}\left\{1 - \frac{y_r^4 - 6y_r^2\left(y_r^2 - 1/4\right)^2 + \left(y_r^2 - 1/4\right)^4}{\left(y_r^2 + 1/4\right)^4}\right\} = \sum_{r=1}^{\infty}\frac{y_r^2 - 8y_r^4 + 16y_r^6}{\left(y_r^2 + 1/4\right)^4} = \sum_{r=1}^{\infty}\frac{y_r^2\left(4 - 16y_r^2\right)^2}{4^2\left(y_r^2 + 1/4\right)^4} $$

So, (4.1) and (4.2) also satisfy Li's Criterion.

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Alien's Mathematics