

Proof of Riemann Hypothesis for Dirichlet Beta by Factorization

Abstract

(1) Completed Dirichlet Beta Function $\omega(z)$ and $\omega(1-z)$ are factored into the Hadamard products as follows.

$$\omega(z) = \prod (1-z/\rho) \quad , \quad \omega(1-z) = \prod (1-(1-z)/\rho)$$

(2) $\omega(z) = \omega(1-z)$ holds on the whole complex plane..

(3) Despite (1) and (2), $\prod (1-z/\rho) \neq \prod (1-(1-z)/\rho)$ in general.

(4) $\prod (1-z/\rho) = \prod (1-(1-z)/\rho)$ holds on the whole complex plane if and only if all $Re(\rho)$ are $1/2$.

Thus, only if all $Re(\rho)$ are $1/2$, the four-step logic $\prod (1-z/\rho) = \omega(z) = \omega(1-z) = \prod (1-(1-z)/\rho)$ complete on the whole complex plane, and the Riemann Hypothesis holds.

Introduction

Functions studied in this paper

In this paper, we study the Dirichlet Beta Function $\beta(z)$ and the Completed Dirichlet Beta Function $\omega(z)$, defined as follows.

$$(0.0) \quad \beta(z) = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + \dots = \frac{1}{4^z} \left\{ \zeta\left(z, \frac{1}{4}\right) - \zeta\left(z, \frac{3}{4}\right) \right\}$$

$$(0.1) \quad \omega(z) = \left(\frac{2}{\sqrt{\pi}} \right)^{1+z} \Gamma\left(\frac{1+z}{2} \right) \beta(z)$$

In addition, it is known that these zeros are equivalent in the critical strip ($0 < Re(z) < 1$).

Notation for zeros of $\beta(z)$ and $\omega(z)$

The zeros ρ of the $\beta(z)$ and the $\omega(z)$ described in this paper as follows.

(1) Complex number notation

$$\sum_{k=1}^{\infty} \frac{1}{\rho_k} \quad , \quad \prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right)$$

(2) Real and imaginary parts notation

Putting $\rho_{2r-1} = x_r - iy_r$, $\rho_{2r} = x_r + iy_r$ $r=1, 2, 3, \dots$ ($y_r > 0$),

$$\sum_{r=1}^{\infty} \left(\frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r} \right) \quad , \quad \prod_{r=1}^{\infty} \left(1 - \frac{z}{x_r - iy_r} \right) \left(1 - \frac{z}{x_r + iy_r} \right)$$

1 Two Hadamard Products

Theorem 1.1 (Hadamard product)

Let the Completed Dirichlet Beta Function $\omega(z)$ be as follows.

$$(0.1) \quad \omega(z) = \left(\frac{2}{\sqrt{\pi}} \right)^{1+z} \Gamma\left(\frac{1+z}{2} \right) \beta(z)$$

Then $\omega(z)$ and $\omega(1-z)$ are factorized by their zeros ρ_k $k=1, 2, 3, \dots$ respectively as follows:

$$(1.1) \quad \omega(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right)$$

$$(1.2) \quad \omega(1-z) = \prod_{k=1}^{\infty} \left(1 - \frac{1-z}{\rho_k} \right)$$

Proof

According to Formula 5.1.1 in my paper " **05 Factorization of Completed Dirichlet Beta** ", $\omega(z)$ is expressed as the Hadamard product as follows.

$$(9.1) \quad \omega(z) = e^{\left(\frac{3 \log \pi}{2} - \frac{\gamma}{2} - \log 2 - 4 \log \Gamma\left(\frac{3}{4} \right) \right) z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right) \cdot e^{\sum_{k=1}^{\infty} \frac{z}{\rho_k}}$$

Here, let

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{z}{x_r - iy_r} \right) \left(1 - \frac{z}{x_r + iy_r} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{2x_r z}{x_r^2 + y_r^2} + \frac{z^2}{x_r^2 + y_r^2} \right)$$

$$\sum_{k=1}^{\infty} \frac{z}{\rho_k} = \sum_{r=1}^{\infty} \left(\frac{z}{x_r - iy_r} + \frac{z}{x_r + iy_r} \right) = \sum_{r=1}^{\infty} \frac{2x_r z}{x_r^2 + y_r^2}$$

Then, (9.1) becoms

$$(9.1') \quad \omega(z) = e^{\left(\frac{3 \log \pi}{2} - \frac{\gamma}{2} - \log 2 - 4 \log \Gamma\left(\frac{3}{4} \right) \right) z} \prod_{r=1}^{\infty} \left(1 - \frac{2x_r z}{x_r^2 + y_r^2} + \frac{z^2}{x_r^2 + y_r^2} \right) \cdot e^{\sum_{r=1}^{\infty} \frac{2x_r z}{x_r^2 + y_r^2}}$$

Further, if we express $x_n + iy_n$ $n=1, 2, 3, \dots$ whose real parts are $1/2$ as $1/2 \pm iy_r$ $r=1, 2, 3, \dots$ and those whose real parts are not $1/2$ as $1/2 \pm \alpha_s \pm i\beta_s$ ($0 < \alpha_s < 1/2$) $s=1, 2, 3, \dots$, then (9.1') becomes

$$(9.1'') \quad \omega(z) = e^{\left(\frac{3 \log \pi}{2} - \frac{\gamma}{2} - \log 2 - 4 \log \Gamma\left(\frac{3}{4} \right) \right) z} \prod_{r=1}^{\infty} \left(1 - \frac{z}{1/4 + y_r^2} + \frac{z^2}{1/4 + y_r^2} \right) \cdot e^{\sum_{r=1}^{\infty} \frac{z}{1/4 + y_r^2}}$$

$$\times \prod_{s=1} \left\{ 1 - \frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2 + \beta_s^2} + \frac{z^2}{(1/2-\alpha_s)^2 + \beta_s^2} \right\} \cdot e^{\sum_{s=1} \frac{(1-2\alpha_s)z}{(1/2-\alpha_s)^2 + \beta_s^2}}$$

$$\times \prod_{s=1} \left\{ 1 - \frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2 + \beta_s^2} + \frac{z^2}{(1/2+\alpha_s)^2 + \beta_s^2} \right\} \cdot e^{\sum_{s=1} \frac{(1+2\alpha_s)z}{(1/2+\alpha_s)^2 + \beta_s^2}}$$

Substituting $z=1$ for both sides of (9.1') and (9.1'') ,

$$(9.2) \quad \omega(1) = 1 = e^{\frac{3 \log \pi}{2} - \frac{\gamma}{2} - \log 2 - 4 \log \Gamma\left(\frac{3}{4}\right)} \prod_{r=1}^{\infty} \left(1 - \frac{2x_r - 1}{x_r^2 + y_r^2}\right) \cdot e^{\sum_{r=1}^{\infty} \frac{2x_r}{x_r^2 + y_r^2}}$$

$$(9.2'') \quad \omega(1) = 1$$

$$= e^{\frac{3 \log \pi}{2} - \frac{\gamma}{2} - \log 2 - 4 \log \Gamma\left(\frac{3}{4}\right)} \times \prod_{s=1}^{\infty} \left\{1 + \frac{2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2}\right\} \left\{1 - \frac{2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2}\right\} \\ \times e^{\sum_{r=1}^{\infty} \frac{1}{1/4 + y_r^2} + \sum_{s=1}^{\infty} \left\{\frac{1 - 2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2} + \frac{1 + 2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2}\right\}}$$

From these,

$$(9.3) \quad \prod_{r=1}^{\infty} \left(1 - \frac{2x_r - 1}{x_r^2 + y_r^2}\right) = \prod_{s=1}^{\infty} \left\{1 + \frac{2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2}\right\} \left\{1 - \frac{2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2}\right\}$$

$$(9.4) \quad e^{\frac{2x_r}{x_r^2 + y_r^2}} = e^{\sum_{r=1}^{\infty} \frac{1}{1/4 + y_r^2} + \sum_{s=1}^{\infty} \left\{\frac{1 - 2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2} + \frac{1 + 2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2}\right\}}$$

Here, conveniently,

$$\left\{1 + \frac{2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2}\right\} \left\{1 - \frac{2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2}\right\} \\ = 1 + \frac{2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2} - \frac{2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2} - \frac{2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2} \frac{2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2} \\ = 1 + \frac{2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2} \frac{2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2} - \frac{2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2} \frac{2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2} \\ = 1$$

Then, (9.3), (9.4) becom

$$(9.3') \quad \prod_{n=1}^{\infty} \left(1 - \frac{2x_n - 1}{x_n^2 + y_n^2}\right) = 1 \quad \left\{\text{i.e. } \prod_{r=1}^{\infty} \left(1 - \frac{1}{x_r - iy_r}\right) \left(1 - \frac{1}{x_r + iy_r}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{1}{\rho_k}\right) = 1\right\}$$

$$(9.4') \quad \sum_{n=1}^{\infty} \frac{2x_n}{x_n^2 + y_n^2} = \sum_{r=1}^{\infty} \frac{1}{1/4 + y_r^2} + \sum_{s=1}^{\infty} \left\{\frac{1 + 2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2} + \frac{1 - 2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2}\right\}$$

Substituting (9.3') for (9.2'),

$$\omega(1) = 1 = e^{\frac{3 \log \pi}{2} - \frac{\gamma}{2} - \log 2 - 4 \log \Gamma\left(\frac{3}{4}\right)} \cdot e^{\sum_{r=1}^{\infty} \frac{2x_r}{x_r^2 + y_r^2}} = e^{\frac{3 \log \pi}{2} - \frac{\gamma}{2} - \log 2 - 4 \log \Gamma\left(\frac{3}{4}\right)} e^{\sum_{k=1}^{\infty} \frac{1}{\rho_k}}$$

Since $1 = e^0$,

$$(9.5) \quad \sum_{k=1}^{\infty} \frac{1}{\rho_k} = 4 \log \Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3 \log \pi}{2} = 0.07778398 \dots$$

Therefore,

$$e^{\sum_{k=1}^{\infty} \frac{z}{\rho_k}} = e^{\left(4 \log \Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3 \log \pi}{2}\right)z}$$

Substituting this for the right side of (9.1),

$$\omega(z) = e^{\left(\frac{3 \log \pi}{2} - \frac{\gamma}{2} - \log 2 - 4 \log \Gamma\left(\frac{3}{4}\right)\right)z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k}\right) \cdot e^{\left(4 \log \Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3 \log \pi}{2}\right)z}$$

i.e.

$$(0.1) \quad \omega(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k}\right)$$

Since both sides are holomorphic on the whole complex plane, replacing z with $1-z$

$$(0.2) \quad \omega(1-z) = \prod_{k=1}^{\infty} \left(1 - \frac{1-z}{\rho_k}\right)$$

Q.E.D.

Note

When $\omega(z) = 1 + A_1 z^1 + A_2 z^2 + A_3 z^3 + \dots$, according to Vieta's formulas, the following holds.

$$(9.5) \quad \sum_{k=1}^{\infty} \frac{1}{\rho_k} = -A_1 = -\left(\frac{3 \log \pi}{2} - \frac{\gamma}{2} - \log 2 - 4 \log \Gamma\left(\frac{3}{4}\right)\right) = -(-0.07778398\dots)$$

2 Functional Equation

Theorem 2.1 (Functional Equation)

Let the Completed Dirichlet Beta Function $\omega(z)$ be as follows.

$$(0.1) \quad \omega(z) = \left(\frac{2}{\sqrt{\pi}} \right)^{1+z} \Gamma\left(\frac{1+z}{2} \right) \beta(z)$$

Then, the following expression holds on the whole complex plane .

$$(2.1) \quad \omega(z) = \omega(1-z)$$

Proof

The following functional equation is known for the Completed Dirichlet Beta Function $\omega(z)$.

$$\beta(z) = \left(\frac{2}{\pi} \right)^{1-z} \cos \frac{\pi z}{2} \Gamma(1-z) \beta(1-z) \quad z \neq 1, 2, 3, \dots$$

Here,

$$\cos \frac{\pi z}{2} = \frac{\pi}{\Gamma\left(\frac{1+z}{2} \right) \Gamma\left(\frac{1-z}{2} \right)}$$

Substituting this for the above,

$$\Gamma\left(\frac{1+z}{2} \right) \beta(z) = \left(\frac{2}{\pi} \right)^{1-z} \frac{\pi \Gamma(1-z)}{\Gamma\left(\frac{1-z}{2} \right)} \beta(1-z)$$

Further,

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2} \right) \implies \frac{\Gamma(1-z)}{\Gamma\left(\frac{1-z}{2} \right)} = \frac{2^{-z}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \frac{1-z}{2} \right)$$

Substituting this for the above,

$$\begin{aligned} \Gamma\left(\frac{1+z}{2} \right) \beta(z) &= \left(\frac{2}{\pi} \right)^{1-z} \frac{2^{-z} \pi}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \frac{1-z}{2} \right) \beta(1-z) \\ &= \frac{2^{1-z}}{(\sqrt{\pi})^{1-z}} \frac{2^{-z} \sqrt{\pi}}{(\sqrt{\pi})^{1-z}} \Gamma\left(\frac{1}{2} + \frac{1-z}{2} \right) \beta(1-z) \\ &= \left(\frac{2}{\sqrt{\pi}} \right)^{1-z} \left(\frac{2}{\sqrt{\pi}} \right)^{-z} \Gamma\left(\frac{1}{2} + \frac{1-z}{2} \right) \beta(1-z) \end{aligned}$$

Multiplying both sides by $(2/\sqrt{\pi})^{1+z}$,

$$\left(\frac{2}{\sqrt{\pi}} \right)^{1+z} \Gamma\left(\frac{1}{2} + \frac{z}{2} \right) \beta(z) = \left(\frac{2}{\sqrt{\pi}} \right)^{2-z} \Gamma\left(\frac{1}{2} + \frac{1-z}{2} \right) \beta(1-z)$$

i.e.

$$(2.1) \quad \omega(z) = \omega(1-z)$$

Since $\omega(z)$ is holomorphic on the whole complex plane, (2.1) also holds on the whole complex plane.

Q.E.D.

3 Inequality of two Hadamard Products

From Theorem 1.1, $\omega(z)$ and $\omega(1-z)$ are factorized by their zeros ρ_k $k=1, 2, 3, \dots$ respectively as follows:

$$(1.1) \quad \omega(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right)$$

$$(1.2) \quad \omega(1-z) = \prod_{k=1}^{\infty} \left(1 - \frac{1-z}{\rho_k} \right)$$

And, from Theorem 2.1, the following equation holds on the whole complex plane.

$$(2.1) \quad \omega(z) = \omega(1-z)$$

The question here is whether the following equation holds on the whole complex plane.

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right) = \prod_{k=1}^{\infty} \left(1 - \frac{1-z}{\rho_k} \right)$$

Putting $\rho_{2r-1} = x_r - iy_r$, $\rho_{2r} = x_r + iy_r$ $r=1, 2, 3, \dots$,

$$\prod_{r=1}^{\infty} \left(1 - \frac{z}{x_r - iy_r} \right) \left(1 - \frac{z}{x_r + iy_r} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{1-z}{x_r - iy_r} \right) \left(1 - \frac{1-z}{x_r + iy_r} \right)$$

Expanding the () () on both sides,

$$\prod_{r=1}^{\infty} \left(1 - \frac{2x_r z}{x_r^2 + y_r^2} + \frac{z^2}{x_r^2 + y_r^2} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{2x_r - 1}{x_r^2 + y_r^2} - \frac{2(1-x_r)z}{x_r^2 + y_r^2} + \frac{z^2}{x_r^2 + y_r^2} \right)$$

Observing this, both sides are different series generally. That is, this equality does not hold on the whole complex plane generally.

Remark

The above means that the following four-step logic does not hold unconditionally.

$$\prod (1-z/\rho) = \omega(z) \quad , \quad \omega(z) = \omega(1-z) \quad , \quad \omega(1-z) = \prod (1-(1-z)/\rho)$$

↓

$$\prod (1-z/\rho) = \prod (1-(1-z)/\rho)$$

4 Equality Conditions for two Hadamard Products

In this chapter, we will find the necessary and sufficient condition for $\prod (1-z/\rho) = \prod (1-(1-z)/\rho)$, thereby proving that the Riemann hypothesis for the Dirichlet Beta Function $\beta(z)$ holds as a theorem.

Theorem 4.1 (Functional Equation for Hadamard Products)

Let Completed Dirichlet Beta Function $\omega(z)$ and $\omega(1-z)$ are factorized by their zeros ρ_k $k=1, 2, 3, \dots$ as follows.

$$(1.1) \quad \omega(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right)$$

$$(1.2) \quad \omega(1-z) = \prod_{k=1}^{\infty} \left(1 - \frac{1-z}{\rho_k} \right)$$

Then, if and only if $Re(\rho_k) = 1/2$ $k=1, 2, 3, \dots$, The following equation holds on the whole complex plane.

$$(4.1) \quad \prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right) = \prod_{k=1}^{\infty} \left(1 - \frac{1-z}{\rho_k} \right)$$

Proof

Express the zeros ρ_k as follows, with separate real and imaginary parts:

$$\rho_{2r-1} = x_r - iy_r, \quad \rho_{2r} = x_r + iy_r \quad r=1, 2, 3, \dots \quad (y_r > 0)$$

Then (4.1) becomes

$$(4.1') \quad \prod_{r=1}^{\infty} \left(1 - \frac{z}{x_r - iy_r} \right) \left(1 - \frac{z}{x_r + iy_r} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{1-z}{x_r - iy_r} \right) \left(1 - \frac{1-z}{x_r + iy_r} \right)$$

I. Sufficiency

If $x_r = 1/2$ $r=1, 2, 3, \dots$, (4.1') becoms

$$(4.1'') \quad \prod_{r=1}^{\infty} \left(1 - \frac{z}{1/2 - iy_r} \right) \left(1 - \frac{z}{1/2 + iy_r} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{1-z}{1/2 - iy_r} \right) \left(1 - \frac{1-z}{1/2 + iy_r} \right)$$

Let us find the roots of the pair on both sides. then,

$$\text{Left side} \quad 1 - \frac{z}{1/2 - iy_r} = 0, \quad 1 - \frac{z}{1/2 + iy_r} = 0 \quad r=1, 2, 3, \dots$$

$$\text{Right side} \quad 1 - \frac{1-z}{1/2 - iy_r} = 0, \quad 1 - \frac{1-z}{1/2 + iy_r} = 0 \quad r=1, 2, 3, \dots$$

From these,

$$\text{Left side} \quad z = 1/2 - iy_r, \quad z = 1/2 + iy_r \quad r=1, 2, 3, \dots$$

$$\text{Right side} \quad z = 1/2 + iy_r, \quad z = 1/2 - iy_r \quad r=1, 2, 3, \dots$$

The zeros on the left and right sides are crosswise coincident. This means that the left and right sides are the same.

In fact, (4.1'') can be rewritten as follows.

$$(4.1''') \quad \prod_{r=1}^{\infty} \left(1 - \frac{z}{1/4 + y_r^2} + \frac{z^2}{1/4 + y_r^2} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{z}{1/4 + y_r^2} + \frac{z^2}{1/4 + y_r^2} \right)$$

This holds clearly on the whole complex plane. So, (4.1) holds on the whole complex plane.

II. Necessity

(1) When neither x_r nor y_r overlap

Let $\rho_{2r-1} = x_r - iy_r$, $\rho_{2r} = x_r + iy_r$ $r=1, 2, 3, \dots$. Then

$$(0.4') \quad \prod_{r=1}^{\infty} \left(1 - \frac{z}{x_r - iy_r} \right) \left(1 - \frac{z}{x_r + iy_r} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{1-z}{x_r - iy_r} \right) \left(1 - \frac{1-z}{x_r + iy_r} \right)$$

Let us find the roots of the pair on both sides. then,

$$\text{Left side } 1 - \frac{z}{x_r - iy_r} = 0 \quad , \quad 1 - \frac{z}{x_r + iy_r} = 0 \quad r=1, 2, 3, \dots$$

$$\text{Right side } 1 - \frac{1-z}{x_r - iy_r} = 0 \quad , \quad 1 - \frac{1-z}{x_r + iy_r} = 0 \quad r=1, 2, 3, \dots$$

From these,

$$\text{Left side } z = x_r - iy_r \quad , \quad z = x_r + iy_r \quad r=1, 2, 3, \dots$$

$$\text{Right side } z = 1 - x_r + iy_r \quad , \quad z = 1 - x_r - iy_r \quad r=1, 2, 3, \dots$$

Since both sides have to be equal, make equations where the signs and symbols of the imaginary parts match.

Then,

$$x_r - iy_r = 1 - x_r - iy_r \quad r=1, 2, 3, \dots$$

$$x_r + iy_r = 1 - x_r + iy_r \quad r=1, 2, 3, \dots$$

From these,

$$x_r = 1 - x_r \quad r=1, 2, 3, \dots$$

$$\text{i.e. } x_r = 1/2 \quad r=1, 2, 3, \dots$$

(2) When x_r overlap

For x_r to overlap, at least two sets of conjugate zeros are required.

For ease of viewing, swap the order so that x_1 and x_2 overlap but there are no overlap after x_3 .

Substituting x_2 for x_1 , ρ_k $k = 1, \dots, 4$ become:

$$\rho_1 = x_1 - iy_1 \quad , \quad \rho_2 = x_1 + iy_1 \quad , \quad \rho_3 = x_1 - iy_2 \quad , \quad \rho_4 = x_1 + iy_2$$

Then,

$$\text{Left side } \prod_{k=1}^4 \left(1 - \frac{z}{\rho_k} \right) = \left(1 - \frac{z}{x_1 - iy_1} \right) \left(1 - \frac{z}{x_1 + iy_1} \right) \left(1 - \frac{z}{x_1 - iy_2} \right) \left(1 - \frac{z}{x_1 + iy_2} \right)$$

$$\text{Right side } \prod_{k=1}^4 \left(1 - \frac{1-z}{\rho_k} \right) = \left(1 - \frac{1-z}{x_1 - iy_1} \right) \left(1 - \frac{1-z}{x_1 + iy_1} \right) \left(1 - \frac{1-z}{x_1 - iy_2} \right) \left(1 - \frac{1-z}{x_1 + iy_2} \right)$$

Let us find the roots of the pairs on both sides. then,

$$\text{Left side } 1 - \frac{z}{x_1 - iy_1} = 0 \quad , \quad 1 - \frac{z}{x_1 + iy_1} = 0 \quad , \quad 1 - \frac{z}{x_1 - iy_2} = 0 \quad , \quad 1 - \frac{z}{x_1 + iy_2} = 0$$

$$\text{Right side } 1 - \frac{1-z}{x_1 - iy_1} = 0 \quad , \quad 1 - \frac{1-z}{x_1 + iy_1} = 0 \quad , \quad 1 - \frac{1-z}{x_1 - iy_2} = 0 \quad , \quad 1 - \frac{1-z}{x_1 + iy_2} = 0$$

From these,

$$\text{Left side } z = x_1 - iy_1 \quad , \quad z = x_1 + iy_1 \quad , \quad z = x_1 - iy_2 \quad , \quad z = x_1 + iy_2$$

$$\text{Right side } z = 1 - x_1 + iy_1 \quad , \quad z = 1 - x_1 - iy_1 \quad , \quad z = 1 - x_1 + iy_2 \quad , \quad z = 1 - x_1 - iy_2$$

Since both sides of each pair have to be equal, make equations where the signs and symbols of the imaginary parts match. Then,

$$x_1 - iy_1 = 1 - x_1 - iy_1 \quad , \quad x_1 - iy_2 = 1 - x_1 - iy_2$$

$$x_1 + iy_1 = 1 - x_1 + iy_1 \quad , \quad x_1 + iy_2 = 1 - x_1 + iy_2$$

From these,

$$x_1 = 1 - x_1$$

i.e. $x_1 = 1/2$

As the result,

Left side $z = 1/2 - iy_1 \quad , \quad z = 1/2 + iy_1 \quad , \quad z = 1/2 - iy_2 \quad , \quad z = 1/2 + iy_2$

Right side $z = 1/2 + iy_1 \quad , \quad z = 1/2 - iy_1 \quad , \quad z = 1/2 + iy_2 \quad , \quad z = 1/2 - iy_2$

That is, $z = 1/2 \pm iy_1 \quad , \quad z = 1/2 \pm iy_2$ are double roots respectively

(3) When y_r overlap

For y_r to overlap, at least two sets of conjugate zeros are required.

For ease of viewing, swap the order so that y_1 and y_2 overlap but there are no overlap after y_3 .

Substituting y_2 for y_1 , $\rho_k \quad k = 1, \dots, 4$ become:

$$\rho_1 = x_1 - iy_1 \quad , \quad \rho_2 = x_1 + iy_1 \quad , \quad \rho_3 = x_2 - iy_1 \quad , \quad \rho_4 = x_2 + iy_1$$

Then,

Left side $\prod_{k=1}^4 \left(1 - \frac{z}{\rho_k} \right) = \left(1 - \frac{z}{x_1 - iy_1} \right) \left(1 - \frac{z}{x_1 + iy_1} \right) \left(1 - \frac{z}{x_2 - iy_1} \right) \left(1 - \frac{z}{x_2 + iy_1} \right)$

Right side $\prod_{k=1}^4 \left(1 - \frac{1-z}{\rho_k} \right) = \left(1 - \frac{1-z}{x_1 - iy_1} \right) \left(1 - \frac{1-z}{x_1 + iy_1} \right) \left(1 - \frac{1-z}{x_2 - iy_1} \right) \left(1 - \frac{1-z}{x_2 + iy_1} \right)$

If the following calculation gives $x_1 \neq x_2$, then these are probably zeros outside the critical line.

Let us find the roots of the pairs on both sides. then,

Left side $1 - \frac{z}{x_1 - iy_1} = 0 \quad , \quad 1 - \frac{z}{x_1 + iy_1} = 0 \quad , \quad 1 - \frac{z}{x_2 - iy_1} = 0 \quad , \quad 1 - \frac{z}{x_2 + iy_1} = 0$

Right side $1 - \frac{1-z}{x_1 - iy_1} = 0 \quad , \quad 1 - \frac{1-z}{x_1 + iy_1} = 0 \quad , \quad 1 - \frac{1-z}{x_2 - iy_1} = 0 \quad , \quad 1 - \frac{1-z}{x_2 + iy_1} = 0$

From these,

Left side $z = x_1 - iy_1 \quad , \quad z = x_1 + iy_1 \quad , \quad z = x_2 - iy_1 \quad , \quad z = x_2 + iy_1$

Right side $z = 1 - x_1 + iy_1 \quad , \quad z = 1 - x_1 - iy_1 \quad , \quad z = 1 - x_2 + iy_1 \quad , \quad z = 1 - x_2 - iy_1$

Since both sides of each pair have to be equal, make equations where the signs and symbols of the imaginary parts match. Then,

$$x_1 - iy_1 = 1 - x_1 - iy_1 \quad , \quad x_2 - iy_1 = 1 - x_2 - iy_1$$

$$x_1 + iy_1 = 1 - x_1 + iy_1 \quad , \quad x_2 + iy_1 = 1 - x_2 + iy_1$$

From these,

$$x_1 = 1 - x_1 \quad , \quad x_2 = 1 - x_2$$

i.e. $x_1 = x_2 = 1/2$

As the result,

Left side $z = 1/2 - iy_1 \quad , \quad z = 1/2 + iy_1 \quad , \quad z = 1/2 - iy_1 \quad , \quad z = 1/2 + iy_1$

Right side $z = 1/2 + iy_1 \quad , \quad z = 1/2 - iy_1 \quad , \quad z = 1/2 + iy_1 \quad , \quad z = 1/2 - iy_1$

That is, $z = 1/2 \pm iy_1$ are quadruple roots. The possibility of zeros outside the critical line disappears.

(4) When both x_r and y_r overlap

For these to overlap, at least two sets of conjugate zeros are required.

For ease of viewing, swap the order so that there are no overlap after x_3, y_3 .

Substituting x_2, y_2 for x_1, y_1 , ρ_k $k = 1, \dots, 4$ become:

$$\rho_1 = x_1 - iy_1, \rho_2 = x_1 + iy_1, \rho_3 = x_1 - iy_1, \rho_4 = x_1 + iy_1$$

Using a similar way to (3), $x_1 = 1/2$, and $z = 1/2 \pm iy_1$ are quadruple roots.

As the result, if and only if $Re(\rho_k) = 1/2$ $k=1, 2, 3, \dots$, (4.1) holds on the whole complex plane.

Q.E.D.

Another Proof

Express the zeros ρ_k as follows, with separate real and imaginary parts:

$$\rho_{2r-1} = x_r - iy_r, \rho_{2r} = x_r + iy_r \quad r=1, 2, 3, \dots \quad (y_r > 0)$$

Then (4.1) becomes

$$(4.1') \quad \prod_{r=1}^{\infty} \left(1 - \frac{z}{x_r - iy_r} \right) \left(1 - \frac{z}{x_r + iy_r} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{1-z}{x_r - iy_r} \right) \left(1 - \frac{1-z}{x_r + iy_r} \right)$$

I. Sufficiency

If $x_r = 1/2$ $r=1, 2, 3, \dots$, (4.1') becomes

$$(4.1'') \quad \prod_{r=1}^{\infty} \left(1 - \frac{z}{1/2 - iy_r} \right) \left(1 - \frac{z}{1/2 + iy_r} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{1-z}{1/2 - iy_r} \right) \left(1 - \frac{1-z}{1/2 + iy_r} \right)$$

Expanding the () on both sides,

$$(4.1''') \quad \prod_{r=1}^{\infty} \left(1 - \frac{z}{1/4 + y_r^2} + \frac{z^2}{1/4 + y_r^2} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{z}{1/4 + y_r^2} + \frac{z^2}{1/4 + y_r^2} \right)$$

This holds clearly on the whole complex plane. So, (4.1) holds on the whole complex plane.

The important thing here is that these zeros actually exist. (e.g. $1/2 + i 6.0209489 \dots$)

For these existing zeros, the following is completed from Theorem 1.1 and Theorem 2.1 .

$$\omega(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right) = \prod_{k=1}^{\infty} \left(1 - \frac{1-z}{\rho_k} \right) = \omega(1-z)$$

As the result, the following equation holds from Vieta's formula (9.5) .

$$(9.5'') \quad \sum_{r=1}^{\infty} \left(\frac{1}{1/2 - iy_r} + \frac{1}{1/2 + iy_r} \right) = 4 \log \Gamma \left(\frac{3}{4} \right) + \frac{\gamma}{2} + \log 2 - \frac{3 \log \pi}{2} = 0.07778398 \dots$$

II. Necessity

Currently, no nontrivial zeros outside the critical line have been found. But now, we will assume that such zeros exist. It is known that a set of such zeros should consist of the following four.

$$1/2 - \alpha_s \pm i\beta_s, 1/2 + \alpha_s \pm i\beta_s \quad (0 < \alpha_s < 1/2, \beta_s > 0)$$

Then, according to Vieta's formula (9.5), the following equation has to hold.

$$\begin{aligned} \sum_{r=1}^{\infty} \left(\frac{1}{1/2 - iy_r} + \frac{1}{1/2 + iy_r} \right) + \sum_{s=1}^{\infty} \left(\frac{1}{1/2 - \alpha_s - i\beta_s} + \frac{1}{1/2 - \alpha_s + i\beta_s} \right) \\ + \sum_{s=1}^{\infty} \left(\frac{1}{1/2 + \alpha_s - i\beta_s} + \frac{1}{1/2 + \alpha_s + i\beta_s} \right) = 4 \log \Gamma \left(\frac{3}{4} \right) + \frac{\gamma}{2} + \log 2 - \frac{3 \log \pi}{2} \end{aligned}$$

i.e.

$$(9.5'') \quad \sum_{r=1}^{\infty} \left(\frac{1}{1/2 - iy_r} + \frac{1}{1/2 + iy_r} \right) + \sum_{s=1}^{\infty} \left\{ \frac{1 - 2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2} + \frac{1 + 2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2} \right\}$$

$$= 4 \log \Gamma \left(\frac{3}{4} \right) + \frac{\gamma}{2} + \log 2 - \frac{3 \log \pi}{2} = 0.07778398 \dots$$

However, since $0 < 1 - 2\alpha_s < 1 + 2\alpha_s < 2$ for $0 < \alpha_s < 1/2$,

$$\sum_{s=1}^{\infty} \left\{ \frac{1 - 2\alpha_s}{(1/2 - \alpha_s)^2 + \beta_s^2} + \frac{1 + 2\alpha_s}{(1/2 + \alpha_s)^2 + \beta_s^2} \right\} > 0$$

Therefore, (9.5'') contradicts (9.5'). Thus, there should be no zeros outside the critical line within the critical strip.

As the result, (0.4) is holds only if $Re(\rho_k) = 1/2 \quad k=1, 2, 3, \dots$.

Q.E.D.

Theorem 4.2 (Riemann)

When $\rho_k \quad k=1, 2, 3, \dots$ are non-trivial zeros of the Dirichlet Beta Function $\beta(z)$,

$$Re(\rho_k) = 1/2 \quad k=1, 2, 3, \dots$$

Proof

As first, the non-trivial zeros ρ_k of the Dirichlet beta function $\beta(z)$ are equivalent to the zeros of the completed Dirichlet beta function $\omega(z)$.

At these zeros, the functional equation has to hold not only for the functions but also for the Hadamard products.

That is, the followings have to hold on the whole complex plane:

$$\omega(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\rho_k} \right) = \prod_{k=1}^{\infty} \left(1 - \frac{1-z}{\rho_k} \right) = \omega(1-z)$$

Theorem 4.1 proves that the necessary and sufficient condition for this is $Re(\rho_k) = 1/2 \quad k=1, 2, 3, \dots$.

Therefore, the Riemann hypothesis for the Dirichlet Beta Function $\beta(z)$ holds as a theorem.

Q.E.D.

Note

The Dirichlet Beta function $\beta(z)$ is a kind of Dirichlet L -function. As expected from the proof in this paper, it seems that the Riemann Hypothesis can also be proved for Dirichlet L -function.

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Kano Kono
Hiroshima, Japan

Alien's Mathematics